

(March 10, 2015)

Series representations for bivariate time-changed Lévy models*

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Abstract: In this paper, we analyze a Lévy model based on two popular concepts - subordination and Lévy copulas. More precisely, we consider a two-dimensional Lévy process such that each component is a time-changed (subordinated) Brownian motion and the dependence between subordinators is described via some Lévy copula. We prove a series representation for our model, which can be efficiently used for simulation purposes, and provide some practical examples based on real data.

MSC 2010 subject classifications: Primary 60G51; secondary 62F99.

Keywords and phrases: Lévy copula, time-changed Lévy process, subordination.

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1. Introduction

Copula is with no doubt the most popular tool for describing the dependence between two random variables. The popularity is partially based on the fact

*This study (research grant No 14-05-0007) was supported by the National Research University-Higher School of Economics' Academic Fund Program in 2014.

that the dependence between any random variables can be modelled by some copula. This fact is known as Sklar's theorem, which states that for any two random variables Y_1 and Y_2 there exists a *copula* C (a two-dimensional real-valued distribution function with domain $[0, 1]^2$ and uniform margins) such that

$$\mathbb{P}\{Y_1 \leq u_1, Y_2 \leq u_2\} = C\left(\mathbb{P}\{Y_1 \leq u_1\}, \mathbb{P}\{Y_2 \leq u_2\}\right), \quad (1)$$

for any $u_1, u_2 \geq 0$. We refer to Cherubini et al. (2004), Joe (1997), Nelsen (2006) for a comprehensive overview of the copula theory.

Now let us switch from random variables to stochastic processes and try to describe dependence between components of some two-dimensional *Lévy process* $\vec{X}(t) = (X_1(t), X_2(t))$, that is, of some cadlag process with independent and stationary increments. Applying Sklar's theorem for any fixed time moment t , we get that the dependence between X_1 and X_2 can be described by some copula C_t , i.e.,

$$\mathbb{P}\{X_1(t) \leq u_1, X_2(t) \leq u_2\} = C_t\left(\mathbb{P}\{X_1(t) \leq u_1\}, \mathbb{P}\{X_2(t) \leq u_2\}\right), \quad (2)$$

for any $u_1, u_2 \geq 0$. Nevertheless, the direct application of the representation (2) to stochastic modeling has a couple of drawbacks. First, it turns out that the copula C_t in most cases essentially depends on t , see Tankov (2004) for examples. Second, since the distribution of $\vec{X}(t)$ is infinitely divisible, (2) is possible only for some subclasses of copulas. In other words, the class C_t depends on the class of marginal laws $X_i(t)$.

To avoid such difficulties, researchers are trying to characterize the dependence between the components of Lévy process in *the time-independent fashion*. One of the most popular approaches for this characterization is the so-called Lévy copula (defined below), which was introduced by Tankov (2003), and later studied by Barndorff-Nielsen and Lindner (2004), Cont and Tankov (2004), Kallsen and Tankov (2006), and others. Among many papers in this field, we would like to emphasize some articles about statistical inference for Lévy copulas (mainly with applications to insurance), which include some basic ideas that are widely used in statistical research on this topic, in particular, in the statistical analysis in the current paper - Esmaili and Klüppelberg (2010 and 2011), Avanzi, Cassar and Wong (2011), Bücher and Vetter (2013).

The main objective of this article is the application of the Lévy copula approach to a class of stochastic processes, known as *time-changed Lévy processes*. In the one-dimensional case, the time-changed Lévy process is defined as $Y_s = L_{\mathcal{T}(s)}$, where L is a Lévy process and \mathcal{T} is a non-negative, non-decreasing stochastic process with $\mathcal{T}(0) = 0$ referred as *stochastic time change* or simply *stochastic clock*. If the process \mathcal{T} is also a Lévy process, then it is called the *subordinator*, and the process Y_s is usually referred as the subordinated process. The economical interpretation of the time change is based on the idea that the "business" time $\mathcal{T}(s)$ may run faster than the physical time in some periods, for instance, when the amount of transactions is high, see Clark (1973), Ané and Geman (2000), Veraart and Winkel (2010).

In this paper, we consider one natural generalization of the aforementioned model to the two-dimensional case. Our construction is based on the so-called *multivariate subordination*, introduced by Barndorff-Nielsen, Pederson and Sato (2001). In particular, we prove the series representation of the processes from our class, which allows to simulate the processes with given characteristics, and show an application of this representation to real data.

The paper is organized as follows. In the next two sections, we shortly explain the notion of Lévy copula and the idea of stochastic change of time. Afterwards, in Section 4, we introduce our model and discuss some properties of it. Our main results are given in Section 5, where we also provide some examples. In the last section, we apply our model to some stock prices.

2. Lévy copulas

The construction of the Lévy copula is based on the concept of tail integrals.

Definition 2.1. For a one-dimensional Lévy process Z with Lévy measure ν_Z , tail integral is defined as

$$U_Z(x) := \begin{cases} \nu_Z(x, +\infty), & \text{if } x > 0, \\ -\nu_Z(-\infty, x), & \text{if } x < 0. \end{cases}$$

Definition 2.1 can be equivalently written as

$$U_Z(x) := (-1)^{s(x)} \nu_Z(I(x)),$$

where

$$I(x) := \begin{cases} (x, +\infty), & \text{if } x > 0, \\ (-\infty, x), & \text{if } x < 0, \end{cases} \quad \text{and} \quad s(x) := \begin{cases} 2, & \text{if } x > 0, \\ 1, & \text{if } x < 0. \end{cases}$$

The reason for this definition is that in the case of infinite measure ν_Z , $U_Z(A)$ is infinite for any set A which contains 0. Analogously, for a two-dimensional process $\vec{Z} = (Z_1, Z_2)$ with Lévy measure $\nu_{\vec{Z}}$,

$$U_{\vec{Z}}(x_1, x_2) := (-1)^{s(x_1)+s(x_2)} \nu_{\vec{Z}}(I(x_1) \times I(x_2)),$$

and this definition is also correct for any real x_1 and x_2 .

Definition 2.2. A two-dimensional Lévy copula is a function from $\bar{\mathbb{R}}^2$ to $\bar{\mathbb{R}}$ such that

1. F is grounded, that is, $F(u_1, u_2) = 0$ if $u_i = 0$ for at least one $i = 1, 2$.
2. F is 2-increasing.
3. F has uniform margins, that is, $F(u, \infty) = F(\infty, u) = u$.
4. $F(u_1, u_2) \neq \infty$ for $(u_1, u_2) \neq (\infty, \infty)$.

The main result on Lévy copulas is an analogue of the Sklar theorem for ordinary copulas which states that for any two-dimensional Lévy process \vec{Z} with tail integral $U_{\vec{Z}}$ and marginal tail integrals U_{Z_1} and U_{Z_2} , there exists a Lévy copula F such that

$$U_{\vec{Z}}(x_1, x_2) = F(U_{Z_1}(x_1), U_{Z_2}(x_2)), \tag{3}$$

and vice versa, for any Lévy copula F and any one-dimensional Lévy process with tail integrals U_{Z_1} and U_{Z_2} there exists a two-dimensional Lévy process with tail integral $U_{\vec{Z}}$ given by (3) with marginal tail integrals U_{Z_1} and U_{Z_2} . The first part of this theorem can be easily verified for the case when the one-dimensional Lévy measures are infinite and have no atoms, because in this case the Lévy copula is equal to

$$F(u_1, u_2) = U\left(U_{Z_1}^{-1}(u_1), U_{Z_2}^{-1}(u_2)\right), \tag{4}$$

where U is the tail integral of the Lévy measure of $\vec{X}(t)$, see [20].

