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EQUICONTINUOUS SWEEPING PROCESSES

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Abstract. We prove that the sweeping process on a "regular" class of convex sets is equicontinuous. Classes of polyhedral sets with a given finite set of normal vectors are regular, as well as classes of uniformly strictly convex sets. Regularity is invariant to certain operations on classes of convex sets such as intersection, finite union, arithmetic sum and affine transformation.

1. **Introduction.** The sweeping process is an input-output operator, where the input is a variable closed convex set $Z(t) \subseteq \mathbb{R}^n$, $t \in I = [0,T]$ (all the results are easily adaptable to $I = [0,+\infty)$), and the output is the position of a "lazy" point $x(t) \in \mathbb{R}^n$ that must remain within $Z(t)$ but tries (locally in time) to minimize the distance passed. This mathematical model finds applications in the theory of elastic-plastic deformations, queueing processes in information networks, macroeconomics, etc. For the history of sweeping process theory and some applications see, for instance, [12, 4, 14, 11, 10, 2, 5, 1, 6, 7, 8] and the review [9].

Here we study the problem of equicontinuity of the sweeping process, that is, of the uniform continuity of the output as the function of the initial point of the output and of the set-valued input in the L_∞-metric on I. We prove that, if the variable convex set (the input) takes values in a "regular" class of closed convex sets (see [13]), then the sweeping process is equicontinuous.

Note that we use an alternative definition of a solution of the sweeping process as the limit of the catching-up procedure. This solution coincides with the classical one whenever the classical solution exists.

2. Sweeping process.

2.1. **Discrete time.** We begin with the discrete-time case. Suppose that a finite sequence Z of closed convex sets $Z_i \subseteq \mathbb{R}^n$, $i = 1,\dots,m$, is given as the (set-valued) input of the process, and $x_0 \in \mathbb{R}^N$ is the initial point of the output. The output $x = (x_1,\dots,x_m)$ is then found step by step as a finite sequence of points x_i, $i = 1,\dots,m$, where $x_i = P_{Z_i}(x_{i-1})$ is the orthogonal projection of x_{i-1} on the set Z_i. Henceforth we assume that $x_0 \in Z_1$, that is, $x_0 = x_1$.

Thus, in discrete time the sweeping process $\{Z, x_0\} \to \{x\}$ is uniquely defined. For $I = [0,T]$ (continuous time), the output $x(t)$ of the process $\{Z(t), x_0\}$ will be

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defined as the limit of the outputs of discrete-time processes that approximate $Z(t)$. The existence of this limit is not guaranteed even if $Z(t)$ is Hausdorff-continuous.

2.2. **Continuous time.** In the continuous-time case the input is a time-dependent closed convex set $Z(t)$, $t \in I = [0, T]$. Again, $x_0 \in Z(0)$ is given.

In order to construct the solution (output) $x(t)$ of the process, we approximate it by discrete-time solutions. Each finite partition $F = \{0 = t_1 < \cdots < t_m = T\}$ of I generates a discrete time approximation of the process $Z(\cdot)$ as follows.

We set $Z_i^F = Z(t_i)$ for $i = 1, \ldots, m$, and find the output $x^F(t_i) = x_i^F$, $i = 1, \ldots, m$, of the corresponding discrete-time sweeping process. Then we interpolate linearly the function $x^F(t_i)$ on the intervals $[t_i, t_{i+1}]$, $i = 1, \ldots, m-1$. Hence, for any partition F, the continuous (actually, piecewise linear) output $x^F(t)$, $t \in I$, is defined.

Denote by $\mathcal{F}$ the set of all finite partitions of I such that $0, T \in F$. Recall that the order relation by inclusion (that is, $F' \succeq F$ whenever $F \subseteq F'$) generates the *directed set* structure (see [3]) on the set of finite partitions, that is, any pair F, F' of partitions has an upper bound (for instance, $F \cup F'$).

We pass to the limit $x(t)$ as the partition F refines indefinitely. This is a well-known *catching-up procedure* that approximates "classical solutions" of the sweeping process in many cases, see [12, 9]. It is important here that for "regular" sweeping processes, the classical solution, if it exists, coincides with the limit of the catching-up procedure. This will be clarified in Section 6.

