Journal of Algebra 346 (2011) 189–226

Contents lists available at SciVerse ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Irreducible representations of Yangians

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ARTICLE INFO

Article history:
Received 20 January 2011
Available online 13 September 2011
Communicated by Vera Serganova

Keywords:
Yangians
Intertwining operators

ABSTRACT

We give explicit realizations of irreducible representations of the Yangian of the general linear Lie algebra and of its twisted analogues, corresponding to symplectic and orthogonal Lie algebras. In particular, we develop the fusion procedure for twisted Yangians. For the non-twisted Yangian, this procedure goes back to the works of Cherednik.

0. Introduction

Take the general linear Lie algebra $\mathfrak{gl}_n$ over the complex field $\mathbb{C}$ and consider the corresponding polynomial current Lie algebra $\mathfrak{gl}_n[x]$. The Yangian $\mathrm{Y}(\mathfrak{gl}_n)$ is a deformation of the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_n[x])$ in the class of Hopf algebras. Irreducible finite-dimensional representations of the associative algebra $\mathrm{Y}(\mathfrak{gl}_n)$ were classified by Drinfeld [D2]. In Sections 1.1 and 1.2 of the present article we recall this classification, together with the original definition of the Yangian $\mathrm{Y}(\mathfrak{gl}_n)$ coming from the theory of quantum integrable systems.

The additive group $\mathbb{C}$ acts on $\mathrm{Y}(\mathfrak{gl}_n)$ by Hopf algebra automorphisms, see (1.2) for the definition of automorphism corresponding to the element $t \in \mathbb{C}$. The associative algebra $\mathrm{Y}(\mathfrak{gl}_n)$ contains $\mathrm{U}(\mathfrak{gl}_n)$ as a subalgebra, and admits a homomorphism onto $\mathrm{U}(\mathfrak{gl}_n)$ identical on this subalgebra; see (1.4). By pulling the representation of $\mathrm{U}(\mathfrak{gl}_n)$ on an exterior power of $\mathbb{C}^n$ back through the homomorphism (1.4), and then back through any automorphism (1.2), we get an irreducible representation of $\mathrm{Y}(\mathfrak{gl}_n)$ called *fundamental*.

The work [D2] provided only a parametrization of all irreducible finite-dimensional representations of $\mathrm{Y}(\mathfrak{gl}_n)$. The results of Akasaka and Kashiwara [AK] showed that up to an automorphism of $\mathrm{Y}(\mathfrak{gl}_n)$ of the form (1.3), each of these representations arises as a quotient of a tensor product of fundamental ones; see also the earlier works of Chari and Pressley [CP] and Cherednik [C2]. This fact can also be derived from the results of Nazarov and Tarasov [NT1]; further results were obtained by Chari [C]. Note that the works [AK] and [C] dealt with representations of quantum affine algebras. For connec-

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2011.08.011

tions to the representation theory of Yangians see the work of Molev, Tolstoy and Zhang [MTZ] and the more recent work of Gautam and Toledano-Laredo [GT].

Now let $\mathfrak{g}_n$ be one of the two Lie algebras $\mathfrak{so}_n$, $\mathfrak{sp}_n$. Regard $\mathfrak{g}_n$ as the Lie subalgebra of $\mathfrak{gl}_n$ preserving a non-degenerate bilinear form on the vector space $\mathbb{C}^n$, symmetric in the case $\mathfrak{g}_n = \mathfrak{so}_n$, or alternating in the case $\mathfrak{g}_n = \mathfrak{sp}_n$. For any $n \times n$ matrix X let $\widetilde{X}$ be the conjugate of X relative to this form. As an associative algebra, the twisted Yangian $\mathrm{Y}(\mathfrak{g}_n)$ is a deformation of the universal enveloping algebra of the *twisted* polynomial current Lie algebra

$$\left\{X(x) \in \mathfrak{gl}_n[x]\colon\ X(-x) = -\widetilde{X}(x)\right\}.$$

This is not a Hopf algebra deformation. However, $\mathrm{Y}(\mathfrak{g}_n)$ is a one-sided coideal subalgebra in the Hopf algebra $\mathrm{Y}(\mathfrak{gl}_n)$. Moreover, $\mathrm{Y}(\mathfrak{g}_n)$ contains $\mathrm{U}(\mathfrak{g}_n)$ as a subalgebra, and has a homomorphism onto $\mathrm{U}(\mathfrak{g}_n)$ identical on this subalgebra.

The definition of the twisted Yangian $\mathrm{Y}(\mathfrak{g}_n)$ was given by Olshanski [O] with some help from the second author of the present article. This definition was motivated by the works of Cherednik [C1] and Sklyanin [S] on the quantum integrable systems with boundary conditions. Irreducible finite-dimensional representations of the associative algebra $\mathrm{Y}(\mathfrak{g}_n)$ have been classified by Molev, see the recent book [M]. In Sections 1.4 and 1.5 we recall this classification, together with the definition of the twisted Yangian $\mathrm{Y}(\mathfrak{g}_n)$.

A new approach to the representation theory of the Yangian $\mathrm{Y}(\mathfrak{gl}_n)$ and of its twisted analogues $\mathrm{Y}(\mathfrak{so}_n)$, $\mathrm{Y}(\mathfrak{sp}_n)$ was developed by Khoroshkin, Nazarov and Vinberg in [KN,KNV]. In particular, it was proved in [KN] that up to an automorphism of $\mathrm{Y}(\mathfrak{sp}_n)$ of the form (1.13), any irreducible finite-dimensional representation of $\mathrm{Y}(\mathfrak{sp}_n)$ is a quotient of some tensor product of fundamental representations of $\mathrm{Y}(\mathfrak{gl}_n)$. The tensor product is regarded as a representation of $\mathrm{Y}(\mathfrak{sp}_n)$ by restriction from $\mathrm{Y}(\mathfrak{gl}_n)$. Similar result was also proved in [KN] for those representations of $\mathrm{Y}(\mathfrak{so}_n)$ which integrate from $\mathfrak{so}_n \subset \mathrm{Y}(\mathfrak{so}_n)$ to the special orthogonal group SO_n. Moreover, the work [KN] provided new proofs of the above mentioned results of [AK] for $\mathrm{Y}(\mathfrak{gl}_n)$. We summarize the results of [KN] in Sections 1.3 and 1.6 of the present article.

The realizations of irreducible representations of $\mathrm{Y}(\mathfrak{gl}_n)$ and $\mathrm{Y}(\mathfrak{g}_n)$ in [AK] and [KN] were not quite explicit. To make them more explicit is the main aim of the present article. In Sections 2.6 and 2.7 we give explicit formulas for intertwining operators of tensor products of fundamental representations of $\mathrm{Y}(\mathfrak{gl}_n)$, such that the quotients by the kernels of these operators realize all irreducible finite-dimensional representations of $\mathrm{Y}(\mathfrak{gl}_n)$, up to automorphisms (1.3). In Sections 3.4, 3.5, 3.6 we give analogues of these formulas for $\mathrm{Y}(\mathfrak{g}_n)$.

To obtain the irreducible representations of $\mathrm{Y}(\mathfrak{gl}_n)$ and $\mathrm{Y}(\mathfrak{g}_n)$ as quotients of tensor products of fundamental representations, in [KN] we imposed certain conditions on the tensor products; see Theorems 1.1 and 1.2 below. Some of these conditions are not necessary. For $\mathrm{Y}(\mathfrak{gl}_n)$ the condition (1.10) did not appear in [AK] and [NT1], but did appear in [C2]. In Section 2.8 we remove this condition from Theorem 1.1 by using the results of Section 2.5. In Section 3.7 we remove the conditions (1.17), (1.18), (1.19) from Theorem 1.2 for $\mathrm{Y}(\mathfrak{g}_n)$ by using the results of Section 3.3. Thus we extend the results of [KN] for both $\mathrm{Y}(\mathfrak{gl}_n)$ and $\mathrm{Y}(\mathfrak{g}_n)$, up to the level of [AK] and [NT1] for $\mathrm{Y}(\mathfrak{gl}_n)$.

1. Representations of Yangians

1.1. First consider the *Yangian* $\mathrm{Y}(\mathfrak{gl}_n)$ of the Lie algebra $\mathfrak{gl}_n$. This is a complex unital associative algebra with a family of generators $T_{ij}^{(1)}, T_{ij}^{(2)}, \ldots$ where $i, j = 1, \ldots, n$. Defining relations for them can be written using the series

$$T_{ij}(x) = \delta_{ij} + T_{ij}^{(1)} x^{-1} + T_{ij}^{(2)} x^{-2} + \cdots$$

where x is a formal parameter. Let y be another formal parameter. Then the defining relations in the associative algebra $\mathrm{Y}(\mathfrak{gl}_n)$ can be written as

$$(x-y)\big[T_{ij}(x), T_{kl}(y)\big] = T_{kj}(x)T_{il}(y) - T_{kj}(y)T_{il}(x). \tag{1.1}$$

The algebra $\mathrm{Y}(\mathfrak{gl}_1)$ is commutative. By (1.1) for any $t \in \mathbb{C}$ the assignments

$$T_{ij}(x) \mapsto T_{ij}(x-t) \tag{1.2}$$

define an automorphism of the algebra $\mathrm{Y}(\mathfrak{gl}_n)$. Here each of the formal power series $T_{ij}(x-t)$ in $(x-t)^{-1}$ should be re-expanded in x^{-1}. Every assignment (1.2) is a correspondence between the respective coefficients of series in x^{-1}. Relations (1.1) also show that for any formal power series $f(x)$ in x^{-1} with coefficients from $\mathbb{C}$ and leading term 1, the assignments

$$T_{ij}(x) \mapsto f(x)T_{ij}(x) \tag{1.3}$$

define an automorphism of the algebra $\mathrm{Y}(\mathfrak{gl}_n)$. The subalgebra consisting of all elements of $\mathrm{Y}(\mathfrak{gl}_n)$ which are invariant under every automorphism of the form (1.3), is called the *special Yangian* of $\mathfrak{gl}_n$, and is denoted by $\mathrm{SY}(\mathfrak{gl}_n)$. Two representations of the algebra $\mathrm{Y}(\mathfrak{gl}_n)$ are called *similar* if they differ by an automorphism of the form (1.3).

Let $E_{ij} \in \mathfrak{gl}_n$ be the standard matrix units. By (1.1) the assignments

$$T_{ij}(x) \mapsto \delta_{ij} + E_{ij}x^{-1} \tag{1.4}$$

define a homomorphism of unital associative algebras $\mathrm{Y}(\mathfrak{gl}_n) \to \mathrm{U}(\mathfrak{gl}_n)$. There is also an embedding $\mathrm{U}(\mathfrak{gl}_n) \to \mathrm{Y}(\mathfrak{gl}_n)$, defined by mapping $E_{ij} \mapsto T_{ij}^{(1)}$. So $\mathrm{Y}(\mathfrak{gl}_n)$ contains the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_n)$ as a subalgebra. The homomorphism (1.4) is evidently identical on the subalgebra $\mathrm{U}(\mathfrak{gl}_n) \subset \mathrm{Y}(\mathfrak{gl}_n)$.

The Yangian $\mathrm{Y}(\mathfrak{gl}_n)$ is a Hopf algebra over the field $\mathbb{C}$. In particular, the comultiplication $\Delta : \mathrm{Y}(\mathfrak{gl}_n) \to \mathrm{Y}(\mathfrak{gl}_n) \otimes \mathrm{Y}(\mathfrak{gl}_n)$ is defined by the assignments

$$\Delta : T_{ij}(x) \mapsto \sum_{k=1}^{n} T_{ik}(x) \otimes T_{kj}(x). \tag{1.5}$$

When taking tensor products of $\mathrm{Y}(\mathfrak{gl}_n)$-modules, we will use (1.5). The counit homomorphism $\mathrm{Y}(\mathfrak{gl}_n) \to \mathbb{C}$ is defined by the assignments $T_{ij}(x) \mapsto \delta_{ij}$. By using this homomorphism, one defines the *trivial* representation of $\mathrm{Y}(\mathfrak{gl}_n)$.

Further, let $T(x)$ be the $n \times n$ matrix whose i, j entry is the series $T_{ij}(x)$. The antipodal map $\mathrm{Y}(\mathfrak{gl}_n) \to \mathrm{Y}(\mathfrak{gl}_n)$ is defined by mapping $T(x) \mapsto T(x)^{-1}$. Here each entry of the inverse matrix $T(x)^{-1}$ is a formal power series in x^{-1} with coefficients from the algebra $\mathrm{Y}(\mathfrak{gl}_n)$, and the assignment $T(x) \mapsto T(x)^{-1}$ is a correspondence between the respective matrix entries.

The special Yangian $\mathrm{SY}(\mathfrak{gl}_n)$ is a Hopf subalgebra of $\mathrm{Y}(\mathfrak{gl}_n)$. Moreover, it is isomorphic to the Yangian $\mathrm{Y}(\mathfrak{sl}_n)$ of the special linear Lie algebra $\mathfrak{sl}_n \subset \mathfrak{gl}_n$ studied in [D1,D2]. For the proofs of these facts see [M, Section 1.8].

1.2. Up to their equivalence and similarity, the irreducible finite-dimensional representations of the associative algebra $\mathrm{Y}(\mathfrak{gl}_n)$ are parametrized by sequences of $n-1$ monic polynomials $P_1(x), \dots, P_{n-1}(x)$ with complex coefficients. In particular, $P_1(x) = \cdots = P_{n-1}(x) = 1$ for the trivial representation of $\mathrm{Y}(\mathfrak{gl}_n)$.

This parametrization was given by Drinfeld [D2, Theorem 2]. In the present article we will use another parametrization. It can be obtained by combining the results of Arakawa and Suzuki [AS] with those of [D1]. Namely, consider the Lie algebra $\mathfrak{gl}_m$. Let $\mathfrak{t}_m$ be a Cartan subalgebra of $\mathfrak{gl}_m$. Up to equivalence and similarity, we will parametrize the non-trivial irreducible finite-dimensional representations of $\mathrm{Y}(\mathfrak{gl}_n)$ by $m = 1, 2, \ldots$ and by certain orbits in $\mathfrak{t}_m^* \times \mathfrak{t}_m^*$ under the diagonal shifted action of the Weyl group of $\mathfrak{gl}_m$.

Pick the Cartan subalgebra $\mathfrak{t}_m$ of $\mathfrak{gl}_m$ with the basis $E_{11}, \ldots, E_{mm}$ of the diagonal matrix units. For any weight $\lambda \in \mathfrak{t}_m^*$ define the sequence $\lambda_1, \ldots, \lambda_m$ of its labels by setting $\lambda_a = \lambda(E_{aa})$ for $a = 1, \ldots, m$. In particular, for the half-sum $\rho \in \mathfrak{t}_m^*$ of the positive roots we have $\rho_a = m/2 - a + 1/2$. The Weyl group of $\mathfrak{gl}_m$ is the symmetric group S_m. It acts on the Cartan subalgebra $\mathfrak{t}_m$ by permuting the basis vectors $E_{11}, \ldots, E_{mm}$. Hence it acts on any weight $\lambda \in \mathfrak{t}_m^*$ by permuting its labels. The shifted action of $w \in S_m$ on $\mathfrak{t}_m^*$ is given by

$$w \circ \lambda = w(\lambda + \rho) - \rho. \tag{1.6}$$

For our parametrization we will be using only the orbits of the pairs $(\lambda, \mu) \in \mathfrak{t}_m^* \times \mathfrak{t}_m^*$ such that all labels of the weight $\nu = \lambda - \mu$ belong to $\{1, \ldots, n-1\}$. Given any such a pair (λ, μ) define a sequence $P_1(x), \ldots, P_{n-1}(x)$ of monic polynomials with complex coefficients, as follows. For each $i = 1, \ldots, n-1$ put

$$P_i(x) = \prod_{\nu_a = i} (x - \mu_a - \rho_a) \tag{1.7}$$

where $\mu_1, \ldots, \mu_m$ and $\nu_1, \ldots, \nu_m$ are the labels of μ and ν respectively. Note that the simultaneous shifted action of the group S_m on the weights λ and μ gives the usual permutational action of S_m on the labels of $\mu + \rho$ and of ν. Therefore each polynomial (1.7) depends only on the S_m-orbit of the pair $(\lambda, \mu) \in \mathfrak{t}_m^* \times \mathfrak{t}_m^*$. Moreover, the orbit is determined by the polynomials (1.7) uniquely. Furthermore, any sequence of monic polynomials $P_1(x), \ldots, P_{n-1}(x)$ of the total degree m with complex coefficients arises in this way.

In the next subsection, to each of these orbits we will attach an irreducible $\mathrm{Y}(\mathfrak{gl}_n)$-module. Its Drinfeld polynomials $P_1(x), \ldots, P_{n-1}(x)$ will be given by (1.7). For the definition of the Drinfeld polynomials of an arbitrary irreducible finite-dimensional $\mathrm{Y}(\mathfrak{gl}_n)$-module used here see [KN, Section 5.1].

1.3. For $k = 0, 1, \ldots, n$ consider the exterior power $\Lambda^k(\mathbb{C}^n)$ of the defining $\mathfrak{gl}_n$-module $\mathbb{C}^n$. Using the homomorphism (1.4), regard it as a module over the Yangian $\mathrm{Y}(\mathfrak{gl}_n)$. For $t \in \mathbb{C}$ denote by Φ_t^k the $\mathrm{Y}(\mathfrak{gl}_n)$-module obtained by pulling the $\mathrm{Y}(\mathfrak{gl}_n)$-module $\Lambda^k(\mathbb{C}^n)$ back through the automorphism (1.2).

Now take any $(\lambda, \mu) \in \mathfrak{t}_m^* \times \mathfrak{t}_m^*$ such that all labels of the weight $\nu = \lambda - \mu$ belong to the set $\{1, \ldots, n-1\}$. Let the weights λ and μ vary so that ν is fixed. For the Cartan subalgebra $\mathfrak{t}_m$ of $\mathfrak{gl}_m$ the weight μ is *generic* if $\mu_a - \mu_b \notin \mathbb{Z}$ whenever $a \neq b$. Consider the tensor product of $\mathrm{Y}(\mathfrak{gl}_n)$-modules

$$\Phi^{\nu_1}_{\mu_1 + \rho_1 + \frac{1}{2}} \otimes \cdots \otimes \Phi^{\nu_m}_{\mu_m + \rho_m + \frac{1}{2}}. \tag{1.8}$$

It is known that the $\mathrm{Y}(\mathfrak{gl}_n)$-module (1.8) is irreducible if (but not only if) the weight μ is generic, see [NT2, Theorem 4.8] for a more general result. Moreover, if (but not only if) μ is generic then all the $\mathrm{Y}(\mathfrak{gl}_n)$-modules obtained from (1.8) by permuting the tensor factors, are equivalent to each other. In particular, for every generic μ there is a unique non-zero $\mathrm{Y}(\mathfrak{gl}_n)$-intertwining operator

$$\Phi^{\nu_1}_{\mu_1 + \rho_1 + \frac{1}{2}} \otimes \cdots \otimes \Phi^{\nu_m}_{\mu_m + \rho_m + \frac{1}{2}} \to \Phi^{\nu_m}_{\mu_m + \rho_m + \frac{1}{2}} \otimes \cdots \otimes \Phi^{\nu_1}_{\mu_1 + \rho_1 + \frac{1}{2}} \tag{1.9}$$

corresponding to the permutation of tensor factors of the maximal length. We will denote this operator by $I(\mu)$; it is unique up to a multiplier from $\mathbb{C} \setminus \{0\}$.

For all generic weights μ, the source vector spaces of the operators $I(\mu)$ are the same. The target vector spaces of all $I(\mu)$ also coincide with each other. Hence $I(\mu)$ is a function of μ taking values in the space of linear operators

$$\Lambda^{\nu_1}\big(\mathbb{C}^n\big) \otimes \cdots \otimes \Lambda^{\nu_m}\big(\mathbb{C}^n\big) \to \Lambda^{\nu_m}\big(\mathbb{C}^n\big) \otimes \cdots \otimes \Lambda^{\nu_1}\big(\mathbb{C}^n\big)$$

between tensor products of exterior powers of $\mathbb{C}^n$. The multipliers from $\mathbb{C} \setminus \{0\}$ can be chosen so that $I(\mu)$ becomes a rational function of μ, see Section 2 for a particular choice. Any such a choice allows to determine intertwining operators (1.9) for those non-generic μ where $I(\mu)$ is regular. Thus the weight μ need not be generic anymore, so that the $\mathrm{Y}(\mathfrak{gl}_n)$-module (1.8) may be reducible. This way of determining intertwining operators between $\mathrm{Y}(\mathfrak{gl}_n)$-modules goes back to the work of Cherednik [C2] and is commonly called the *fusion procedure*.

For the Cartan subalgebra $\mathfrak{t}_m$ of $\mathfrak{gl}_m$ the weight λ will be called *dominant* if $\lambda_a - \lambda_b \neq -1, -2, \ldots$ whenever $a < b$. The pair $(\lambda, \mu) \in \mathfrak{t}_m^* \times \mathfrak{t}_m^*$ will be called *good* if the weight $\lambda + \rho$ is dominant and moreover

$$\nu_a \geqslant \nu_b \quad \text{whenever} \quad \lambda_a + \rho_a = \lambda_b + \rho_b \quad \text{and} \quad a < b. \tag{1.10}$$

The orbit of any $(\lambda, \mu) \in \mathfrak{t}_m^* \times \mathfrak{t}_m^*$ under the shifted action of S_m on $\mathfrak{t}_m^*$ does contain a good pair. Here we assume only that $\nu_1, \ldots, \nu_m \in \{1, \ldots, n-1\}$. In its present form, the next theorem was proved in [KN, Section 5.1]. It will be generalized in Section 2.8 of the present article.