3. Time-changed Lévy models

As it was already mentioned in the introduction, the time-changed Lévy process in the one-dimensional case is defined as

$$Y_s = L_{\mathcal{T}(s)}, \tag{5}$$

where L is a Lévy process, and $\mathcal{T}(s)$ - a non-negative, non-decreasing stochastic process with $\mathcal{T}(0) = 0$. This class of models has strong mathematical background based on the so-called Monroe theorem [22], which stands that any semimartingale can be represented as a time-changed Brownian motion (that is, in the form (5) with L equal to the Brownian motion W) and vice versa, any time-changed Brownian motion is a semimartingale. Various aspects of this theory are discussed in [5] and [11]. Nevertheless, the first part of the Monroe theorem doesn't hold if one introduces any of the following additional assumptions:

1. Processes W and \mathcal{T} are independent. This assumption is widely used in the statistical literature and is quite convenient for both theoretical and practical purposes.
2. Time change process \mathcal{T} is itself a Lévy process; such processes are known as *subordinators*. In this case any resulting process Y_s is also a Lévy process, which is usually called a *subordinated process*.

These drawbacks of the time-changed Brownian motion lead to the idea of considering more general model (5) with any Lévy process instead of the Brownian motion and introducing the assumption that the processes \mathcal{T} and L are independent. This model has been attracting attention of many researches, see, e.g., [7], [8] [10], [12], [26].

Nevertheless, there is no clear understanding in the literature how to extend this model to the two-dimensional case. The most popular construction is to

consider the model (5) with a two-dimensional Lévy process \vec{L} and to provide a time change in each component with the same process \mathcal{T} , see [25].

Interestingly enough, in the case when \vec{L} is a Brownian motion, the correlation coefficient between subordinated processes is upper bounded by the correlation coefficient between the components of the Brownian motion, see [16]. Moreover, these coefficients coincides in some cases, see [15].

4. Two-dimensional subordinated processes

In this section, we introduce a two-dimensional generalization of the model (5). This generalization is based on the notion of the two-dimentional subordinator, which we define below.

Definition 4.1. A two-dimensional subordinator $\vec{T}(s) = (T_1(s), T_2(s))$ is a Lévy process in \mathbb{R}^2 such that both components T_1 and T_2 are one-dimensional subordinators.

Evidently, \vec{T} is a two-dimensional subordinator if its margins T_1 and T_2 are independent. Another example is given by the following statement (see [27]): if T_1, T_2, T_3 are 3 independent subordinators, then the processes

$$(T_1(s) + T_3(s), T_2(s) + T_3(s))$$

and

$$(T_1(T_3(s)), T_2(T_3(s)))$$

are two-dimensional subordinators. To the best of our knowledge, general criteria providing necessary and sufficient conditions for two-dimensional process with marginal positive Lévy processes to be a two-dimensional subordinator, are not known in the literature.

Consider now a two-dimensional Lévy process $\vec{L}(t) = (L_1(t), L_2(t))$ with independent components and a two-dimensional subordinator $\vec{\mathcal{T}}(s) = (\mathcal{T}_1(s), \mathcal{T}_2(s))$ such that $\mathcal{T}_i(s)$ is independent of $L_i(s)$ for any $i = 1, 2$. Define the subordinated process by composition

$$\vec{X}(s) = (X_1(s), X_2(s)) := (L_1(\mathcal{T}_1(s)), L_2(\mathcal{T}_2(s))). \tag{6}$$

This construction, known as multivariate subordination, was firstly considered in [4]. The next theorem sheds some light to the characteristics of such processes.

Theorem 4.2. Let $W_i, i = 1, 2$ be two independent one-dimensional Brownian motions and let $\vec{\mathcal{T}}(s) = (\mathcal{T}_1(s), \mathcal{T}_2(s))$ be a two-dimensional subordinator with Lévy triplet $(\vec{\rho}, 0, \eta)$, where $\vec{\rho} = (\rho_1, \rho_2)$ with $\rho_i \geq 0, i = 1, 2$ and η is a Lévy measure in \mathbb{R}_+^2 .

Denote by $\text{diag}(x, y)$ with $x, y \in \mathbb{R}$ a two-dimensional diagonal matrix with the values x and y on the diagonal.

Then

- the process

$$\vec{X}(s) := \left(W_1(\mathcal{T}_1(s)), W_2(\mathcal{T}_2(s)) \right) \quad (7)$$

is a two-dimensional Lévy process;

- the Lévy triplet of the process \vec{X} is given by

$$\left(\vec{0}, \text{diag}(\rho_1, \rho_2), \nu \right),$$

where the Lévy measure ν is defined as

$$\nu(B) := \int_{\mathbb{R}_+^2} \mu\left(B; \text{diag}(y_1, y_2)\right) \eta(dy_1, dy_2), \quad B \subset \mathbb{R}^2,$$

and $\mu(B; A) := \mathbb{P}\{\xi_A \in B\}$ is the probability that a random variable ξ_A with zero mean and covariance matrix A belongs to the set B .

Proof. This theorem is essentially proven in [4]. □

5. Series representation for subordinated processes

In this section, we apply the result by Rosinsky [24] to our setup.

Theorem 5.1. *Let $\vec{X}(s)$ be a two-dimensional Lévy process constructed by multivariate subordination of the Brownian motion, see Theorem 4.2 for notation. Denote by $F(u, v)$ a positive Lévy copula between $\mathcal{T}_1(s)$ and $\mathcal{T}_2(s)$. Assume that $F(u, v)$ is continuous and the mixed derivative $\partial^2 F(u, v)/\partial u \partial v$ exists in \mathbb{R}_+^2 . Moreover, assume that there exists a density function $p^*(\cdot)$ and a function $f^*(u, x) : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$, such that*

1. for any $u, x > 0$,

$$\int_{-\infty}^{f^*(u, x)} p^*(z) dz = \frac{\partial F(u, x)}{\partial u}; \quad (8)$$

2. the function $f^*(u, x)$ monotonically increases in x for any fixed u , and moreover the equation

$$f^*(u, x) = y$$

has a solution in closed form with respect to x for any $y > 0$; we denote this solution by $h^*(u, y)$.

