On families of lagrangian tori on hyperkähler manifolds

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1 Introduction

Recall that an irreducible holomorphic symplectic manifold is a simply-connected compact Kähler manifold X such that the space of holomorphic 2-forms $H^{2,0}(X)$ is generated by a symplectic, that is, nowhere degenerate, form σ. It is well-known that such manifolds are exactly the compact hyperkähler manifolds from differential geometry. There are two series of known examples, namely the Hilbert scheme $Hilb^{[n]}(X)$ of length n subschemes of a $K3$ surface X and the Kummer variety associated to an abelian surface A (recall that this is defined as the fiber over zero of the summation map $\Sigma : Hilb^{[n]}(A) \to A$), and two sporadic examples in dimensions 6 and 10 due to O'Grady. All known irreducible holomorphic symplectic manifolds are deformations of those.

In order to study the classification problem for irreducible holomorphic symplectic manifolds and their geometry, it is important to understand how such an X can fiber over lower-dimensional varieties. Several results are known in this direction. The first one, very striking, has been obtained by D. Matsushita:

Theorem 1 *([M1],[M2]) Let X be an irreducible holomorphic symplectic variety and let $f : X \to B$ be a map with connected fibers from X onto a normal complex space B, $0 < dim(B) < n$. Then all fibers of f are lagrangian (in particular, they are of dimension n), and the smooth fibers are tori*[1].

Strong restrictions on B are known. First, the results of Varouchas ([V1],[V2]) imply the existence of a smooth Kähler model B' of B, and B' does not carry any nonzero holomorphic 2-form (since its lift to X would not be proportional to σ), so Kodaira's embedding theorem shows that B' is projective.

[1]More precisely, this is proved for X and B projective in [M1] and it is remarked in [M2] that the arguments work in the Kähler case (that is, with X an arbitrary irreducible holomorphic symplectic manifold and B a normal Kähler space). In general, by [V2] B is birational to a Kähler manifold B', and it follows from the Kodaira embedding theorem that B' is projective. Replacing $f : X \to B$ by $f' : X \dashrightarrow B'$ and considering an ample H' on B', one shows (as in [AC], Prop. 3.1) that $(f')^*H'$ is isotropic with respect to the Beauville-Bogomolov form. One then proves as in [M1] that a general fiber of f', and therefore of f, is lagrangian. By [M1bis], f is equidimensional, and therefore B is normal Kähler by [V1].

Much more is true. Matsushita proved that the base B of such a fibration is "very similar" to $\mathbb{P}^n$. Later, Hwang has shown in [H] that if B is smooth, then indeed $B \cong \mathbb{P}^n$ (the general case is open).

In [AC] we have partially extended this to the meromorphic setting. Since there are always meromorphic fibrations by complete intersections of hypersurfaces if X is projective, we assumed that the general fiber of $f : X \dashrightarrow B$ is not of general type; in this case, we have proved, among other things, that $dim(B) \leq n$ (proposition 3.4 of [AC]).

When X is non-projective, our assumption on the fibers is automatically satisfied since the total space of a fibration with Moishezon base and general type fibers is Moishezon, see [U], 2.10.

In fact the general fiber of f is not of general type if and only if f is defined by sections of a line bundle with zero Beauville-Bogomolov square ([AC], Proposition 3.1). This is related to a famous and difficult conjecture about lagrangian fibrations:

Conjecture 2 *("Lagrangian conjecture"): Let L be a nontrivial nef line bundle on X, such that $q(L) = 0$. Then some power of L is base-point-free, that is, the sections of some power of L define a holomorphic lagrangian fibration.*

Under the additional assumption that X is covered by curves C such that $LC = 0$, this has been proved by Matsushita [M3]. The main idea of the proof is to use the *nef reduction* from [BCE...], which in this case yields a non-trivial meromorphic fibration.

Recently, Beauville has asked the following question:

Question 3 *Let X be an irreducible holomorphic symplectic variety and $A \subset X$ a lagrangian torus. Is it true that A is a fiber of a (meromorphic) lagrangian fibration?*

This is plausible since it is known that the deformations of A in X are unobstructed and that the smooth ones are again lagrangian tori. The symplectic form defines an isomorphism between the normal bundle of a lagrangian torus and its cotangent bundle, so this bundle is trivial. Hence, locally in a neighbourhood of A, there is a lagrangian fibration, and the question is whether it globalizes. Another reformulation is as follows: one knows that through a general point of X passes a finite number d of deformations of A, and one wants to know whether $d = 1$.

