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On Lower and on Sharp Asymptotic Estimates of Solutions of Emden–Fowler-Type Equations

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Abstract—Emden–Fowler-type equations of arbitrary order are considered. Lower and sharp asymptotic estimates of the nonoscillating continuable solutions of these equations are established.

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1. INTRODUCTION

Consider the equation of the following form:

$$\begin{gathered} y^{(n)} = p(x)|y|^{\sigma} \operatorname{sgn} y, \qquad n \ge 2, \quad \sigma > 1, \\ y = y(x), \quad p(x) \in C^0, \quad x, y \in \mathbb{R}^1, \quad p(x) \ne 0. \end{gathered} \tag{1}$$

For $n = 2$ and $p(x) = \pm x^{\beta}$, $x > 0$, $\beta = \text{const}$, this is the well known Emden–Fowler equation (see, for example, [1]), related to the study of a number of physical processes.

Definition 1. A solution $y(x)$ of Eq. (1) is said to be *continuable to the right* (*to the left*) if it is defined in a neighborhood of $+\infty$ ($-\infty$).

Definition 2. A nontrivial solution $y(x)$ of Eq. (1) is said to be *oscillating to the right* (*to the left*) if, for any x belonging to its domain of definition, there exists an $\widetilde{x} > x$ ($\widetilde{x} < x$) such that $y(\widetilde{x}) = 0$.

Solutions that are not continuable (oscillating) in any particular direction will be called *noncontinuable* (*nonoscillating*) in that direction.

Asymptotic estimates of noncontinuable nonoscillating solutions of Eq. (1) were given in [2] and [3].

In the present paper, we consider nonoscillating continuable to the right (to the left) solutions of Eq. (1) and establish asymptotic estimates of such solutions as $x \to \pm\infty$.

It is known (see [4] as well as [5, Lemma 2]) that, under the following condition:

$$|p(x)| \ge c x^{-n}, \qquad c = \text{const} > 0, \quad x \ge x_0 > 0, \tag{2}$$

for $(-1)^n p(x) > 0$, Eq. (1) has nontrivial continuable (to the right) nonoscillating solutions. Further, for any such solution, the derivatives $y^{(i)}(x)$, $0 \le i \le n-1$, are monotone functions and the following condition holds:

$$y^{(i)}(x) y^{(i+1)}(x) < 0, \quad \lim_{x \to +\infty} y^{(i)}(x) = 0, \qquad 0 \le i \le n-1. \tag{3}$$

If $(-1)^n p(x) < 0$, then, under the condition (2), Eq. (1) has no nontrivial continuable (to the right) nonoscillating solutions (see [6] as well as [5, Theorems 3 and 4]).

The following theorem [3] establishes upper asymptotic estimates of continuable (to the right) nonoscillating solutions.

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Theorem 1. *If, in Eq.* (1)*, the condition*

$$(-1)^n p(x) \geq c_1 x^{-m}, \qquad x \geq x_0 > 0, \quad c_1, m = \text{const}, \quad c_1 > 0, \quad m \leq n, \tag{4}$$

holds, then the nontrivial continuable (*to the right*) *nonoscillating* (*to the right*) *solutions of this equation satisfy the following estimates*:

- *if* $n > m$*, then*

$$|y(x)| \leq D x^{(m-n)/(\sigma-1)}, \qquad D = \text{const} > 0, \quad x \geq x_0; \tag{5}$$

- *if* $n = m$*, then*

$$|y(x)| \leq D |\ln x|^{-1/(\sigma-1)}, \qquad D = \text{const} > 0, \quad x \geq x_0 + 1. \tag{6}$$

Theorem 2 given below contains lower asymptotic estimates of these solutions.