Next, define a two-dimensional stochastic process $\vec{Z}(s) = (Z_1(s), Z_2(s))$:

$$Z_1(s) := \sum_{i=1}^{\infty} \sqrt{U_1^{(-1)}(\Gamma_i)} \cdot G_i^{(1)} \cdot I\{R_i \leq s\}, \quad (9)$$

$$Z_2(s) := \sum_{i=1}^{\infty} G_i^{(2)} \sqrt{U_2^{(-1)}\left(h^*(\Gamma_i, G_i^{(3)})\right)} \cdot I\{R_i \leq s\} \quad (10)$$

where U_1 and U_2 are tail integrals of the subordinators T_1 and T_2 resp., $U_1^{(-1)}$ and $U_2^{(-1)}$ are their generalized inverse functions, that is,

$$U_i^{(-1)}(y) = \inf \{x > 0 : U_i(x) < y\}, \quad i = 1, 2, \quad y \in \mathbb{R}_+,$$

Γ_i is an independent sequence of jump times of a standard Poisson process, $G_i^{(1)}, G_i^{(2)}$ - are two sequences of i.i.d. standard normal r.v., $G_i^{(3)}$ - sequence of i.i.d. random variables with density $p^*(\cdot)$, R_i - sequence of i.i.d. r.v., uniformly distributed on $[0, 1]$, and all sequences of r.v. are independent of each other. Then

$$\vec{X}(s) \stackrel{\mathcal{L}}{=} \vec{Z}(s), \quad \forall s \in [0, 1].$$

Examples.

1. Consider the positive Clayton-Lévy copula

$$F_C(u, v) = (u^{-\theta} + v^{-\theta})^{-1/\theta}$$

with some $\theta > 0$. First derivative of this function is equal to

$$\frac{\partial F_C(u, v)}{\partial u} = \frac{(v/u)^{\theta+1}}{(1 + (v/u)^\theta)^{(1+\theta)/\theta}}.$$

Motivated by this representation, we suggest to define the density function $p^*(z)$ as

$$p^*(z) = \frac{\partial}{\partial z} \left\{ \frac{z^{\theta+1}}{(1 + z^\theta)^{(1+\theta)/\theta}} \right\} = \frac{\partial}{\partial z} \left\{ \frac{1}{(z^{-\theta} + 1)^{(1+\theta)/\theta}} \right\},$$

and the function $f^*(u, x) := x/u$. Both conditions on the functions p^* and f^* are fulfilled.

2. Note that the same arguments can be applied to any sufficiently smooth homogeneous Lévy copula, that is, to any copula such that

$$F_H(ku, kv) = kF_H(u, v), \quad \forall u, v, k > 0, \tag{11}$$

see Remark 5.2 about the difference between ordinary copulas and Lévy copulas. Note that taking the derivatives with respect to u from both parts of (11), yields

$$\frac{\partial}{\partial u} F_H(ku, kv) = \frac{\partial}{\partial u} F_H(u, v) = \frac{\partial}{\partial r_1} F_H(r_1, r_2) \Big|_{\substack{r_1=1 \\ r_2=v/u}},$$

and therefore one can define

$$p^*(z) = \frac{\partial}{\partial z} \left\{ \frac{\partial}{\partial r_1} F_H(r_1, r_2) \Big|_{\substack{r_1=1 \\ r_2=z}} \right\}, \quad f^*(u, x) := x/u.$$

For the description of the class of homogeneous Lévy copulas we refer to Section 4 from [3].

3. Moreover, we can apply the same approach for any mixtures of homogeneous Lévy copulas. In fact, consider the function

$$F_M(u, v) = \sum_{r=1}^n \beta_r F_r(u, v),$$

where β_1, \dots, β_n are positive numbers such that $\sum_{r=1}^n \beta_r = 1$ and $F_r(u, v)$ are homogeneous Lévy copulas for any $r = 1..n$. It's easy to see that $F_M(u, v)$ is a homogeneous Lévy copula. As it was shown in the previous example, we can take $f^*(u, x) := x/u$. Note also that in the case of mixture model,

$$p^*(z) = \sum_{r=1}^n \beta_r p_r^*(z),$$

where by $p_r^*(\cdot)$ we denote the density functions such that

$$\int_{-\infty}^{x/u} p_r^*(z) dz = \frac{\partial F_r(u, x)}{\partial u}.$$

Proof. (of Theorem 5.1) Since the Lévy copula F is sufficiently smooth, we can differentiate both parts in (3) and get that

$$\begin{aligned} \nu(B) &= \int_{\mathbb{R}_+^2} \mu(B; \text{diag}(y_1, y_2)) \left. \frac{\partial^2 F}{\partial r_1 \partial r_2} \right|_{\substack{r_1=U_1(y_1) \\ r_2=U_2(y_2)}} d(U_1(y_1)) d(U_2(y_2)) \\ &= \int_{\mathbb{R}_+^2} \mu(B; \text{diag}(U_1^{-1}(r_1), U_2^{-1}(r_2))) \frac{\partial^2 F(r_1, r_2)}{\partial r_1 \partial r_2} dr_1 dr_2, \end{aligned}$$

see Proposition 5.8 from [15]. Our general aim is to apply the result by Rosinsky [24], which is nicely formulated as Theorem 6.2 in [15]. Comparison of this result and the statement of our theorem leads to the idea to find a function $H : (0, +\infty) \times S \rightarrow \mathbb{R}^2$, where S is a measurable space, such that

$$\nu(B) = \int_{\mathbb{R}_+} \mathbb{P} \left\{ \vec{H}(r, \vec{D}) \in B \right\} dr, \quad \forall B \in \mathcal{B}(\mathbb{R}^2), \quad (12)$$

where \vec{D} is a random vector from S .

First note that it is sufficient to consider the sets $B = B_1 \times B_2$, where $B_1 = [x, \infty)$, $B_2 = [y, \infty)$, $x, y \in \mathbb{R}$. For such B ,

$$\mu(B; \text{diag}(U_1^{-1}(r_1), U_2^{-1}(r_2))) = \mu(B_1; U_1^{-1}(r_1)) \cdot \mu(B_2; U_2^{-1}(r_2)),$$

where by $\mu(\sigma, B)$ we denote the one-dimensional normal distribution with zero mean and variance equal to σ (since there is no risk of confusion, we use the same letter for 2-dimensional and 1-dimensional distributions). Therefore,

$$\nu(B) = \int_{\mathbb{R}_+} \mathbb{E}_{\mathcal{L}(r_1)} \left[\mu(B_2; U_2^{-1}(\cdot)) \right] \mu(B_1; U_1^{-1}(r_1)) dr_1, \quad (13)$$

where by

$$\mathbb{E}_{\mathcal{L}(r_1)} \left[\mu(B_2; U_2^{-1}(\cdot)) \right] = \int_{\mathbb{R}_+} \mu(B_2; U_2^{-1}(r_2)) \frac{\partial}{\partial r_2} \left(\frac{\partial F(r_1, r_2)}{\partial r_1} \right) dr_2$$

we denote the mathematical expectation with respect to the measure $\mathcal{L}(r_1)$ with the distribution function $\tilde{F}(r_2) = \partial F(r_1, r_2)/\partial r_1$ (the statement that $\tilde{F}(r_2)$ is in fact a distribution function is proven in [15], Lemma 5.3). By the well-known Fubini theorem,

$$\mathbb{E}_{\mathcal{L}(r_1)} \left[\mu(B_2; U_2^{-1}(\cdot)) \right] = \int_{B_2} g(v; r_1) dv,$$

where

$$g(\cdot; r_1) = \int_{\mathbb{R}_+} p_1(\cdot; U_2^{-1}(r_2)) \frac{\partial^2 F(r_1, r_2)}{\partial r_1 \partial r_2} dr_2, \quad r_1 > 0, \quad (14)$$

and $p_1(\cdot; U_2^{-1}(r_2))$ is the density of the normal distribution with zero-mean and variance equal to $U_2^{-1}(r_2)$, and $\partial^2 F(r_1, r_2)/(\partial r_1 \partial r_2)$ is the density function corresponding to the distribution function $\tilde{F}(r_2)$.

