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A new approach to quantum theory of multimode coupled parametric processes

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Abstract

A new approach is proposed to solve the quantum evolution problem for a system with an arbitrary number of coupled optical parametric processes. Our method is based on the canonical transformations which define the evolution of the system in the Heisenberg picture. This theory overcomes the difficulties arising in the Wei–Norman method. The application of the approach developed is illustrated with the example of generation of a three-mode entangled light field.

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1. Introduction

Coupled (or concurrent) optical parametric interactions are of interest as a method of obtaining multipartite entanglement. Coupled nonlinear optical interactions can be implemented, for example, in periodically or aperiodically nonlinear optical crystals located inside or outside the cavity. In such processes, entanglement of modes with different frequencies is produced. It gives the possibility to execute teleportation, storing and information processing in various frequency ranges in the quantum communication network. Coupled interactions in nonlinear optical crystals are conventionally described using the undepleted pump wave approximation that allows us to obtain quite a simple solution for the Bose operators of interacting waves. In order to obtain the state vector of such multimode fields, the Wei–Norman method [1] is a standard tool which is based on transformation of the unitary propagator of a system to a product of exponential operators that results in the need for the solution of a set of nonlinear differential equations. Evolution of the state vector of three-mode entangled states produced by coupled nonlinear interactions is derived in [2–5]. However, for a larger number of modes finding the analytical solution of nonlinear differential equations posess insuperable difficulties.

In this paper, we suggest a comprehensive quantum theory for coupled multimode light fields, generated via coupled parametric interactions. The approach elaborated allows one to overcome the difficulties arising in the nonlinear Wei–Norman method and to find the state vector with an arbitrary number of parametrically interacting modes. In sections 2 and 3, we derive equations for the canonical transformations and show the relationship between them and evolution in the Schrödinger picture. In section 4, we investigate the evolution of the state vector produced by three coupled parametric processes. The summary and conclusions are given in section 5.

2. The interaction Hamiltonian and canonical transformations of Bose operators

Consider the Hamiltonian operator describing the nonlinear optical interactions:

$$H = (\imath\hbar/2)\sum_{i,j=1}^{n}(a_i^\dagger A_{ij}a_j^\dagger - 2\imath a_i^\dagger B_{ij}a_j - a_i A_{ij}^* a_j), \qquad (1)$$

where A_{ij} are the symmetric $n\times n$ matrix elements. Here $*$ denotes complex conjugation, B_{ij} are the Hermitian $n\times n$ matrix elements, $a_i^\dagger$ (a_i) is the creation (annihilation) operators with the standard commutation relationships $[a_i, a_j^\dagger] = \delta_{ij}$, $[a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0$ and δ_{ij} is Kronecker's symbol. The A_{ij} elements are responsible for the parametric down-conversion and the B_{ij} elements for the parametric up-conversion. The case $B_{ij} = 0$ is studied in [6].

The unitary evolution operator of the system (1) is given by

$$U(t) = \mathrm{e}^{-\frac{\imath}{\hbar}Ht}. \tag{2}$$

We derive first the canonical transformation for this system:

$$\begin{aligned}\mathbf{a(t)} &= U^\dagger(t)\mathbf{a}U(t) = \mathbf{\Phi(t)a} + \mathbf{\Psi(t)a}^\dagger,\\ \mathbf{a}^\dagger\mathbf{(t)} &= U^\dagger(t)\mathbf{a}^\dagger U(t) = \mathbf{\Phi^*(t)a}^\dagger + \mathbf{\Psi^*(t)a},\end{aligned} \tag{3}$$

where $\mathbf{a}^\dagger = (a_1^\dagger, \ldots, a_n^\dagger)^{\mathrm{T}}$, $\mathbf{a} = (a_1, \ldots, a_n)^{\mathrm{T}}$ are columns of size n, and $\mathbf{\Phi(t)}$, $\mathbf{\Psi(t)}$ are the $n\times n$ matrices.