Theorem 2. *If, in Eq.* (1)*, condition* (2) *and the condition*

$$0 < (-1)^n p(x) \leq c_1 x^{-m}, \qquad x \geq x_0 > 0, \quad c_1 = \text{const} > 0, \quad 0 \leq m \leq n, \tag{7}$$

hold, then the nontrivial continuable (*to the right*) *nonoscillating* (*to the right*) *solutions satisfy the following estimates*:

- *if* $n > m$*, then*

$$|y(x)| \geq D x^{(m-n)/(\sigma-1)}, \qquad D = \text{const} > 0, \quad x \geq x_0, \tag{8}$$

- *if* $n = m$*, then*

$$|y(x)| \geq D |\ln x|^{-1/(\sigma-1)}, \qquad D = \text{const} > 0, \quad x \geq x_0 + 1. \tag{9}$$

Now consider an important case describing the asymptotics of the solutions of Eq. (1) in which the function $p(x)$ has order-sharp power:

$$c_1 x^{-m} \leq (-1)^n p(x) \leq c_2 x^{-m}, \qquad x \geq x_0 > 0, \quad c_1, c_2 = \text{const} > 0, \quad m \leq n. \tag{10}$$

This case also includes the classical Emden–Fowler equation of arbitrary order.

Theorem 3. *If, in Eq.* (1)*, condition* (10) *holds, then its nontrivial continuable* (*to the right*) *nonoscillating* (*to the right*) *solutions satisfy the following estimates*:

- *if* $n > m$*, then*

$$D_1 x^{(m-n)/(\sigma-1)} \leq |y(x)| \leq D_2 x^{(m-n)/(\sigma-1)}, \qquad D_1, D_2 = \text{const} > 0, \quad x \geq x_0 > 0, \tag{11}$$

- *if* $n = m$*, then*

$$D_1 |\ln x|^{-1/(\sigma-1)} \leq |y(x)| \leq D_2 |\ln x|^{-1/(\sigma-1)}, \qquad D_1, D_2 = \text{const} > 0, \quad x \geq x_0 + 1. \tag{12}$$

Note that, in (10), in contrast to (7), it is not assumed that $m \geq 0$.

Assertions contained in Theorems 1–3 carry over in a natural way to continuable (to the left) nonoscillating solutions of Eq. (1).

2. EXAMPLES

Example 1. Consider the equation

$$y^{(n)} = x^{-n+\beta}|y|^{\sigma}\operatorname{sgn} y, \qquad \sigma > 1, \quad \beta > 0. \tag{13}$$

Here and below, we use the terminology adopted in [7] and [8]. The Newton polygon of this equation is the closed interval $[Q_1, Q_2]$, $Q_1 = (-n, 1)$, $Q_2 = (-n+\beta, \sigma)$, with the vector $[1, \mu]$, $\mu = \beta(1-\sigma)^{-1}$ serving as the normal to it. The reduced equation corresponding to this interval coincides with the complete equation (13). Using recommendations from [7], we obtain the solution of Eq. (13) as a power function, $y(x) = cx^{\mu}$, $x > 0$. By substitution, we obtain

$$c = ((-1)^n \mu(\mu-1)\cdots(\mu-n+1))^{1/(\sigma-1)}.$$

Thus, the function $y(x) = cx^{\mu}$, $x > 0$, is a solution of Eq. (13) for which estimates (11) become equalities.

Example 2. Consider the equation

$$y^{(4)} = x^{-4}y^2. \tag{14}$$

Its Newton polygon is the closed interval $[Q_1, Q_2]$, $Q_1 = (-4, 1)$, $Q_2 = (-4, 2)$, with vector $(1, 0)$ serving as the normal to it. By the change of variable $t = \ln(x)$, Eq. (14) can be rewritten

$$y_t^{(4)} - 6y_t''' + 11y_t'' - 6y_t' = y^2. \tag{15}$$

To the right edge of the Newton polygon of this equation corresponds the reduced equation $-6y_t' = y^2$ whose solution is of the form $y = y_0(t) = 6t^{-1}$. Replacing $y = z + y_0(t)$, we wsee that Eq. (15) takes the form

$$z_t^{(4)} - 6z_t''' + 11z_t'' - 6z_t' = z^2 + 12t^{-1}z + O(t^{-3}). \tag{16}$$

An analysis of this equation shows that its solution is

$$z(t) = o(t^{-1-\delta}), \qquad \delta > 0$$

(we omit the proof of this fact). This means that there exists a solution of Eq. (14) having the form $y = 6(\ln x)^{-1}(1 + o((\ln x)^{-\delta}))$ and, therefore, satisfying estimate (12).