Note that g is a density function, see Remark 5.3. Changing the variables we get

$$g(\cdot; r_1) = \int_{\mathbb{R}_+} p_1(\cdot; \tilde{r}_2) \frac{\partial^2 F(r_1, r_2)}{\partial r_1 \partial r_2} \Big|_{r_2=U_2(\tilde{r}_2)} d(U_2(\tilde{r}_2)), \quad r_1 > 0.$$

The last expression yields that $g(\cdot; r_1)$ is in fact a variance mixture of the normal distribution (see [6] or [21]). This in particular gives that the random variable

$$\xi = \eta_1 \sqrt{\eta_2}$$

has a distribution with density $g(\cdot; r_1)$, where η_1 has standard normal distribution, and η_2 - distribution with density

$$\begin{aligned} \check{p}(\cdot; r_1) &= - \frac{\partial^2 F(r_1, r_2)}{\partial r_1 \partial r_2} \Big|_{r_2=U_2(\cdot)} U_2'(\cdot) \\ &= \frac{\partial^2 F(r_1, r_2)}{\partial r_1 \partial r_2} \Big|_{r_2=U_2(\cdot)} |U_2'(\cdot)|, \end{aligned}$$

that is, the density of the random variable $U_2^{-1}(\eta_3)$, where η_3 has a distribution function $\tilde{F}(r_2)$. Since (8) holds, we get that

$$\frac{\partial^2 F(r_1, r_2)}{\partial r_1 \partial r_2} = \frac{\partial f^*(r_1, r_2)}{\partial r_2} p^*(f^*(r_1, r_2)),$$

and therefore η_3 has the same distribution as $h^*(r_1, \eta_4)$, where η_4 has distribution with density $p^*(\cdot)$.

Finally we get the following representation for the Lévy measure ν :

$$\nu(B) = \int_{\mathbb{R}_+} \left[\int_{B_1} p_1(u; U_1^{-1}(r_1)) du \cdot \mathbb{P}\left\{ \eta_1 \sqrt{U_2^{-1}(h^*(r_1, \eta_4))} \in B_2 \right\} \right] dr_1.$$

This representation motivates to define the function \vec{H} by

$$\vec{H}(r, \vec{D}) = \left(\begin{array}{c} \sqrt{U_1^{-1}(r) \cdot D_1} \\ D_2 \sqrt{U_2^{-1}(h^*(r, D_3))} \end{array} \right),$$

with $\vec{D} = (D_1, D_2, D_3)$, where D_1, D_2 have standard normal distribution, and D_3 has a distribution with density function $p^*(\cdot)$. This observation completes the proof. □

Remark 5.2. *In the context of ordinary copulas, it is common to introduce the homogeneous copula $C_H^{(k)}$ of order k by*

$$C_H^{(k)}(ku, kv) = k^\alpha C_H(u, v), \quad \forall k, u, v > 0, \tag{15}$$

see, e.g., [23]. Substituting $u = v = 1$, we get $C_H(k, k) = k^\alpha$. Therefore, taking into account the Fréchet bounds, we arrive at the inequality

$$\max(2k - 1, 0) \leq k^\alpha \leq k,$$

which yields that $\alpha \in [1, 2]$. Moreover, it turns out that the class of homogeneous ordinary copulas coincides with Cuadras-Augé family. More precisely,

$$C_H^{(k)}(u, v) = (\min(u, v))^{2-\alpha} (uv)^{\alpha-1}, \quad u, v \in [0, 1],$$

see Theorem 3.4.2 from [23]. Returning to Lévy copulas, we realize that similar to (15) equality

$$F_H(ku, kv) = k^\alpha F_H(u, v), \quad \forall k, u, v > 0$$

is possible only in case $\alpha = 1$. In fact, taking limit as $v \rightarrow \infty$, we get the equality $ku = k^\alpha u, \forall u$, which leads to trivial conclusion $\alpha = 1$. This argument yields the definition of homogeneous Lévy copula (11).

Remark 5.3. *Let us shortly show that the function $g(r_1; \cdot)$ defined by (14) is a density function for any r_1 . In fact, as it was mentioned before, the function $\tilde{F}(r_2) = \partial F(r_1, r_2) / \partial r_1$ is a distribution function, and moreover $\partial^2 F(r_1, r_2) / \partial r_1 \partial r_2$ is the density function of this distribution. Therefore, $g(r_1; \cdot) \geq 0$, and*

$$\begin{aligned} \int_{\mathbb{R}} g(r_1; v) dv &= \int_{\mathbb{R}_+} \left[\int_{\mathbb{R}} p_1(U_2^{-1}(r_2); v) dv \right] \frac{\partial^2 F(r_1, r_2)}{\partial r_1 \partial r_2} dr_2 \\ &= \int_{\mathbb{R}_+} \frac{\partial^2 F(r_1, r_2)}{\partial r_1 \partial r_2} dr_2 = 1. \end{aligned}$$

Remark 5.4. *It is a worth mentioning that the right way to truncate series in (9)-(10) is to fix some r and keep $N(r) = \inf_i \{\Gamma_i \geq r\}$ terms, see [15] for details.*

6. Empirical analysis

In this section, we consider the following model:

$$\vec{X}(s) := \left(\widetilde{W}_1(\mathcal{T}_1(s)), \widetilde{W}_2(\mathcal{T}_2(s)) \right) \tag{16}$$

where

$$\widetilde{W}_i(t) = \mu_i t + \sigma_i W_i(t), \quad i = 1, 2, \tag{17}$$

$W_1(t), W_2(t)$ are two independent Brownian motions, $\mu_1, \mu_2 \in \mathbb{R}$, $\sigma_1, \sigma_2 \in \mathbb{R}_+$, and $(\mathcal{T}_1(s), \mathcal{T}_2(s))$ is a two-dimensional subordinator. The dependence between $\mathcal{T}_1(s)$ and $\mathcal{T}_2(s)$ is described via some Lévy copula $F(\cdot, \cdot; \delta)$ which belongs to a class parametrized by $\delta \in \mathbb{R}$. The marginal subordinators $\mathcal{T}_1(s)$ and $\mathcal{T}_2(s)$ belong to some parametric class of Lévy processes. The corresponding parameters are denoted by θ_1 and θ_2 .

In what follows, we will apply this model to the modeling of stock returns. In this context, $\vec{X}(s)$ represents the returns of two stocks traded on the Nasdaq, and $(\mathcal{T}_1(s), \mathcal{T}_2(s))$ are cumulative numbers of trades of these stocks. Our approach can be considered as a generalization of the paper [1], where the one-dimensional time-changed Brownian motion is used for representing one-dimensional stock returns. In [1], it is shown that the cumulative number of trades is a good approximation of business time. More precisely, the authors showed that the theoretical moments of the subordinator almost perfectly coincide with the empirical moments of cumulative number of trades.