Theorem 1.1. *For the fixed weight $\nu = \lambda - \mu$ the multipliers from $\mathbb{C} \setminus \{0\}$ of the operators* (1.9) *for all generic weights μ can be chosen so that the rational function $I(\mu)$ is regular and non-zero whenever the pair (λ, μ) is good. Then for any good pair (λ, μ) the quotient of* (1.8) *by the kernel of operator $I(\mu)$ is an irreducible* $\mathrm{Y}(\mathfrak{gl}_n)$*-module with the Drinfeld polynomials given by* (1.7)*.*

1.4. Let $\mathfrak{g}_n$ be one of the two Lie algebras $\mathfrak{so}_n$, $\mathfrak{sp}_n$. We will regard $\mathfrak{g}_n$ as the Lie subalgebra of $\mathfrak{gl}_n$ preserving a non-degenerate bilinear form $\langle\,,\rangle$ on $\mathbb{C}^n$, symmetric in the case $\mathfrak{g}_n = \mathfrak{so}_n$, or alternating in the case $\mathfrak{g}_n = \mathfrak{sp}_n$. In the latter case, the positive integer n has to be even. When considering $\mathfrak{so}_n$, $\mathfrak{sp}_n$ simultaneously we will use the following convention. Whenever the double sign $\pm$ or $\mp$ appears, the upper sign will correspond to the case of a symmetric form on $\mathbb{C}^n$ so that $\mathfrak{g}_n = \mathfrak{so}_n$. The lower sign will correspond to the case of an alternating form on $\mathbb{C}^n$ so that $\mathfrak{g}_n = \mathfrak{sp}_n$.

Let $\widetilde{T}(x)$ be the conjugate of the matrix $T(x)$ relative to the form $\langle\,,\rangle$ on $\mathbb{C}^n$. An involutive automorphism of the algebra $\mathrm{Y}(\mathfrak{gl}_n)$ is defined by the assignment

$$T(x) \mapsto \widetilde{T}(-x). \tag{1.11}$$

This assignment is a correspondence between respective matrix entries. Now consider the matrix product $S(x) = \widetilde{T}(-x)T(x)$. Its ij entry is the series

$$S_{ij}(x) = \sum_{k=1}^{n} \widetilde{T}_{ik}(-x)T_{kj}(x) = \delta_{ij} + S_{ij}^{(1)}x^{-1} + S_{ij}^{(2)}x^{-2} + \cdots \tag{1.12}$$

with coefficients from the algebra $\mathrm{Y}(\mathfrak{gl}_n)$. The *twisted Yangian* corresponding to the Lie algebra $\mathfrak{g}_n$ is the subalgebra of $\mathrm{Y}(\mathfrak{gl}_n)$ generated by the coefficients $S_{ij}^{(1)}, S_{ij}^{(2)}, \ldots$ where $i, j = 1, \ldots, n$. We denote this subalgebra by $\mathrm{Y}(\mathfrak{g}_n)$.

The algebras $\mathrm{Y}(\mathfrak{so}_n)$ corresponding to different choices of the symmetric form $\langle\,,\rangle$ on $\mathbb{C}^n$ are isomorphic to each other, and so are the algebras $\mathrm{Y}(\mathfrak{sp}_n)$ corresponding to different choices of the alternating form $\langle\,,\rangle$ on $\mathbb{C}^n$. These isomorphisms can be described explicitly, see for instance [M, Corollary 2.3.2].

The subalgebra $\mathrm{Y}(\mathfrak{g}_n) \cap \mathrm{SY}(\mathfrak{gl}_n)$ of $\mathrm{Y}(\mathfrak{gl}_n)$ is denoted by $\mathrm{SY}(\mathfrak{g}_n)$, and is called the *special twisted Yangian* corresponding to $\mathfrak{g}_n$. The automorphism (1.3) of $\mathrm{Y}(\mathfrak{gl}_n)$ determines an automorphism of $\mathrm{Y}(\mathfrak{g}_n)$ which maps

$$S(x) \mapsto f(x)f(-x)S(x). \tag{1.13}$$

The subalgebra $\mathrm{SY}(\mathfrak{g}_n)$ of $\mathrm{Y}(\mathfrak{g}_n)$ consists of the elements fixed by all such automorphisms. Two representations of the algebra $\mathrm{Y}(\mathfrak{g}_n)$ are called *similar* if they differ by such an automorphism.

There is an analogue for $\mathrm{Y}(\mathfrak{g}_n)$ of the homomorphism $\mathrm{Y}(\mathfrak{gl}_n) \to \mathrm{U}(\mathfrak{gl}_n)$ defined by (1.4). Namely, one can define a homomorphism $\mathrm{Y}(\mathfrak{g}_n) \to \mathrm{U}(\mathfrak{g}_n)$ by

$$S_{ij}(x) \mapsto \delta_{ij} + \frac{E_{ij} - \widetilde{E}_{ij}}{x \pm \frac{1}{2}} \tag{1.14}$$

where $\widetilde{E}_{ij}$ is the conjugate of the matrix unit $E_{ij} \in \mathfrak{gl}_n$ relative to the form $\langle\,,\rangle$ on $\mathbb{C}^n$. This can be proved by using the defining relations for the generators $S_{ij}^{(1)}, S_{ij}^{(2)}, \ldots$ which we do not reproduce here; see [M, Proposition 2.1.2] for the proof. Further, there is an embedding $\mathrm{U}(\mathfrak{g}_n) \to \mathrm{Y}(\mathfrak{g}_n)$ defined by mapping

$$E_{ij} - \widetilde{E}_{ij} \mapsto S_{ij}^{(1)}.$$

Hence the twisted Yangian $\mathrm{Y}(\mathfrak{g}_n)$ contains the universal enveloping algebra $\mathrm{U}(\mathfrak{g}_n)$ as a subalgebra. The homomorphism $\mathrm{Y}(\mathfrak{g}_n) \to \mathrm{U}(\mathfrak{g}_n)$ defined by (1.14) is evidently identical on the subalgebra $\mathrm{U}(\mathfrak{g}_n) \subset \mathrm{Y}(\mathfrak{g}_n)$.

The twisted Yangian $\mathrm{Y}(\mathfrak{g}_n)$ is not only a subalgebra of $\mathrm{Y}(\mathfrak{gl}_n)$, it is also a right coideal of the coalgebra $\mathrm{Y}(\mathfrak{gl}_n)$ relative to the comultiplication (1.5). Indeed, by the definition of the series (1.12) we get

$$\Delta\big(S_{ij}(x)\big) = \sum_{k,l=1}^{n} S_{kl}(x) \otimes \widetilde{T}_{ik}(-x)T_{lj}(x).$$

Therefore

$$\Delta\big(\mathrm{Y}(\mathfrak{g}_n)\big) \subset \mathrm{Y}(\mathfrak{g}_n) \otimes \mathrm{Y}(\mathfrak{gl}_n).$$

Hence by taking a tensor product of an $\mathrm{Y}(\mathfrak{g}_n)$-module with an $\mathrm{Y}(\mathfrak{gl}_n)$-module we get another $\mathrm{Y}(\mathfrak{g}_n)$-module.

The *trivial* representation of the algebra $\mathrm{Y}(\mathfrak{g}_n)$ is defined by restricting the counit homomorphism $\mathrm{Y}(\mathfrak{gl}_n) \to \mathbb{C}$ to the subalgebra $\mathrm{Y}(\mathfrak{g}_n) \subset \mathrm{Y}(\mathfrak{gl}_n)$. Under this representation $S_{ij}(x) \mapsto \delta_{ij}$. Note that restricting any representation of $\mathrm{Y}(\mathfrak{gl}_n)$ to the subalgebra $\mathrm{Y}(\mathfrak{g}_n)$ amounts to taking the tensor product of that representation of $\mathrm{Y}(\mathfrak{gl}_n)$ with the trivial representation of the algebra $\mathrm{Y}(\mathfrak{g}_n)$.

A parametrization of irreducible finite-dimensional representations of the algebra $\mathrm{Y}(\mathfrak{g}_n)$ was given by Molev, see [M, Chapter 4]. In the present article we will use another parametrization, introduced in [KN]. In the next subsection we will establish a correspondence between the two parametrizations. Let us call an $\mathrm{Y}(\mathfrak{so}_n)$-module *integrable* if the action of the Lie algebra $\mathfrak{so}_n \subset \mathrm{Y}(\mathfrak{so}_n)$ on it integrates to an action of the complex Lie group SO_n. When working with the algebra $\mathrm{Y}(\mathfrak{so}_n)$, we will be considering the integrable representations only.

1.5. First recall the parametrization from [M]. Write $n = 2l$ or $n = 2l+1$ depending on whether n is even or odd. If $\mathfrak{g}_n = \mathfrak{sp}_n$ then n has to be even. Up to equivalence and similarity, the irreducible finite-dimensional modules of the algebra $\mathrm{Y}(\mathfrak{sp}_n)$ are parametrized by sequences of l monic polynomials $Q_1(x), \ldots, Q_l(x)$ with complex coefficients, where the last polynomial $Q_l(x)$ is even. Further, if n is even then the integrable irreducible finite-dimensional $\mathrm{Y}(\mathfrak{so}_n)$-modules are parametrized by the same sequences of polynomials, and by an extra parameter $\delta \in \{+1, -1\}$ in the case when $Q_l(0) = 0$.

If n is odd then $\mathfrak{g}_n = \mathfrak{so}_n$. The integrable irreducible finite-dimensional modules of the algebra $\mathrm{Y}(\mathfrak{so}_n)$ with odd n are parametrized by sequences of l monic polynomials $Q_1(x), \dots, Q_l(x)$ with complex coefficients, but without any further conditions on the polynomial $Q_l(x)$. For the trivial module of the algebra $\mathrm{Y}(\mathfrak{g}_n)$ with arbitrary $\mathfrak{g}_n$ we always have $Q_1(x) = \dots = Q_l(x) = 1$, and there is no extra parameter δ then, even if $\mathfrak{g}_n = \mathfrak{so}_n$ and $n = 2l$.

Now let $\mathfrak{f}_m = \mathfrak{sp}_{2m}$ if $\mathfrak{g}_n = \mathfrak{sp}_n$, and let $\mathfrak{f}_m = \mathfrak{so}_{2m}$ if $\mathfrak{g}_n = \mathfrak{so}_n$. Let $\mathfrak{h}_m$ be a Cartan subalgebra of the reductive Lie algebra $\mathfrak{f}_m$. The Weyl group of $\mathfrak{sp}_{2m}$ is isomorphic to the hyperoctahedral group $H_m = S_m \ltimes \mathbb{Z}_2^m$. The Weyl group of $\mathfrak{so}_{2m}$ is isomorphic to a subgroup of H_m of index two. In the latter case the action of the Weyl group on $\mathfrak{h}_m$ can be extended by a diagram automorphism of SO_{2m} of order two, so that the extended Weyl group is still isomorphic to H_m. Instead of $Q_1(x), \dots, Q_l(x)$ for any $\mathfrak{g}_n$ we will use certain orbits in $\mathfrak{h}_m^* \times \mathfrak{h}_m^*$ under the diagonal shifted action of the group H_m.

When working with the Lie algebra $\mathfrak{f}_m$ it will be convenient to label the standard basis vectors of $\mathbb{C}^{2m}$ by the indices $-m, \dots, -1, 1, \dots, m$. Let a, b be any pair of these indices. Let $E_{ab} \in \mathfrak{gl}_{2m}$ be the corresponding matrix unit. Choose the antisymmetric bilinear form on $\mathbb{C}^{2m}$ so that the subalgebra $\mathfrak{f}_m \subset \mathfrak{gl}_{2m}$ preserving this bilinear form is spanned by the elements

$$F_{ab} = E_{ab} - \operatorname{sign}(ab) \cdot E_{-b,-a} \quad \text{or} \quad F_{ab} = E_{ab} - E_{-b,-a}$$

for $\mathfrak{f}_m = \mathfrak{sp}_{2m}$ or $\mathfrak{f}_m = \mathfrak{so}_{2m}$ respectively. Choose the Cartan subalgebra $\mathfrak{h}_m$ of $\mathfrak{f}_m$ with the basis $F_{11}, \dots, F_{mm}$. For any weight $\lambda \in \mathfrak{h}_m^*$ define the sequence $\lambda_1, \dots, \lambda_m$ of its labels by $\lambda_a = \lambda(F_{aa})$ for $a = 1, \dots, m$. For the half-sum $\rho \in \mathfrak{h}_m^*$ of positive roots of $\mathfrak{f}_m$ we have $\rho_a = -a$ if $\mathfrak{f}_m = \mathfrak{sp}_{2m}$, and $\rho_a = 1 - a$ if $\mathfrak{f}_m = \mathfrak{so}_{2m}$. Here the positive roots are chosen as in [KN, Section 4.1].

The group H_m acts on the Cartan subalgebra $\mathfrak{h}_m$ by permuting the basis vectors chosen above, and by multiplying any of them by -1. The group H_m also acts on the dual space $\mathfrak{h}_m^*$. The shifted action of any element $w \in H_m$ is given by the universal formula (1.6). Let $\kappa \in \mathfrak{h}_m^*$ be the weight of $\mathfrak{f}_m$ such that $\kappa_a = n/2$ for $a = 1, \dots, m$. Instead of the sequences of polynomials $Q_1(x), \dots, Q_l(x)$ for our parametrization of irreducible $\mathrm{Y}(\mathfrak{g}_n)$-modules we use the orbits of the pairs $(\lambda, \mu) \in \mathfrak{h}_m^* \times \mathfrak{h}_m^*$ such that all labels of the weight $\nu = \lambda - \mu + \kappa$ belong to the set $\{1, \dots, n-1\}$. Note that the definitions of ν here and in the case of the Yangian $\mathrm{Y}(\mathfrak{gl}_n)$ are different.

Given any such a pair (λ, μ) define a sequence $Q_1(x), \dots, Q_l(x)$ of monic polynomials as follows. For any $\mathfrak{g}_n$ and each $i = 1, \dots, l$ put

$$Q_i(x) = \prod_{\nu_a = i} (x + \mu_a + \rho_a) \cdot \prod_{\nu_a = n-i} (x - \mu_a - \rho_a). \tag{1.15}$$

Here $\mu_1, \dots, \mu_m$ and $\nu_1, \dots, \nu_m$ are the labels of the weights μ and ν of $\mathfrak{f}_m$. Note that if n is even then $l = n - l$, so that $Q_l(x)$ is an even polynomial then.

The simultaneous shifted action of the subgroup $S_m \subset H_m$ on the weights λ and μ gives a permutational action of S_m on the labels of $\mu + \rho$ and of ν. Further, for any $a = 1, \dots, m$ multiplying the basis vector $F_{aa} \in \mathfrak{h}_m$ by -1 results in changing the labels $\mu_a + \rho_a$ and ν_a to respectively $-\mu_a - \rho_a$ and $n - \nu_a$. Therefore each of polynomials (1.15) depends only on the H_m-orbit of the pair $(\lambda, \mu) \in \mathfrak{h}_m^* \times \mathfrak{h}_m^*$. Moreover, the orbit is determined by these polynomials uniquely. Furthermore, any sequence of monic polynomials $Q_1(x), \dots, Q_l(x)$ of total degree m with complex coefficients arises in this way, provided that for an even n the last polynomial $Q_l(x)$ is also even.

In the next subsection, to each of these orbits we will attach an irreducible $\mathrm{Y}(\mathfrak{g}_n)$-module, unless $\mathfrak{g}_n = \mathfrak{so}_n$ with even n and $\mu_a + \rho_a = 0$ for some a. In the latter case, to such an orbit we will attach two irreducible $\mathrm{Y}(\mathfrak{g}_n)$-modules, not equivalent to each other. For these two, we will have $\delta = 1$ and $\delta = -1$.

In any case, the polynomials $Q_1(x), \dots, Q_l(x)$ of the attached modules will be given by (1.15). For the definition of $Q_1(x), \dots, Q_l(x)$ for any irreducible finite-dimensional $\mathrm{Y}(\mathfrak{g}_n)$-module see [KN, Sections 5.3, 5.4, 5.5].

1.6. For $k = 0, 1, \dots, n$ let us denote by Φ_t^{-k} the $\mathrm{Y}(\mathfrak{gl}_n)$-module obtained by pulling the $\mathrm{Y}(\mathfrak{gl}_n)$-module Φ_t^k back through the automorphism (1.11). Note that due to Lemma 2.3 the $\mathrm{Y}(\mathfrak{gl}_n)$-module Φ_t^{-k} is similar to the module Φ_{1-t}^{n-k}.

Take any $(\lambda, \mu) \in \mathfrak{h}_m^* \times \mathfrak{h}_m^*$ such that all labels of the weight $\nu = \lambda - \mu + \kappa$ belong to the set $\{1, \dots, n-1\}$. Let the weights λ and μ vary so that ν is fixed. For the Cartan subalgebra $\mathfrak{h}_m$ of $\mathfrak{f}_m$ the weight μ is *generic* if $\mu_a - \mu_b \notin \mathbb{Z}$ and $\mu_a + \mu_b \notin \mathbb{Z}$ whenever $a \neq b$, and $2\mu_a \notin \mathbb{Z}$ for any a.

Consider the tensor product of $\mathrm{Y}(\mathfrak{gl}_n)$-modules (1.8) where μ, ν and ρ are now weights of $\mathfrak{f}_m$, not of $\mathfrak{gl}_m$ as before. It is known that the restriction of the $\mathrm{Y}(\mathfrak{gl}_n)$-module (1.8) to the subalgebra $\mathrm{Y}(\mathfrak{g}_n) \subset \mathrm{Y}(\mathfrak{gl}_n)$ is irreducible if (but not only if) the weight μ of $\mathfrak{f}_m$ is generic, see [KN, Theorems 5.3, 5.4, 5.5]. Moreover, if (but not only if) μ is generic then all the $\mathrm{Y}(\mathfrak{g}_n)$-modules obtained from (1.8) by permuting the tensor factors and by replacing any tensor factor Φ_t^k by Φ_t^{-k}, are equivalent to each other. In particular, for each generic weight μ of $\mathfrak{f}_m$ there is a unique non-zero $\mathrm{Y}(\mathfrak{g}_n)$-intertwining operator

$$\Phi^{\nu_1}_{\mu_1+\rho_1+\frac{1}{2}} \otimes \dots \otimes \Phi^{\nu_m}_{\mu_m+\rho_m+\frac{1}{2}} \to \Phi^{-\nu_1}_{\mu_1+\rho_1+\frac{1}{2}} \otimes \dots \otimes \Phi^{-\nu_m}_{\mu_m+\rho_m+\frac{1}{2}} \tag{1.16}$$

corresponding to the element of the group H_m of the maximal length. We will denote this operator by $J(\mu)$; it is unique up to a multiplier from $\mathbb{C} \setminus \{0\}$.

For all generic weights μ of $\mathfrak{f}_m$, the source and the target vector spaces of the operators $J(\mu)$ are the same tensor product of exterior powers of $\mathbb{C}^n$

$$\Lambda^{\nu_1}(\mathbb{C}^n) \otimes \dots \otimes \Lambda^{\nu_m}(\mathbb{C}^n).$$

Hence $J(\mu)$ is a function of μ taking values in the space of linear operators on this tensor product. The multipliers from $\mathbb{C} \setminus \{0\}$ can be chosen so that $J(\mu)$ becomes a rational function of μ, see Section 3 for a particular choice. Any such a choice allows to determine intertwining operators (1.16) for those non-generic weights μ where the function $J(\mu)$ is regular. The next theorem summarizes the results of [KN, Sections 5.3, 5.4, 5.5]. It will be generalized in Section 3.7 of the present article.

For the Lie algebra $\mathfrak{f}_m$ the weight λ is *dominant* if $\lambda_a - \lambda_b \neq -1, -2, \dots$ and $\lambda_a + \lambda_b \neq 1, 2, \dots$ for all $a < b$, with an extra condition that $\lambda_a \neq 1, 2, \dots$ in the case $\mathfrak{f}_m = \mathfrak{sp}_{2m}$. The pair $(\lambda, \mu) \in \mathfrak{h}_m^* \times \mathfrak{h}_m^*$ is called *good* if the weight $\lambda + \rho$ is dominant and

$$\nu_a \geqslant \nu_b \quad \text{whenever} \quad \lambda_a - \lambda_b + \rho_a - \rho_b = 0 \quad \text{and} \quad a < b, \tag{1.17}$$

$$\nu_a + \nu_b \leqslant n \quad \text{whenever} \quad \lambda_a + \lambda_b + \rho_a + \rho_b = 0 \quad \text{and} \quad a < b, \tag{1.18}$$

with an extra condition that in the case $\mathfrak{f}_m = \mathfrak{sp}_{2m}$

$$2\nu_a \leqslant n \quad \text{whenever} \quad \lambda_a + \rho_a = 0. \tag{1.19}$$

The orbit of any $(\lambda, \mu) \in \mathfrak{h}_m^* \times \mathfrak{h}_m^*$ under the shifted action of H_m on $\mathfrak{h}_m^*$ does contain a good pair. Here we assume only that $\nu_1, \dots, \nu_m \in \{1, \dots, n-1\}$. Note that for $\mathfrak{f}_m = \mathfrak{so}_{2m}$ a good pair is contained already in the orbit of any (λ, μ) under the shifted action of the Weyl group, which is a subgroup of H_m of index two. However, this fact will not be used in the present article.