3. PROOFS OF THE THEOREMS

Here and below, to show that some quantity is bounded by any constant, we shall use the so-called "universal" constant $D > 0$, assuming that $D + D = D$ and $D^{\beta} = D$, $\beta > 0$.

Theorem 1 was proved in [3]. To prove Theorem 2, we shall need the following lemmas.

Lemma 1. *If, in Eq.* (1), *conditions* (2) *and* (7) *hold, then its continuable* (*to the right*) *positive solutions $y = y(x)$ satisfy the following estimates*:

$$\begin{gathered}(-1)^{n+q-k}y^{(n-k)}y^{(q)} \le (-1)^{q-k}Dy^{\sigma}y^{(q-k)}x^{-m},\\ k \le q \le n-k, \quad k \ge 0, \quad x \ge x_0 > 0, \quad D = \text{const} > 0.\end{gathered} \tag{17}$$

The following assertion is an obvious consequence of Lemma 1.

Lemma 2. *If, in Eq.* (1), *conditions* (2) *and* (7) *hold, then its continuable* (*to the right*) *positive solutions $y = y(x)$ satisfy the following estimates*:

$$(-1)^n y^{(n-q)}y^{(q)} \le Dy^{\sigma+1}x^{-m}, \qquad 0 \le q \le n, \quad x \ge x_0 > 0, \quad D = \text{const} > 0. \tag{18}$$

Proof of Lemma 1. We shall argue by induction on k. For $k=0$ and any $0\le q\le n$, estimates (17) follow from Eq. (1) and conditions (3) and (7). Now suppose that the required estimate holds for some $k=K>0$, and let us prove its validity for $k=K+1$ and the corresponding values from $K+1\le q\le n-K-1$. This will again be proved by induction from the larger values of q down to the smaller. Let us, first, prove (17) for $k=K+1$ and $q=n-K-1$. By the inductive assumption, we have

$$(-1)^{2n-2K-1}y^{(n-K)}y^{(n-K-1)}\le(-1)^{n-2K-1}Dy^{\sigma}y^{(n-2K-1)}x^{-m}.$$

Integrating this inequality over the interval $[x,x_1]$, $x_0\le x\le x_1$, we obtain

$$H(x)-H(x_1)\le D(G(x)-G(x_1))+(-1)^nD\int_x^{x_1}y^{(n-2K-2)}(y^{\sigma}x^{-m})'\,dx\le D(G(x)-G(x_1)),$$

$$H(x)=(y^{(n-K-1)})^2,\qquad G(x)=(-1)^ny^{(n-2K-2)}y^{\sigma}x^{-m}.$$

Here we have taken into account the fact the inequality

$$(-1)^ny^{(n-2K-2)}(y^{\sigma}x^{-m})'\le0.$$

Letting x_1 tend to $+\infty$, we obtain the required estimate (17) for $q=n-(K+1)$.