When X is non-projective, the affirmative answer has been given by Greb, Lehn and Rollenske in [GLR]. Moreover, they have proved that if the pair (X, A) can be deformed to a pair (X', A') where X' is non-projective and A' is still a lagrangian torus on X', then the answer is also positive (because the lagrangian fibration on X' deforms back to X). Finally, they have observed that by deformation theory of hyperkähler manifolds, the existence of a non-projective deformation of the pair (X, A) is equivalent to the existence of an effective divisor D on X such that its restriction to A is zero.

In [A], the first author has given a very simple solution of Beauville's problem in dimension 4, by combining a lemma from linear algebra with the above-mentioned proposition from [AC].

In [HW], the existence of an effective divisor D restricting trivially to A is established, so that the affirmative answer to question 3 is obtained in full generality. The argument, which proceeds by the study of the monodromy action on the total space of the family of deformations of A, uses some highly non-trivial finite group theory.

This solution of Beauville's question, as well as several other important proofs in holomorphic symplectic geometry, depends in an essential way on deforming from projective to non-projective data.

The purpose of this note is to give another, very simple, proof in the non-projective case (the original one from [GLR] uses results about algebraic reduction of hyperkähler manifolds from [COP] and involves some case-by-case analysis) and to explain a possible algebro-geometric approach to Beauville's question (not relying on deformations or on group theory). We aim to explicitely construct a linear system which, possibly after some flops, would give us a Lagrangian fibration $f : X \to B$ with generic fiber meeting the generic member or our Lagrangian family A along a positive-dimensional subvariety. Then hopefully one can show that this implies that A is a fiber of f [2].

Unfortunately our algebro-geometric argument does not, at the moment, give a general answer: we can produce a lagrangian fibration in a special case (corollary 8), and only a weaker statement (theorem 7) is obtained in general.

2 The non-projective case

Let us start with the following observation, which is an immediate consequence of the argument in [C2], Prop. 2.1.

Proposition 4 *A smooth lagrangian subvariety of an irreducible holomorphic symplectic manifold is projective.*

By [C1], there is an almost holomorphic fibration ϕ_S associated to any compact and covering family Z_s, $s \in S$, of subvarieties in X: the fiber of ϕ_S through a sufficiently general point $x \in X$ consists of all points y such that there is a chain of subvarieties from S joining x and y. Moreover, again by [C1], if Z_s are projective varieties, then so are the fibers of ϕ_S.

Throughout the paper, we shall also need the following simple lemma, which may be seen as an analogue of our problem for tori:

[2] For instance, let $f : X \to B$ and $g : X \to C$ be two Lagrangian fibrations. Then either $f = g$, or the generic fibers of f and g have finite intersections. This is seen by using the fact that otherwise the Beauville-Bogomolov form vanishes on the subspace generated by divisors coming from B and C.

Lemma 5 *Let Z be an irreducible subvariety of a complex torus T. Assume that an open neighborhood U of Z admits a proper holomorphic fibration $g_U : U \to V$ having Z as one of its fibers. Then Z is a complex subtorus of T, so that the local fibration yields a global fibration $g : T \to B$.*

Proof: The generic fiber of g_U is a smooth connected complex submanifold Z' of T. By adjunction, its canonical bundle is trivial. By Ueno's theorem ([U], theorem 10.3), Z' is a complex torus, thus so is Z. Indeed, Z is a translate of Z', since the translates of Z' form a connected component of the Barlet-Chow space of T.

Theorem 6 *Let X be a non-algebraic irreducible holomorphic symplectic manifold and A a lagrangian torus on X. Then X admits an almost holomorphic lagrangian fibration such that A is a fiber.*

Proof: Consider the family of deformations $A_t,\ t \in T$ of A and the fibration ϕ_T associated to A_t (note that T is compact since X is compact Kähler). We claim that its relative dimension is n, that is, A_t is a fiber of ϕ_T for general t. Indeed, the relative dimension is obviously at least n. If it is $2n$, that is, ϕ_T is a constant map, then X, being a fiber of ϕ_T, is projective because the A_t are projective. If it is strictly between n and $2n$, consider, for a general x, all tori $A_1, \dots, A_d$ passing through x. By assumption, $d \geq 2$. The tori $A_1, \dots, A_d$ are contained in the fiber F_x of ϕ_T through x, in particular, the dimension of the subspace $V_x \subset T_{X,x}$ generated by $T_{A_1,x}, \dots, T_{A_d,x}$ is strictly less than $2n$.