The convergence is understood in the sense of Moore-Smith, that is, $x^F(\cdot)$ converges to $x(\cdot)$ if, for any $\varepsilon > 0$, there exists a partition $F \in \mathcal{F}$ such that

$$\sup_{t \in I} \|x^{F'}(t) - x(t)\| < \varepsilon \quad \text{whenever} \quad F' \succeq F.$$

We will prove that, in the "regular" case, a unique limit $x(t)$ exists for a Hausdorff-continuous input $Z(t)$ and $x(\cdot)$ depends equicontinuously on the input $Z(\cdot)$, in the L_∞-metric.

Note that the assumption of regularity is essential since there exist Hausdorff-continuous inputs $Z(t)$ for which the limit $x(t)$ does not exist, see the last example of Section 2.4.

2.3. **Continuity.** By $d(A, B)$ we denote the Hausdorff distance between the sets $A, B \subseteq \mathbb{R}^n$:

$$d(A, B) = \max\{e(A, B), e(B, A)\}, \quad \text{where} \quad e(A, B) = \sup_{x \in A} \inf_{y \in B} \|x - y\|.$$

The L_∞-distance between the inputs $Z(\cdot)$ and $Z'(\cdot)$ on I will be defined as

$$d_I(Z(\cdot), Z'(\cdot)) = \sup_{t \in I} d(Z(t), Z'(t)),$$

and between the outputs $x(\cdot)$ and $x'(\cdot)$ as

$$d_I(x(\cdot), x'(\cdot)) = \sup_{t \in I} \|x(t) - x'(t)\|.$$

As follows from the results of [10], the sweeping process in a neighborhood of a compact-valued Lipschitz-continuous input $Z(t)$, $t \in I$, such that $\operatorname{int}(Z(t)) \neq \emptyset$, $t \in I$, is a continuous operator in the above sense.

It is also known that, for special classes of $Z(t)$, this result can be strengthened. For instance, if $Z(t) = Z + u(t)$ for some closed convex $Z \subseteq \mathbb{R}^n$ and continuous

$u(t) : I \to \mathbb{R}^n$, and if Z is a polyhedral set, then the map $u(\cdot) \to x(\cdot)$ (provided x_0 is given) is a Lipschitz continuous map from $C(I, \mathbb{R}^n)$ to itself, see [14, 5, 1, 7].

This assertion can be also extended to polyhedral inputs of the form

$$Z(t) = \{z \in \mathbb{R}^n : \langle d_i, z \rangle \leq u_i(t), \quad i = 1, \dots, m\},$$

where $u_i(t)$ are continuous scalar functions.

By equicontinuity we will understand the following property. For each $\varepsilon > 0$ there is a $\delta > 0$ such that the inequalities

$$\|x_0 - x_0'\| < \delta \quad \text{and} \quad d_I(Z(\cdot), Z'(\cdot)) < \delta \tag{1}$$

imply

$$d_I(x(\cdot), x'(\cdot)) < \varepsilon. \tag{2}$$

In what follows we will prove the equicontinuity property of "regular" classes $\mathcal{Z}$ of closed convex sets. Namely, we will prove that, for each $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon, \mathcal{Z}) > 0$ such that (1) implies (2) for all Hausdorff-continuous $Z(\cdot), Z'(\cdot)$ that satisfy $Z(t), Z'(t) \in \mathcal{Z}$ for all $t \in I$.

2.4. **Examples.** In general, the sweeping process is not continuous as a map from $\{x_0, Z(\cdot)\}$ to $x(\cdot)$, even in a neighborhood of a constant input $Z(t) \equiv Z$. Say, this is the case for $Z = [0, 1] \times \{0\}$ in $\mathbb{R}^2$. Indeed, by small oscillations of the variable set $Z'(t)$ in an arbitrarily small neighborhood of Z we may force the output $x'(t)$ to move, say, from $x_0 = (0, 0)$ to $(1, 0)$, while for the process $\{x_0, Z(t)\}$ the output $x(t)$ is constant, that is, $x(t) \equiv x_0$.

For instance, let $Z'(t)$ be the triangle $[(0, -h(t)), (1, -h(t)), (1, \delta/2 - h(t)]$. If $h(t)$ makes a large number of oscillations between 0 and δ as t changes from 0 to T, and if $h(T) = 0$, then $x(T) = (1, 0)$.