Now suppose that (17) is valid for $k=K+1$, $q=Q$; let us prove this inequality in the case $k=K+1$, $q=Q-1$. By the inductive assumption, estimate (17) holds for $k=K$, $q=Q-1$ and $k=K+1$, $q=Q$:

$$\begin{gathered}(-1)^Ay^{(n-K)}y^{(Q-1)}\le(-1)^BDy^{\sigma}y^{(Q-K-1)}x^{-m},\\(-1)^Ay^{(n-K-1)}y^{(Q)}\le(-1)^BDy^{\sigma}y^{(Q-K-1)}x^{-m},\\A=n+Q-K-1,\qquad B=Q-K-1.\end{gathered}\tag{19}$$

Integrating the first of these inequalities over the interval $[x,x_1]$, $x_0\le x\le x_1$, and using the second inequality from (19), we obtain

$$\widetilde H(x)-\widetilde H(x_1)\le D\big(\widetilde G(x)-\widetilde G(x_1)\big)+(-1)^{B+1}D\int_x^{x_1}y^{(Q-K-2)}(y^{\sigma}x^{-m})'\,dx\le D(\widetilde G(x)-\widetilde G(x_1)),$$

$$\widetilde H(x)=(-1)^{A+1}y^{(n-K-1)}y^{(Q-1)},\qquad \widetilde G(x)=(-1)^{B+1}y^{\sigma}y^{(Q-K-2)}x^{-m}.$$

Here we have taken into account the fact that

$$(-1)^iy^{(i)}(y^{\sigma}x^{-m})'\le0,\qquad 0\le i\le n.$$

Letting x_1 tend to $+\infty$, we obtain the required estimate (17) for $k=K+1$, $q=Q-1$. The inductive assertion is proved. The proof of Lemma 1 is complete. □

In the proofs of Theorems 2 and 3, without loss of generality, we shall consider only positive solutions of Eq. (1).

Proof of Theorem 2. We introduce the following notation:

$$Y_{q,h}=(y^{(h)})^{q_h}(y^{(h-1)})^{q_{h-1}}\cdots(y')^{q_1}y^{q_0},\qquad q=(q_h,\dots,q_0),$$

$$q_j\ge0,\quad 0\le j\le h-1,\quad q_h>0,\quad |q|=\sum_{j=0}^{h}q_j,\quad \omega(q)=\sum_{j=0}^{h}jq_j,$$

where h and q_j are integers.

The proof of the theorem is based on the following estimate:

$$\begin{gathered}(-1)^{n+\omega(q)-k}y^{(n-k)}Y_{q,h}\le Dy^{\sigma+1}x^{-m}\sum(-1)^{\omega(\widetilde q)}Y_{\widetilde q,\widetilde h},\\|q|\ge1,\quad \omega(q)\ge k\ge h\ge1,\quad k\le n-1,\quad D=\text{const}>0;\end{gathered}\tag{20}$$

the sum on the right-hand side is taken over all $\widetilde q,\widetilde h$ such that $\widetilde h\le k$, $\omega(\widetilde q)=\omega(q)-k$, and $|\widetilde q|=|q|-1$.

The proof of estimate (20) will be given later, but now we assume that it is valid. Consider the following values of the parameters: $k = n-1$, $h = 1$, $q = (n-1, 0)$. Estimate (20) implies the inequality

$$(-y')^n \le Dx^{-m} y^{\sigma+n-1};$$

integrating it, we obtain estimates (8) and (9), which proves Theorem 2.

Let us now pass to the proof of estimate (20). For $k = 1$, it obviously follows from (18), where $q = 1$. Let us argue by induction. Suppose that inequality (20) holds for $1 \le k \le K$, and let us prove it for $k = K+1$. This fact will also be proved by induction on decreasing h. Note that, for $h = k$, where k, $1 \le k \le n-1$, is an arbitrary integer, estimate (20) obviously follows from (18), where $q = k$, as well as from the inequality $(-1)^{\omega(q)} Y_{q,h} > 0$, which holds for any h, $1 \le h \le n-1$. Suppose that (20) holds for $H < h \le k = K+1$, and let us prove this inequality for $h = H$.