Theorem 1.2. *For the fixed $\nu = \lambda - \mu + \kappa$ the multipliers from $\mathbb{C} \setminus \{0\}$ of the operators* (1.16) *for all generic weights μ can be chosen so that the rational function $J(\mu)$ is regular and non-zero whenever the pair (λ, μ) is good. Then for any good pair (λ, μ) the quotient of* (1.8) *by the kernel of the operator $J(\mu)$ is an irreducible $\mathrm{Y}(\mathfrak{g}_n)$-module, unless $\mathfrak{g}_n = \mathfrak{so}_n$ with even n and $\mu_a + \rho_a = 0$ for some index a. In the latter case the quotient of* (1.8) *by the kernel of $J(\mu)$ is a direct sum of two irreducible $\mathrm{Y}(\mathfrak{g}_n)$-modules, not equivalent to each other. For any $\mathfrak{g}_n$ the polynomials $Q_1(x), \dots, Q_l(x)$ of the irreducible $\mathrm{Y}(\mathfrak{g}_n)$-modules occurring as above are given by* (1.15).

2. Intertwining operators

2.1. In this subsection we develop the formalism of R-matrices; it will be used to produce explicit formulas for intertwining operators (1.9) and (1.16) over $\mathrm{Y}(\mathfrak{gl}_n)$ and $\mathrm{Y}(\mathfrak{g}_n)$ respectively. Let P denote the linear operator on $(\mathbb{C}^n)^{\otimes 2}$ exchanging the two tensor factors. The *Yang R-matrix* is the rational function of a variable x

$$R(x) = 1 - Px^{-1}$$

taking values in

$$\mathrm{End}\big(\mathbb{C}^n\big)^{\otimes 2} = \big(\mathrm{End}\,\mathbb{C}^n\big)^{\otimes 2}.$$

It satisfies the *Yang–Baxter equation* in $(\mathrm{End}\,\mathbb{C}^n)^{\otimes 3}$

$$R_{12}(x)R_{13}(x+y)R_{23}(y) = R_{23}(y)R_{13}(x+y)R_{12}(x). \tag{2.1}$$

As usual, the subscripts in (2.1) indicate different embeddings of the algebra $(\mathrm{End}\,\mathbb{C}^n)^{\otimes 2}$ to $(\mathrm{End}\,\mathbb{C}^n)^{\otimes 3}$ so that $R_{12}(x) = R(x) \otimes 1$ and $R_{23}(y) = 1 \otimes R(y)$.

We have

$$P = \sum_{i,j=1}^{n} E_{ij} \otimes E_{ji}.$$

Denote

$$\overline{P} = \sum_{i,j=1}^{n} E_{ij} \otimes E_{ij} \quad \text{and} \quad \widetilde{P} = \sum_{i,j=1}^{n} \widetilde{E}_{ij} \otimes E_{ji}. \tag{2.2}$$

Then put

$$\overline{R}(x) = 1 - \overline{P}x^{-1} \quad \text{and} \quad \widetilde{R}(x) = 1 - \widetilde{P}x^{-1}.$$

The values of the function $\overline{R}(x)$ are obtained from those of $R(x)$ by applying the matrix transposition to the first tensor factor of $(\mathrm{End}\,\mathbb{C}^n)^{\otimes 2}$. The values of the function $\widetilde{R}(x)$ are obtained from those of $R(x)$ by applying to the first tensor factor of $(\mathrm{End}\,\mathbb{C}^n)^{\otimes 2}$ the conjugation with respect to the form $\langle\,,\rangle$. Now (2.1) and the relation $PR(x)P = R(x)$ imply that

$$\overline{R}_{12}(x+y)\overline{R}_{13}(x)R_{23}(y) = R_{23}(y)\overline{R}_{13}(x)\overline{R}_{12}(x+y), \tag{2.3}$$

$$\widetilde{R}_{12}(x+y)\widetilde{R}_{13}(x)R_{23}(y) = R_{23}(y)\widetilde{R}_{13}(x)\widetilde{R}_{12}(x+y). \tag{2.4}$$

Finally, denote

$$\widehat{P} = \sum_{i,j=1}^{n} \widetilde{E}_{ij} \otimes E_{ij}$$

and put

$$\widehat{R}(x) = 1 - \widehat{P}x^{-1}.$$

The values of the function $\widehat{R}(x)$ are obtained from those of $\overline{R}(x)$ by applying to the first tensor factor of $(\operatorname{End}\mathbb{C}^n)^{\otimes 2}$ the conjugation with respect to $\langle\,,\rangle$. Therefore the relation (2.3) implies that

$$\widehat{R}_{12}(x)\widehat{R}_{13}(x+y)R_{23}(y)=R_{23}(y)\widehat{R}_{13}(x+y)\widehat{R}_{12}(x). \tag{2.5}$$

Note that $P\overline{R}(x)P=\overline{R}(x)$ and $P\widetilde{R}(x)P=\widetilde{R}(x)$. Further, due to $\overline{P}^2=n\overline{P}$ we have $\overline{R}(x)\overline{R}(n-x)=1$. By using the latter relation, (2.3) also implies that

$$\overline{R}_{12}(n-x-y)\widehat{R}_{13}(x)\widetilde{R}_{23}(y)=\widetilde{R}_{23}(y)\widehat{R}_{13}(x)\overline{R}_{12}(n-x-y). \tag{2.6}$$

Observe that $R(1)=1-P$. A direct calculation now shows that

$$R_{12}(x)R_{13}(x+1)R_{23}(1)=\bigl(1-(P_{12}+P_{13})x^{-1}\bigr)(1-P_{23}). \tag{2.7}$$

In particular, when $y=1$, the rational function of x at either side of (2.1) has no pole at $x=-1$. Similarly, when $y=1$, the rational function of x at either side of (2.3) or (2.4) has no pole at $x=-1$. Moreover,

$$\widetilde{R}_{12}(x+1)\widetilde{R}_{13}(x)R_{23}(1)=\bigl(1-(\widetilde{P}_{12}+\widetilde{P}_{13})x^{-1}\bigr)(1-P_{23}). \tag{2.8}$$

2.2. Now consider the representation of $\mathrm{Y}(\mathfrak{gl}_n)$ obtained by pulling the defining representation of $\mathfrak{gl}_n$ back through the homomorphism (1.4), and then back through the automorphism (1.2) of $\mathrm{Y}(\mathfrak{gl}_n)$. The resulting $\mathrm{Y}(\mathfrak{gl}_n)$-module has been denoted by Φ_t^1. Note that under this representation

$$T(x)\mapsto \overline{R}(t-x). \tag{2.9}$$

Here on the left we regard $n\times n$ matrices with entries from the algebra $\mathrm{Y}(\mathfrak{gl}_n)$ as elements of $\operatorname{End}\mathbb{C}^n\otimes \mathrm{Y}(\mathfrak{gl}_n)$; we will always do so in this and in the next section. Our explicit formulas for intertwining operators (1.9) and (1.16) are based on the following simple and well-known lemma, first appeared in [KRS].

Lemma 2.1. *For any $k=1,\dots,n$ and $t\in\mathbb{C}$ the $\mathrm{Y}(\mathfrak{gl}_n)$-module Φ_t^k appears as the submodule of*

$$\Phi^1_{t+k-1}\otimes\cdots\otimes\Phi^1_{t+1}\otimes\Phi^1_t \tag{2.10}$$

with the underlying subspace

$$\Lambda^k\bigl(\mathbb{C}^n\bigr)\subset\bigl(\mathbb{C}^n\bigr)^{\otimes k}. \tag{2.11}$$

Proof. First consider the standard action of the Lie algebra $\mathfrak{gl}_n$ on the vector space $(\mathbb{C}^n)^{\otimes k}$. Turn this vector space into an $\mathrm{Y}(\mathfrak{gl}_n)$-module, by pulling the action of $\mathfrak{gl}_n$ back through the homomorphism (1.4) and then back through the automorphism (1.2) of $\mathrm{Y}(\mathfrak{gl}_n)$. Under the resulting representation of $\mathrm{Y}(\mathfrak{gl}_n)$

$$T(x)\mapsto 1+(\overline{P}_{01}+\cdots+\overline{P}_{0,k-1}+\overline{P}_{0k})(x-t)^{-1}, \tag{2.12}$$

see (2.2). Here we use the subscripts $0,\dots,k-1,k$ rather than $1,\dots,k,k+1$ to label the tensor factors of $(\operatorname{End}\mathbb{C}^n)^{\otimes(k+1)}$. The $\mathrm{Y}(\mathfrak{gl}_n)$-module Φ_t^k is defined by restricting the above described action of $\mathrm{Y}(\mathfrak{gl}_n)$ to the subspace (2.11).

Next consider the action of $\mathrm{Y}(\mathfrak{gl}_n)$ on (2.10). By (1.5) and (2.9), then

$$T(x) \mapsto \overline{R}_{01}(t+k-1-x) \cdots \overline{R}_{0,k-1}(t+1-x)\overline{R}_{0k}(t-x).$$

The latter product is a rational function of x, valued in $(\operatorname{End}\mathbb{C}^n)^{\otimes(k+1)}$. It tends to 1 when $x \to \infty$, and has a simple pole at $x = t$ with the residue

$$P_{01} + \cdots + P_{0,k-1} + P_{0k}.$$

Further, by an observation made after (2.7) for any $i = 1, \ldots, k-1$ the product

$$\overline{R}_{01}(t+k-1-x) \cdots \overline{R}_{0,k-1}(t+1-x)\overline{R}_{0k}(t-x)(1-P_{i,i+1})$$

has no pole at $x = t+k-i$. Hence the restriction of the action of $\mathrm{Y}(\mathfrak{gl}_n)$ on (2.10) to the subspace (2.11) coincides with the restriction of the action of $\mathrm{Y}(\mathfrak{gl}_n)$ on the vector space $(\mathbb{C}^n)^{\otimes k}$ described by the assignment (2.12). $\square$

The assignment (1.11) defines a coalgebra anti-automorphism of $\mathrm{Y}(\mathfrak{gl}_n)$. This immediately implies another lemma, to be used when working with $\mathrm{Y}(\mathfrak{g}_n)$.

Lemma 2.2. *The* $\mathrm{Y}(\mathfrak{gl}_n)$*-module obtained by pulling* (2.10) *back through the automorphism* (1.11) *is equivalent to*

$$\Phi_t^{-1} \otimes \Phi_{t+1}^{-1} \otimes \cdots \otimes \Phi_{t+k-1}^{-1}.$$

The linear operator on $(\mathbb{C}^n)^{\otimes k}$ *reversing the order of tensor factors intertwines the two equivalent modules.*

For $k = 0, 1, \ldots, n$ denote respectively by $\dot{\Phi}_t^k$ and $\dot{\Phi}_t^{-k}$ the $\mathrm{Y}(\mathfrak{gl}_n)$-modules obtained by pulling Φ_t^k and Φ_t^{-k} back through the automorphism (1.3) where

$$f(x) = \frac{x-t}{x-t+1}.$$

Lemma 2.3. *The* $\mathrm{Y}(\mathfrak{gl}_n)$*-modules* Φ_t^{-k} *and* $\dot{\Phi}_{1-t}^{n-k}$ *are mutually equivalent.*

Proof. Let $e_1, \ldots, e_n$ be the standard basis vectors of $\mathbb{C}^n$. Denote by x_i the operator of left multiplication by the element e_i in the exterior algebra $\Lambda(\mathbb{C}^n)$. Let ∂_i be the corresponding operator of left derivation on $\Lambda(\mathbb{C}^n)$. Note that

$$x_i \partial_j + \partial_j x_i = \delta_{ij}.$$

For each $k = 0, 1, \ldots, n$ the action of the algebra $\mathrm{Y}(\mathfrak{gl}_n)$ on its module Φ_t^k is defined by the homomorphism $\mathrm{Y}(\mathfrak{gl}_n) \to \operatorname{End}(\Lambda(\mathbb{C}^n))$ which maps

$$T_{ij}(x) \mapsto \delta_{ij} + \frac{x_i \partial_j}{x-t}. \tag{2.13}$$

It suffices to prove Lemma 2.3 for any choices of the symmetric and of the alternating form $\langle\,,\rangle$ on $\mathbb{C}^n$. For the proof only, choose the form as follows. Put $\theta_i = -1$ if $\mathfrak{g}_n = \mathfrak{sp}_n$ and $i > n/2$; otherwise put $\theta_i = 1$. Set $\langle e_i, e_j\rangle = \theta_i \delta_{\tilde{\imath} j}$ where we write $\tilde{\imath} = n - i + 1$ for short. Then

$$\widetilde{E}_{ij} = \theta_i \theta_j E_{\tilde{\jmath}\tilde{\imath}}. \tag{2.14}$$

Using (1.11) and (2.13), the action of $\mathrm{Y}(\mathfrak{gl}_n)$ on Φ_t^{-k} is then defined by

$$T_{ij}(x) \mapsto \delta_{ij} - \frac{\theta_i \theta_j x_{\tilde{j}} \partial_{\tilde{\imath}}}{x+t}. \tag{2.15}$$

On the other hand, the action of the algebra $\mathrm{Y}(\mathfrak{gl}_n)$ on its module $\dot{\Phi}_{1-t}^{n-k}$ is defined by the homomorphism $\mathrm{Y}(\mathfrak{gl}_n) \to \mathrm{End}(\Lambda(\mathbb{C}^n))$ which maps

$$T_{ij}(x) \mapsto \frac{x+t-1}{x+t}\left(\delta_{ij} + \frac{x_i \partial_j}{x+t-1}\right) = \delta_{ij} - \frac{\partial_j x_i}{x+t}. \tag{2.16}$$

By comparing the right-hand sides of the assignments (2.15) and (2.16), the equivalence of the $\mathrm{Y}(\mathfrak{gl}_n)$-modules $\dot{\Phi}_{1-t}^{n-k}$ and Φ_t^{-k} can be realized by the linear map of the underlying vector spaces $\Lambda^{n-k}(\mathbb{C}^n) \to \Lambda^k(\mathbb{C}^n)$:

$$e_{i_1} \wedge \cdots \wedge e_{i_{n-k}} \mapsto (\theta_{j_1} e_{\tilde{j}_1}) \wedge \cdots \wedge (\theta_{j_k} e_{\tilde{j}_k})$$

where

$$e_{i_1} \wedge \cdots \wedge e_{i_{n-k}} \wedge e_{j_1} \wedge \cdots \wedge e_{j_k} = e_1 \wedge \cdots \wedge e_n.$$

Indeed, for any indices $i, j = 1, \ldots, n$ this map intertwines the operators $\partial_j x_i$ and $\theta_i \theta_j x_{\tilde{j}} \partial_{\tilde{\imath}}$ on the vector spaces $\Lambda^{n-k}(\mathbb{C}^n)$ and $\Lambda^k(\mathbb{C}^n)$ respectively. $\square$

2.3. Once again take any weights λ and μ of $\mathfrak{gl}_m$ such that all the labels of the weight $\nu = \lambda - \mu$ belong to the set $\{1, \ldots, n-1\}$. Put $N = \nu_1 + \cdots + \nu_m$ and split the sequence $1, \ldots, N$ to the consecutive segments of lengths $\nu_1, \ldots, \nu_m$. Hence the ath segment is the sequence of numbers

$$p = \nu_1 + \cdots + \nu_{a-1} + i \quad \text{where} \quad i = 1, \ldots, \nu_a. \tag{2.17}$$

Then put

$$x_p = \mu_a + \rho_a + \tfrac{1}{2} + \nu_a - i. \tag{2.18}$$

Let

$$P_\nu : \left(\mathbb{C}^n\right)^{\otimes \nu_1} \otimes \cdots \otimes \left(\mathbb{C}^n\right)^{\otimes \nu_m} \to \left(\mathbb{C}^n\right)^{\otimes \nu_m} \otimes \cdots \otimes \left(\mathbb{C}^n\right)^{\otimes \nu_1}$$

be the linear operator on $(\mathbb{C}^n)^{\otimes N}$ reversing the order of the tensor factors $\mathbb{C}^n$ by segments of lengths $\nu_1, \ldots, \nu_m$ in their sequence. Within any segment, the order of tensor factors is not changed. Then $P_\nu = P$ if $m = 2$ and $\nu_1 = \nu_2 = 1$.

Now let the weight $\mu \in \mathfrak{t}_m^*$ vary while the weight ν is fixed. Let $1', \ldots, N'$ be the sequence obtained from $1, \ldots, N$ by reversing the order of the terms by the segments of lengths $\nu_1, \ldots, \nu_m$ introduced above. Within every segment, the order of terms is not changed. Let $1'', \ldots, N''$ be the sequence obtained from $1, \ldots, N$ by reversing the order of terms within the segments. The order of the segments themselves is not changed now. Take the ordered product

$$B(\mu) = \prod_{(p,q)}^{\longrightarrow} R_{pq}(x_q - x_p) \tag{2.19}$$

where $p < q$ and they belong to different segments of the sequence $1, \dots, N$. Here the pair (p, q) precedes (r, s) if $p < r$ or if $p = r$ and q precedes s in the sequence $1'', \dots, N''$. Note that $B(\mu)$ is a rational function of μ without poles at generic weights of $\mathfrak{gl}_m$.

Proposition 2.4. *Suppose that the weight μ of $\mathfrak{gl}_m$ is generic. Then $P_\nu B(\mu)$ is an intertwining operator of* $\mathrm{Y}(\mathfrak{gl}_n)$*-modules*

$$\Phi^1_{x_1} \otimes \cdots \otimes \Phi^1_{x_N} \to \Phi^1_{x_{1'}} \otimes \cdots \otimes \Phi^1_{x_{N'}}. \tag{2.20}$$

Proof. Under the action of $\mathrm{Y}(\mathfrak{gl}_n)$ on the source module in (2.20),

$$T(x) \mapsto \overline{R}_{01}(x_1 - x) \cdots \overline{R}_{0N}(x_N - x). \tag{2.21}$$

Like in the proof of Lemma 2.1, here we use the subscripts $0, 1, \dots, N$ rather than $1, 2, \dots, N+1$ to label the tensor factors of $(\operatorname{End} \mathbb{C}^n)^{\otimes (N+1)}$. Similarly, under the action of $\mathrm{Y}(\mathfrak{gl}_n)$ on the target module in (2.20),

$$T(x) \mapsto \overline{R}_{01}(x_{1'} - x) \cdots \overline{R}_{0N}(x_{N'} - x). \tag{2.22}$$

Denote by X and X' the right-hand sides of the assignments (2.21) and (2.22) respectively. By using the relation (2.3) repeatedly, we get

$$P_\nu B(\mu) X = P_\nu \overline{R}_{01'}(x_{1'} - x) \cdots \overline{R}_{0N'}(x_{N'} - x) B(\mu) = X' P_\nu B(\mu).$$

The equality of the left- and of the right-hand sides here proves the claim. $\square$

2.4. Let A_k be the operator of antisymmetrization on $(\mathbb{C}^n)^{\otimes k}$, normalized so that $A_k^2 = A_k$. The subspace (2.11) is the image of A_k. The ordered product

$$\prod^{\longrightarrow}_{(i,j)} R_{ij}(j - i) = k! A_k \tag{2.23}$$

where $1 \leqslant i < j \leqslant k$ and the pairs (i, j) are ordered lexicographically. The formula (2.23) has appeared in [KRS] but was already known to Jucys [J].

Let $e_1, \dots, e_n$ be the standard basis vectors of $\mathbb{C}^n$. For each $k = 1, \dots, n$ consider the vector $\varphi_k = e_1 \wedge \cdots \wedge e_k \in \Lambda^k(\mathbb{C}^n)$. Using the embedding (2.11),

$$\varphi_k = A_k(e_1 \otimes \cdots \otimes e_k).$$

The next proposition is known; see [N, Theorem 2] for a more general result. It is still instructive to give a proof here, as it will be used later on.

Proposition 2.5. *For any generic weight μ of $\mathfrak{gl}_m$ the vector $\varphi_{\nu_1} \otimes \cdots \otimes \varphi_{\nu_m}$ is an eigenvector of the operator $B(\mu)$ on $(\mathbb{C}^n)^{\otimes N}$ with the eigenvalue*

$$\prod_{1 \leqslant a < b \leqslant m} \begin{cases} \dfrac{\lambda_a - \lambda_b + \rho_a - \rho_b + \nu_b}{\mu_a - \mu_b + \rho_a - \rho_b} & \text{if} \quad \nu_a \leqslant \nu_b; \\[2ex] \dfrac{\lambda_a - \lambda_b + \rho_a - \rho_b + \nu_b}{\lambda_a - \lambda_b + \rho_a - \rho_b} & \text{if} \quad \nu_a \geqslant \nu_b. \end{cases} \tag{2.24}$$

Proof. This proposition immediately follows from its particular case of $m = 2$. Let us consider this case only. Then we have

$$
\begin{aligned}
B(\mu)(\varphi_{\nu_1} \otimes \varphi_{\nu_2}) = & \prod_{i=1,\dots,\nu_1}^{\longrightarrow} \left(\prod_{j=1,\dots,\nu_2}^{\longleftarrow} R_{i,\nu_1+j}(x_{\nu_1+j} - x_i) \right) \\
& \times (A_{\nu_1} \otimes A_{\nu_2})(e_1 \otimes \cdots \otimes e_{\nu_1} \otimes e_1 \otimes \cdots \otimes e_{\nu_2}) \\
= & (A_{\nu_1} \otimes A_{\nu_2}) \prod_{i=1,\dots,\nu_1}^{\longleftarrow} \left(\prod_{j=1,\dots,\nu_2}^{\longrightarrow} R_{i,\nu_1+j}(x_{\nu_1+j} - x_i) \right) \qquad (2.25) \\
& \times (e_1 \otimes \cdots \otimes e_{\nu_1} \otimes e_1 \otimes \cdots \otimes e_{\nu_2}) \qquad (2.26)
\end{aligned}
$$

where the last equality is obtained by using the formula (2.23) and by applying (2.1) repeatedly. The reversed arrow over the product symbol indicates that the factors corresponding to the running index are arranged from right to left.