Let us also give an example of Hausdorff-continuous input $Z(t)$ for which there is no solution of sweeping process. We modify the previous example as follows. The interval $[0, T]$ is partitioned into an infinite number of intervals $[0, T/2]$, $[T/2, 3T/4]$, etc. Then, at the first interval we take $\varepsilon_1 = 1$ and construct the input $Z_1(t)$ that is ε_1-proximal to Z, and that forces $x(T/2)$ to take value $(1, 0)$ for each initial value $x(0)$.

Then we take $\varepsilon_2 = \varepsilon_1/2$ and construct the input $Z_2(t)$ on $[T/2, 3T/4]$ that is ε_2-proximal to Z, and such that $Z_2(T/2) = Z_1(T/2)$ and $x(3T/4) = (0, 0)$, and so on. We also set $Z(T) = Z = [(0, 0), (1, 0)]$. As a result, we construct a continuous input $Z(t)$ on $[0, T]$ for which the value of $x(T)$ is undefined.

3. **Regular classes.**

3.1. **Definition.** Let $\mathcal{Z}$ be some class of closed convex sets $Z \subseteq \mathbb{R}^n$.

Definition 3.1. The class $\mathcal{Z}$ is *regular* if, for any $h \in X$, there exists an $\varepsilon > 0$ such that $Z \in \mathcal{Z}$, $x \in Z$ and $d(x + h, Z) \leq \varepsilon$ imply $h \in K_Z(x)$, where $K_Z(x)$ is the tangent cone to Z at x. Recall that

$$K_Z(x) = \overline{\cup_{y \in Z, \alpha > 0} \alpha(y - x)}.$$

Equivalently,

Definition 3.2. The class $\mathcal{Z}$ is regular if, for each vector $h \in \mathbb{R}^n$, there is an $\varepsilon > 0$ such that, whenever $Z \in \mathcal{Z}$, $x \in Z$, it follows from $[x, x + h] \cap Z = \{x\}$ that $d(x + h, Z) \geq \varepsilon$ and from $[x, x - h] \cap Z = \{x\}$ that $d(x - h, Z) \geq \varepsilon$.

The equivalence of these definitions follows from a simple assertion:

Lemma 3.3. *If $x \in Z$ and $x + \alpha h \notin Z$ then $h \notin K_Z(y)$, where $y = P_Z(x + \alpha h)$.*

Proof. Denote $g = x + \alpha h - y$. By definition of the projection, $\langle g, y - x\rangle \geq 0$, that is, $\langle \alpha h, g\rangle \geq \langle g, g\rangle > 0$. However, $\langle g, f\rangle \leq 0$ for each $f \in K_Z(y)$. □

We will also use the following equivalent definition of regularity.

Definition 3.4. A class $\mathcal{Z}$ of convex closed sets in $\mathbb{R}^n$ is regular if, for any $h \in \mathbb{R}^n$, there exists an $\varepsilon = \varepsilon(h) > 0$ such that $Z \in \mathcal{Z}$, $x \in Z$ and $d(x + h, Z) \leq \varepsilon$ imply $x + h/2 \in Z$.

3.2. **Examples.** The class of all singletons $Z = \{x\}$ is regular. A half-space $H = \{x \in \mathbb{R}^n : \langle x, h\rangle \leq 0\}$ is regular. A class of uniformly strictly convex sets is regular. Namely, if, for any $\alpha > 0$ there exists a $\beta = \beta(\alpha) > 0$ such that $x, y \in Z \subseteq \mathcal{Z}$ and $\|x - y\| \geq \alpha$ imply $\frac{x+y}{2} + B_\beta \subseteq Z$, then $\mathcal{Z}$ is regular. Here B_β is the ball $\{x \in \mathbb{R}^n : \|x\| \leq \beta\}$.

Indeed, if $x \in Z$ and $d(x + h, Z) < \varepsilon$, then $x, x + h' \in Z$ for some h' such that $\|h - h'\| < \varepsilon$. Hence $\|h'\| > \|h\| - \varepsilon$ and $x + h'/2 + B_{\beta(\|h'\|)} \subseteq Z$. If $\beta(\|h'\|) \geq \varepsilon/2$, then $x + h/2 \in Z$ since $\|h'/2 - h/2\| \leq \varepsilon/2$.

Respectively, it suffices to require that the inequalities

$$\|h'\| \geq \frac{\|h\|}{2} \quad \text{and} \quad \beta\left(\frac{\|h\|}{2}\right) \geq \frac{\varepsilon}{2}$$

hold. Therefore, it suffices to assign $\varepsilon(h) = 2\beta(\frac{1}{2}\|h\|)$.