Our simulation study consists of three steps.

1. *Estimation of the Lévy copula between one-dimensional subordinators.* First, we assume some parametric structure of the Lévy copula between $\mathcal{T}_1(s)$ and $\mathcal{T}_2(s)$ and estimate the parameters of this structure. Our estimation procedure is motivated by the research [17], [18] and is described below in Section 6.2.
2. *Estimation of the parameters of the processes $\widetilde{W}_1(t)$ and $\widetilde{W}_2(t)$.* On this stage, we apply methodology described in [1] to the processes $\widetilde{W}_1(t)$ and $\widetilde{W}_2(t)$, and get the estimators of the parameters $\mu_1, \mu_2, \sigma_1, \sigma_2$. This part of the empirical analysis is given in Section 6.3.
3. *Applying simulation techniques.* Since we already have the estimates of the Lévy copula between the subordinators and the parameters $\mu_1, \mu_2, \sigma_1, \sigma_2$, we can apply the main theoretical result of the current paper presented in Theorem 5.1. The algorithm is described in Section 6.4. Some graphs are given in the appendix.

6.1. Description of the data

We examine 10- and 30-minutes Cisco, Intel and Microsoft prices traded on the Nasdaq over the period from the 25. August 2014 till the 21. November 2014. For each equity we have time variable, price variable and number of trades. The length of time series is 832 observations for 30- minutes data and 2496 observations for 10- minutes data. Before the analysis, last observations of each trading day were deleted due to the abnormally small number of trades. We suppose that it is related to the microstructure of the market.

Some descriptive statistics of the data are given in Tables 3 and 4 (see Appendix A).

6.2. Step 1. Lévy copula estimation

The techniques for the parametric estimation of Lévy copula are not well-described in the literature and are mainly known for compound poisson processes (see [2], [17], [18]). In this research, we also assume that subordinators $\mathcal{T}_1(t)$ and $\mathcal{T}_2(t)$ are compound poisson processes (CPP) with positive jumps, that is,

$$\mathcal{T}_1(t) = \sum_{i=1}^{N_1(t)} X_i, \quad \mathcal{T}_2(t) = \sum_{j=1}^{N_2(t)} Y_j, \tag{18}$$

where X_i and Y_i are i.i.d random variables with densities $f_1(x; \theta_1)$ and $f_2(x; \theta_2)$ with support in \mathbb{R}_+ , $N_1(t)$ and $N_2(t)$ are Poisson processes with intensities λ_1 and λ_2 resp.

At first glance, (18) seems to be a strong assumption. However, the Lévy processes with truncated jumps, that is processes in the form

$$J_t = \sum_{\substack{0 \leq s \leq t \\ |\Delta Z_s| \geq c}} \Delta Z_s, \quad \Delta Z_s = Z_s - Z_{s-},$$

where Z is a (multi-dimensional) Lévy process, c is a positive constant, are compound poisson processes (see [15], [25]). Moreover, the truncation of the jumps is a well-used procedure for different simulation and estimation techniques, and is a quite natural tool in the context of stock data (see [15], [18]).

Two-dimensional CPP could be represented in the following way:

$$\begin{aligned} \mathcal{T}_1(t) &= \mathcal{T}_1^\perp(t) + \mathcal{T}_1^\parallel(t), \\ \mathcal{T}_2(t) &= \mathcal{T}_2^\perp(t) + \mathcal{T}_2^\parallel(t), \end{aligned}$$

where \mathcal{T}_1^\perp , $\mathcal{T}_2^\perp(t)$, $(\mathcal{T}_1^\parallel(t), \mathcal{T}_2^\parallel(t))$ are independent compound Poisson processes, and moreover

- the Lévy measure of the process $\mathcal{T}_1^\perp(t)$ is equal to $\eta(\cdot, 0)$, that is, the component $\mathcal{T}_1^\perp(t)$ represents the jumps of the process $\mathcal{T}_1(t)$ which occur independently of the process $\mathcal{T}_2(t)$;
- analogously, the Lévy measure of the process $\mathcal{T}_2^\perp(t)$ is equal to $\eta(0, \cdot)$, that is, the component $\mathcal{T}_2^\perp(t)$ represents the jumps of the process $\mathcal{T}_2(t)$ which occur independently of the process $\mathcal{T}_1(t)$;
- the Lévy measure of the joint process $(\mathcal{T}_1^\parallel(t), \mathcal{T}_2^\parallel(t))$ is equal to $\eta(B)$ for all sets B such that $(0, y)$ and $(x, 0)$ doesn't belong to B for any real x, y , that is, $(\mathcal{T}_1^\parallel(t), \mathcal{T}_2^\parallel(t))$ represents the simultaneous jumps of the processes $\mathcal{T}_1(t)$ and $\mathcal{T}_2(t)$.

Shortly speaking, we decompose the 2-dimensional CPP into the jump independent parts $(\mathcal{T}_1^\perp, \mathcal{T}_2^\perp)$ and jump dependent parts $(\mathcal{T}_1^\parallel, \mathcal{T}_2^\parallel)$. In other words, first part represents positive jumps only in one coordinate and second part represents positive jumps in both coordinates. We denote by $\Pi_1^\perp, \Pi_2^\perp, \Pi^\parallel$ the Lévy measures of the processes $\mathcal{T}_1^\perp, \mathcal{T}_2^\perp, (\mathcal{T}_1^\parallel, \mathcal{T}_2^\parallel)$ resp., by $\lambda_1^\perp, \lambda_2^\perp, \lambda^\parallel$ - the intensities of the corresponding processes, and by $n_1^\perp, n_2^\perp, n^\parallel$ - the total number of jumps occurring only in the observed processes up to some fixed time T .

The characterisitic function of two-dimensional CPP can be decomposed as follows (see [15]):

$$\begin{aligned}
 E[\exp(iz_1\mathcal{T}_1(t) + iz_2\mathcal{T}_2(t))] = & \\
 \exp \left\{ t \int_R (\exp(iz_1x) - 1) \Pi_1^\perp(dx) + t \int_R (\exp(iz_2y) - 1) \Pi_2^\perp(dy) + \right. & \\
 \left. + t \int_{R^2} (\exp(iz_1x + iz_2y) - 1) \Pi^\parallel(dx \times dy) \right\} = & \\
 E [\exp(iz_1\mathcal{T}_1^\perp(t))] E [\exp(iz_2\mathcal{T}_2^\perp(t))] E [\exp(iz_1\mathcal{T}_1^\parallel(t) + iz_2\mathcal{T}_2^\parallel(t))] . &
 \end{aligned}$$