For $k = K+1$, $h = H$ we must prove the inequality

$$\begin{gathered}(-1)^{n+\omega(q)-K-1} y^{(n-K-1)} Y_{q,H} \le D y^{\sigma+1} x^{-m} \sum (-1)^{\omega(\widetilde{q})} Y_{\widetilde{q},\widetilde{h}},\\ |q| \ge 1, \quad \omega(q) \ge K+1 > H \ge 1, \quad K+1 \le n-1,\\ D = \mathrm{const} > 0, \quad \widetilde{h} \le K+1, \quad |\widetilde{q}| = |q|-1, \quad \omega(\widetilde{q}) = \omega(q) - K - 1.\end{gathered} \tag{21}$$

By the induction assumption, the following inequality holds:

$$(-1)^{n+\omega(q)-K} y^{(n-K)} Y_{q,H} \le D y^{\sigma+1} x^{-m} \sum (-1)^{\omega(\widetilde{\widetilde{q}})} Y_{\widetilde{\widetilde{q}},\widetilde{\widetilde{h}}}, \qquad D = \mathrm{const} > 0. \tag{22}$$

Here the values of q are the same as in (21) and the sum on the right-hand side is taken over all $\widetilde{\widetilde{q}}, \widetilde{\widetilde{h}}$ such that

$$\widetilde{\widetilde{h}} \le K, \qquad \omega(\widetilde{\widetilde{q}}) = \omega(q) - K, \qquad |\widetilde{\widetilde{q}}| = |q| - 1.$$

Integrating this inequality over the interval $[x, x_1]$, $x_0 \le x \le x_1$, we obtain

$$\begin{gathered}H(x_1) - H(x) - \int_x^{x_1} F(x)\,dx \le D \int_x^{x_1} y^{\sigma+1} x^{-m} \sum (-1)^{\omega(\widetilde{\widetilde{q}})} Y_{\widetilde{\widetilde{q}},\widetilde{\widetilde{h}}}\,dx,\\ H(x) = (-1)^{n+\omega(q)-K} y^{(n-K-1)} Y_{q,H}, \quad F(x) = (-1)^{n+\omega(q)-K} y^{(n-K-1)} (Y_{q,H})',\\ D = \mathrm{const} > 0.\end{gathered} \tag{23}$$

Let us show that the induction assumption implies the following inequality:

$$\int_x^{x_1} F(x)\,dx \le D \int_x^{x_1} y^{\sigma+1} x^{-m} \sum (-1)^{\omega(\widetilde{\widetilde{q}})} Y_{\widetilde{\widetilde{q}},\widetilde{\widetilde{h}}}\,dx, \qquad D = \mathrm{const} > 0. \tag{24}$$

Here the notation is the same as in (22).

Estimate (24) will be proved by induction on decreasing $\omega(q)$. To do this, along with the monomial $Y_{q,H}$, we consider all monomials $Y_{q^*,H}$ for which $|q^*| = |q|$. The greatest value of $\omega(q^*)$ for these q^* will be $H|q|$, and it will be attained at $q^* = q^1 = (|q|, 0, \dots, 0)$. In that case,

$$(Y_{q^1,H})' = |q| y^{(H+1)} (y^{(H)})^{|q|-1}$$

and, by the induction assumption,

$$\begin{gathered}(-1)^{n+\omega(q^1)-K-1} y^{(n-K-1)} (Y_{q^1,H})' \le D y^{\sigma+1} x^{-m} \sum (-1)^{\omega(\widetilde{\widetilde{q}})} Y_{\widetilde{\widetilde{q}},\widetilde{\widetilde{h}}},\\ D = \mathrm{const} > 0, \quad \widetilde{\widetilde{h}} \le K+1, \quad |\widetilde{\widetilde{q}}| = |q| - 1, \quad \omega(\widetilde{\widetilde{q}}) = \omega(q^1) - K - 1.\end{gathered}$$

Thus, for the monomial $Y_{q^1,H}$, inequality (24) is established.