Using the property of the commutation relation conservation, we obtain (see [6])

$$\mathbf{\Phi\Phi}^\dagger - \mathbf{\Psi\Psi}^\dagger = \mathbf{\Phi}^*\mathbf{\Phi}^{\mathrm{T}} - \mathbf{\Psi}^*\mathbf{\Psi}^{\mathrm{T}} = \mathbf{I}, \tag{4}$$

$$\mathbf{\Phi\Psi}^{\mathrm{T}} - \mathbf{\Psi\Phi}^{\mathrm{T}} = \mathbf{\Phi}^*\mathbf{\Psi}^\dagger - \mathbf{\Psi}^*\mathbf{\Phi}^\dagger = \mathbf{0}. \tag{5}$$

Let us derive equations for the matrices of canonical transformations. The equations of motion for the creation and annihilation operators take the forms

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}a_i(t) &= \frac{\imath}{\hbar}U^\dagger(t)[H, a_i]U(t) = -\imath B_{ij}a_j(t) + A_{ij}a_j^\dagger(t)\\ &= (A_{ij}\Psi^*_{jm} - \imath B_{ij}\Phi_{jm})a_m + (A_{ij}\Phi^*_{jm} - \imath B_{ij}\Psi_{jm})a_m^\dagger,\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}a_i^\dagger(t) &= \frac{\imath}{\hbar}U^\dagger(t)[H, a_i^\dagger]U(t) = \imath B_{ij}^{\mathrm{T}}a_j^\dagger(t) + A_{ij}^\dagger a_j(t)\\ &= (A_{ij}^\dagger\Phi_{jm} + \imath B_{ij}^{\mathrm{T}}\Psi^*_{jm})a_m + (A_{ij}^\dagger\Psi_{jm} + \imath B_{ij}^{\mathrm{T}}\Phi^*_{jm})a_m^\dagger,\end{aligned}$$

whereas by differentiating the expression (3) we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}a_i(t) &\equiv \dot{a}_i(t) = (\dot{\Phi})_{ij}a_j + (\dot{\Psi})_{ij}a_j^\dagger,\\ \frac{\mathrm{d}}{\mathrm{d}t}a_i^\dagger(t) &\equiv \dot{a}_i^\dagger(t) = (\dot{\Psi}^*)_{ij}a_j + (\dot{\Phi}^*)_{ij}a_j^\dagger.\end{aligned}$$

Consequently, the evolution equations for the matrices $\mathbf{\Phi}$, $\mathbf{\Psi}$ are

$$\begin{aligned}\dot{\mathbf{\Phi}} = -\imath\mathbf{B\Phi} + \mathbf{A\Psi}^*, \quad \dot{\Psi}^* = \imath\mathbf{B}^{\mathrm{T}}\mathbf{\Psi}^* + \mathbf{A}^\dagger\mathbf{\Phi},\\ \mathbf{\Phi}(0) = \mathbf{I}, \quad \mathbf{\Psi}^*(0) = \mathbf{0},\end{aligned} \tag{6}$$

with the initial conditions at $t = 0$.

The solution of equation (6) allows one to describe the evolution of a multimode system which obeys equation (3) in the Heisenberg picture. However, the analysis of a number of characteristics of the quantum system is more convenient in the Schrödinger picture associated with the evolution of the system state vector.

3. The ordered formula for a unitary operator

To study the quantum system in the Schrödinger picture, it is convenient to represent the unitary evolution operator of the quantum system in the normally ordered form

$$\begin{aligned}U(t) = \exp(-\tfrac{1}{2}a_i^\dagger Q_{ij}(t)a_j^\dagger)\exp(a_i^\dagger P_{ij}(t)a_j)\\ \times\exp(\tfrac{1}{2}a_i R_{ij}(t)a_j + s(t)),\end{aligned} \tag{7}$$

where Q_{ij}, R_{ij} and P_{ij} are the elements of $n\times n$ matrices, $\mathbf{Q}$, $\mathbf{R}$ are the symmetric matrices and $s(t)$ is a scalar function of t and plays the role of a normalizing constant. The summation is assumed over indices which occur twice.