3.3. **Main properties of regular classes.** The regularity property is invariant with respect to some operations on convex sets.

Theorem 3.5. *If $\mathcal{A}$ and $\mathcal{B}$ are regular classes, then*

- *$\mathcal{A} \cup \mathcal{B}$ is regular (the union),*
- *$\mathcal{C} = \{A \cap B : A \in \mathcal{A}, B \in \mathcal{B}\}$ is regular (the intersection),*
- *$M\mathcal{A}$ is regular for any affine mapping M (affine transformation),*
- *$\mathcal{D} = \{A + B : A \in \mathcal{A},\ B \in \mathcal{B}\}$ is regular (the arithmetic sum).*

Proof. The union is regular because we can take

$$\varepsilon_{\mathcal{A}\cup\mathcal{B}}(h) = \min\{\varepsilon_{\mathcal{A}}(h), \varepsilon_{\mathcal{A}}(h)\}.$$

The regularity of intersections follows immediately from Definition 3.4 since $d(x + h, \cap_{\gamma\in\Gamma} A_\gamma) < \varepsilon$ implies $d(x + h, A_\gamma) < \varepsilon$ for all $\gamma \in \Gamma$.

Let us now prove the regularity of $M\mathcal{A}$, where M is a linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$. Suppose the contrary. Let, for some $h \in \mathbb{R}^m$, there exist a sequence of $x_i \in MA_i \in \mathcal{A}$ such that $x_i + h/2 \notin MA_i$ and $d(x_i + h, MA_i) \to 0$ as $i \to \infty$.

Let $y_i \in MA_i$ be a sequence of points such that $\|x_i + h - y_i\| \to 0$ as $i \to \infty$. Since $x_i, y_i \in MA_i$, there exist $p_i, q_i \in A_i$ such that $x_i = Mp_i$, $y_i = Mq_i$. We assume that, for a given p_i, the corresponding q_i is chosen such that

$$\|q_i - p_i\| = \min_{q\in Z_i : Mq = y_i} \|q - p_i\|$$

(the minimum is attained because A_i is closed). Now we consider two cases.

Suppose first that

$$\liminf_{i\to\infty} \|q_i - p_i\| < +\infty.$$

Then there exists a subsequence of i (let it be the whole sequence without loss of generality) such that

$$\lim_{i\to\infty} (q_i - p_i) = g \in \mathbb{R}^n. \tag{3}$$

Then, clearly, $h = Mg$. Hence, $p_i + g/2 \notin A_i$ (otherwise $x_i + h/2 \in MA_i$). Thus, by the regularity of $\mathcal{A}$, the inequality

$$d(q_i + g, A_i) \geq \varepsilon(g)$$

holds for each $i = 1, 2, \dots$, which is a contradiction with (3).

Now let

$$\lim_{i\to\infty} \|q_i - p_i\| = +\infty. \tag{4}$$

Passing, if necessary, to a subsequence, we denote

$$f = \lim_{i\to\infty} \frac{q_i - p_i}{\|q_i - p_i\|}. \tag{5}$$

It follows that $Mf = 0$. By the choice of q_i we conclude that $q_i - \alpha f \notin A_i$ for all $\alpha > 0$ (otherwise there exists a point $q_i - \alpha f \in A_i$ which is closer to p_i than q_i, and, since $M(q_i - \alpha f) = y_i$, we have a contradiction).

Then $q_i - f/2 \notin A_i$ and, by the regularity assumption,

$$d(q_i - f, A_i) \geq \varepsilon(f).$$

By convexity of A_i we conclude that

$$d(q_i - \|q_i - p_i\| f, A_i) \geq \varepsilon(f)\|q_i - p_i\|,$$

but this is a contradiction with (5) and (4). It remains to verify that MA is closed whenever $A \in \mathcal{A}$. The proof is nearly the same as the proof of the second case above.

Finally, let us prove that $\mathcal{D} = \mathcal{A}+\mathcal{B}$ is regular. First, let us note that the product $\mathcal{A}\times\mathcal{B}$ is regular (this product consists of the sets $\{(x, y) \in \mathbb{R}^{2n} : x \in A, y \in B\}$ for all pairs $A \in \mathcal{A}$, $B \in \mathcal{B}$).