First suppose that $\nu_1 \leqslant \nu_2$. Arguing like in the proof of Lemma 2.1 we can always rewrite the product displayed in line (2.25) as

$$
(A_{\nu_1} \otimes A_{\nu_2}) \prod_{i=1,\dots,\nu_1}^{\longleftarrow} \left(1 - (x_{\nu_1+\nu_2} - x_i)^{-1} \sum_{j=1}^{\nu_2} P_{i,\nu_1+j} \right)
$$

where any sum over $j = 1, \dots, \nu_2$ clearly commutes with the operator $1 \otimes A_{\nu_2}$ on $(\mathbb{C}^n)^{\otimes(\nu_1+\nu_2)}$. But for $\nu_1 \leqslant \nu_2$ the operators $(1 \otimes A_{\nu_2})P_{i,\nu_1+j}$ with $i \neq j$ annihilate the vector (2.26) while the operators P_{i,ν_1+i} do not change it. Hence applying the operator (2.25) to (2.26) gives the vector $\varphi_{\nu_1} \otimes \varphi_{\nu_2}$ multiplied by

$$
\prod_{i=1,\dots,\nu_1} \left(1 - (x_{\nu_1+\nu_2} - x_i)^{-1} \right) = \frac{\lambda_1 - \lambda_2 + \rho_1 - \rho_2 + \nu_2}{\mu_1 - \mu_2 + \rho_1 - \rho_2}.
$$

Next suppose that $\nu_1 \geqslant \nu_2$. We can always rewrite the product (2.25) as

$$
(A_{\nu_1} \otimes A_{\nu_2}) \prod_{j=1,\dots,\nu_2}^{\longrightarrow} \left(\prod_{i=1,\dots,\nu_1}^{\longleftarrow} R_{i,\nu_1+j}(x_{\nu_1+j} - x_i) \right).
$$

Arguing like in the proof of Lemma 2.1 we can rewrite the latter product as

$$
(A_{\nu_1} \otimes A_{\nu_2}) \prod_{j=1,\dots,\nu_2}^{\longrightarrow} \left(1 - (x_{\nu_1+j} - x_1)^{-1} \sum_{i=1}^{\nu_1} P_{i,\nu_1+j} \right)
$$

where any sum over $i = 1, \dots, \nu_1$ clearly commutes with the operator $A_{\nu_1} \otimes 1$ on $(\mathbb{C}^n)^{\otimes(\nu_1+\nu_2)}$. But for $\nu_1 \geqslant \nu_2$ the operators $(A_{\nu_1} \otimes 1)P_{i,\nu_1+j}$ with $i \neq j$ annihilate the vector (2.26) while the operators P_{i,ν_1+i} do not change it. Hence applying the operator (2.25) to (2.26) gives the vector $\varphi_{\nu_1} \otimes \varphi_{\nu_2}$ multiplied by

$$
\prod_{j=1,\dots,\nu_2} \left(1 - (x_{\nu_1+j} - x_1)^{-1} \right) = \frac{\lambda_1 - \lambda_2 + \rho_1 - \rho_2 + \nu_2}{\lambda_1 - \lambda_2 + \rho_1 - \rho_2}.
$$

This observation completes the proof of Proposition 2.5. $\square$

2.5. Using the relations (2.1) and $R(1) = 1 - P$ repeatedly, one demonstrates that for any generic weight μ of $\mathfrak{gl}_m$ the operator $B(\mu)$ preserves the subspace

$$\Lambda^{\nu_1}\left(\mathbb{C}^n\right) \otimes \cdots \otimes \Lambda^{\nu_m}\left(\mathbb{C}^n\right) \subset \left(\mathbb{C}^n\right)^{\otimes N}, \tag{2.27}$$

see the proof of Proposition 2.5 above. Moreover, we have another proposition.

Proposition 2.6. *For any generic weight μ of $\mathfrak{gl}_m$ the restriction of $B(\mu)$ to the subspace* (2.27) *coincides with that of the operator*

$$\prod_{1 \leqslant a<b \leqslant m}^{\longrightarrow} \left(1 + \sum_{d>0} \sum_{\substack{i_1,\dots,i_d \\ j_1,\dots,j_d}} \prod_{k=1}^{d} \frac{P_{i_k j_k}}{\lambda_a - \lambda_b + \rho_a - \rho_b + \nu_b - k}\right) \tag{2.28}$$

where we order the pairs (a, b) lexicographically while $i_1, \dots, i_d$ and $j_1, \dots, j_d$ are pairwise distinct numbers respectively from the ath and bth segments of the sequence $1, \dots, N$ taken so that different are all corresponding sets of d pairs

$$(i_1, j_1), \dots, (i_d, j_d). \tag{2.29}$$

Proof. Note that as the indices $i_1, \dots, i_d$ and $j_1, \dots, j_d$ are pairwise distinct, we have $d \leqslant \nu_a$ and $d \leqslant \nu_b$ for any non-zero summand in the brackets in (2.28).

The proposition immediately follows from its particular case of $m = 2$. Let us consider this case only. Then (2.28) is an operator on $(\mathbb{C}^n)^{\otimes(\nu_1+\nu_2)}$ equal to

$$\sum_{d \geqslant 0} \sum_{\substack{i_1,\dots,i_d \\ j_1,\dots,j_d}} \prod_{k=1}^{d} \frac{P_{i_k j_k}}{x_k - x_{\nu_1+\nu_2}} \tag{2.30}$$

where $i_1, \dots, i_d$ and $j_1, \dots, j_d$ are pairwise distinct numbers taken respectively from $1, \dots, \nu_1$ and $\nu_1 + 1, \dots, \nu_1 + \nu_2$ so that different are all the corresponding sets of d pairs (2.29). Here we use the equalities

$$x_{\nu_1+\nu_2} = \mu_2 + \rho_2 + \tfrac{1}{2} \quad \text{and} \quad x_k = \mu_1 + \rho_1 + \tfrac{1}{2} + \nu_1 - k$$

for any $k \leqslant \nu_1$. We also assume that 1 is the only term in (2.30) with $d = 0$. On the other hand, for $m = 2$ by definition we have

$$B(\mu) = \prod_{i=1,\dots,\nu_1}^{\longrightarrow} \left(\prod_{j=1,\dots,\nu_2}^{\longleftarrow} R_{i,\nu_1+j}(x_{\nu_1+j} - x_i)\right). \tag{2.31}$$

Let us relate two operators on the vector space $(\mathbb{C}^n)^{\otimes(\nu_1+\nu_2)}$ by the symbol $\equiv$ if their actions coincide on the subspace

$$\Lambda^{\nu_1}\left(\mathbb{C}^n\right) \otimes \Lambda^{\nu_2}\left(\mathbb{C}^n\right) \subset \left(\mathbb{C}^n\right)^{\otimes(\nu_1+\nu_2)}. \tag{2.32}$$

We will establish the relation $\equiv$ between (2.30) and (2.31) by induction on ν_1.

If $\nu_1 > 1$ then we assume the latter relation holds for $\nu_1 - 1$ instead of ν_1; if $\nu_1 = 1$ then we are not making any assumption. Arguing like in the proof of Lemma 2.1 and then using the induction assumption, (2.31) is related by $\equiv$ to

$$\overrightarrow{\prod_{i=1,\dots,\nu_1-1}}\left(\overleftarrow{\prod_{j=1,\dots,\nu_2}} R_{i,\nu_1+j}(x_{\nu_1+j} - x_i)\right) \times \left(1 + \sum_{j=1}^{\nu_2} \frac{P_{\nu_1,\nu_1+j}}{x_{\nu_1} - x_{\nu_1+\nu_2}}\right)$$

$$\equiv \left(\sum_{d\geqslant 0}\sum_{\substack{i_1,\dots,i_d\\ j_1,\dots,j_d}}\prod_{k=1}^{d}\frac{P_{i_k j_k}}{x_k - x_{\nu_1+\nu_2}}\right) \times \left(1 + \sum_{j=1}^{\nu_2} \frac{P_{\nu_1,\nu_1+j}}{x_{\nu_1} - x_{\nu_1+\nu_2}}\right)$$

where $i_1, \dots, i_d$ and $j_1, \dots, j_d$ are distinct indices taken from $1, \dots, \nu_1 - 1$ and $\nu_1 + 1, \dots, \nu_1 + \nu_2$, respectively. We assume that all corresponding sets of d pairs (2.29) are different. The right-hand side of the last relation equals

$$\sum_{d\geqslant 0}\sum_{\substack{i_1,\dots,i_d\\ j_1,\dots,j_d}}\prod_{k=1}^{d}\frac{P_{i_k j_k}}{x_k - x_{\nu_1+\nu_2}} \tag{2.33}$$

$$+\sum_{d>0}\sum_{\substack{i_1,\dots,i_d\\ j_1,\dots,j_d}}\sum_{l=1}^{d}\left(\prod_{k=1}^{d}\frac{P_{i_k j_k}}{x_k - x_{\nu_1+\nu_2}}\right)\frac{P_{\nu_1 j_l}}{x_{\nu_1} - x_{\nu_1+\nu_2}} \tag{2.34}$$

$$+\sum_{d\geqslant 0}\sum_{\substack{i_1,\dots,i_d\\ j_1,\dots,j_d}}\sum_{j}\left(\prod_{k=1}^{d}\frac{P_{i_k j_k}}{x_k - x_{\nu_1+\nu_2}}\right)\frac{P_{\nu_1,\nu_1+j}}{x_{\nu_1} - x_{\nu_1+\nu_2}} \tag{2.35}$$

where the index j is taken from $1, \dots, \nu_2$ but $\nu_1 + j \neq j_1, \dots, j_d$ however.

Consider the sum displayed in the line (2.34). Here we have

$$\left(\prod_{k=1}^{d} P_{i_k j_k}\right) P_{\nu_1 j_l} = \left(\prod_{k\neq l} P_{i_k j_k}\right) P_{i_l j_l} P_{\nu_1 j_l} = \left(\prod_{k\neq l} P_{i_k j_k}\right) P_{\nu_1 j_l} P_{i_k \nu_1} \equiv -\left(\prod_{k\neq l} P_{i_k j_k}\right) P_{\nu_1 j_l}$$

where the right-hand side does not involve the index i_l. Now let us fix a number $j \in \{1, \dots, \nu_2\}$ and take any set of d pairs (2.29) such that one of the pairs contains the number $\nu_1 + j$. Then this number has the form of j_l for some index l. If the set of the other $d - 1$ pairs (i_k, j_k) with $k \neq l$ is also fixed, then i_l ranges over a set of cardinality $\nu_1 - d$, namely over the fixed set

$$\{1, \dots, \nu_1 - 1\} \setminus \{i_1, \dots, i_{l-1}, i_{l+1}, \dots, i_d\}.$$

Now let us perform the summation over the indices i_l, j_l and l in (2.34) first of all the running indices. After that, rename the running indices $i_{l+1}, \dots, i_d$ and $j_{l+1}, \dots, j_d$ respectively by $i_l, \dots, i_{d-1}$ and $j_l, \dots, j_{d-1}$. By the arguments given in the previous paragraph, the sum (2.34) gets related by $\equiv$ to the sum

$$\sum_{d>0}\sum_{\substack{i_1,\dots,i_{d-1}\\ j_1,\dots,j_{d-1}}}\sum_{j}\left(\prod_{k=1}^{d-1}\frac{P_{i_k j_k}}{x_k - x_{\nu_1+\nu_2}}\right)\frac{d-\nu_1}{x_d - x_{\nu_1+\nu_2}}\,\frac{P_{\nu_1,\nu_1+j}}{x_{\nu_1} - x_{\nu_1+\nu_2}} \tag{2.36}$$

where $i_1, \dots, i_{d-1}$ and $j_1, \dots, j_d$ are distinct indices taken from $1, \dots, \nu_1 - 1$ and $\nu_1 + 1, \dots, \nu_1 + \nu_2$ respectively so that different are all sets of $d-1$ pairs

$$(i_1, j_1), \dots, (i_{d-1}, j_{d-1})$$

while the index j is taken from $1, \dots, \nu_2$ but $\nu_1 + j \neq j_1, \dots, j_{d-1}$ however.

Replace the running index $d \geqslant 0$ in (2.35) by $d-1$ where $d > 0$. We get

$$\sum_{d>0} \sum_{\substack{i_1,\dots,i_{d-1}\\ j_1,\dots,j_{d-1}}} \sum_j \left(\prod_{k=1}^{d-1} \frac{P_{i_k j_k}}{x_k - x_{\nu_1+\nu_2}} \right) \frac{P_{\nu_1, \nu_1+j}}{x_{\nu_1} - x_{\nu_1+\nu_2}} \tag{2.37}$$

with the same assumptions on the running indices as in (2.36). By adding up together the sums (2.36) and (2.37), we get

$$\sum_{d>0} \sum_{\substack{i_1,\dots,i_{d-1}\\ j_1,\dots,j_{d-1}}} \sum_j \left(\prod_{k=1}^{d-1} \frac{P_{i_k j_k}}{x_k - x_{\nu_1+\nu_2}} \right) \frac{P_{\nu_1, \nu_1+j}}{x_d - x_{\nu_1+\nu_2}} \tag{2.38}$$

by the equality $x_d + d = x_{\nu_1} + \nu_1$. The sum of (2.33) and (2.38) equals (2.30). Thus we have made the induction step. $\square$

By Lemma 2.1 and by Proposition 2.4 the restriction of operator $P_\nu B(\mu)$ to the subspace (2.27) is an $\mathrm{Y}(\mathfrak{gl}_n)$-intertwining operator (1.9). Let $I(\mu)$ be this restriction divided by the rational function (2.24). Then by Proposition 2.5

$$I(\mu) : \varphi_{\nu_1} \otimes \cdots \otimes \varphi_{\nu_m} \mapsto \varphi_{\nu_m} \otimes \cdots \otimes \varphi_{\nu_1}. \tag{2.39}$$

Theorem 2.7. *For any fixed weight $\nu = \lambda - \mu$ the rational function $I(\mu)$ is regular at any point $\mu \in \mathfrak{t}_m^*$ where the weight $\lambda + \rho$ is dominant.*

The operator-valued rational function $I(\mu)$ does not vanish at any point $\mu \in \mathfrak{t}_m^*$ due to the normalization (2.39). The regularity of $I(\mu)$ was proved in [KN] for all μ where the pair (λ, μ) is good; our Theorem 2.7 is more general.

In next two subsections, we give two proofs of Theorem 2.7. Each of them provides an explicit formula for the operator $I(\mu)$ whenever $\lambda + \rho$ is dominant. We give two proofs, because the resulting formulas for $I(\mu)$ are quite different.

However, in both proofs we will use the following observation. Suppose that the weight $\lambda + \rho$ of $\mathfrak{gl}_m$ is dominant. It means that

$$\lambda_a - \lambda_b + \rho_a - \rho_b \neq -1, -2, \dots \quad \text{whenever} \quad a < b. \tag{2.40}$$

Then

$$\lambda_a - \lambda_b + \rho_a - \rho_b + \nu_b \neq 0$$

since ν_b is a positive integer. So the rational function (2.24) does not vanish then. Moreover, then any factor of the product (2.24) with $\nu_a \geqslant \nu_b$ has a simple pole as a function of $\lambda_a - \lambda_b + \rho_a - \rho_b$ at the zero point.

2.6. In this subsection we will derive Theorem 2.7 from Proposition 2.6. Let us consider the factor of product (2.28) corresponding to any pair of indices $a < b$. There $k = 1, \dots, d$. Since the indices $i_1, \dots, i_d$ and $j_1, \dots, j_d$ are pairwise distinct, we may assume that $d \leqslant \nu_a$ and $d \leqslant \nu_b$ in (2.28). If $\nu_a < \nu_b$ then $k < \nu_b$. Therefore any factor of (2.28) corresponding to $a < b$ with $\nu_a < \nu_b$ is regular at any point $\mu \in \mathfrak{t}_m^*$ where the weight $\lambda + \rho$ is dominant.

If $\nu_a \geqslant \nu_b$ then the denominator in (2.28) is zero if and only if $k = d = \nu_b$ and $\lambda_a + \rho_a = \lambda_b + \rho_b$. Hence any factor in the product (2.28) with $\nu_a \geqslant \nu_b$ has a simple pole as a function of $\lambda_a - \lambda_b + \rho_a - \rho_b$, at the zero point. Moreover, the residue of the latter function at the zero point equals

$$\frac{1}{(\nu_b - 1)!} \sum_{i_1, \dots, i_{\nu_b}} P_{i_1 1} \cdots P_{i_{\nu_b} \nu_b}$$

where the indices $i_1, \dots, i_{\nu_b}$ are distinct and taken from $1, \dots, \nu_a$.

Using the observation on (2.24) made at the end of the previous subsection, we now complete the proof of Theorem 2.7 for any dominant weight $\lambda + \rho$.

Note that Proposition 2.5, like Theorem 2.7 above, can also be derived from Proposition 2.6; see the proof of [KN, Proposition 4.4]. Further, Proposition 2.6 and Theorem 2.7 both can be proved by using [KN, Sections 1.4 and 4.4].

2.7. In this subsection we will give another proof of Theorem 2.7. It provides a multiplicative formula for the operator $I(\mu)$ with dominant $\lambda + \rho$. Consider the product (2.19). It is taken over the pairs (p, q) where $p < q$ while p and q belong to different segments of the sequence $1, \dots, N$; see (2.17). Let a and b be the numbers of these two segments, so that $a < b$. Let us now rearrange the pairs (p, q) in the product (2.19) as follows.

The new order on the pairs (p, q) will be an extension of the lexicographical order on the corresponding pairs (a, b). To define the extension, we have to order the pairs (p, q) corresponding to a given (a, b). Take another pair (r, s) such that the indices r and s belong to the segments a and b, that is to the same segments as the indices p and q respectively. For $\nu_a < \nu_b$, the pair (p, q) will precede (r, s) if $p < r$ or if $p = r$ and $q > s$. For $\nu_a \geqslant \nu_b$, the pair (p, q) will precede (r, s) if $q > s$ or if $q = s$ and $p < r$. By exchanging commuting factors in (2.19), this rearrangement does not alter the value of the ordered product.

Let i and j be the numbers of the indices p and q within their segments, so that by definition we have the equalities (2.18) and

$$x_q = \mu_b + \rho_b + \tfrac{1}{2} + \nu_b - j. \tag{2.41}$$

Consider the factor $R_{pq}(x_q - x_p)$ in the reordered product (2.19). As a function of μ, this factor has a pole at $x_p = x_q$. The latter equation can be written as

$$\lambda_a - \lambda_b + \rho_a - \rho_b = i - j \tag{2.42}$$

using (2.18) and (2.41). Hence if $x_p = x_q$ while $\lambda + \rho$ is dominant, then $i \geqslant j$.

First, suppose that $\nu_a < \nu_b$. If $i \geqslant j$, then $j < \nu_b$ since $i \leqslant \nu_a$. Then the index q is not the last in its segment, so that the pair $(p, q+1)$ immediately precedes (p, q) in the new ordering. Consider the pairs which follow (p, q) in the new ordering. Take the product of the factors in (2.19) corresponding to the latter pairs, and multiply it on the right by

$$R_{q,q+1}(x_q - x_{q+1}) = R_{q,q+1}(1) = 1 - P_{q,q+1}. \tag{2.43}$$

Using (2.1), the resulting product is also divisible by (2.43) on the left. Due to (2.7) we can therefore replace in (2.19) the product of two adjacent factors

$$R_{p,q+1}(x_{q+1}-x_p)R_{pq}(x_q-x_p) \quad \text{by} \quad 1-(P_{pq}+P_{p,q+1})/(x_{q+1}-x_p)$$

without changing the restriction of the operator (2.19) to the subspace (2.27). But the replacement does not have a pole at $x_p = x_q$. So the factors in (2.19) corresponding to the pairs (p,q) with $\nu_a < \nu_b$ do not increase the order of the pole of $I(\mu)$ at any point μ such that the weight $\lambda + \rho$ is dominant.

Next, suppose that $\nu_a \geqslant \nu_b$ while $i > 1$, so that the index p is not the first in its segment. Then the pair $(p-1,q)$ immediately precedes (p,q) in the new ordering. Consider the pairs following (p,q). Take the product of the factors in (2.19) corresponding to the latter pairs. Multiply it on the right by

$$R_{p-1,p}(x_{p-1}-x_p) = R_{p-1,p}(1) = 1 - P_{p-1,p}. \tag{2.44}$$

Using (2.1), the resulting product is also divisible by (2.44) on the left. Due to (2.7) we can now replace in (2.19) the product of two adjacent factors

$$R_{p-1,q}(x_q-x_{p-1})R_{pq}(x_q-x_p) \quad \text{by} \quad 1-(P_{p-1,q}+P_{pq})/(x_q-x_{p-1})$$

without changing the restriction of (2.19) to (2.27). The replacement has no pole at $x_p = x_q$. So the factors in (2.19) corresponding to the pairs (p,q) with $\nu_a \geqslant \nu_b$ and $i > 1$ do not increase the order of the pole of $I(\mu)$ at any point μ.