Using the Baker–Hausdorff formula, we obtain the following relationships:

$$\begin{aligned}a_k(t) = U^\dagger(t)a_kU(t) = (\mathrm{e}^P)_{kl}a_l - Q_{km}(\mathrm{e}^{-P^{\mathrm{T}}})_{ml}a_l^\dagger\\ + Q_{km}(\mathrm{e}^{-P^{\mathrm{T}}})_{mj}R_{jl}a_l,\end{aligned} \tag{8}$$

$$a_k^\dagger(t) = U^\dagger(t)a_k^\dagger U(t) = (\mathrm{e}^{-P^{\mathrm{T}}})_{kl}a_l^\dagger - (\mathrm{e}^{-P^{\mathrm{T}}})_{kj}R_{jl}a_l.$$

Comparing (8) with (3), we bring out the relationship between the matrices $\mathbf{\Phi}$, $\mathbf{\Psi}$ and the matrices $\mathbf{Q}$, $\mathbf{P}$, $\mathbf{R}$:

$$\begin{aligned}\mathbf{\Phi}^* = \mathrm{e}^{-P^{\mathrm{T}}}, \quad \mathbf{\Psi}^* = -\mathrm{e}^{-P^{\mathrm{T}}}\mathbf{R},\\ \mathbf{\Phi} = \mathrm{e}^P + \mathbf{Q}\mathrm{e}^{-P^{\mathrm{T}}}\mathbf{R}, \quad \mathbf{\Psi} = -\mathbf{Q}\mathrm{e}^{-P^{\mathrm{T}}}.\end{aligned}$$

From here we obtain

$$\begin{aligned}\mathbf{Q(t)} &= -\mathbf{\Psi}(t)(\mathbf{\Phi}^*(t))^{-1},\\ \mathbf{P(t)} &= -\ln(\mathbf{\Phi}^\dagger(t)),\\ \mathbf{R}(t) &= -(\mathbf{\Phi}^*(t))^{-1}\mathbf{\Psi}^*(t).\end{aligned} \tag{9}$$

Therefore, the expansion of the evolution operator (7) is unambiguously defined by the matrices of canonical transformation.

The equations for the matrices $\mathbf{Q}$, $\mathbf{P}$, $\mathbf{R}$ can be directly related to the matrices $\mathbf{A}$, $\mathbf{B}$. For this purpose, we consider the derivative of the operator $\mathrm{e}^{G(t)}$ for the case $[G(t), \dot{G}(t)] \neq 0$. In this case Feynman [7] has applied the formula which connects $\dot{G}(t)$ and the left derivative of the operator exponent $\dot{G}^{\mathrm{L}}(t)$:

$$\begin{aligned}\dot{G}^{\mathrm{L}}(t) &\overset{\mathrm{def}}{=} \left(\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{G(t)}\right)\mathrm{e}^{-G(t)}\\ &= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_0^1 \mathrm{d}s\,\frac{\mathrm{d}}{\mathrm{d}s}\,\mathrm{e}^{s(G(t)+\Delta t\dot{G}(t))}\mathrm{e}^{-sG(t)}\\ &= \int_0^1 \mathrm{d}s\,\mathrm{e}^{sG(t)}\dot{G}(t)\mathrm{e}^{-sG(t)}.\end{aligned}$$

We use this formula for the left derivative of $\mathrm{e}^{a_i^\dagger P_{ij}(t)a_j}$ because in the general case $[\mathbf{P(t)}, \dot{\mathbf{P}}\mathbf{(t)}] \neq \mathbf{0}$. Therefore, we have

$$a_i^\dagger\dot{P}_{ij}^{\mathrm{L}}(\tau)a_j = a_i^\dagger\Big(\int_0^1 \mathrm{d}\sigma\,\exp(\sigma\mathbf{P}(\tau))\dot{\mathbf{P}}\exp(-\sigma\mathbf{P}(\tau))\Big)_{ij}a_j.$$