Let $h = (h_1, h_2)$. Note that the inequality $d(h, A\times B) \leq \varepsilon$ implies $d(h_1, A) \leq \varepsilon$ and $d(h_2, B) \leq \varepsilon$. Now the regularity of $\mathcal{A}\times\mathcal{B}$ follows straightaway from the definition of regularity.

It remains to note that $A + B$ is the image of $A\times B$ under the linear map $M : (x, y) \to x + y$ that acts from $\mathbb{R}^{2n}$ to $\mathbb{R}^n$. The assertion now follows from the regularity of $M\mathcal{D}$. □

3.4. **Some regular and irregular classes.** As a consequence of Theorem 3.5, for a given finite set $\{h_1, \dots, h_K\}$ of unit vectors in $\mathbb{R}^n$, the class of polyhedral sets of the form

$$Z = \{z \in \mathbb{R}^n : \langle z, h_i\rangle \leq a_i, \quad i = 1, \dots, K\}$$

is regular.

Also, if $\mathcal{Z}$ is regular, then the class of neighborhoods $Z + B_a$, $0 \leq a \leq A < \infty$, $Z \in \mathcal{Z}$, is regular since the class of balls $\{B_a : 0 \leq a \leq A < \infty\}$ is regular as a uniformly strictly convex class.

More examples of regular classes can be constructed by means of operations on sets from Theorem 3.5. It can be easily seen that in $\mathbb{R}^2$ any compact convex set is regular.

On the other hand, if $n \geq 3$, there exist non-regular compact convex sets, for instance, the truncated cone $Z = \{(x, y, z) : z \geq 0, \ \sqrt{x^2 + y^2} + z \leq 1\}$ in $\mathbb{R}^3$. Indeed, set $h = (1, 0, 1)$ and $p = (\varepsilon - 1, \sqrt{2\varepsilon - \varepsilon^2}, 0)$. Then $p + h/2 \notin Z$ but $d(p + h, Z) \to 0$ as $\varepsilon \to 0$.

Modifying this example we may construct an irregular set with smooth boundary. Its dual set is, hence, strictly convex and is an example of a regular set whose dual is irregular.

3.5. **Polyhedral norm.** Let $\mathcal{Z}$ be a regular class with the symmetric function $\varepsilon(h)$ as in Definition 3.2. By S^{n-1} we denote the unit sphere in $\mathbb{R}^n$. The open balls $B^0(h,\varepsilon(h))$, $h \in S^{n-1}$, cover the sphere S^{n-1} (compact set). Hence, we can find a finite symmetric subset $Q = \{v_1, \dots, v_K\} \subseteq S^{n-1}$ such that the balls $B^0(v,\varepsilon(v))$, $v \in Q$, still cover S^{n-1}.

Denote $R(v) = \{g \in S^{n-1} : \|v - g\| = \varepsilon(v)\}$. There is a unique hyperplane $H(v)$ that contains $R(v)$. This hyperplane is orthogonal to v. Actually,

$$H(v) = \{z \in \mathbb{R}^n : \langle z, v\rangle = c(v)\}, \quad \text{where} \quad c(v) = 1 - \frac{\varepsilon(v)^2}{2}.$$

Let us consider the polyhedron G bounded by the hyperplanes $H(v)$, $v \in Q$:

$$G = \{z \in \mathbb{R}^n : |\langle z, v\rangle| \le c(v), \quad v \in Q\}.$$

Now we define the polyhedral norm

$$\|x\|_G = \inf\{a > 0 : \frac{1}{a}x \in G\}.$$

This norm will be used for the proof of equicontinuity of regular sweeping processes.

3.6. **Some properties of the polyhedral norm.** By construction,

$$\|x\|_G > \|x\| \quad \text{if} \quad x \neq 0. \tag{6}$$

For each $h \in S^{n-1}$, let us define the vector $v(h) \in Q$ such that

$$\langle h, v(h)\rangle = \max_{v \in Q} \frac{\langle h, v\rangle}{c(v)}.$$

Such a choice is unique if h intersects the boundary of G at the relative interior of a face of maximal dimension $n-1$. Otherwise we chose a single vector $v \in Q$ among several alternatives (does not matter which one, but we will require that $v(h) = -v(-h)$, $h \in S^{n-1}$).