Last, suppose that $\nu_a \geqslant \nu_b$ and $i = 1$. If $x_p = x_q$ then $\lambda_a + \rho_a = \lambda_b + \rho_b$ and $j = 1$ whenever the weight $\lambda + \rho$ is dominant, due to (2.42). The observation on (2.24) made at the end of Section 2.5 now proves Theorem 2.7.

Note that all the replacements in the product (2.19) described above can be made simultaneously. Hence our argument provides an explicit formula for the operator $I(\mu)$ whenever $\lambda + \rho$ is dominant. See also [GNP, Section 2.3].

2.8. In this subsection we will generalize Theorem 1.1. Theorem 2.7 allows us to determine the intertwining operator $I(\mu)$ of the $\mathrm{Y}(\mathfrak{gl}_n)$-modules (1.9) for any $\mu \in \mathfrak{t}_m^*$, provided the weight $\lambda + \rho$ is dominant. Our generalization of Theorem 1.1 is based on the following lemma. For any index $c = 1, \dots, m-1$ let $s_c \in S_m$ be the transposition of c and $c+1$. Here the symmetric group S_m acts on the numbers $1, \dots, m$ by their permutations. The latter correspond to permutations of the basis vectors $E_{11}, \dots, E_{mm}$ of $\mathfrak{t}_m$.

Lemma 2.8. *Fix $c > 0$ and suppose that both $\lambda + \rho$ and $s_c(\lambda + \rho)$ are dominant. Then the images of the intertwining operators $I(\mu)$ and $I(s_c \circ \mu)$ corresponding to the pairs (λ, μ) and $(s_c \circ \lambda, s_c \circ \mu)$ are equivalent as $\mathrm{Y}(\mathfrak{gl}_n)$-modules.*

Proof. Note that in this lemma the intertwining operator $I(s_c \circ \mu)$ corresponds to the weight $s_c \circ \lambda$, not to λ. Moreover, the source and target $\mathrm{Y}(\mathfrak{gl}_n)$-modules of this operator are different from those of the operator $I(\mu)$ in general. This difference should not cause any confusion however.

Let $\check\lambda$ and $\check\mu$ and $\check\rho$ be the weights of $\mathfrak{gl}_2$ with the labels λ_c, λ_{c+1} and μ_c, μ_{c+1} and ρ_c, ρ_{c+1} respectively. The weights $\check\lambda + \check\rho$ and $s_1(\check\lambda + \check\rho)$ of $\mathfrak{gl}_2$ are dominant. By using Theorem 2.7 with $m = 2$ we get the $\mathrm{Y}(\mathfrak{gl}_n)$-intertwining operators

$$\Phi^{\nu_c}_{\mu_c+\rho_c+\frac{1}{2}} \otimes \Phi^{\nu_{c+1}}_{\mu_{c+1}+\rho_{c+1}+\frac{1}{2}} \to \Phi^{\nu_{c+1}}_{\mu_{c+1}+\rho_{c+1}+\frac{1}{2}} \otimes \Phi^{\nu_c}_{\mu_c+\rho_c+\frac{1}{2}}, \tag{2.45}$$

$$\Phi^{\nu_{c+1}}_{\mu_{c+1}+\rho_{c+1}+\frac{1}{2}} \otimes \Phi^{\nu_c}_{\mu_c+\rho_c+\frac{1}{2}} \to \Phi^{\nu_c}_{\mu_c+\rho_c+\frac{1}{2}} \otimes \Phi^{\nu_{c+1}}_{\mu_{c+1}+\rho_{c+1}+\frac{1}{2}}. \tag{2.46}$$

These two operators are inverse to each other. This assertion can be proved first for any generic weight μ of $\mathfrak{gl}_m$, either by a direct calculation employing the definition (2.19), or by observing that for any generic μ all the double tensor products above are $\mathrm{Y}(\mathfrak{gl}_n)$-irreducible while (2.45) and (2.46) respectively map

$$\varphi_{\nu_c} \otimes \varphi_{\nu_{c+1}} \mapsto \varphi_{\nu_{c+1}} \otimes \varphi_{\nu_c} \quad \text{and} \quad \varphi_{\nu_{c+1}} \otimes \varphi_{\nu_c} \mapsto \varphi_{\nu_c} \otimes \varphi_{\nu_{c+1}}.$$

Then the assertion extends to all μ such that $\lambda + \rho$ and $s_c(\lambda + \rho)$ are dominant.

Now denote by I the operator which acts on the tensor product of cth and $(c+1)$th factors of (1.8) as the intertwining operator (2.45), and which acts trivially on all other $m-2$ tensor factors of (1.8). Arguing like above, that is either performing a direct calculation, or using the irreducibility of the source and target $\mathrm{Y}(\mathfrak{gl}_n)$-modules in (1.9) for any generic μ, we get the relation

$$I(\mu) = I I(s_c \circ \mu) I.$$

It proves the lemma, since I is invertible and intertwines $\mathrm{Y}(\mathfrak{gl}_n)$-modules. □

For any $\lambda \in \mathfrak{t}_m^*$ denote by S_λ the subgroup of S_m consisting of all elements w such that $w \circ \lambda = \lambda$. Let $\mathcal{O}$ be any orbit of the shifted action of the subgroup $S_\lambda \subset S_m$ on $\mathfrak{t}_m^*$. If $\nu_1, \ldots, \nu_m \in \{1, \ldots, n-1\}$ for at least one weight $\mu \in \mathcal{O}$, then every $\mu \in \mathcal{O}$ satisfies the same condition. Suppose this is the case for $\mathcal{O}$. If $\lambda + \rho$ is dominant, then there is at least one weight $\mu \in \mathcal{O}$ such that the pair (λ, μ) is good. Theorem 1.1 generalizes due to the following proposition.

Proposition 2.9. *If $\lambda + \rho$ is dominant, then for all $\mu \in \mathcal{O}$ the images of the corresponding operators $I(\mu)$ are equivalent to each other as* $\mathrm{Y}(\mathfrak{gl}_n)$*-modules.*

Proof. Take any $w \in S_\lambda$ and any reduced decomposition $w = s_{c_l} \cdots s_{c_1}$. It can be derived from [B, Corollary VI.1.2] that the weight $s_{c_k} \cdots s_{c_1}(\lambda + \rho)$ of $\mathfrak{gl}_m$ is dominant for each $k = 1, \ldots, l$. Proposition 2.9 now follows by applying Lemma 2.8 repeatedly. Note that this proposition can also be proved by using the results of Zelevinsky [Z, Theorem 6.1] together with those of [AS,D1]. □

Thus all assertions of Theorem 1.1 will remain valid if we replace the good pair there by any pair (λ, μ) such that the weight $\lambda + \rho$ of $\mathfrak{gl}_m$ is dominant. However, we still assume that $\nu_1, \ldots, \nu_m \in \{1, \ldots, n-1\}$ for the latter pair.

3. More intertwining operators

3.1. Now let λ and μ be any weights of the Lie algebra $\mathfrak{f}_m = \mathfrak{sp}_{2m}$ or $\mathfrak{f}_m = \mathfrak{so}_{2m}$ such that all labels of the weight $\nu = \lambda - \mu + \kappa$ are in $\{1, \ldots, n-1\}$. We will keep using the notation (2.17), (2.18), (2.19). But now λ_a, μ_a, ν_a and ρ_a with $a = 1, \ldots, m$ are labels of weights of $\mathfrak{f}_m$. Recall that $\kappa_a = n/2$ by definition.

Let the weight $\mu \in \mathfrak{h}_m^*$ vary while ν is fixed. Determine the rational function $B(\mu)$ by the same formula (2.19) as for $\mathrm{Y}(\mathfrak{gl}_n)$. Also take the ordered product

$$C(\mu) = \prod_{(p,q)}^{\longleftarrow} \widetilde{R}_{pq}(n - x_p - x_q) \tag{3.1}$$

where $1 \leqslant p < q \leqslant N$ and the pairs (p, q) are ordered lexicographically. Here the reversed arrow indicates that the factors corresponding to these pairs are arranged from right to left. Note that $C(\mu)$ is a rational function of $\mu \in \mathfrak{h}_m^*$ without poles at the generic weights μ of $\mathfrak{f}_m$.

Take the sequence $1'', \ldots, N''$ introduced in the previous subsection. Let Q_ν be the linear operator on $(\mathbb{C}^n)^{\otimes N}$ which for each $p = 1, \ldots, N$ exchanges the tensor factors $\mathbb{C}^n$ labeled by p and p''. Then $Q_\nu = P$ if $m = 1$ and $\nu_1 = 2$.

Proposition 3.1. *Suppose the weight μ of $\mathfrak{f}_m$ is generic. Then $Q_\nu B(\mu)C(\mu)$ is an intertwining operator of* $\mathrm{Y}(\mathfrak{g}_n)$*-modules*

$$\Phi^1_{x_1} \otimes \cdots \otimes \Phi^1_{x_N} \to \Phi^{-1}_{x_{1''}} \otimes \cdots \otimes \Phi^{-1}_{x_{N''}}. \tag{3.2}$$

Proof. Under the action of $\mathrm{Y}(\mathfrak{gl}_n)$ on the source module in (3.2),

$$S(x) \mapsto \widehat{R}_{0N}(x_N + x) \cdots \widehat{R}_{01}(x_1 + x)\overline{R}_{01}(x_1 - x) \cdots \overline{R}_{0N}(x_N - x).$$

Let us denote by Y the right-hand side of this assignment. Here we use the subscripts $0, 1, \ldots, N$ rather than $1, 2, \ldots, N+1$ to label the tensor factors of $(\operatorname{End}\mathbb{C}^n)^{\otimes(N+1)}$, like we did in the proof of Lemma 2.1. Further, let us denote

$$Y' = \overline{R}_{01}(x_1 - x) \cdots \overline{R}_{0N}(x_N - x)\widehat{R}_{0N}(x_N + x) \cdots \widehat{R}_{01}(x_1 + x).$$

By using the relation (2.6) repeatedly, we get the equality $C(\mu)Y = Y'C(\mu)$. Here we also use the relation $P\widetilde{R}(x)P = \widetilde{R}(x)$.

Under the action of $\mathrm{Y}(\mathfrak{g}_n)$ on the target module in (3.2),

$$S(x) \mapsto \overline{R}_{0N}(x_{N''} - x) \cdots \overline{R}_{01}(x_{1''} - x)\widehat{R}_{01}(x_{1''} + x) \cdots \widehat{R}_{0N}(x_{N''} + x).$$

Denote by Y'' the right-hand side of the latter assignment. Observe that $k'' = (N - k + 1)'$ for each $k = 1, \ldots, N$. Therefore we can write

$$Y'' = \overline{R}_{0N}(x_{1'} - x) \cdots \widehat{R}_{01}(x_{N'} - x)\widehat{R}_{01}(x_{N'} + x) \cdots \widehat{R}_{0N}(x_{1'} + x).$$

But by using the relations (2.3) and (2.5) repeatedly, we get

$$\begin{aligned} Q_\nu B(\mu)Y' &= Q_\nu \overline{R}_{01'}(x_{1'} - x) \cdots \overline{R}_{0N'}(x_{N'} - x)\widehat{R}_{0N'}(x_{N'} + x) \cdots \widehat{R}_{01'}(x_{1'} + x)B(\mu) \\ &= Y'' Q_\nu B(\mu); \end{aligned}$$

see also the proof of Proposition 2.4. Thus we get the equality

$$Q_\nu B(\mu)C(\mu)Y = Y'' Q_\nu B(\mu)C(\mu),$$

which proves the claim. $\square$

3.2. Observe that $\widetilde{P}P = \pm\widetilde{P}$. Thus $\widetilde{P}R(1) = 0$ in the case $\mathfrak{g}_n = \mathfrak{so}_n$. Hence in this case the restriction of the operator $\widetilde{P}_{pq}$ to the subspace (2.27) is zero for any two distinct indices p, q from the same segment of the sequence $1, \ldots, N$. Here we mean the segments of lengths $\nu_1, \ldots, \nu_m$ as defined in Section 2.3.

In the case $\mathfrak{g}_n = \mathfrak{so}_n$ the relation (2.4) now implies that when considering the restriction of the operator $C(\mu)$ to the subspace (2.27), we can skip those factors in the product (3.1) which correspond to the pairs (p, q) where both p and q belong to the same segment: skipping does not change the restriction. In particular, in the case $\mathfrak{g}_n = \mathfrak{so}_n$ the restriction of the operator $C(\mu)$ to the subspace (2.27) does not have a pole if $\mu_a - \mu_b \notin \mathbb{Z}$ and $\mu_a + \mu_b \notin \mathbb{Z}$ whenever $a \neq b$; the condition $2\mu_a \notin \mathbb{Z}$ is not needed here.

Let us give an analogue of Proposition 2.5 for $C(\mu)$. Except in the proof of Lemma 2.3, we worked with any symmetric or alternating non-degenerate bilinear form $\langle\,,\rangle$ on $\mathbb{C}^n$ so far. Choose the form as in the proof, so that (2.14) holds. The elements $E_{ij} - \widetilde{E}_{ij}$ with $i \leqslant j$ span a Borel subalgebra of $\mathfrak{g}_n \subset \mathfrak{gl}_n$ then; the elements $E_{ii} - \widetilde{E}_{ii}$ span the corresponding Cartan subalgebra of $\mathfrak{g}_n$.

Proposition 3.2. *For any generic weight μ of $\mathfrak{f}_m$ the vector $\varphi_{\nu_1} \otimes \cdots \otimes \varphi_{\nu_m}$ is an eigenvector of the operator $C(\mu)$ on $(\mathbb{C}^n)^{\otimes N}$. The eigenvalue is the product*

$$\prod_{1 \leqslant a<b \leqslant m} \begin{cases} \dfrac{\lambda_a+\lambda_b+\rho_a+\rho_b}{\mu_a+\mu_b+\rho_a+\rho_b} & \text{if} \quad \nu_a+\nu_b \geqslant n; \\ 1 & \text{if} \quad \nu_a+\nu_b \leqslant n, \end{cases} \tag{3.3}$$

multiplied in the case of $\mathfrak{f}_m = \mathfrak{sp}_{2m}$ by the product

$$\prod_{1 \leqslant a \leqslant m} \begin{cases} \dfrac{\lambda_a+\rho_a}{\mu_a+\rho_a} & \text{if} \quad 2\nu_a \geqslant n; \\ 1 & \text{if} \quad 2\nu_a \leqslant n. \end{cases} \tag{3.4}$$

Proof. This proposition immediately follows from its particular cases of $m=1$ and $m=2$. We will consider these two cases only. First suppose that $m=1$. In this case, for $\mathfrak{g}_n = \mathfrak{so}_n$ the restriction of the operator $C(\mu)$ to the subspace (2.27) is the identity operator; see the observation made in the very beginning of the present subsection. Suppose that $\mathfrak{g}_n = \mathfrak{sp}_n$. Put $\varepsilon = (-1)^{[\nu_1/2]}$. Then

$$\begin{aligned} C(\mu)\varphi_{\nu_1} &= \overleftarrow{\prod_{i=1,\ldots,\nu_1-1}} \left(\overleftarrow{\prod_{j=i+1,\ldots,\nu_1}} \widetilde{R}_{ij}(n-x_i-x_j) \right) \times \varepsilon A_{\nu_1}(e_{\nu_1} \otimes \cdots \otimes e_1) \\ &= \varepsilon A_{\nu_1} \overrightarrow{\prod_{i=1,\ldots,\nu_1-1}} \left(\overrightarrow{\prod_{j=i+1,\ldots,\nu_1}} \widetilde{R}_{ij}(n-x_i-x_j) \right)(e_{\nu_1} \otimes \cdots \otimes e_1) \end{aligned} \tag{3.5}$$

where the latter equality is obtained by using (2.4) and (2.23). We will prove by induction on ν_1 that the vector (3.5) equals φ_{ν_1} multiplied by the scalar (3.4) where $m=1$. Recall that $\nu_1 = \lambda_1 - \mu_1 + n/2$. If $\nu_1 \leqslant n/2$ then here we have

$$\widetilde{P}_{ij}(e_{\nu_1} \otimes \cdots \otimes e_1) = 0$$

by our choice of the form $\langle\, , \rangle$ on $\mathbb{C}^n$. Hence then the vector (3.5) equals φ_{ν_1} as required. We will also use that equality for $\nu_1 \leqslant n/2$ as the induction base.

Let $\nu_1 > n/2$. In particular, then $\nu_1 > 1$ because n is even. The induction assumption then implies that

$$\begin{aligned} &(1 \otimes A_{\nu_1-1}) \overrightarrow{\prod_{i=2,\ldots,\nu_1-1}} \left(\overrightarrow{\prod_{j=i+1,\ldots,\nu_1}} \widetilde{R}_{ij}(n-x_i-x_j) \right)(e_{\nu_1} \otimes e_{\nu_1-1} \otimes \cdots \otimes e_1) \\ &\quad = u e_{\nu_1} \otimes A_{\nu_1-1}(e_{\nu_1-1} \otimes \cdots \otimes e_1) \end{aligned}$$

where

$$u = \frac{\lambda_1+\rho_1-1}{\mu_1+\rho_1}.$$

We use the inequality $2(\nu_1-1) \geqslant n$ which follows from $\nu_1 > n/2$, because n is even. Arguing like in the proof of Lemma 2.1 and using the relation (2.8) we get

$$(1 \otimes A_{\nu_1 - 1}) \prod_{j=2,\ldots,\nu_1}^{\longrightarrow} \widetilde{R}_{1j}(n - x_1 - x_j) = (1 \otimes A_{\nu_1 - 1}) \left(1 - (n - x_1 - x_2)^{-1} \sum_{j=2}^{\nu_1} \widetilde{P}_{1j} \right).$$

Further, by our choice of the form $\langle\, , \rangle$ on $\mathbb{C}^n$ the vector

$$A_{\nu_1} \widetilde{P}_{1j} (e_{\nu_1} \otimes \cdots \otimes e_1)$$

is equal to

$$2 A_{\nu_1} (e_{\nu_1} \otimes \cdots \otimes e_1)$$

if $j = 2\nu_1 - n$, and is equal to zero for any other $j \geqslant 2$. By writing (3.5) as

$$\varepsilon A_{\nu_1} \prod_{j=2,\ldots,\nu_1}^{\longrightarrow} \widetilde{R}_{1j}(n - x_1 - x_j) \times \prod_{i=2,\ldots,\nu_1 - 1}^{\longrightarrow} \left(\prod_{j=i+1,\ldots,\nu_1}^{\longrightarrow} \widetilde{R}_{ij}(n - x_i - x_j) \right) (e_{\nu_1} \otimes \cdots \otimes e_1)$$

we now see that the vector (3.5) is equal to φ_{ν_1} multiplied by the scalar

$$\frac{n - x_1 - x_2 - 2}{n - x_1 - x_2} u = \frac{\lambda_1 + \rho_1}{\mu_1 + \rho_1}.$$

Thus we have finished the proof of Proposition 3.2 in the case $m = 1$, and will now suppose that $m = 2$. Then by the definition (3.1) the operator $C(\mu)$ is the ordered product of $\widetilde{R}_{pq}(n - x_p - x_q)$ over the pairs (p, q) where $1 \leqslant p < q \leqslant \nu_1 + \nu_2$; the arrangement of the pairs is reversed lexicographical. Without changing the product, we can rearrange these pairs as follows. From left to right, first come the pairs (p, q) where $\nu_1 < p < q \leqslant \nu_1 + \nu_2$, second come the pairs where $1 \leqslant p \leqslant \nu_1$ and $\nu_1 < q \leqslant \nu_1 + \nu_2$; third come the pairs where $1 \leqslant p < q \leqslant \nu_1$. Within each of the three groups, the arrangement of the pairs (p, q) is still reversed lexicographical.

Consider the two products of $\widetilde{R}_{pq}(n - x_p - x_q)$, over the first and the third group of pairs (p, q). The already settled case $m = 1$ implies that $\varphi_{\nu_1} \otimes \varphi_{\nu_2}$ is an eigenvector for each of these two products. If $\mathfrak{g}_n = \mathfrak{so}_n$ then each of the two corresponding eigenvalues is 1. If $\mathfrak{g}_n = \mathfrak{sp}_n$ then the product of the two eigenvalues is (3.4) where $m = 2$. For $\mathfrak{g}_n = \mathfrak{so}_n, \mathfrak{sp}_n$ we will show that $\varphi_{\nu_1} \otimes \varphi_{\nu_2}$ is an eigenvector of the product of $\widetilde{R}_{pq}(n - x_p - x_q)$ over the second group of pairs, with the eigenvalue (3.3) where $m = 2$. This will settle the case $m = 2$.