To find equations for the matrices $\mathbf{Q}$, $\mathbf{P}$, $\mathbf{R}$ and the function $s(t)$, we apply the Baker–Hausdorff formula:

$$\begin{aligned}&\exp(-\tfrac{1}{2}a_i^\dagger Q_{ij}a_j^\dagger)a_k^\dagger\dot{P}_{kl}^{\mathrm{L}}a_l\exp(\tfrac{1}{2}a_i^\dagger Q_{ij}a_j^\dagger)\\ &\quad= a_k^\dagger\dot{P}_{kl}^{\mathrm{L}}a_l + \tfrac{1}{2}(a_k^\dagger(\dot{P}^{\mathrm{L}})_{km}Q_{ml}a_l^\dagger + a_l^\dagger Q_{lm}(\dot{P}^{\mathrm{L}})^{\mathrm{T}}_{mk}a_k^\dagger),\\ &\quad\exp(a_i^\dagger P_{ij}a_j)a_k\dot{R}_{kl}a_l\exp(-a_i^\dagger P_{ij}a_j) = a_i(\mathrm{e}^{-P^{\mathrm{T}}}\dot{R}\,\mathrm{e}^{-P})_{ij}a_j,\\ &\quad\exp(-\tfrac{1}{2}a_k^\dagger Q_{kl}a_l^\dagger)a_i(\mathrm{e}^{-P^{\mathrm{T}}}\dot{R}\,\mathrm{e}^{-P})_{ij}a_j\exp(\tfrac{1}{2}a_k^\dagger Q_{kl}(t)a_l^\dagger)\\ &\quad= a_i(\mathrm{e}^{-P^{\mathrm{T}}}\dot{R}\,\mathrm{e}^{-P})_{ij}a_j + a_m^\dagger(Q\mathrm{e}^{-P^{\mathrm{T}}}\dot{R}\,\mathrm{e}^{-P}Q)_{mn}a_n^\dagger\\ &\quad+ 2a_m^\dagger(Q\mathrm{e}^{-P^{\mathrm{T}}}\dot{R}\,\mathrm{e}^{-P})_{mn}a_n + \mathrm{Tr}(\mathrm{e}^{-P^{\mathrm{T}}}\dot{R}\,\mathrm{e}^{-P}Q).\end{aligned} \tag{10}$$

Then, differentiating the evolution equation in the forms (2) and (7) and taking into account (10), we collect the coefficients at $a_i^\dagger a_j^\dagger$, $a_i a_j$, $a_i^\dagger a_j$. As a result the following equations are obtained:

$$\begin{aligned}
&\mathbf{A}+\dot{\mathbf{Q}}-\dot{\mathbf{P}}^{\mathrm{L}}\mathbf{Q}-\mathbf{Q}(\dot{\mathbf{P}}^{\mathrm{L}})^{\mathrm{T}}-\mathbf{Q}(\mathrm{e}^{-\mathbf{P}^{\mathrm{T}}}\dot{\mathbf{R}}\mathrm{e}^{-\mathbf{P}})\mathbf{Q}=\mathbf{0},\\
&\mathrm{e}^{-\mathbf{P}^{\mathrm{T}}}\dot{\mathbf{R}}\mathrm{e}^{-\mathbf{P}}+\mathbf{A}^\dagger=\mathbf{0},\\
&\imath\mathbf{B}+\dot{\mathbf{P}}^{\mathrm{L}}+\mathbf{Q}\mathrm{e}^{-\mathbf{P}^{\mathrm{T}}}\dot{\mathbf{R}}\mathrm{e}^{-\mathbf{P}}=0,\\
&\dot{s}+\tfrac{1}{2}\mathbf{Tr}(\mathrm{e}^{-\mathbf{P}^{\mathrm{T}}}\dot{\mathbf{R}}\mathrm{e}^{-\mathbf{P}}\mathbf{Q})=\mathbf{0},
\end{aligned}$$

which can be reduced to the equivalent form

$$\begin{aligned}
&\mathbf{A}+\dot{\mathbf{Q}}+\imath\mathbf{B}\mathbf{Q}+\imath\mathbf{Q}\mathbf{B}^{\mathrm{T}}-\mathbf{Q}\mathbf{A}^\dagger\mathbf{Q}=\mathbf{0},\\
&\mathrm{e}^{-\mathbf{P}^{\mathrm{T}}}\dot{\mathbf{R}}\mathrm{e}^{-\mathbf{P}}+\mathbf{A}^\dagger=\mathbf{0},\\
&\imath\mathbf{B}+\dot{\mathbf{P}}^{\mathrm{L}}-\mathbf{Q}\mathbf{A}^\dagger=\mathbf{0},\\
&\dot{s}+\tfrac{1}{2}\mathbf{Tr}(-\mathbf{A}^\dagger\mathbf{Q})=\mathbf{0}.
\end{aligned}\tag{11}$$