In [17], Esmaili and Kluppelberg applied the MLE approach to estimate the parameters in this model. According to the Theorem 4.1 from [17], the likelihood of the bivariate compound poisson process is given by:

$$L(\lambda_1, \lambda_2, \theta_1, \theta_2, \delta) = I_1 \cdot I_2 \cdot I_3 \tag{19}$$

where

$$\begin{aligned}
 I_1 &= (\lambda_1)^{n_1^\perp} \exp(-\lambda_1^\perp T) \\
 &\quad \cdot \prod_{i=1}^{n_1^\perp} \left[f_1(\tilde{x}_i, \theta_1) \left(1 - \frac{\partial}{\partial u} F(u, \lambda_2; \delta) \Big|_{u=\lambda_1 \bar{F}_1(\tilde{x}_i; \theta_1)} \right) \right], \\
 I_2 &= (\lambda_2)^{n_2^\perp} \exp(-\lambda_2^\perp T) \\
 &\quad \cdot \prod_{i=1}^{n_2^\perp} \left[f_2(\tilde{y}_i, \theta_2) \left(1 - \frac{\partial}{\partial v} F(\lambda_1, v; \delta) \Big|_{v=\lambda_2 \bar{F}_2(\tilde{y}_i; \theta_2)} \right) \right], \\
 I_3 &= (\lambda_1 \lambda_2)^{n^\parallel} \exp(-\lambda^{n^\parallel} T) \\
 &\quad \cdot \prod_{i=1}^{n^\parallel} \left[f_1(x_i; \theta_1) f_2(y_i; \theta_2) \cdot \frac{\partial^2}{\partial u \partial v} F(u, v, \delta) \Big|_{\substack{u=\lambda_1 \bar{F}_1(x_i; \theta_1), \\ v=\lambda_2 \bar{F}_2(y_i; \theta_2)}} \right].
 \end{aligned}$$

where \tilde{x}_i and \tilde{y}_i are jumps in the first and the second component occurring at different time, x and y are jumps for both components occurring at the same time, $\bar{F}_i(\cdot; \theta_i) := \int_{\cdot}^{+\infty} f_i(u; \theta_i) du$, $i = 1, 2$.

Formula (19) is based on a couple of simple facts, which directly follows from our construction. First, note that distance between jump moments for both components are independent and identically distributed exponential random variables with parameters λ_1 and λ_2 . Second, one can show that

$$U_i =: U_i^\perp + U_i^\parallel, \tag{20}$$

where U_i , $i = 1, 2$ - marginal tail integral, U_i^\perp and U_i^\parallel are one-dimensional tail integrals of independent and dependent parts. Third, it is a worth mentioning that

$$\lambda^\parallel = C(\lambda_1, \lambda_2; \delta). \tag{21}$$

From (20) and properties of two-dimensional tail integral and Lévy copula we get that for any positive x, y

$$\begin{aligned}
 \lambda_1^\perp(x)(1 - F_1^\parallel(x)) &= \lambda_1 \bar{F}_1 - C(\lambda_1 \bar{F}_1(x), \lambda_2; \delta) \\
 \lambda_2^\perp(x)(1 - F_2^\parallel(x)) &= \lambda_2 \bar{F}_2(x) - C(\lambda_1, \lambda_2 \bar{F}_2(x); \delta) \\
 \lambda^\parallel F(x, y) &= C(\lambda_1 \bar{F}_1(x), \lambda_2 \bar{F}_2(y); \delta).
 \end{aligned}$$

For numerical example, we model the process of two-dimensional cumulative number of trades as a compound poisson process with exponential jumps. Denote by θ_1 and θ_2 the parameters of the jump sizes densities, that is,

$$f_i(x; \theta_i) = \theta_i \exp(-\theta_i x) \quad \text{for } x > 0, \quad i = 1, 2$$

is the jump density for the i -th component. The dependence between \mathcal{T}_1 and \mathcal{T}_2 is described by a Clayton Lévy copula with parameter δ . Clayton copula is

TABLE 1

Estimated parameters						
Pair	θ_1	θ_2	δ	λ_1	λ_2	Log likelihood value
30-minutes returns						
Csco vs Int	0,29	0,14	2,21	24,91	14,69	5161,43
Csco vs Msf	0,23	0,14	2,71	16,39	17,21	5196,93
Int vs Msf	0,14	0,17	2,38	14,41	24,68	5579,32
10-minutes returns						
Csco vs Int	0,85	0,43	1,76	74,18	48,60	10299,11
Csco vs Msf	0,71	0,42	2,11	52,66	55,78	10406,96
Int vs Msf	0,42	0,49	2,00	46,22	72,48	11511,90

homogeneous copula and perfectly fits the conditions of the Theorem 5.1 (see Example 1 on page 7).

The likelihood function of the continuously observed two-dimensional CPP process $(\mathcal{T}_1(t), \mathcal{T}_2(t))$ can be written in the following form, assuming that jumps occur at each moment for both components (there are no time intervals without trades in our data due to the fact that Cisco, Intel and Microsoft are liquid securities):

$$\begin{aligned}
 &L(\lambda_1, \lambda_2, \theta_1, \theta_2, \delta) \\
 &= ((1 + \delta)\theta_1\theta_2(\lambda_1\lambda_2)^{\delta+1})^n \exp \left\{ -\lambda T - (1 + \delta) \left(\theta_1 \sum_{i=1}^n x_i + \theta_2 \sum_{i=1}^n y_i \right) \right\} \\
 &\quad \cdot \prod_{i=1}^n (\lambda_1^\delta \exp(-\theta_1 \delta x_i) + \lambda_2^\delta \exp(-\theta_2 \delta y_i))^{-\frac{1}{\delta}-2}. \quad (22)
 \end{aligned}$$

The results of the numerical optimization of $\ln L(\lambda_1, \lambda_2, \theta_1, \theta_2, \delta)$ are presented in the Table 1.

6.3. Step 2. Estimation in time-changed model

Let us shortly recall the model of stochastic time change in one-dimensional case. Denote by P_t a equity price at moment t, and the returns by

$$Y_t = \log \left(\frac{P_t}{P_{t-1}} \right). \quad (23)$$

The main idea of the pioneer research [1] is to show that

$$Y_t = \widetilde{W}(\tau(t)), \quad (24)$$

where $\widetilde{W}(t) = \mu t + \sigma W(t)$ with Brownian motion W_t , and $\tau(t)$ is the cumulative number of trades up to time t . It is a well-known fact that on one side, financial

TABLE 2

Pair	μ_1	μ_2	σ_1^2	σ_2^2
30-minutes data				
cscoc intc	1,94E-08	7,25E-09	3,45E-09	4,93E-09
cscoc msft	2,31E-08	1,16E-08	4,10E-09	3,05E-09
intc msft	7,33E-09	1,00E-08	4,99E-09	3,84E-10
10-minutes data				
cscoc intc	-6,00E-09	-7,64E-10	5,17E-10	9,56E-10
cscoc msft	-7,05E-09	-2,63E-09	6,07E-10	4,14E-10
intc msft	-7,85E-10	-2,35E-09	9,82E-10	3,69E-10

returns generally are not normally distributed (e.g. they are fat tailed), and on the other side, normality hypothesis is very convenient tool in finance, e.g. in mean-variance paradigm. In this context, formula (24) shows that returns are in some sense normal in business time, which differs from calendar time.