From the fact that in a neighbourhood of a general torus, our family is a fibration, we easily deduce that if $x \in X$ is general, the tori $A_1, \dots, A_d$ are not tangent to each other at x, meaning that their intersection has only one component Z_x through x and $n \geq dim(Z_x) = dim T_{Z_x,x} = dim \cap_i T_{A_i,x}$. But $\cap_i T_{A_i,x}$ is exactly the σ-orthogonal to V_x and thus it is strictly positive-dimensional, meaning that so is the component of the intersection of A_i through x. One thus obtains a meromorphic fibration of X by such components; call them E_x. If one knows that E_x is not of general type, then by Proposition 3.4 of [AC] one concludes that $dim(E_x) = n$. By definition of E_x, this means that $d = 1$ and the family A_t fibers X, q.e.d..

The fact that E_x is not of general type is easily deduced by induction on d: indeed we know that the family A_t gives a local fibration near its general member, so the same must be true for intersections $A_t \cap A_s$ for fixed general A_s and varying general A_t intersecting A_s. But by lemma 5, a torus can only be locally fibered in subtori. If x is general, then so are A_1 and A_2, so the component of $A_1 \cap A_2$ through x is a torus. Continuing in this way, we conclude that also $A_1 \cap \dots \cap A_d$ is a torus. This finishes the proof.

3 The projective case: some results

Consider the family of lagrangian tori $A_t, t \in T$, which are deformations of a certain lagrangian torus $A \subset X$. Recall that these deformations cover X, and that there exists a number d such that exactly d members of this family pass through a generic point of X. Assuming that this does not yield a meromorphic fibration, that is, $d > 1$, we are going to construct two large families of subvarieties of X of complementary dimensions e and $2n - e$, such that the intersection number of the corresponding cycles in the cohomology is zero.

Recall that the intersection $A_s \cap A_t$ has no zero-dimensional components (this follows from the fact that our family is a local fibration, see [A], lemma 1). For fixed t, the tori intersecting A_t form a finite number of irreducible families $S_1, \dots, S_N \subset T$. Fix one of them and call it S: for $s \in S$ general, $A_s \cap A_t$ is an equidimensional ([A], lemma 1) union of disjoint subtori by lemma 5. We shall see very soon that the all these subtori are translates of each other in A_t.

Set $e = e_S$ the dimension of $A_s \cap A_t$ for general $s \in S$. Denote by $E_{s,t}$ a component of $A_s \cap A_t$, and by $Z_{s,t}$ the union of A_s, $s \in S$, which is thus of dimension $2n - e$.

Our main result is the following theorem.

Theorem 7 $[E_{s,t}] \cdot [Z_{s,t}] = 0$ *in the cohomologies of* X.

Before proving the theorem, let us state a corollary which partially answers Beauville's question in the case $e = 1$ (i.e. in the case when there exists an S as above such that $e_S = 1$). Notice that the "opposite" case when $e = n - 1$ for all S has a completely elementary treatment ([A], remark 4). Unfortunately we did not succeed in proving an analogue of this corollary for arbitrary e.

Corollary 8 *Suppose* $e = 1$. *Then for some* $m \geq 0$, *the linear system* $|mZ| = |mZ_{s,t}|$ *gives an almost holomorphic map* $\phi : X \dashrightarrow B$, *which has a holomorphic model* $\phi' : X' \to B'$ *(and therefore is a lagrangian fibration).*

Proof: This would follow at once from theorem 7 and the main theorem of [M3] if the divisor Z had been nef: indeed X is covered by curves $C = E_{s,t}$ with $CZ = 0$. In general, one cannot affirm that Z is nef. But it is mobile, that is, has no base components. Such a divisor on a hyperkähler manifold can be made nef by a sequence of flops (see [M4]). That is, there is a birational transformation $h : X \dashrightarrow X'$ which is an isomorphism in codimension one, and such that h_*Z is nef.

To see that the sections of some power of h_*Z give a lagrangian fibration $\phi' : X' \to B'$, it suffices now to remark that a general member of the family of deformations of $E_{s,t}$ does not intersect the locus $W \subset X$ where h is not an isomorphism. This follows easily from the observation that a neighbourhood $U_s \subset X$ of A_s is fibered not only by lagrangian tori which are small deformations of A_s, but also by deformations of the subtorus $E_{s,t}$: the local lagrangian fibration $f_s : U_s \to B_s$ factors through $g_s : U_s \to D_s$ which has $E_{s,t}$ for a fiber. The locus W, being of codimension at least two, cannot dominate U_s and so we can deform $E_{s,t}$ away from it and then take the image by h to obtain a dominating family of curves which do not intersect h_*Z. With all this done, we conclude by [M3].