Now let (24) be valid for all monomials $Y_{q^*,H}$ for which $\omega(q^*) > L$, where L is an integer, $K+1 \le L < H|q|$. Now consider the monomial $Y_{q^*,H}$ for which $\omega(q^*) = L$. Estimate (24) for such a monomial follows from the fact that its derivative $(Y_{q^*,H})'$ is the sum of the monomials Y_{q^2,H^1} for which $\omega(q^2) = \omega(q^*) + 1$ and either $H^1 = H$ or $H^1 = H + 1$. In both cases, the assertion obviously follows from the inductive assumption. Inequality (24) is proved.

Considering the integral on the right-hand side of (24) and noting that $\omega(\widetilde{\widetilde{q}}) \ge 1$, we obtain

$$\int_x^{x_1} y^{\sigma+1} x^{-m} \sum (-1)^{\omega(\widetilde{\widetilde{q}})} Y_{\widetilde{\widetilde{q}},\widetilde{\widetilde{h}}}\,dx$$
$$\le G(x_1) - G(x) - D\int_x^{x_1} (-1)^{\omega(\widehat{q})+1} Y_{\widehat{q},\widehat{h}} (y^{\sigma+1}x^{-m})'\,dx \le G(x_1) - G(x),$$
$$G(x) = Dy^{\sigma+1}x^{-m}\sum(-1)^{\omega(\widehat{q})+1}Y_{\widehat{q},\widehat{h}}, \quad \widehat{h} \le H, \quad |\widehat{q}| = |q| - 1, \quad \omega(\widehat{q}) = \omega(q) - K - 1.$$

Here we have taken into account the inequalities $(y^{\sigma+1}x^{-m})' < 0$ and $(-1)^{\omega(\widehat{q})}Y_{\widehat{q},\widehat{h}} > 0$.

Thus, in view of (24), after the transformations indicated above, inequality (23) becomes

$$\begin{gathered} H(x_1) - H(x) \le G(x_1) - G(x), \qquad H(x) = (-1)^{n+\omega(q)-K} y^{(n-K-1)} Y_{q,H}, \\ G(x) = Dy^{\sigma+1}x^{-m}\sum(-1)^{\omega(\widehat{q})+1}Y_{\widehat{q},\widehat{h}}, \\ \widehat{h} \le H, \quad |\widehat{q}| = |q| - 1, \quad \omega(\widehat{q}) = \omega(q) - K - 1, \quad D = \text{const} > 0. \end{gathered} \tag{25}$$

In (25), letting $x_1 \to +\infty$, we obtain the required inequality (21). Theorem 2 is proved. □

Remark. In the proof of Theorems 2, we did not use estimate (2) explicitly, but used only the properties (3) of the solution $y(x)$ of Eq. (1) and estimate (7) for the function $p(x)$.

Proof of Theorem 3. For $0 \le m \le n$, Theorem 3 follows from Theorems 1 and 2, while, for $m < 0$, the right-hand side of estimates (11) and (12) follows from Theorem 1. It remains to prove the left-hand side of (11) for $m < 0$.

Without loss of generality, we assume that $y(x) > 0$. Define the number

$$\sigma_1 = (\sigma n - m)(n - m)^{-1}.$$

Obviously, $1 < \sigma_1 < \sigma$ for $m < 0$.

The function $y(x)$ satisfies the equation

$$y^{(n)} = p_1(x)y^{\sigma_1}, \qquad p_1(x) = p(x)(y(x))^{\sigma-\sigma_1}. \tag{26}$$

It follows from (7) and the right-hand side of estimate (5) that

$$0 < (-1)^n p_1(x) \le D_3, \qquad D_3 = \text{const} > 0.$$

Applying Theorem 2 (noting the remark to its proof) to Eq. (26), we find that the solution $y(x)$ satisfies the estimate

$$y(x) \ge Dx^{-n/(\sigma_1-1)} = Dx^{(m-n)/(\sigma-1)}, \qquad D = \text{const} > 0.$$

The proof of Theorem 3 is complete. □

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