Returning to our model (16)-(17), we now consider the problem of statistical estimation of the parameters $\mu_1, \mu_2, \sigma_1, \sigma_2$. This task can separately solved for both components of the vector $\vec{X}(s)$ by the method of moments. Assuming that τ is a CPP with intensity λ and jumps distributed by exponential law with parameter θ , we get

$$E[Y_t] = \mu \frac{\lambda t}{\theta}, \quad \text{Var}[Y_t] = \frac{\sigma^2 \lambda t}{\theta} + \frac{2\mu^2 \lambda t}{\theta^2}.$$

Solving the system of equations

$$E[Y_t] = \widehat{E}[Y_t], \quad \text{Var}[Y_t] = \widehat{\text{Var}}[Y_t],$$

we arrive at the following estimates of the parameters μ and σ^2 :

$$\hat{\mu} = \frac{\theta \widehat{E}[Y_t]}{\lambda t}, \quad \hat{\sigma}^2 = \frac{\widehat{\text{Var}}[Y_t] - 2\hat{\mu}^2 \lambda t / \theta}{\lambda t}.$$

Estimation results are presented in the Table 1.

6.4. Step 3. Simulation techniques

Here we show the performance of our approach introduced in Theorem 5.1. Our goal is to simulate two-dimensional time-changed Lévy process:

$$\begin{aligned} \vec{X}(s) &= \left(\widetilde{W}_1(\mathcal{T}_1(s)), \widetilde{W}_2(\mathcal{T}_2(s)) \right) \\ &= \left(\mu_1 \mathcal{T}_1(s) + \sigma_1 W_1(\mathcal{T}_1(s)), \mu_2 \mathcal{T}_2(s) + \sigma_2 W_2(\mathcal{T}_2(s)) \right). \end{aligned}$$

Our simulation algorithm consists of the following steps:

1. Model an independent sequence of jump times of a standard Poisson process Γ_i :

$$\Gamma_i = \sum_{j=1}^i T_j < r,$$

where r determines the truncation level, T_j is a standard exponential random variable.

2. Model k independent standard normal random variables $G_i^{(1)}$ and $G_i^{(2)}$, where $i = 1, \dots, k$.
3. Model k independent uniform random variables R_i on $[0, 1]$, where $i = 1, \dots, k$.
4. Model k independent random variables $G_i^{(3)}$ with distribution function $F(z)$, which is equal to

$$F(z) = \frac{1}{(z^{-\theta} + 1)^{\frac{(1+\theta)}{\theta}}} \tag{25}$$

by the method of inverse function, that is $G_i^{(3)} \equiv F^{-1}(\xi_i)$, where ξ_i are independent uniform random variables on $[0, 1]$.

5. Model subordinated Brownian motions by (truncated) series representation:

$$\begin{aligned} Z_1(s) &:= \sum_{i=1}^k \sqrt{U_1^{-1}(\Gamma_i)} \cdot G_i^{(1)} \cdot I\{R_i \leq s\}, \\ Z_2(s) &:= \sum_{i=1}^k G_i^{(2)} \sqrt{U_2^{-1}(h^*(\Gamma_i, G_i^{(3)}))} \cdot I\{R_i \leq s\}, \end{aligned}$$

where the generalized inverse functions of U_i , $i = 1, 2$ have the form

$$U_i^{(-1)}(x) = \begin{cases} -\frac{1}{\theta_i} \log\left(\frac{x}{\lambda_i}\right), & \text{for } x \leq \lambda_i, \\ 0, & \text{for } x > \lambda_i, \end{cases} \tag{26}$$

and $h^*(x, y)$ is equal to $h^*(x, y) = xy$.

6. Model two-dimensional subordinator $(\mathcal{T}_1(s), \mathcal{T}_2(s))$ with Clayton-Lévy copula and compound poisson margins (with exponential jumps) by series representation (see [15], algorithm 6.13).
7. Resulting trajectory is a linear transform of subordinator and subordinated brownian motion:

$$X_1(s) = \hat{\mu}_1 \mathcal{T}_1(s) + \hat{\sigma}_1^2 Z_1(s) \tag{27}$$

$$X_2(s) = \hat{\mu}_2 \mathcal{T}_2(s) + \hat{\sigma}_2^2 Z_2(s) \tag{28}$$

Typical trajectories of simulated processes are presented in the Appendix. Figures 1 and 4 display trajectories for time-changed brownian motions modeled by Theorem 5.1 for 30 and 10 minutes data. Figures 2 and 5 show typical trajectories for suborninators modelled as compound poisson process with exponential jumps for 30 and 10 minutes data. Finally, Figures 3 and 6 display resulting trajectories for the two-dimensional process \vec{X} calculated by (27)-(28).

6.5. Further research

One interesting question, which was not addressed before, is to compare the Lévy copula between simulated process $(\widetilde{W}_1(\mathcal{T}_1(s)), \widetilde{W}_2(\mathcal{T}_2(s)))$ and copula between subordinators $(\mathcal{T}_1(s), \mathcal{T}_2(s))$. This question is motivated by the paper [16], where some relations between the corresponding correlation coefficients are given.

In this paper, we would like to visually compare the copulas. The nonparametric estimation of the Lévy copula between Lévy processes $X^{(1)}$ and $X^{(2)}$ has been recently studied in [9]. The proposed estimator for any $x_1 \geq 0$ and $x_2 \geq 0$ is equal to

$$\hat{F}(x_1, x_2) = \sum_{k=1}^n I\{\widehat{U}_{1,n}(\Delta_k^n X^{(1)}) \leq x_1, \widehat{U}_{2,n}(\Delta_k^n X^{(2)}) \leq x_2\}, \tag{29}$$

where by

$$\Delta_k^n X^{(i)} = X_{k\Delta_n}^{(i)} - X_{(k-1)\Delta_n}^{(i)}, \quad k = 1..n, \quad i = 1, 2,$$

we denote the increments of the processes $X^{(1)}$ and $X^{(2)}$, and

$$\widehat{U}_{n,i}(x) = \frac{1}{n\Delta_n} \sum_{k=1}^n I\{\Delta_k^n X^{(i)} \geq x\}, \quad i = 1, 2, \tag{30}$$

are the non-parametric estimators of the tail integrals of the underlined Lévy processes.

We applied this methodology to the simulated process $\widetilde{W}_1(\mathcal{T}_1(s))$ and $\widetilde{W}_2(\mathcal{T}_2(s))$ and got the Lévy-copula estimate (see Figure 7).