By construction, $\|h - v(h)\| \le \varepsilon(v(h))$. From (6) we conclude that there exists an $\alpha = \alpha(G,Q) > 0$ such that

$$\|h - v(h)\| < \varepsilon(v(h)) - \alpha \quad \text{for each} \quad h \in S^{n-1}. \tag{7}$$

Additionally,

Lemma 3.6. *If* $\langle z, v(h)\rangle \ge 0$ *then* $\|h + z\|_G \ge \|h\|_G$.

This assertion follows from the fact that $h/\|h\|_G$ belongs to the $(n-1)$-dimensional face of G, orthogonal to $v(h)$.

Let us also extend the map $v(\cdot)$ to $\mathbb{R}^n \backslash \{0\}$ by homogeneity. For any $h \neq 0$ we set $v(h) = \|h\| v(\frac{h}{\|h\|})$.

4. **Equicontinuity in discrete time.**

4.1. **Projections.**

The discrete-time sweeping process is defined as a sequence of projections onto closed convex sets. We assume all these sets to be taken from a regular class $\mathcal{Z}$ ($\mathcal{Z}$-processes). We will compare the solutions x_i and x'_i of $\mathcal{Z}$-processes Z and Z', respectively.

Denote $h_i = x'_i - x_i$. Let us study the dynamics of h_i, $i = 1, \ldots$. To begin with, consider two sets $Z, Z' \in \mathcal{Z}$ and two points $x, x' \in \mathbb{R}^n$. Denote $y = P_Z(x)$, $y' = P_{Z'}(x')$, $g = x' - x$, $h = y' - y$. Suppose that $\|h\| \geq 1$. We will compare $\|g\|_G$ and $\|h\|_G$.

Lemma 4.1. *If $d(Z, Z') \leq \alpha\|h\|$ then $\|h\|_G \leq \|g\|_G$,*

where α is defined by (7).

Proof. Since $\|h\| \geq 1$, (7) implies

$$\|h - v(h)\| < \|h\|(\varepsilon(v(h)) - \alpha).$$

Then, since $y' = y + h \in Z'$ and, hence, $d(y + h, Z) \leq \alpha\|h\|$, we get the inequalities

$$d(y + v(h), Z) < \alpha\|h\| + \|h\|(\varepsilon(v(h)) - \alpha) = \|h\|\varepsilon(v(h)). \tag{8}$$

and, respectively

$$d(y' - v(h), Z') < \|h\|\varepsilon(v(h)). \tag{9}$$

It follows from (8), (9) that $[y, y + v(h)] \cap Z \neq \{y\}$ and $[y', y' - v(h)] \cap Z' \neq \{y'\}$. For instance, $y + \alpha v(h) \in Z$ for some $\alpha > 0$. Since $y = P_Z(x)$, we have $\langle x - y, (y + \alpha v(h)) - y\rangle \leq 0$, that is, $\langle x - y, v(h)\rangle \leq 0$. Similarly, $\langle x' - y', v(h)\rangle \geq 0$. Hence,

$$\langle h, v(h)\rangle \leq \langle g, v(h)\rangle.$$

From Lemma 3.6 we conclude then that $\|h\|_G \leq \|g\|_G$. □

4.2. **Main theorem.**

Theorem 4.2. *The regular discrete-time sweeping process is equicontinuous.*

Proof. First, from the inequality

$$\|P_Z(x) - P_Z(y)\| \leq \|x - y\|$$

we conclude that the outputs x_i, y_i of the processes $\{Z_i, x_0\}$ and $\{Z_i, x'_0\}$ (the same input, different initial points) are non-diverging, that is, $\|x_i - y_i\| \leq \|x_{i-1} - y_{i-1}\|$ for all $i = 1, \ldots, K$. Hence, it suffices to consider the case $x_0 = x'_0$.

Let us consider the difference $h_i = x'_i - x_i$ between the outputs of two processes Z and Z'. It follows from Lemma 4.1 that, if $d(Z_i, Z'_i) < \alpha$ then $\|h_i\|_G \leq \|h_{i-1}\|_G$ whenever $\|h_i\| \geq 1$.