Use the induction on ν_1. Denote the last mentioned product by Z, so that

$$Z = \prod_{i=1,\ldots,\nu_1}^{\longleftarrow} \left(\prod_{j=1,\ldots,\nu_2}^{\longleftarrow} \widetilde{R}_{i,\nu_1 + j}(n - x_i - x_{\nu_1 + j}) \right) \tag{3.6}$$

by definition. By using (2.4) and (2.23), we get the equality

$$\begin{aligned} &Z (A_{\nu_1} \otimes A_{\nu_2}) (e_{\nu_1} \otimes \cdots \otimes e_1 \otimes e_{\nu_2} \otimes \cdots \otimes e_1) \\ &\quad = (A_{\nu_1} \otimes A_{\nu_2}) \prod_{i=1,\ldots,\nu_1}^{\longrightarrow} \left(\prod_{j=1,\ldots,\nu_2}^{\longrightarrow} \widetilde{R}_{i,\nu_1 + j}(n - x_i - x_{\nu_1 + j}) \right) \\ &\qquad \times (e_{\nu_1} \otimes \cdots \otimes e_1 \otimes e_{\nu_2} \otimes \cdots \otimes e_1). \end{aligned} \tag{3.7}$$

If $\nu_1 + \nu_2 \leqslant n$ then by our choice of the form $\langle\,,\rangle$ on $\mathbb{C}^n$

$$\widetilde{P}_{i,\nu_1+j}(e_{\nu_1} \otimes \cdots \otimes e_1 \otimes e_{\nu_2} \otimes \cdots \otimes e_1) = 0$$

for i, j as above. Hence then the product (3.6) acts on the vector $\varphi_{\nu_1} \otimes \varphi_{\nu_2}$ as the identity. We will also use this result for $\nu_1 + \nu_2 \leqslant n$ as the induction base.

Let $\nu_1 + \nu_2 > n$. Then $\nu_1 > 1$ because $\nu_2 < n$. By the induction assumption

$$\begin{aligned}
&(1 \otimes A_{\nu_1-1} \otimes A_{\nu_2}) \prod_{i=2,\ldots,\nu_1}^{\longrightarrow} \left(\prod_{j=1,\ldots,\nu_2}^{\longrightarrow} \widetilde{R}_{i,\nu_1+j}(n - x_i - x_{\nu_1+j}) \right) \\
&\quad \times (e_{\nu_1} \otimes e_{\nu_1-1} \otimes \cdots \otimes e_1 \otimes e_{\nu_2} \otimes \cdots \otimes e_1) \\
&= v e_{\nu_1} \otimes (A_{\nu_1-1} \otimes A_{\nu_2})(e_{\nu_1-1} \otimes \cdots \otimes e_1 \otimes e_{\nu_2} \otimes \cdots \otimes e_1)
\end{aligned}$$

where

$$v = \frac{\lambda_1 + \lambda_2 + \rho_1 + \rho_2 - 1}{\mu_1 + \mu_2 + \rho_1 + \rho_2}.$$

Arguing like in the proof of Lemma 2.1 and using the relation (2.8) we get the equality in the algebra $(\operatorname{End}\mathbb{C}^n)^{\otimes(\nu_1+\nu_2)}$

$$(1 \otimes A_{\nu_2}) \prod_{j=1,\ldots,\nu_2}^{\longrightarrow} \widetilde{R}_{1,\nu_1+j}(n - x_1 - x_{\nu_1+j}) = (1 \otimes A_{\nu_2}) \left(1 - (n - x_1 - x_{\nu_1+1})^{-1} \sum_{j=1}^{\nu_2} \widetilde{P}_{1,\nu_1+j} \right).$$

Further, by our choice of the form $\langle\,,\rangle$ on $\mathbb{C}^n$ the vector

$$(A_{\nu_1} \otimes A_{\nu_2}) \widetilde{P}_{1q}(e_{\nu_1} \otimes \cdots \otimes e_1 \otimes e_{\nu_2} \otimes \cdots \otimes e_1)$$

is equal to

$$(A_{\nu_1} \otimes A_{\nu_2})(e_{\nu_1} \otimes \cdots \otimes e_1 \otimes e_{\nu_2} \otimes \cdots \otimes e_1)$$

if $j = \nu_1 + \nu_2 - n$, and is equal to zero for any other j. Writing (3.7) as

$$(A_{\nu_1} \otimes A_{\nu_2}) \prod_{j=1,\ldots,\nu_2}^{\longrightarrow} \widetilde{R}_{1,\nu_1+j}(n - x_1 - x_{\nu_1+j}) \times \prod_{i=2,\ldots,\nu_1}^{\longrightarrow} \left(\prod_{j=1,\ldots,\nu_2}^{\longrightarrow} \widetilde{R}_{i,\nu_1+j}(n - x_1 - x_{\nu_1+j}) \right)$$

we now see that $\varphi_{\nu_1} \otimes \varphi_{\nu_2}$ is an eigenvector for (3.6) with the eigenvalue

$$\frac{n - x_1 - x_{\nu_1+1} - 1}{n - x_1 - x_{\nu_1+1}} v = \frac{\lambda_1 + \lambda_2 + \rho_1 + \rho_2}{\mu_1 + \mu_2 + \rho_1 + \rho_2}.$$

This observation completes the proof of Proposition 3.2. $\square$

3.3. Using the relations (2.1), (2.4) and $R(1) = 1 - P$ one shows that for any generic weight μ of $\mathfrak{f}_m$ the operator $C(\mu)$ preserves the subspace (2.27), see the proof of Proposition 3.2. Moreover, we have another proposition. It can be used to give another proof of Proposition 3.2, see the proof of [KN, Proposition 4.6].

Proposition 3.3. *For any generic weight μ of $\mathfrak{f}_m$ the restriction of $C(\mu)$ to the subspace* (2.27) *coincides with that of the operator*

$$\prod_{1\leqslant a\leqslant b\leqslant m}^{\longleftarrow} \begin{cases} 1+\sum\limits_{d>0}\sum\limits_{\substack{i_1,\dots,i_d\\ j_1,\dots,j_d}}\prod\limits_{k=1}^{d}\dfrac{\widetilde{P}_{i_k j_k}}{\lambda_a+\lambda_b+\rho_a+\rho_b-k}, & \text{if}\quad a<b;\\[2ex] 1+\sum\limits_{d>0}\sum\limits_{\substack{i_1,\dots,i_d\\ j_1,\dots,j_d}}\prod\limits_{k=1}^{d}\dfrac{\widetilde{P}_{i_k j_k}}{2(\lambda_a+\rho_a-k)} & \text{if}\quad a=b,\ \mathfrak{f}_m=\mathfrak{sp}_{2m};\\[2ex] 1 & \text{if}\quad a=b,\ \mathfrak{f}_m=\mathfrak{so}_{2m}; \end{cases}$$

here the pairs (a,b) are ordered lexicographically. If $a<b$ then $i_1,\dots,i_d$ and $j_1,\dots,j_d$ are distinct numbers from the ath and bth segments of the sequence $1,\dots,N$ respectively, taken so that different are all the sets (2.29)*. If $a=b$ then $i_1, j_1,\dots,i_d, j_d$ are pairwise distinct numbers from the ath segment of the sequence $1,\dots,N$ taken so that different are all the sets of d unordered pairs*

$$\{i_1, j_1\},\dots,\{i_d, j_d\}. \tag{3.8}$$

Proof. This proposition immediately follows from its particular cases of $m=1$ and $m=2$. We will consider these two cases only. First suppose that $m=1$. We have already observed that then for $\mathfrak{g}_n=\mathfrak{so}_n$ the restriction of the operator $C(\mu)$ to the subspace (2.27) is the identity operator. Suppose $\mathfrak{g}_n=\mathfrak{sp}_n$. Then in the second displayed line in Proposition 3.3 we have the operator on $(\mathbb{C}^n)^{\otimes\nu_1}$

$$\sum_{d\geqslant 0}\sum_{\substack{i_1,\dots,i_d\\ j_1,\dots,j_d}}\prod_{k=1}^{d}\frac{\widetilde{P}_{i_k j_k}}{2x_k-n-1} \tag{3.9}$$

where $i_1, j_1,\dots,i_d, j_d$ are pairwise distinct and taken from $1,\dots,\nu_1$ so that different are all the sets of d unordered pairs (3.8). Here we use the equality

$$x_k=\lambda_1+\rho_1+(n+1)/2-k \tag{3.10}$$

for any $k\leqslant\nu_1$. We also assume that 1 is the only term in (3.9) with $d=0$. On the other hand, for $m=1$ we can write

$$C(\mu)=\prod_{j=2,\dots,\nu_1}^{\longleftarrow}\left(\prod_{i=1,\dots,j-1}^{\longleftarrow}\widetilde{R}_{ij}(n-x_i-x_j)\right). \tag{3.11}$$

Let us now relate two operators on the vector space $(\mathbb{C}^n)^{\otimes\nu_1}$ by the symbol $\equiv$ if their actions coincide on the subspace

$$\Lambda^{\nu_1}\left(\mathbb{C}^n\right)\subset\left(\mathbb{C}^n\right)^{\otimes\nu_1}. \tag{3.12}$$

We will establish the relation $\equiv$ between (3.9) and (3.11) by induction on ν_1.

If $\nu_1 > 1$ then we assume the latter relation holds for $\nu_1 - 1$ instead of ν_1; if $\nu_1 = 1$ then we are not making any assumption. Arguing like in the proof of Lemma 2.1 and then using the induction assumption, (3.11) is related by $\equiv$ to

$$\left(1 + \sum_{i=1}^{\nu_1 - 1} \frac{\widetilde{P}_{i\nu_1}}{x_1 + x_{\nu_1} - n}\right) \times \prod_{j=2,\dots,\nu_1-1}^{\longleftarrow} \left(\prod_{i=1,\dots,j-1}^{\longleftarrow} \widetilde{R}_{ij}(n - x_i - x_j)\right)$$
$$\equiv \left(1 + \sum_{i=1}^{\nu_1 - 1} \frac{\widetilde{P}_{i\nu_1}}{x_1 + x_{\nu_1} - n}\right) \times \left(\sum_{d \geqslant 0} \sum_{\substack{i_1,\dots,i_d \\ j_1,\dots,j_d}} \prod_{k=1}^{d} \frac{\widetilde{P}_{i_k j_k}}{2x_k - n - 1}\right)$$

where $i_1, j_1, \dots, i_d, j_d$ are distinct and taken from $1, \dots, \nu_1 - 1$ so that different are all corresponding sets (3.8). The right-hand side of the last relation equals

$$\sum_{d \geqslant 0} \sum_{\substack{i_1,\dots,i_d \\ j_1,\dots,j_d}} \prod_{k=1}^{d} \frac{\widetilde{P}_{i_k j_k}}{2x_k - n - 1} \tag{3.13}$$

$$+ \sum_{d>0} \sum_{l=1}^{d} \sum_{\substack{i_1,\dots,i_d \\ j_1,\dots,j_d}} \frac{\widetilde{P}_{i_l \nu_1} + \widetilde{P}_{j_l \nu_1}}{x_1 + x_{\nu_1} - n} \prod_{k=1}^{d} \frac{\widetilde{P}_{i_k j_k}}{2x_k - n - 1} \tag{3.14}$$

$$+ \sum_{d \geqslant 0} \sum_{\substack{i_1,\dots,i_d \\ j_1,\dots,j_d}} \sum_{i} \frac{\widetilde{P}_{i \nu_1}}{x_1 + x_{\nu_1} - n} \prod_{k=1}^{d} \frac{\widetilde{P}_{i_k j_k}}{2x_k - n - 1} \tag{3.15}$$

where i is taken from $1, \dots, \nu_1 - 1$ and is different from $i_1, j_1, \dots, i_d, j_d$.

Consider the sum displayed in the line (3.14). Here we have

$$\widetilde{P}_{i_l \nu_1} \prod_{k=1}^{d} \widetilde{P}_{i_k j_k} = \left(\prod_{k \neq l} \widetilde{P}_{i_k j_k}\right) \widetilde{P}_{i_l \nu_1} \widetilde{P}_{i_l j_l} = \left(\prod_{k \neq l} \widetilde{P}_{i_k j_k}\right) \widetilde{P}_{i_l \nu_1} P_{j_l \nu_1} \equiv -\left(\prod_{k \neq l} \widetilde{P}_{i_k j_k}\right) \widetilde{P}_{i_l \nu_1}$$

where the right-hand side does not involve j_l. Similarly, in (3.14) we have

$$\widetilde{P}_{j_l \nu_1} \prod_{k=1}^{d} \widetilde{P}_{i_k j_k} \equiv -\left(\prod_{k \neq l} \widetilde{P}_{i_k j_k}\right) \widetilde{P}_{j_l \nu_1}$$

where the right-hand side does not involve the index i_l.

Now fix a number $i \in \{1, \dots, \nu_1 - 1\}$ and take any set of d pairs (3.8) such that one of the pairs contains the number i. Then $i = i_l$ or $i = j_l$ for some l. Let j be the element of the pair $\{i_l, j_l\}$ different from i, so that $j = j_l$ or $j = i_l$ respectively. If the set of the $d - 1$ pairs $\{i_k, j_k\}$ with $k \neq l$ is also fixed, then j ranges over a set of cardinality $\nu_1 - 2d$, namely over the fixed set

$$\{1, \dots, \nu_1 - 1\} \setminus \{i_1, j_1, \dots, i_{l-1}, j_{l-1}, i, i_{l+1}, j_{l+1}, \dots, i_d, j_d\}.$$

Let us perform the summation over the indices i_l, j_l and l in (3.14) first of all the running indices. After that rename the running indices $i_{l+1}, \dots, i_d$ and $j_{l+1}, \dots, j_d$ respectively by $i_l, \dots, i_{d-1}$ and $j_l, \dots, j_{d-1}$. By the arguments given in the previous two paragraphs, the sum (3.14) gets related by $\equiv$

to

$$\sum_{d>0}\sum_{\substack{i_1,\dots,i_{d-1}\\ j_1,\dots,j_{d-1}}}\sum_{i}\frac{2d-\nu_1}{2x_d-n-1}\,\frac{\widetilde{P}_{i\nu_1}}{x_1+x_{\nu_1}-n}\prod_{k=1}^{d-1}\frac{\widetilde{P}_{i_kj_k}}{2x_k-n-1} \tag{3.16}$$

where i and $i_1, j_1, \dots, i_{d-1}, j_{d-1}$ are distinct indices taken from $1, \dots, \nu_1 - 1$ so that different are all the sets of $d-1$ unordered pairs

$$\{i_1, j_1\}, \dots, \{i_{d-1}, j_{d-1}\}.$$

Replace the running index $d \geqslant 0$ in (3.15) by $d-1$ where $d > 0$. We get

$$\sum_{d>0}\sum_{\substack{i_1,\dots,i_{d-1}\\ j_1,\dots,j_{d-1}}}\sum_{i}\frac{\widetilde{P}_{i\nu_1}}{x_1+x_{\nu_1}-n}\prod_{k=1}^{d-1}\frac{\widetilde{P}_{i_kj_k}}{2x_k-n-1} \tag{3.17}$$

with the same assumptions on the running indices as in (3.16). By adding up together the sums (3.16) and (3.17), we get

$$\sum_{d>0}\sum_{\substack{i_1,\dots,i_{d-1}\\ j_1,\dots,j_{d-1}}}\sum_{i}\frac{\widetilde{P}_{i\nu_1}}{2x_d-n-1}\prod_{k=1}^{d-1}\frac{\widetilde{P}_{i_kj_k}}{2x_k-n-1} \tag{3.18}$$

by the equality

$$2x_d + 2d = x_1 + 1 + x_{\nu_1} + \nu_1.$$

The sum of (3.13) and (3.18) equals (3.9). This makes the induction step. Thus we have finished the proof of Proposition 3.3 in the case $m = 1$.

Now let $m = 2$. We will begin considering this case in the same way as we did it in the proof of Proposition 3.3. Namely, by the definition (3.1) the operator $C(\mu)$ is the ordered product of the factors $\widetilde{R}_{pq}(n - x_p - x_q)$ over the pairs (p, q) where $1 \leqslant p < q \leqslant \nu_1 + \nu_2$; the arrangement of the pairs is reversed lexicographical. Without changing the product, we can rearrange these pairs as follows. From left to right, first come the pairs (p, q) where $\nu_1 < p < q \leqslant \nu_1 + \nu_2$, second come the pairs where $1 \leqslant p \leqslant \nu_1$ and $\nu_1 < q \leqslant \nu_1 + \nu_2$; third come the pairs where $1 \leqslant p < q \leqslant \nu_1$. Within each of the three groups, the arrangement of the pairs (p, q) is still reversed lexicographical.

Consider the two products of $\widetilde{R}_{pq}(n - x_p - x_q)$, taken over the first and the third group of pairs (p, q). If $\mathfrak{g}_n = \mathfrak{so}_n$ then the restriction of each of the two products to the subspace (2.32) is 1. If $\mathfrak{g}_n = \mathfrak{sp}_n$ then the already settled case of $m = 1$ implies that the restrictions of the two products to (2.32) coincide with that of the sum in the second displayed line in Proposition 3.3, where $a = 1$ and $a = 2$ respectively. Now consider the product of $\widetilde{R}_{pq}(n - x_p - x_q)$ over the second group of pairs. We have denoted this product by Z, see (3.6). For $\mathfrak{g}_n = \mathfrak{so}_n, \mathfrak{sp}_n$ we will show that the operator Z has the same restriction to (2.32) as the sum in the first displayed line in Proposition 3.3, where $a = 1$ and $b = 2$. This will settle the case of $m = 2$.

Recall (2.31). There for any fixed i we arrange the factors corresponding to the indices $j = 1, \dots, \nu_2$ from right to left. That is, we arrange from left to right the factors corresponding to $j = \nu_2, \dots, 1$. The numbers x_{ν_1+j} in (2.31) with $j = \nu_2, \dots, 1$ then make a sequence increasing by 1. This is the sequence

$$x_{\nu_1+\nu_2}, \dots, x_{\nu_1+1}. \tag{3.19}$$

It was only the increasing by 1 property of (3.19) that we used to prove that the restrictions of the operators (2.30) and (2.31) to the subspace (2.32) are equal. Hence we can replace the sequence (3.19) in this equality by any other sequence of length ν_2 that is increasing by 1. As a replacement, let us use the sequence

$$n - x_{\nu_1+1}, \ldots, n - x_{\nu_1+\nu_2}.$$

Since $k \leqslant \nu_1$ in (2.30), then we get the equality in $(\operatorname{End} \mathbb{C}^n)^{\otimes(\nu_1+\nu_2)}$

$$\left(\sum_{d \geqslant 0} \sum_{\substack{i_1,\ldots,i_d \\ j_1,\ldots,j_d}} \prod_{k=1}^{d} \frac{P_{i_k j_k}}{x_k + x_{\nu_1+1} - n}\right)(A_{\nu_1} \otimes A_{\nu_2}) \tag{3.20}$$

$$= \prod_{i=1,\ldots,\nu_1}^{\longrightarrow} \left(\prod_{j=1,\ldots,\nu_2}^{\longleftarrow} R_{i,\nu_1+j}(n - x_i - x_{\nu_1+\nu_2-j+1}) \right) \times (A_{\nu_1} \otimes A_{\nu_2}). \tag{3.21}$$

The indices $i_1, \ldots, i_d$ and $j_1, \ldots, j_d$ in (3.20) have the same range as in the first displayed line in Proposition 3.3. By applying to (3.20) the operator conjugation relative to the form $\langle\,,\rangle$ in each of the first ν_1 tensor factors of $(\operatorname{End} \mathbb{C}^n)^{\otimes(\nu_1+\nu_2)}$ we get the product

$$(A_{\nu_1} \otimes 1)\left(\sum_{d \geqslant 0} \sum_{\substack{i_1,\ldots,i_d \\ j_1,\ldots,j_d}} \prod_{k=1}^{d} \frac{\widetilde{P}_{i_k j_k}}{x_k + x_{\nu_1+1} - n}\right)(1 \otimes A_{\nu_2})$$

$$= \left(\sum_{d \geqslant 0} \sum_{\substack{i_1,\ldots,i_d \\ j_1,\ldots,j_d}} \prod_{k=1}^{d} \frac{\widetilde{P}_{i_k j_k}}{x_k + x_{\nu_1+1} - n}\right)(A_{\nu_1} \otimes A_{\nu_2}). \tag{3.22}$$

The sum over $d \geqslant 0$ in (3.22) coincides with the sum in the first displayed line in Proposition 3.3 for $a = 1$ and $b = 2$, by the equalities (3.10) for $k \leqslant \nu_1$ and

$$x_{\nu_1+1} = \lambda_2 + \rho_2 + (n-1)/2.$$

Note that the product (3.20) commutes with the operator on $(\mathbb{C}^n)^{\otimes(\nu_1+\nu_2)}$ reversing the order of the last ν_2 tensor factors. Let us conjugate the product (3.21) by this operator. The result is the product

$$\prod_{i=1,\ldots,\nu_1}^{\longrightarrow} \left(\prod_{j=1,\ldots,\nu_2}^{\longleftarrow} R_{i,\nu_1+\nu_2-j+1}(n - x_i - x_{\nu_1+\nu_2-j+1}) \right) \times (A_{\nu_1} \otimes A_{\nu_2})$$

$$= \prod_{i=1,\ldots,\nu_1}^{\longrightarrow} \left(\prod_{j=1,\ldots,\nu_2}^{\longrightarrow} R_{i,\nu_1+j}(n - x_i - x_{\nu_1+j}) \right) \times (A_{\nu_1} \otimes A_{\nu_2})$$

$$= (A_{\nu_1} \otimes 1) \times \prod_{i=1,\ldots,\nu_1}^{\longleftarrow} \left(\prod_{j=1,\ldots,\nu_2}^{\longrightarrow} R_{i,\nu_1+j}(n - x_i - x_{\nu_1+j}) \right) \times (1 \otimes A_{\nu_2}).$$

By applying to the last displayed line the operator conjugation relative to $\langle\,,\rangle$ in each of the first ν_1 tensor factors of $(\operatorname{End} \mathbb{C}^n)^{\otimes(\nu_1+\nu_2)}$ we get the product

$$Z \times (A_{\nu_1} \otimes A_{\nu_2}). \tag{3.23}$$

The equality of (3.20) and (3.21) implies the equality of (3.22) and (3.23). The latter equality settles the case of $m = 2$. □

The operators $B(\mu)$ and $C(\mu)$ preserve the subspace (2.27). Due to Lemmas 2.1, 2.2 and Propositions 2.4, 3.1 the restriction of operator $B(\mu)C(\mu)$ to this subspace is an $\mathrm{Y}(\mathfrak{g}_n)$-intertwining operator (1.16). Let $J(\mu)$ be this restriction divided by the rational functions (2.24) and (3.3), and in the case of $\mathfrak{g}_n = \mathfrak{sp}_n$ also divided by the rational function (3.4). Then by Propositions 2.5 and 3.2

$$J(\mu) : \varphi_{\nu_1} \otimes \cdots \otimes \varphi_{\nu_m} \mapsto \varphi_{\nu_1} \otimes \cdots \otimes \varphi_{\nu_m}. \tag{3.24}$$

Theorem 3.4. *For a fixed weight $\nu = \lambda - \mu + \kappa$ the rational function $J(\mu)$ is regular at any point $\mu \in \mathfrak{h}_m^*$ where the weight $\lambda + \rho$ is dominant.*

The operator-valued rational function $J(\mu)$ does not vanish at any point $\mu \in \mathfrak{h}_m^*$ due to the normalization (3.24). The regularity of $J(\mu)$ was proved in [KN] for all μ where the pair (λ, μ) is good; our Theorem 3.4 is more general.