Taking into account the relationship $\mathrm{Tr}(\int_0^1 \mathrm{d}s\, \mathrm{e}^{s\mathbf{P}}\dot{\mathbf{P}}\mathrm{e}^{-s\mathbf{P}})=\mathrm{Tr}(\dot{\mathbf{P}})$, we rewrite the last equation of the system (11) as a simple ordinary differential equation:

$$\dot{s}-\tfrac{1}{2}\mathbf{Tr}(\imath\mathbf{B})-\tfrac{1}{2}\mathbf{Tr}(\dot{\mathbf{P}})=\mathbf{0}.\tag{12}$$

The initial conditions for the system (11) are the following:

$$\mathbf{Q(0)}=\mathbf{0},\quad \mathbf{P(0)}=\mathbf{0},\quad \mathbf{R(0)}=\mathbf{0},\quad s(0)=0,$$

and equation (12) implies that

$$s=\tfrac{1}{2}\mathbf{t}\mathrm{Tr}(\imath\mathbf{B})+\tfrac{1}{2}\mathrm{Tr}(\mathbf{P}).\tag{13}$$

Let us show that expressions for $\mathbf{Q},\mathbf{P},\mathbf{R}$ (9) satisfy equation (11):

$$\begin{aligned}
\dot{\mathbf{Q}}&=-\frac{\mathbf{d}}{\mathbf{d}t}(\mathbf{\Psi}(\mathbf{\Phi}^*)^{-1})=-\dot{\mathbf{\Psi}}(\mathbf{\Phi}^*)^{-1}+\mathbf{\Psi}(\mathbf{\Phi}^*)^{-1}\dot{\mathbf{\Phi}}^*(\mathbf{\Phi}^*)^{-1}\\
&=-\mathbf{A}-\mathbf{iBQ}-\imath\mathbf{Q}\mathbf{B}^{\mathrm{T}}+\mathbf{Q}\mathbf{A}^\dagger\mathbf{Q},
\end{aligned}$$

$$\dot{\mathbf{P}}^{\mathrm{L}}=\left(\frac{\mathbf{d}}{\mathbf{dt}}(\mathbf{\Phi}^\dagger)^{-\mathbf{1}}\right)\mathbf{\Phi}^\dagger=-(\mathbf{\Phi}^\dagger)^{-\mathbf{1}}\dot{\mathbf{\Phi}}^\dagger=-\imath\mathbf{B}^{\mathrm{T}}+\mathbf{Q}\mathbf{A}^\dagger,$$

$$\begin{aligned}
\mathrm{e}^{-\mathbf{P}^{\mathrm{T}}}\dot{\mathbf{R}}\mathrm{e}^{-\mathbf{P}}&=\mathbf{\Phi}^*\dot{\mathbf{R}}\mathbf{\Phi}^\dagger=\mathbf{\Phi}^*\frac{\mathbf{d}}{\mathbf{dt}}((\mathbf{\Phi}^*)^{-\mathbf{1}}\mathbf{\Psi}^*)\mathbf{\Phi}^\dagger\\
&=\mathbf{\Phi}^*(\mathbf{\Phi}^*)^{-\mathbf{1}}\dot{\mathbf{\Phi}}^*(\mathbf{\Phi}^*)^{-\mathbf{1}}\mathbf{\Psi}^*\mathbf{\Phi}^\dagger-\mathbf{\Phi}^*(\mathbf{\Phi}^*)^{-\mathbf{1}}\dot{\mathbf{\Psi}}^*\mathbf{\Phi}^\dagger\\
&=-\mathbf{A}^\dagger(\mathbf{\Phi}\mathbf{\Phi}^\dagger-\mathbf{\Psi}(\mathbf{\Phi}^*)^{-\mathbf{1}}\mathbf{\Psi}^*\mathbf{\Phi}^\dagger)=-\mathbf{A}^\dagger.
\end{aligned}$$

The last equality readily follows from (4). Therefore, the unitary operator $U(t)$ can be represented in the form (7). Formulae (9) and (13) allow us to find the unitary operator of the system (1) in the normally ordered form via the canonical transformation matrices.