By obvious scaling we prove also that, for any $\varepsilon > 0$, there exists a $\delta > 0$ such that, if $d(Z_i, Z'_i) < \delta$ then $\|h_i\|_G \leq \|h_{i-1}\|_G$ whenever $\|h_i\| \geq \varepsilon$. Since the norms $\|\cdot\|$ and $\|\cdot\|_G$ are equivalent (as is always the case in a finite-dimensional space), we conclude that $\|h_i\|_G$ never exceeds $\gamma\varepsilon$ for some $\gamma = \gamma(G) > 0$. □

5. **Equicontinuity in continuous time.**

5.1. **Uniqueness.** The output of the continuous-time sweeping process $\{Z(t), x_0\}$ on the time interval I is defined as a limit of discrete-time ones as the finite partition F of I refines indefinitely. This limit exists and is unique if $Z(t)$ is a Hausdorff-continuous regular process.

Indeed, the maximum of Hausdorff distance between $Z^{F'}(t)$ and $Z^{F''}(t)$ taken over all $F', F'' \succeq F$ and all $t \in I$, vanishes as F refines indefinitely. Then, by Theorem 4.2, the maximal distance between the outputs $x^{F'}(t)$ and $x^{F''}(t)$ also vanishes. Therefore the directed set $x^F(\cdot)$ has the Cauchy property and, hence, converges uniformly as F refines to infinity. The limit $x(t)$ is called the output of the continuous-time sweeping process $Z(t)$.

5.2. **Equicontinuity.**

Theorem 5.1. *The regular continuous-time sweeping process is equicontinuous, that is, for any $\varepsilon > 0$ there is a $\delta > 0$ such that $\|x'(t) - x(t)\| < \varepsilon$, $t \in I$, whenever $x_0 = x_0'$ and $d_H(Z(t), Z'(t)) < \varepsilon$, $t \in I$.*

The result is an immediate consequence of Theorem 4.2. Actually, the assertion is true without the assumption of Hausdorff-continuity of $Z(t)$. Obviously, it suffices to require unique solvability of the sweeping process for the class of inputs in consideration.

As a consequence, we prove again that a polyhedral sweeping process is Lipschitz continuous. This is due to the scaling properties of the finite class of half-spaces. Indeed, the polyhedral class $\mathcal{Z}$ is invariant to multiplication by a positive constant α. Moreover, the output of the process $\{\alpha Z(t), \alpha x_0\}$ is equal to $\alpha x(t)$.

Hence, if
$$d(Z(\cdot), Z'(\cdot)) < \delta, \quad \|x_0 - x_0'\| < \delta$$
implies
$$d(x(\cdot) - x'(\cdot)) < \varepsilon,$$
then
$$d(Z(\cdot), Z'(\cdot)) < \alpha\delta, \quad \|x_0 - x_0'\| < \alpha\delta$$
implies
$$d(x(\cdot) - x'(\cdot)) < \alpha\varepsilon.$$

6. **Classical solutions.**

Let us discuss briefly the relation between our definition of the output $x(t)$ and the classical solution of the sweeping process defined by J.-J. Moreau. Namely, we demonstrate that $x(t)$ coincides with the classical solution whenever the latter exists.

Recall that $x(t)$ is the classical solution of the sweeping process $Z(t)$ if $x(t) \in Z(t)$ is absolutely continuous on I and
$$\dot{x}(t) \in -K_{Z(t)}(x(t)) \quad \text{for almost all} \quad t \in I.$$

Let $Z(t)$ be a Hausdorff-continuous regular input with a classical solution $x(t)$. Let us construct a piecewise constant input $Z'(t)$ with the catching-up output $x'(t)$ such that $d(Z(t), Z'(t)) < \varepsilon$ and $\|x(t) - x'(t)\| < \varepsilon$ for all $t \in I$. As a first step, we construct $x'(t)$ as a broken line that is close to $x(t)$ uniformly on I and such that the vector $x'_{i+1} - x'_i = x'(t_{i+1}) - x'(t_i)$ for each linear segment of $x'(\cdot)$ is parallel to $\dot{x}(s_i)$ for some $s_i \in [t_i, t_{i+1}]$.

Then we set $Z'(t) = Z(s_i)$ for all $t \in (t_i, t_{i+1}]$ which guarantees that $x'(t)$ is the output of the input $Z'(t)$. If $x'(\cdot)$ is close enough to $x(\cdot)$ and $\max_i \|x'_{i+1} - x'_i\|$ is small enough, we get the required approximation.

Finally, we let $\varepsilon \to 0$ and use the fact that the corresponding outputs $x'_\varepsilon(t)$ converge to the output of $Z(t)$ uniformly on I. Since they also converge to the classical solution $x(t)$, we conclude that the classical solution coincides with the catching-up one.

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