Below we will give explicit formulas for the operator $J(\mu)$ with dominant $\lambda + \rho$. Theorem 3.4 will follow from these formulas. For related results see the work of Isaev and Molev [IM]. Both Proposition 3.3 and Theorem 3.4 can also be proved by using the arguments from [KN, Sections 1.4 and 4.4].

3.4. Let the weight $\lambda + \rho$ of $\mathfrak{f}_m = \mathfrak{so}_{2m}$ or of $\mathfrak{f}_m = \mathfrak{sp}_{2m}$ be dominant. Then $\lambda_1, \dots, \lambda_m$ obey the inequalities (2.40) where $\rho_1, \dots, \rho_m$ are now the labels of the half-sum of positive roots of $\mathfrak{f}_m$. But for any given $a < b$ the difference $\rho_a - \rho_b = b - a$ is now the same as it was for $\mathfrak{gl}_m$. Moreover, for $\mathfrak{f}_m$ we have the same relation

$$\lambda_a - \lambda_b = \mu_a - \mu_b + \nu_a - \nu_b$$

as we had for $\mathfrak{gl}_m$. Any of our two proofs of Theorem 2.7 now shows that the operator $B(\mu)$ divided by (2.24) has a regular restriction to the subspace (2.27). Moreover, each of the two proofs gives an explicit formula for the restriction.

We will give two parallel proofs of the regularity of restriction to (2.27) of the operator $C(\mu)$ divided by (3.3), and in the case $\mathfrak{f}_m = \mathfrak{sp}_{2m}$ also divided by (3.4). We will keep assuming that the weight $\lambda + \rho$ is dominant. In particular,

$$\lambda_a + \lambda_b + \rho_a + \rho_b \neq 1, 2, \dots \quad \text{whenever} \quad a < b, \tag{3.25}$$

$$\lambda_a + \rho_a \neq 1, 2, \dots \quad \text{if} \quad \mathfrak{f}_m = \mathfrak{sp}_{2m}. \tag{3.26}$$

Then the operator $C(\mu)$ has a regular restriction to (2.27) by Proposition 3.3. Let us now prove the last fact directly, that is without using Proposition 3.3.

Take any pair of indices (p, q) with $p < q$ and consider the corresponding factor $\widetilde{R}_{pq}(n - x_p - x_q)$ of the product (3.1). Let a and b be the numbers of the segments of the sequence $1, \dots, N$ containing the indices p and q respectively. Let i and j be the numbers of the indices p and q within their segments. Then

$$x_p + x_q - n = \lambda_a + \rho_a + \lambda_b + \rho_b - i - j + 1. \tag{3.27}$$

First, suppose that p and q belong to different segments, so that $a < b$. Then the right-hand side of the equality (3.27) is not zero by (3.25). Therefore the factor $\widetilde{R}_{pq}(n - x_p - x_q)$ of the product (3.1) with $a < b$ is regular.

Next, suppose that p and q belong to the same segment, that is $a = b$. If $\mathfrak{f}_m = \mathfrak{so}_{2m}$ then the factor $\widetilde{R}_{pq}(n - x_p - x_q)$ can be skipped without changing the restriction of (3.1) to the subspace (2.27). If

$\mathfrak{f}_m = \mathfrak{sp}_{2m}$ then by (3.26) the right-hand side of the equality (3.27) is not zero whenever the sum $i+j$ is odd. In particular, if $\mathfrak{f}_m = \mathfrak{sp}_{2m}$ then the factor $\widetilde{R}_{pq}(n - x_p - x_q)$ with $q = p+1$ is regular, because for this factor $j = i+1$.

Last, suppose that $q > p+1$ while $a = b$. Then the pair following (p,q) in the reversed lexicographical ordering is $(p, q-1)$. Moreover, then the index $q-1$ belongs to the same segment as p and q. Take the product of the factors in (3.1) corresponding the pairs following $(p, q-1)$. Multiply this product by

$$R_{q-1,q}(x_{q-1} - x_q) = R_{q-1,q}(1) = 1 - P_{q-1,q} \tag{3.28}$$

on the right. By using the relation (2.4) repeatedly, one shows that the resulting product is also divisible by (3.28) on the left. Due to (2.8) we can then replace in (3.1) the product of two adjacent factors

$$\widetilde{R}_{pq}(n - x_p - x_q)\widetilde{R}_{p,q-1}(n - x_p - x_{q-1})$$

by

$$1 - (\widetilde{P}_{pq} + \widetilde{P}_{p,q-1})/(n - x_p - x_{q-1})$$

without changing the restriction of the operator (3.1) to the subspace (2.27). But the replacement is regular at $x_p + x_q = n$. This observation completes our second proof of the regularity of the restriction of $C(\mu)$ to the subspace (2.27), whenever the weight $\lambda + \rho$ of $\mathfrak{f}_m$ is dominant.

Now recall that defining the operator $J(\mu)$ involves dividing $C(\mu)$ by (3.3), and also dividing by (3.4) in the case $\mathfrak{f}_m = \mathfrak{sp}_{2m}$. So we have to consider the zeroes of the rational functions (3.3) and (3.4). In Section 3.5 for $m = 1$ and $\mathfrak{g}_n = \mathfrak{sp}_n$ we will prove that (3.11) annihilates the subspace (3.12) whenever

$$\lambda_1 + \rho_1 = 0 \quad \text{and} \quad 2\nu_1 > n. \tag{3.29}$$

In Section 3.6 for $m = 2$ and both $\mathfrak{g}_n = \mathfrak{so}_n, \mathfrak{sp}_n$ we will prove that the operator (3.6) annihilates the subspace (2.32) whenever

$$\lambda_1 + \lambda_2 + \rho_1 + \rho_2 = 0 \quad \text{and} \quad \nu_1 + \nu_2 > n. \tag{3.30}$$

Theorem 3.4 for any $m \geqslant 1$ will then follow from the definitions (3.3) and (3.4).

3.5. For $m = 1$ and $\mathfrak{g}_n = \mathfrak{sp}_n$ consider the operator (3.11) on the vector space $(\mathbb{C}^n)^{\otimes \nu_1}$. Here the positive integer n is even. For $p = 1, \ldots, \nu_1$ introduce the rational function of $x \in \mathbb{C}$ taking values in the operator algebra $(\operatorname{End}\mathbb{C}^n)^{\otimes \nu_1}$

$$D(x,p) = \prod_{j=2,\ldots,p}^{\longleftarrow} \left(\prod_{i=1,\ldots,j-1}^{\longleftarrow} \widetilde{R}_{ij}(i + j - 2x - 1) \right). \tag{3.31}$$

By (3.10) we have

$$C(\mu) = D(\lambda_1 + \rho_1, \nu_1) \tag{3.32}$$

while $D(x,1) = 1$ by definition. Put $D(x,0) = 1$. Like we did at the beginning of the proof of Proposition 3.3, let us relate two operators on the vector space $(\mathbb{C}^n)^{\otimes \nu_1}$ by the symbol $\equiv$ if their actions on the subspace (3.12) coincide.

Lemma 3.5. *For each* $p = 2, \ldots, \nu_1$ *we have the relation in* $(\operatorname{End}\mathbb{C}^n)^{\otimes \nu_1}$

$$\widetilde{P}_{p-1,p} D(x, p) \equiv \frac{x + n/2 - p + 1}{x - 1} \widetilde{P}_{p-1,p} D(x - 1, p - 2). \tag{3.33}$$

Proof. Using Proposition 3.3 where $m = 1$ whereas the numbers $\lambda_1 + \rho_1$ and ν_1 are replaced by x and p respectively, we obtain the relation

$$D(x, p) \equiv \sum_{d \geqslant 0} \sum_{\substack{i_1,\ldots,i_d \\ j_1,\ldots,j_d}} \prod_{k=1}^{d} \frac{\widetilde{P}_{i_k j_k}}{2(x - k)} \tag{3.34}$$

where $i_1, j_1, \ldots, i_d, j_d$ are pairwise distinct indices from the sequence $1, \ldots, p$ taken so that different are all the sets of d unordered pairs (3.8). Like in the proof of Proposition 3.3 we assume that 1 is the only term in (3.34) with $d = 0$.

For any $d \geqslant 0$ and for any choice of the set of d pairs (3.8) made as above consider the corresponding product $\widetilde{P}_{i_1 j_1} \cdots \widetilde{P}_{i_d j_d}$ showing in (3.34). Multiply the product by $\widetilde{P}_{p-1,p}$ on the left. If neither of the indices $p - 1$, p occurs in the pairs (3.8) then leave the result of multiplication as it is, $\widetilde{P}_{p-1,p} \widetilde{P}_{i_1 j_1} \cdots \widetilde{P}_{i_d j_d}$.

Next, suppose that exactly one of the indices $p - 1$, p occurs in (3.8). We can assume that then $j_d = p - 1$ or $j_d = p$ without further loss of generality. Put $j = p$ or $j = p - 1$ respectively. Then

$$\begin{aligned} \widetilde{P}_{p-1,p} \widetilde{P}_{i_1 j_1} \cdots \widetilde{P}_{i_d j_d} &= \widetilde{P}_{p-1,p} \widetilde{P}_{i_1 j_1} \cdots \widetilde{P}_{i_{d-1} j_{d-1}} P_{i_d j} \\ &\equiv -\widetilde{P}_{p-1,p} \widetilde{P}_{i_1 j_1} \cdots \widetilde{P}_{i_{d-1} j_{d-1}}. \end{aligned} \tag{3.35}$$

Note that in either case, that is $j_d = p - 1$ or $j_d = p$, for any given distinct indices $i_1, j_1, \ldots, i_{d-1}, j_{d-1}$ taken from $1, \ldots, p - 2$ there are exactly $p - 2d$ choices of the index i_d yielding the same term (3.35) where i_d does not occur. Counting both cases, we will get the term (3.35) with multiplicity $2(p - 2d)$.

Finally, suppose that both of the indices $p - 1$, p occur in (3.8). If they occur in the same pair, then without further loss of generality we may assume that $i_d = p - 1$ and $j_d = p$. Then

$$\widetilde{P}_{p-1,p} \widetilde{P}_{i_1 j_1} \cdots \widetilde{P}_{i_d j_d} = n \widetilde{P}_{p-1,p} \widetilde{P}_{i_1 j_1} \cdots \widetilde{P}_{i_{d-1} j_{d-1}}. \tag{3.36}$$

If $p - 1$, p occur in different pairs in (3.8) then without further loss of generality we may assume that $j_{d-1} = p - 1$ and $i_d = p$. Then

$$\begin{aligned} \widetilde{P}_{p-1,p} \widetilde{P}_{i_1 j_1} \cdots \widetilde{P}_{i_d j_d} &= \widetilde{P}_{p-1,p} \widetilde{P}_{i_1 j_1} \cdots \widetilde{P}_{i_{d-2} j_{d-2}} P_{i_{d-1},p} \widetilde{P}_{p,j_d} \\ &\equiv -\widetilde{P}_{p-1,p} \widetilde{P}_{i_1 j_1} \cdots \widetilde{P}_{i_{d-2} j_{d-2}} \widetilde{P}_{i_{d-1} j_d}. \end{aligned} \tag{3.37}$$

Without altering the value of the term (3.37), we can either exchange the pair (i_{d-1}, j_d) with any of the pairs $(i_1, j_1), \ldots, (i_{d-2}, j_{d-2})$ or swap the two indices i_{d-1}, j_d between each other. Counting these together with the initial choice of i_{d-1}, j_d we will get the term (3.37) with multiplicity $2(d - 1)$. Note that here the term (3.37) arises only when $d \geqslant 2$. But for $d = 1$ we have $2(d - 1) = 0$.

Let us multiply the relation (3.34) by $\widetilde{P}_{p-1,p}$ on the left, and then perform the summation over the indices d and $i_1, j_1, \ldots, i_d, j_d$. The terms (3.35), (3.36) occur only when $d \geqslant 1$. Now replace the index $d - 1$ in (3.35), (3.36) by $d \geqslant 0$. After this replacement, the multiplicity of (3.35) will become $2(p - 2d - 2)$.

Further, rename the running index j_d in (3.37) by j_{d-1} and then replace $d - 1$ by d. After this, the corresponding multiplicity is $2d$. Then $d \geqslant 1$. But we can also sum over $d \geqslant 0$, because the multiplicity in the case $d = 0$ is zero.

After these replacements, the product $\widetilde{P}_{p-1,p}D(x,p)$ gets related by $\equiv$ to

$$\sum_{d\geqslant 0}\sum_{\substack{i_1,\dots,i_d\\ j_1,\dots,j_d}}\widetilde{P}_{p-1,p}\left(1+\frac{n-2(p-2d-2)-2d}{2(x-d-1)}\right)\prod_{k=1}^{d}\frac{\widetilde{P}_{i_kj_k}}{2(x-k)}$$

$$=\sum_{d\geqslant 0}\sum_{\substack{i_1,\dots,i_d\\ j_1,\dots,j_d}}\widetilde{P}_{p-1,p}\frac{x+n/2-p+1}{x-d-1}\prod_{k=1}^{d}\frac{\widetilde{P}_{i_kj_k}}{2(x-k)}$$

$$=\sum_{d\geqslant 0}\sum_{\substack{i_1,\dots,i_d\\ j_1,\dots,j_d}}\widetilde{P}_{p-1,p}\frac{x+n/2-p+1}{x-1}\prod_{k=1}^{d}\frac{\widetilde{P}_{i_kj_k}}{2(x-k-1)}$$

where $i_1,j_1,\dots,i_d,j_d$ are distinct indices taken from the sequence $1,\dots,p-2$ such that different are all the sets of d unordered pairs (3.8). But the sum in the last displayed line is related by $\equiv$ to the right-hand side of (3.33). Here we use the relation (3.34) with $x-1$ and $p-2$ instead of x and p respectively. $\square$

From now until the end of this subsection we assume that $2\nu_1>n$. Denote

$$l=\nu_1-n/2.$$

It is a classical fact [W, Section VI.3] that then the subspace (3.12) is contained in the span of the images of all operators on $(\mathbb{C}^n)^{\otimes\nu_1}$ of the form $\widetilde{P}_{i_1j_1}\cdots\widetilde{P}_{i_lj_l}$ where $i_1,j_1,\dots,i_l,j_l$ are any pairwise distinct indices taken from $1,\dots,\nu_1$. To show that $C(\mu)\equiv 0$ under the conditions (3.29), it now suffices to prove

$$\widetilde{P}_{i_1j_1}\cdots\widetilde{P}_{i_lj_l}C(\mu)\equiv 0\quad\text{for}\quad\lambda_1+\rho_1=0\tag{3.38}$$

and for all those $i_1,j_1,\dots,i_l,j_l$. Here we also use the equality $\widetilde{P}^2=n\widetilde{P}$ and the fact that the operator $C(\mu)$ with generic μ preserves the subspace (3.12). Furthermore, due to the latter fact, it suffices to prove the relation (3.38) for any single choice of the indices $i_1,j_1,\dots,i_l,j_l$. Let us choose

$$i_1=\nu_1-2l+1,\quad j_1=\nu_1-2l+2,\quad\dots,\quad i_l=\nu_1-1,\quad j_l=\nu_1.$$

By using (3.32) and then applying Lemma 3.5 repeatedly, namely applying it l times, we get the following relation between rational functions of $\mu\in\mathfrak{h}_1^*$:

$$\widetilde{P}_{\nu_1-2l+1,\nu_1-2l+2}\cdots\widetilde{P}_{\nu_1-1,\nu_1}C(\mu)\equiv\frac{\lambda_1+\rho_1}{\mu_1+\rho_1}\widetilde{P}_{\nu_1-2l+1,\nu_1-2l+2}\cdots\widetilde{P}_{\nu_1-1,\nu_1}D(\mu_1+\rho_1,n-\nu_1).$$

To get the latter relation, we also used the equality $\nu_1-2l=n-\nu_1$. If $\lambda_1+\rho_1=0$ then we have $\mu_1+\rho_1=-l$, and the fraction in the above display equals zero. The last factor in that display is regular at $\lambda_1+\rho_1=0$ by the definition (3.31). This proves (3.38) for our choice of $i_1,j_1,\dots,i_l,j_l$.

As a rational function of μ, the last displayed factor can be replaced by the sum at the right-hand side of the relation (3.34) where $x=\mu_1+\rho_1$ and $p=n-\nu_1$. Arguing like in Section 3.4, we can also provide a multiplicative formula for the value of that factor whenever $\mu_1+\rho_1\neq 1,2,\dots$.

Multiplying the last displayed relation on the left by operators on $(\mathbb{C}^n)^{\otimes\nu_1}$ which permute the ν_1 tensor factors, we obtain analogues of that relation for all other choices of $i_1,j_1,\dots,i_l,j_l$. Therefore in the case $m=1$ and $2\nu_1>n$, the last displayed relation determines the action on the subspace (3.12) for any value of the function $C(\mu)$ divided by (3.4), whenever $\mu_1+\rho_1\neq 1,2,\ldots$.

3.6. For $m=2$ and $\mathfrak{g}_n=\mathfrak{so}_n,\mathfrak{sp}_n$ consider the operator (3.6) on $(\mathbb{C}^n)^{\otimes(\nu_1+\nu_2)}$. For $p=1,\ldots,\nu_1$ and $q=\nu_1+1,\ldots,\nu_1+\nu_2$ introduce the rational function of $x\in\mathbb{C}$ taking values in the algebra $(\operatorname{End}\mathbb{C}^n)^{\otimes(\nu_1+\nu_2)}$

$$D(x,p,q)=\overleftarrow{\prod_{i=1,\ldots,p}}\left(\overleftarrow{\prod_{j=1,\ldots,q-\nu_1}}\widetilde{R}_{ij}(i+j-x-1)\right). \tag{3.39}$$

By (3.10)

$$Z=D(\lambda_1+\lambda_2+\rho_1+\rho_2,\nu_1,\nu_1+\nu_2). \tag{3.40}$$

Put $D(x,p,\nu_1)=D(x,0,q)=1$. Let us relate operators on $(\mathbb{C}^n)^{\otimes(\nu_1+\nu_2)}$ by the symbol $\equiv$ if their actions coincide on the subspace (2.32).

Lemma 3.6. *For any $p=1,\ldots,\nu_1$ and $q=\nu_1+1,\ldots,\nu_1+\nu_2$ we have*

$$\widetilde{P}_{pq}D(x,p,q)\equiv\frac{x+n-p-q+\nu_1+1}{x-1}\widetilde{P}_{pq}D(x-1,p-1,q-1). \tag{3.41}$$

Proof. At the end of the proof of Proposition 3.3 we established the equality of the expressions (3.22) and (3.23). Replacing the numbers $\lambda_1+\lambda_2+\rho_1+\rho_2$ and ν_1, ν_2 in that equality by x and p, $q-\nu_1$ respectively, we get the relation

$$D(x,p,q)\equiv\sum_{d\geqslant 0}\sum_{\substack{i_1,\ldots,i_d\\ j_1,\ldots,j_d}}\prod_{k=1}^{d}\frac{\widetilde{P}_{i_kj_k}}{x-k} \tag{3.42}$$

where $i_1,\ldots,i_d$ and $j_1,\ldots,j_d$ are distinct indices taken respectively from the sequences $1,\ldots,p$ and $\nu_1+1,\ldots,q$ so that different are the corresponding sets of d pairs (2.29). We assume that 1 is the only summand in (3.42) with $d=0$.