In the next section, we show that this approach is rather convenient for the analysis of multimode parametric processes.

4. Three coupled parametric processes, three-mode entangled states

Let us demonstrate the application of our approach to the case of generation of three-mode entangled states. Consider the following coupled nonlinear optical processes of the kind [4]:

$$\begin{aligned}
&\omega_{\mathrm{p}}=\omega_1+\omega_2,\\
&\omega_{2\mathrm{p}}=2\omega_{\mathrm{p}}=\omega_2+\omega_3,\\
&\omega_1+\omega_{\mathrm{p}}=\omega_3,
\end{aligned}\tag{14}$$

where ω_{p}, $\omega_{2\mathrm{p}}$ are the pump frequencies and $\omega_1,\omega_2,\omega_3$, are the generated frequencies.

In (14), the first and third processes are nondegenerate parametric down-conversions and the second one is the parametric up-conversion.

The interaction Hamiltonian of the system under study has the following form:

$$\begin{aligned}
H_{\mathrm{int}}=\imath\hbar[&\beta_1(a_1^\dagger a_2^\dagger-a_1a_2)+\beta_2(a_2^\dagger a_3^\dagger-a_2a_3)\\
&+\gamma(a_1a_3^\dagger-a_1^\dagger a_3)],
\end{aligned}\tag{15}$$

where $a_j^\dagger$, a_j $(j=1,2,3)$ are the creation and annihilation operators of the photons with frequency ω_j. β_j is the nonlinear coupling coefficient responsible for parametric down-conversion, and γ_j is responsible for parametric up-conversion.

Note that this Hamiltonian (15) is of the same type as equation (1). Nonzero elements of the matrices **A** and **B** are $A_{12}=A_{21}=\beta_1$, $A_{23}=A_{32}=\beta_2$, $B_{13}=-B_{31}=-\imath\gamma$.

In this case the solution of the system (6) for the canonical matrices is given by the matrices

$$\mathbf{\Phi(t)}=\begin{pmatrix}M_{11}&0&M_{13}\\0&M_{22}&0\\M_{31}&0&M_{33}\end{pmatrix},\quad \mathbf{\Psi(t)}=\begin{pmatrix}0&M_{12}&0\\M_{21}&0&M_{23}\\0&M_{32}&0\end{pmatrix},\tag{16}$$

where the elements M_{ij} are functions of t

$$\begin{aligned}
&M_{11}(t)=\frac{(\beta_1^2-\gamma^2)C+\beta_2^2}{\Gamma^2},\qquad M_{12}(t)=\frac{\beta_2\gamma(1-C)}{\Gamma^2}-\frac{\beta_1S}{\Gamma},\\
&M_{13}(t)=\frac{\beta_1\beta_2(C-1)}{\Gamma^2}+\frac{\gamma S}{\Gamma},\quad M_{21}(t)=\frac{\beta_2\gamma(C-1)}{\Gamma^2}-\frac{\beta_1S}{\Gamma},\\
&M_{22}(t)=\frac{(\beta_1^2+\beta_2^2)C-\gamma^2}{\Gamma^2},\qquad M_{23}(t)=\frac{\beta_1\gamma(1-C)}{\Gamma^2}-\frac{\beta_2S}{\Gamma},\\
&M_{31}(t)=\frac{\beta_1\beta_2(C-1)}{\Gamma^2}-\frac{\gamma S}{\Gamma},\quad M_{32}(t)=\frac{\beta_1\gamma(C-1)}{\Gamma^2}-\frac{\beta_2S}{\Gamma},\\
&M_{33}(t)=\frac{(\beta_2^2-\gamma^2)C+\beta_1^2}{\Gamma^2},
\end{aligned}\tag{17}$$

where $\Gamma=\sqrt{\beta_1^2+\beta_2^2-\gamma^2}$, $C=\cosh(\Gamma t)$, $S=\sinh(\Gamma t)$. Equations (17) imply the following properties of M_{ij}:

$$\begin{aligned}
&M_{11}M_{33}-M_{13}M_{31}=M_{22},\quad M_{21}M_{33}-M_{23}M_{31}=M_{12},\\
&M_{11}M_{23}-M_{13}M_{21}=M_{32}.
\end{aligned}\tag{18}$$