The proof of the theorem uses several preliminary lemmas related to the following construction ("the characteristic foliation").

Consider the local lagrangian fibration $f_s : U_s \to B_s$ in a neighbourhood of A_s. Then $A_t \cap U_s$ projects onto a codimension e analytic subvariety $C_s \subset U_s$. Take $D^0 = f_s^{-1}(C_s)$. This is the "principal branch" of the subvariety $Z_{s,t}$ in the neighbourhood of A_s. It is clear that we can move $E_{s,t}$ away from D^0 by replacing it on some neighbouring $A_{s'}$ with $s' \notin C_s$. But there are other branches $D^1, \ldots, D^l$ of $Z_{s,t}$ intersecting A_s and all of its neighbours, and we must show that $E_{s,t}$ can be moved away from them, too.

For this, we look at the kernel of the restriction of the symplectic form σ to the smooth part of $Z_{s,t}$: this is a distribution $\mathcal{F}$ of rank e, often called characteristic foliation in the literature.

Lemma 9 *The restriction of $\mathcal{F}$ to A_s is trivial (as a subbundle of T_{A_s}).*

Proof: The symplectic form σ defines in the usual way an isomorphism between the restriction of $\mathcal{F}$ to D_0 and the conormal bundle to D_0 in X: indeed the last one is the kernel of the restriction map from the dual of $T_X|_{D_0}$ to the dual of T_{D_0}. We claim that the restriction of the normal bundle $N_{D_0,X}$ to A_s is trivial. This is clear from the exact sequence

$$0 \to N_{A_s,D_0} \to N_{A_s,X} \to N_{D_0,X}|_{A_s} \to 0 :$$

the first term is a trivial vector bundle since D_0 is a fibration, the second one is trivial because A_s is lagrangian and therefore $N_{A_s,X}$ is isomorphic to the cotangent bundle of A_s, and the third one is trivial because it therefore has e everywhere linearly independent global sections.

Corollary 10 *The distribution $\mathcal{F}$ is tangent to a fibration of $Z_{s,t}$ by deformations of $E_{s,t}$.*

Proof: The restriction of $\mathcal{F}$ to A_s has at least one algebraic leaf, that is, $E_{s,t}$ (being a component of the intersection of lagrangian A_s and A_t, it is orthogonal to both). Since it is trivial as a vector bundle, all leaves are translates of $E_{s,t}$ in A_s. Since A_s is generic, all leaves on $Z_{s,t}$ are algebraic.

The following lemma, which seems to be an explicit geometric analogue of Hwang–Weiss' "pairwise integrability" (though we have obtained it independently around the same time), is crucial.

Lemma 11 *There is only a finite number of $Z_{s,t}$ passing through a general point $x \in X$. In particular, if $A_{t'}$ is a small deformation of A_t which still intersects A_s, then $Z_{s,t} = Z_{s,t'}$ and $A_{t'} \subset Z_{s,t}$. Therefore, through a general translate of $E_{s,t}$ in A_s (and not only in A_t), there is a lagrangian torus contained in $Z_{s,t}$.*

Proof: Indeed, each $Z_{s,t}$ is a union of tori A_s and their degenerations (which do not cover X). Therefore a $Z_{s,t}$ passing through a general x should contain such a torus A_s through x, and its tangent space at x must be σ-orthogonal to that of a subtorus of dimension e in A_s. But there is only a countable number of possibilities for those. Since we are dealing with bounded families, we conclude that there are in fact only finitely many possibilities for $T_x Z_{s,t}$. Since x is general, an application of standard results (either the unicity theorem for solutions of differential equations on a suitable covering of X or Sard's lemma on a suitable fibered product) shows that there is also only a finite number of $Z_{s,t}$ through x.

The second assertion follows from the first since $Z_{s,t}$ through x does not deform. The third one follows from the second by taking closure.

As an immediate corollary we obtain an assertion already announced before the statement of theorem 7:

Corollary 12 *The intersection $A_s \cap A_t$ for s general is a union of translates of $E_{s,t}$ both in A_s and in A_t.*

Indeed, this intersection must be tangent to the kernel of the restriction of σ, and as we have seen above, this kernel is tangent to the fibration given by the translates.