For any $d\geqslant 0$ and for any choice of the set of d pairs (2.29) made as above consider the corresponding product $\widetilde{P}_{i_1j_1}\cdots\widetilde{P}_{i_dj_d}$ showing in (3.42). Multiply the product by $\widetilde{P}_{pq}$ on the left. If neither of the indices p, q occurs in the pairs (2.29) then leave the result of multiplication as it is, that is $\widetilde{P}_{pq}\widetilde{P}_{i_1j_1}\cdots\widetilde{P}_{i_dj_d}$.

Next, suppose that exactly one of the indices p, q occurs in (2.29). We can assume that then $i_d=p$ or $j_d=q$ without further loss of generality. Then

$$\widetilde{P}_{pq}\widetilde{P}_{i_1j_1}\cdots\widetilde{P}_{i_dj_d}\equiv-\widetilde{P}_{pq}\widetilde{P}_{i_1j_1}\cdots\widetilde{P}_{i_{d-1}j_{d-1}}. \tag{3.43}$$

In the first case, that is if $i_d=p$, for any given $i_1,j_1,\ldots,i_{d-1},j_{d-1}$ there are exactly $q-\nu_1-d$ choices of the index j_d yielding the same right-hand side of (3.43), where j_d does not occur. In the second case, that is if $j_d=q$, for any given $i_1,j_1,\ldots,i_{d-1},j_{d-1}$ there are exactly $p-d$ choices of the index i_d yielding the same right-hand side of (3.43), where i_d does not occur. Counting both cases, we will get the term (3.43) with multiplicity $p+q-\nu_1-2d$.

Last suppose p, q both occur in (2.29). If they occur in the same pair, then without further loss of generality we may assume that $i_d=p$ and $j_d=q$. Then

$$\widetilde{P}_{pq}\widetilde{P}_{i_1j_1}\cdots\widetilde{P}_{i_dj_d}=n\widetilde{P}_{pq}\widetilde{P}_{i_1j_1}\cdots\widetilde{P}_{i_{d-1}j_{d-1}}. \tag{3.44}$$

If p, q occur in different pairs in (2.29) then without further loss of generality we may assume that $j_{d-1} = q$ and $i_d = p$. Then

$$\widetilde{P}_{pq}\widetilde{P}_{i_1 j_1} \cdots \widetilde{P}_{i_d j_d} \equiv -\widetilde{P}_{pq}\widetilde{P}_{i_1 j_1} \cdots \widetilde{P}_{i_{d-2} j_{d-2}}\widetilde{P}_{i_{d-1} j_d}. \tag{3.45}$$

Without altering the product at the right-hand side of (3.45), we can exchange the pair (i_{d-1}, j_d) with any of the pairs $(i_1, j_1), \ldots, (i_{d-2}, j_{d-2})$. Counting these together with the initial choice of (i_{d-1}, j_d) we will get the term (3.37) with multiplicity $d-1$. Note that here the term (3.45) arises only when $d \geqslant 2$.

Let us multiply the relation (3.42) by $\widetilde{P}_{pq}$ on the left, and then perform the summation over the indices d and $i_1, j_1, \ldots, i_d, j_d$. The terms (3.43), (3.44) occur only when $d \geqslant 1$. Now replace the index $d-1$ in (3.43), (3.44) by $d \geqslant 0$. After this replacement, the multiplicity of (3.43) becomes $p+q-\nu_1-2d-2$.

Further, let us rename the running index j_d at the right-hand side of (3.45) by j_{d-1}, and then replace $d-1$ by d there. Having done this, the corresponding multiplicity becomes d. Now $d \geqslant 1$. But we can also sum over $d \geqslant 0$, because the multiplicity in the case $d = 0$ is zero.

After these replacements, the product $\widetilde{P}_{pq} D(x, p, q)$ gets related by $\equiv$ to

$$\sum_{d \geqslant 0} \sum_{\substack{i_1, \ldots, i_d \\ j_1, \ldots, j_d}} \widetilde{P}_{pq}\left(1 + \frac{n - (p+q-\nu_1-2d-2) - d}{x-d-1}\right) \prod_{k=1}^{d} \frac{\widetilde{P}_{i_k j_k}}{x-k}$$

$$= \sum_{d \geqslant 0} \sum_{\substack{i_1, \ldots, i_d \\ j_1, \ldots, j_d}} \widetilde{P}_{pq} \frac{x+n-p-q+\nu_1+1}{x-1} \prod_{k=1}^{d} \frac{\widetilde{P}_{i_k j_k}}{x-k-1}$$

where $i_1, \ldots, i_d$ and $j_1, \ldots, j_d$ are pairwise distinct indices taken respectively from $1, \ldots, p-1$ and $\nu_1+1, \ldots, q-1$ so that different are the corresponding sets of d pairs (2.29). The sum in the last displayed line is related by $\equiv$ to the right-hand side of (3.41). Here we use the relation (3.42) with $x-1$, $p-1$, $q-1$ instead of x, p, q respectively. $\square$

From now until the end of this subsection assume that $\nu_1 + \nu_2 > n$. Denote

$$l = \nu_1 + \nu_2 - n.$$

Then the subspace (2.32) is contained in the span of the images of all operators on $(\mathbb{C}^n)^{\otimes(\nu_1+\nu_2)}$ of the form $\widetilde{P}_{i_1 j_1} \cdots \widetilde{P}_{i_l j_l}$ where $i_1, \ldots, i_l$ and $j_1, \ldots, j_l$ are pairwise distinct indices from the sequences $1, \ldots, \nu_1$ and $\nu_1+1, \ldots, \nu_1+\nu_2$ respectively; see for instance the proof of [W, Lemma V.7.B]. To show that $Z \equiv 0$ under the conditions (3.30), it therefore suffices to prove that

$$\widetilde{P}_{i_1 j_1} \cdots \widetilde{P}_{i_l j_l} Z \equiv 0 \quad \text{for} \quad \lambda_1 + \lambda_2 + \rho_1 + \rho_2 = 0 \tag{3.46}$$

and for all those $i_1, j_1, \ldots, i_l, j_l$. Here we also use the fact that for generic $\mu \in \mathfrak{h}_2^*$ the operator Z preserves the subspace (2.32). Due to the latter fact, it suffices to prove (3.46) for any single choice of $i_1, j_1, \ldots, i_l, j_l$. Let us choose

$$i_1 = \nu_1 - l + 1, \quad j_1 = \nu_1 + \nu_2 - l + 1, \quad \ldots, \quad i_l = \nu_1, \quad j_l = \nu_1 + \nu_2.$$

By using (3.40) and then applying Lemma 3.6 repeatedly, namely applying it l times, we get the following relation between rational functions of $\mu \in \mathfrak{h}_2^*$:

$$\widetilde{P}_{\nu_1-l+1,\nu_1+\nu_2-l+1}\cdots\widetilde{P}_{\nu_1,\nu_1+\nu_2}Z$$
$$\equiv\frac{\lambda_1+\lambda_2+\rho_1+\rho_2}{\mu_1+\mu_2+\rho_1+\rho_2}\widetilde{P}_{\nu_1-l+1,\nu_1+\nu_2-l+1}\cdots\widetilde{P}_{\nu_1,\nu_1+\nu_2}D(\mu_1+\mu_2+\rho_1+\rho_2,n-\nu_2,n).$$

If $\lambda_1+\lambda_2+\rho_1+\rho_2=0$ then $\mu_1+\mu_2+\rho_1+\rho_2=-l$, and the fraction in the above display equals zero. But the last factor in that display is regular at $\lambda_1+\lambda_2+\rho_1+\rho_2=0$ by the definition (3.39). This proves the relation (3.46) for our particular choice of the indices $i_1, j_1, \dots, i_l, j_l$.

As rational function of μ, the last factor can be replaced by the sum at right-hand side of the relation (3.42) where $x=\mu_1+\mu_2+\rho_1+\rho_2$ while $p=n-\nu_2$ and $q=n$. Arguing like in Section 3.4, we can also provide a multiplicative formula for the value of that factor whenever $\mu_1+\mu_2+\rho_1+\rho_2\neq 1,2,\dots$.

Further, multiplying the last displayed relation on the left by operators on $(\mathbb{C}^n)^{\otimes(\nu_1+\nu_2)}$ which permute the first ν_1 tensor factors between themselves, and also permute the last ν_2 tensor factors, we get analogues of that relation for all other choices of $i_1, j_1, \dots, i_l, j_l$. So in the case $m=2$ and $\nu_1+\nu_2>n$, the last displayed relation determines the action on the subspace (2.32) for any value of the function Z divided by (3.3), when $\mu_1+\mu_2+\rho_1+\rho_2\neq 1,2,\dots$.

3.7. In this subsection we will generalize Theorem 1.2. Theorem 3.4 allows us to determine the intertwining operator $J(\mu)$ of the $\mathrm{Y}(\mathfrak{g}_n)$-modules (1.16) for any $\mu\in\mathfrak{h}_m^*$, provided $\lambda+\rho$ is dominant. Our generalization of Theorem 1.2 is based on the next two lemmas. For each index $c=1,\dots,m-1$ regard s_c as an element of the group H_m. The action of s_c on $\mathfrak{h}_m$ exchanges the basis vectors F_{cc} and $F_{c+1,c+1}$, leaving all other basis vectors of $\mathfrak{h}_m$ fixed. The first of the two lemmas is an analogue of Lemma 2.8 for the twisted Yangian $\mathrm{Y}(\mathfrak{g}_n)$.

Lemma 3.7. *Fix $c>0$. Suppose that both $\lambda+\rho$ and $s_c(\lambda+\rho)$ are dominant. Then the images of the intertwining operators $J(\mu)$ and $J(s_c\circ\mu)$ corresponding to the pairs (λ,μ) and $(s_c\circ\lambda, s_c\circ\mu)$ are equivalent as $\mathrm{Y}(\mathfrak{g}_n)$-modules.*

Proof. Let $\check{\mu}$ and $\check{\rho}$ be the weights of $\mathfrak{gl}_2$ with the labels μ_c, μ_{c+1} and ρ_c, ρ_{c+1} respectively. Let $\check{\lambda}$ be the weight of $\mathfrak{gl}_2$ with labels $\mu_c+\nu_c$, $\mu_{c+1}+\nu_{c+1}$. Here

$$\mu_c+\nu_c=\lambda_c+n/2 \quad \text{and} \quad \mu_{c+1}+\nu_{c+1}=\lambda_{c+1}+n/2.$$

The dominance of the weights $\lambda+\rho$ and $s_c(\lambda+\rho)$ of $\mathfrak{f}_m$ implies that the weights $\check{\lambda}+\check{\rho}$ and $s_1(\check{\lambda}+\check{\rho})$ of $\mathfrak{gl}_2$ are also dominant. By Theorem 2.7 with $m=2$, we get an intertwining operator of $\mathrm{Y}(\mathfrak{gl}_n)$-modules (2.45). It is invertible, see the proof of Lemma 2.8. Like in that proof, denote by I the operator which acts on the tensor product of cth and $(c+1)$th factors of (1.8) as this intertwining operator (2.45), and which acts trivially on other $m-2$ tensor factors of (1.8).

Similarly, by using Theorem 2.7 with $m=2$ once again, we get an invertible intertwining operator of $\mathrm{Y}(\mathfrak{gl}_n)$-modules

$$\Phi^{-\nu_c}_{\mu_c+\rho_c+\frac{1}{2}}\otimes\Phi^{-\nu_{c+1}}_{\mu_{c+1}+\rho_{c+1}+\frac{1}{2}}\to\Phi^{-\nu_{c+1}}_{\mu_{c+1}+\rho_{c+1}+\frac{1}{2}}\otimes\Phi^{-\nu_c}_{\mu_c+\rho_c+\frac{1}{2}}. \tag{3.47}$$

Now denote by J the operator which acts as (3.47) on the tensor product of cth and $(c+1)$th factors of the target $\mathrm{Y}(\mathfrak{g}_n)$-module in (1.16), and which acts trivially on other $m-2$ tensor factors of the latter module. Arguing like in the end of the proof of Lemma 2.8, that is either performing a direct calculation, or using the irreducibility of the source and target $\mathrm{Y}(\mathfrak{g}_n)$-modules in (1.16) for any generic weight μ of $\mathfrak{f}_m$, we obtain the relation

$$JJ(\mu)=J(s_c\circ\mu)I$$

whenever $\lambda+\rho$ and $s_c(\lambda+\mu)$ are dominant. It proves Lemma 3.7, since I and J are invertible and intertwine $\mathrm{Y}(\mathfrak{g}_n)$-modules by restriction from $\mathrm{Y}(\mathfrak{gl}_n)$. $\square$

Let $s_0 \in H_m$ be the element which acts on $\mathfrak{h}_m$ by mapping F_{11} to $-F_{11}$, and leaves all other basis vectors fixed. Note that in the case of $\mathfrak{f}_m = \mathfrak{so}_{2m}$ the element s_0 belongs to the extended Weyl group, not to the Weyl group proper. Further, in this case the dominance of $s_0(\lambda + \rho)$ is equivalent to that of $\lambda + \rho$.

Lemma 3.8. *Suppose that both the weights $\lambda + \rho$ and $s_0(\lambda + \rho)$ are dominant. Then the images of the intertwining operators $J(\mu)$ and $J(s_0 \circ \mu)$ corresponding to the pairs (λ, μ) and $(s_0 \circ \lambda, s_0 \circ \mu)$ are similar as* $\mathrm{Y}(\mathfrak{g}_n)$*-modules.*

Proof. Let $\check{\lambda}$, $\check{\mu}$, $\check{\rho}$ be the weights of $\mathfrak{f}_1$ with labels λ_1, μ_1, ρ_1 respectively. The weights $\check{\lambda} + \check{\rho}$ and $s_0(\check{\lambda} + \check{\rho})$ of $\mathfrak{f}_1$ are dominant. For $\mathfrak{f}_1 = \mathfrak{sp}_2$ this means that $\lambda_1 \notin \mathbb{Z} \setminus \{1\}$. For $\mathfrak{f}_1 = \mathfrak{so}_2$ any weight is dominant. By using Theorem 3.4 with $m = 1$ we get the intertwining operators of $\mathrm{Y}(\mathfrak{g}_n)$-modules

$$\Phi^{\nu_1}_{\mu_1+\rho_1+\frac{1}{2}} \to \Phi^{-\nu_1}_{\mu_1+\rho_1+\frac{1}{2}}, \tag{3.48}$$

$$\Phi^{n-\nu_1}_{\frac{1}{2}-\mu_1-\rho_1} \to \Phi^{\nu_1-n}_{\frac{1}{2}-\mu_1-\rho_1}. \tag{3.49}$$

For $\mathfrak{f}_1 = \mathfrak{so}_2$ each of these two operators acts as the identity, see remarks made at the beginning of Section 3.2. For $\mathfrak{f}_1 = \mathfrak{sp}_2$ none of these two operators acts as the identity in general, but they are still invertible. The latter assertion can be proved either by direct calculation, or by using the irreducibility of all four $\mathrm{Y}(\mathfrak{g}_n)$-modules in (3.48) and (3.49) for generic $\check{\mu}$, that is for $\mu_1 \notin \mathbb{Z}/2$.

For instance, let us prove the invertibility of (3.48). By applying Lemma 2.3 to both the source and target $\mathrm{Y}(\mathfrak{gl}_n)$-modules in (3.49) we get an intertwining operator

$$\Phi^{-\nu_1}_{\mu_1+\rho_1+\frac{1}{2}} \to \Phi^{\nu_1}_{\mu_1+\rho_1+\frac{1}{2}} \tag{3.50}$$

of $\mathrm{Y}(\mathfrak{sp}_n)$-modules. This operator maps $\varphi_{\nu_1} \mapsto \varphi_{\nu_1}$ as does the operator (3.48). Hence the operators (3.48) and (3.50) are inverse to each other for $\mu_1 \notin \mathbb{Z}/2$, and therefore for any $\mu_1 \in \mathbb{C}$ such that $\lambda_1 \notin \mathbb{Z} \setminus \{1\}$.

By using Lemma 2.3 with $k = \nu_1$ and $t = \mu_1 + \rho_1 + \frac{1}{2}$ we get an invertible intertwining operator

$$\Phi^{-\nu_1}_{\mu_1+\rho_1+\frac{1}{2}} \to \dot{\Phi}^{n-\nu_1}_{\frac{1}{2}-\mu_1-\rho_1} \tag{3.51}$$

of $\mathrm{Y}(\mathfrak{gl}_n)$-modules. It is also an intertwiner of $\mathrm{Y}(\mathfrak{g}_n)$-modules by restriction. Denote by I the operator which acts as the composition of (3.48) with (3.51) on the first tensor factor the source $\mathrm{Y}(\mathfrak{g}_n)$-module in (1.16), and which acts trivially on other $m-1$ tensor factors of the source module.

The intertwiner of $\mathrm{Y}(\mathfrak{g}_n)$-modules (3.49) can also be regarded as that of the $\mathrm{Y}(\mathfrak{g}_n)$-modules

$$\dot{\Phi}^{n-\nu_1}_{\frac{1}{2}-\mu_1-\rho_1} \to \dot{\Phi}^{\nu_1-n}_{\frac{1}{2}-\mu_1-\rho_1} \tag{3.52}$$

where we use the notation introduced immediately before stating Lemma 2.3. Denote by J the operator which acts as the composition of (3.51) with (3.52) on the first tensor factor the target $\mathrm{Y}(\mathfrak{g}_n)$-module in (1.16), and which acts trivially on other $m-1$ tensor factors of the target module.

By definition, $J(s_0 \circ \mu)$ is an intertwining operator of $\mathrm{Y}(\mathfrak{g}_n)$-modules

$$\begin{gathered}\Phi^{n-\nu_1}_{\frac{1}{2}-\mu_1-\rho_1} \otimes \Phi^{\nu_2}_{\mu_2+\rho_2+\frac{1}{2}} \otimes \cdots \otimes \Phi^{\nu_m}_{\mu_m+\rho_m+\frac{1}{2}} \\ \downarrow \\ \Phi^{\nu_1-n}_{\frac{1}{2}-\mu_1-\rho_1} \otimes \Phi^{-\nu_2}_{\mu_2+\rho_2+\frac{1}{2}} \otimes \cdots \otimes \Phi^{-\nu_m}_{\mu_m+\rho_m+\frac{1}{2}}.\end{gathered}$$

Let us now replace the first tensor factors of the above two $\mathrm{Y}(\mathfrak{g}_n)$-modules by

$$\dot{\Phi}^{\,n-\nu_1}_{\frac{1}{2}-\mu_1-\rho_1} \quad \text{and} \quad \dot{\Phi}^{\,\nu_1-n}_{\frac{1}{2}-\mu_1-\rho_1}$$

respectively. The operator $J(s_0 \circ \mu)$ also intertwines the resulting two tensor products as $\mathrm{Y}(\mathfrak{g}_n)$-modules. Take $J(s_0 \circ \mu)$ in its latter capacity. Then arguing like in the end of the proof of Lemma 2.8, that is either performing a direct calculation, or using the irreducibility of the source and target $\mathrm{Y}(\mathfrak{g}_n)$-modules in (1.16) for any generic weight μ of $\mathfrak{f}_m$, we obtain the relation

$$J\,J(\mu) = J(s_0 \circ \mu)I$$

for any dominant weights $\lambda + \rho$ and $s_0(\lambda + \mu)$ of $\mathfrak{f}_m$. This proves Lemma 3.8, because both I and J are invertible and intertwine $\mathrm{Y}(\mathfrak{g}_n)$-modules. □

For any $\lambda \in \mathfrak{h}_m^*$ let H_λ be the subgroup of H_m consisting of all elements w such that $w \circ \lambda = \lambda$. Let $\mathcal{O}$ be an orbit of the shifted action of the subgroup $H_\lambda \subset H_m$ on $\mathfrak{h}_m^*$. If $\nu_1, \ldots, \nu_m \in \{1, \ldots, n-1\}$ for at least one weight $\mu \in \mathcal{O}$, then every $\mu \in \mathcal{O}$ satisfies the same condition. Suppose this is the case for $\mathcal{O}$. If $\lambda + \rho$ is dominant, then there is at least one weight $\mu \in \mathcal{O}$ such that the pair (λ, μ) is good. Theorem 1.2 generalizes due to the following proposition.

Proposition 3.9. *If $\lambda + \rho$ is dominant, then for all $\mu \in \mathcal{O}$ the images of the corresponding operators $J(\mu)$ are similar to each other as* $\mathrm{Y}(\mathfrak{g}_n)$*-modules.*

Proof. Take any $w \in H_\lambda$ and any reduced decomposition $w = s_{c_l} \cdots s_{c_1}$. Here $c_1, \ldots, c_l \geqslant 0$. It can be derived from [B, Corollary VI.1.2] that the weight $s_{c_k} \cdots s_{c_1}(\lambda + \rho)$ of $\mathfrak{f}_m$ is dominant for each $k = 1, \ldots, l$. Applying Lemmas 3.7 and 3.8 now completes the proof of Proposition 3.9. □

Thus all assertions of Theorem 1.2 will remain valid if we replace the good pair there by any pair (λ, μ) such that the weight $\lambda + \rho$ of $\mathfrak{f}_m$ is dominant. However, we still assume that $\nu_1, \ldots, \nu_m \in \{1, \ldots, n-1\}$ for the latter pair.

Acknowledgments

We are grateful to Ivan Cherednik and Vitaly Tarasov for helpful discussions. The first author was supported by the RFBR grant 11-01-00962, by the joint RFBR-Royal Society grant 11-01-92612, and by the Federal Agency for Science and Innovations of the Russian Federation under the contract 14.740.11.0347.

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