Three useful identities, which we use further, follow from (17):

$$\cosh^2\delta_1(t)=\frac{M_{22}^2}{1+M_{32}^2},$$
$$\sinh^2\delta_1(t)=\frac{M_{12}^2}{1+M_{32}^2}, \qquad (19)$$
$$\sinh^2\delta_2(t)=M_{32}^2.$$

According to (3), the evolution of the Bose operators of the system under consideration is described by the equations

$$a_1(t)=M_{11}a_1+M_{12}a_2^\dagger+M_{13}a_3,$$
$$a_2(t)=M_{21}a_1^\dagger+M_{22}a_2+M_{23}a_3^\dagger, \qquad (20)$$
$$a_3(t)=M_{31}a_1+M_{32}a_2^\dagger+M_{33}a_3.$$

Now, let us find the evolution of the state $|\psi(t)\rangle$. Suppose that the initial state is vacuum $|\psi(0)\rangle=|0\rangle_1|0\rangle_2|0\rangle_3$. Then, using (7) we can obtain

$$|\psi(t)\rangle=U(t)|\psi(0)\rangle=\mathrm{e}^{a_i^\dagger(\Psi(t)(\Phi^*(t))^{-1})_{ij}a_j^\dagger+s(t)}|0\rangle_1|0\rangle_2|0\rangle_3,$$

since the last two factors in (7) act as $\mathrm{e}^{s(t)}$ in a vacuum state, where $\mathrm{e}^{s(t)}=M_{22}$ is derived from (13) by using the first equation of (18).

From the properties of M_{ij} (18), we have

$$a_i^\dagger(\Psi(t)(\Phi^*(t))^{-1})_{ij}a_j^\dagger=\frac{M_{12}}{M_{22}}a_1^\dagger a_2^\dagger+\frac{M_{32}}{M_{22}}a_2^\dagger a_3^\dagger.$$

Due to this relationship, the state vector can be transformed to the form

$$|\psi(t)\rangle=M_{22}\sum_{m,n}\left(\frac{M_{12}}{M_{22}}\right)^m\left(\frac{M_{32}}{M_{22}}\right)^n\sqrt{C_{m+n}^m}|m\rangle_1|m+n\rangle_2|n\rangle_3,$$

where $C_l^k=\frac{l!}{k!(l-k)!}$.

Using formula (19), we obtain the explicit form of the system state:

$$|\psi(t)\rangle=\frac{1}{\sqrt{\cosh^2\delta_1\cosh^2\delta_2}}\sum_{m,n}(\tanh\delta_1)^m \times\left(-\frac{\tanh\delta_2}{\cosh\delta_1}\right)^n\sqrt{C_{m+n}^m}|m\rangle_1|m+n\rangle_2|n\rangle_3. \qquad (21)$$

This result agrees with that of [4] obtained by the Wei–Norman method, but in our approach only the system of linear differential equations (6) is solved, while in the Wei–Norman method we must solve coupled nonlinear differential equations.

5. Conclusion

The method presented in this paper allows us to obtain an explicit expression for the evolution of the system state vector and therefore to find the density operator for an arbitrary number of coupled parametric processes. Let us note once again that this approach does not require one to solve a set of coupled nonlinear differential equations as occurs in the Wei–Norman method (see [2, 4], where the problems for two coupled parametric processes were solved). However, this set of nonlinear differential equations is not solved analytically when generation of the multimode entangled states, states with the number of modes more than three, is analyzed. Among four-mode processes, of particular interest is finding a state vector for the process of parametric amplification at low-frequency pumping [8] in which entanglement on frequencies both below and above the pumping frequency is formed. The approach presented in our paper does not face such a difficulty even for a large number of coupled parametric processes.

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