Proof of theorem 7: The subvariety $Z_{s,t}$ is the union of A_s where A_s varies in an irreducible $(n-e)$-parametric family S of tori intersecting A_t. As we have just seen, $A_s \cap A_t$ for s general is a union of translates of $E_{s,t}$ in A_t: our first claim is that the same is true for for any s (but the union can a priori be infinite, that is, of greater dimension than e). This is easily seen from the factorisation of the local fibration $f_s : U_s \to B_s$ through $g_s : U_s \to D_s$ from the proof of corollary 8.

Next, let s be general and $u \in S$ arbitrary. We claim that $A_u \cap A_s$ is again a union of translates of $E_{s,t}$ in A_s. Suppose first that A_u passes through a point $y \in E_{s,t} \subset A_s \cap A_t$. Then by what we have just observed, A_u contains $E_{s,t}$ (indeed $A_u \cap A_t$ is a union of translates of $E_{s,t}$, so if this contains a point $y \in E_{s,t}$, then it contains the whole of $E_{s,t}$). Now we can repeat the same argument supposing that A_u passes through a point y' on a translate E' of $E_{s,t}$ in A_s: indeed by lemma 11 we can find an $A_{t'}$ contained in $Z_{s,t}$ such that $E' = E_{s,t'}$ is a component of the intersection $A_s \cap A_{t'}$ and just replace t by t'. In conclusion, together with each of its points, $A_u \cap A_s$ contains the whole translate of $E_{s,t}$ passing through this point. This proves our second claim.

Now we are able to show that $E_{s,t} \cdot Z_{s,t} = 0$. Recall that we have denoted by D^0 the principal branch of $Z_{s,t}$ around A_s, so that D^0 is a union of fibers of the local fibration f_s, and by $D^1, \dots, D^l$ the other branches. It is clear that a general deformation of $E_{s,t}$ is disjoint from D^0: it suffices to move $E_{s,t}$ to a neighbouring torus not contained in D^0. But now we see that it is disjoint from $D^1, \dots, D^l$ as well: indeed $D^i \cap A_s$ is a union of translates of $E_{s,t}$ and so replacing $E_{s,t}$ by a suitable translate in A_s we can make it disjoint from D^i.

This finishes the proof of theorem 7.

As a final remark, let us notice that our arguments also prove the following statement, which can be of independent interest:

Proposition 13 *Let U resp. V be connected complex manifolds of dimension $2n$ resp. n, and let σ be a holomorphic symplectic form on U. Let $\varphi : U \to V$ be a proper lagrangian fibration with fibers $A_v, v \in V$ (which are then automatically complex tori). Assume that there exists a connected open subset $W \subset U$ and a holomorphic map $\eta : W \to M$ with M a connected complex manifold of dimension n, such that $W_m := \eta^{-1}(m), m \in M$ is a connected closed Lagrangian submanifold of U, that the restriction $\varphi_m : W_m \to V$ of φ to W_m is proper for every m, and that the restriction $\eta_v : W \cap A_v \to M$ is proper for each $v \in V$.*

Then the image $V_m = \varphi(W_m)$ (which is a closed irreducible complex analytic subset of V) is independent of $m \in M$.

More precisely, $\varphi = q \circ \alpha$ with $q : U \to N$ and $\alpha : N \to V$ proper submersions: the fibers of q are connected components of $A_v \cap W_m$ and their translates in A_v. This factorisation induces a smooth foliation F on V such that at each point $v \in V$, F_v is the projection by φ_* of the σ-orthogonal to the tangent space to the fibre of q at any point of A_v, and V_m is a leaf of F for any $m \in M$.

Proof: For v, m generic in $V \times M$, $A_v \cap W_m$ is smooth and (by lemma 5) is a finite union of translated subtori $E_{v,m}$ of A_v. The tangent space to the subvariety $Z_{v,m}$ (defined in the same way as before) at a generic point x of a component of $A_v \cap W_m$ is generated by $T_{A_v,x}$ and $T_{W_m,x}$, and is thus the σ-orthogonal (by the Lagrangian property of A_v and W_m) of $T_{E_{v,m},x}$. By the rigidity (up to translation) of subtori of A_v, this $E_{v,m}$ is, up to translation, independent of m, and $T_{Z_{v,m},x'}$ is thus, for every $x' \in U \cap A_v$, the σ-orthogonal to $TE_{x',v}$, where $E_{x',v}$ is the translate of $E_{v,m}$ through x'. This shows that the tangent space of V_m at v is independent of m, and concludes the proof of the first claim. The rest is similar.

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