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Appendix A: Descriptive statistics

TABLE 3

Variable	Descriptive statistics for number of trades, in thousands							
	mean	stdev	min	max	median	m_2	m_3	m_4
30cscs	5,13	3,12	1,19	33,41	4,24	9,74	74,52	1,357,62
30int	7,83	6,22	2,09	79,24	6,02	38,64	1,093,33	57,900,47
30msf	8,87	5,53	1,46	59,85	7,53	30,54	539,90	205,79,57
10cscs	1,71	1,20	0,26	13,01	1,41	1,44	5,11	36,57
10int	2,61	2,32	0,52	39,43	2,00	5,37	69,65	1,759,46
10msf	2,96	2,11	0,35	39,80	2,43	4,46	42,90	1,022,25

TABLE 4

Returns	Descriptive statistics for returns							
	mean	stdev	min	max	median	m_2	m_3	m_4
30cscs	1,06E-04	4,35E-03	-4,81E-02	3,01E-02	-1,12E-04	1,89E-05	-6,04E-08	1,25E-08
30int	5,00E-05	5,84E-03	-6,81E-02	4,16E-02	0,00E+00	3,40E-05	-2,59E-07	4,42E-08
30msf	9,34E-05	4,97E-03	-5,65E-02	3,86E-02	0,00E+00	2,46E-05	-1,39E-07	2,67E-08
10cscs	-3,36E-05	1,70E-03	-1,17E-02	2,26E-02	0,00E+00	2,90E-06	7,01E-09	2,70E-10
10int	-5,50E-06	2,62E-03	-3,17E-02	5,56E-02	0,00E+00	6,87E-06	5,94E-08	4,86E-09
10msf	-2,22E-05	1,87E-03	-2,00E-02	3,09E-02	-5,50E-06	3,49E-06	8,84E-09	5,28E-10

Appendix B: Graphs

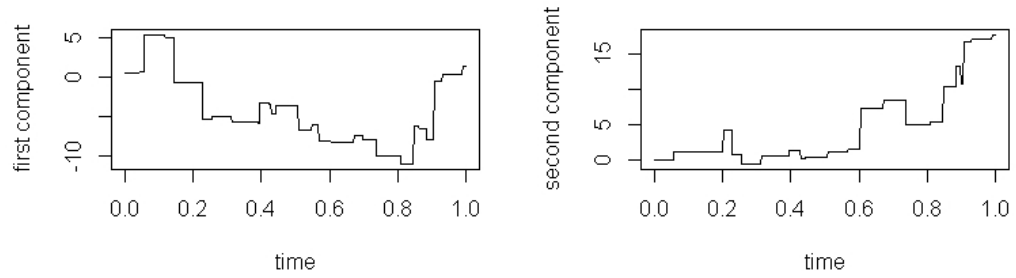


FIG 1. Time-changed brownian motion. Subordinators are CPP with exponential jumps. Parameters are estimated from the Cisco and Intel 30-minutes data

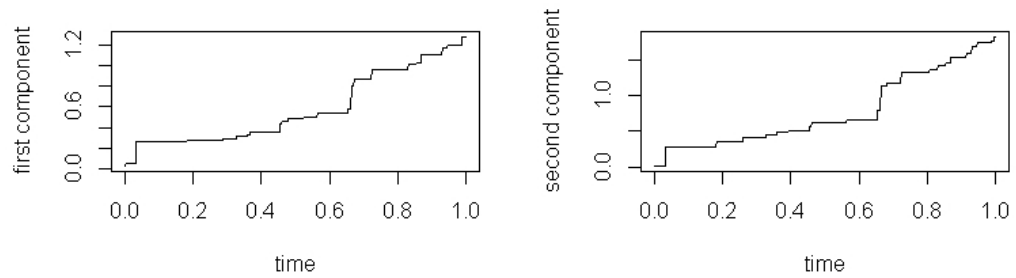


FIG 2. Subordinators for 30 minute data. Parameters are estimated from the Cisco and intel 30-minute data

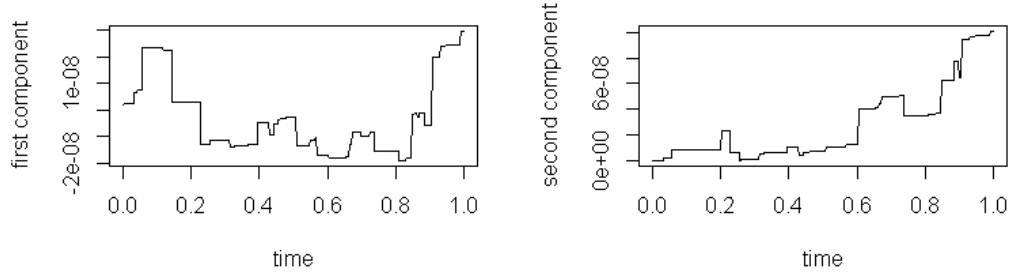


FIG 3. Resulting trajectory of process $Y(t)$ for 30 minute data

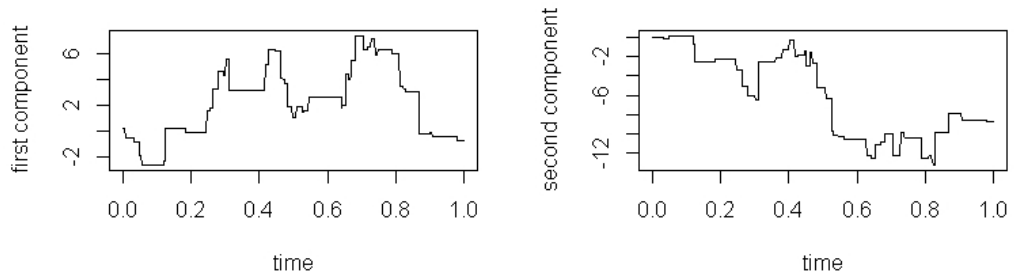


FIG 4. Time-changed brownian motion. Subordinators are CPP with exponential jumps. Parameters are estimated from the Cisco and Microsoft 10-minutes data

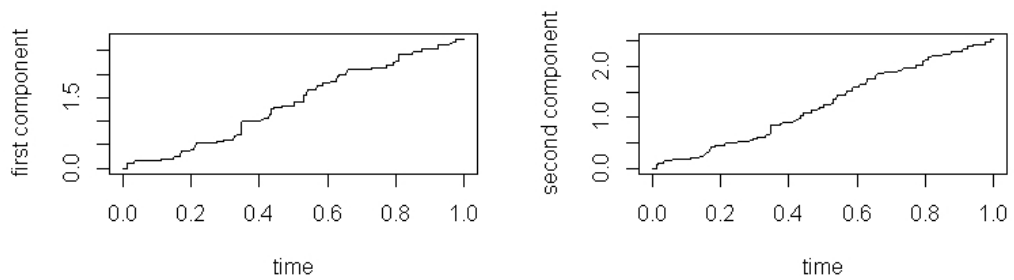


FIG 5. Subordinators for 10-minute data. Parameters are estimated from Cisco and Microsoft 10-minute data

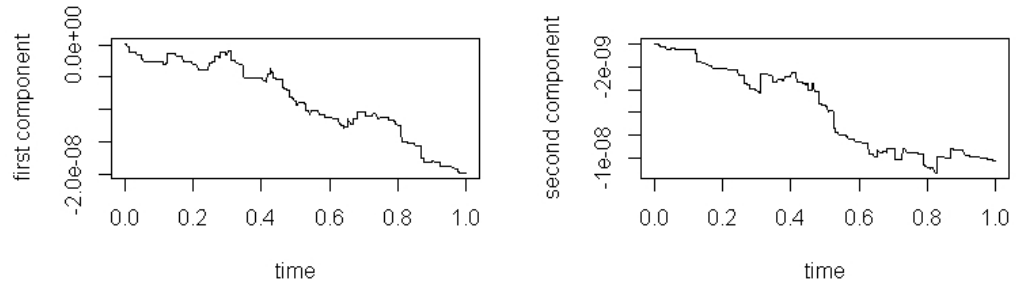


FIG 6. Resulting trajectory of process $Y(s)$ for 10- minute data

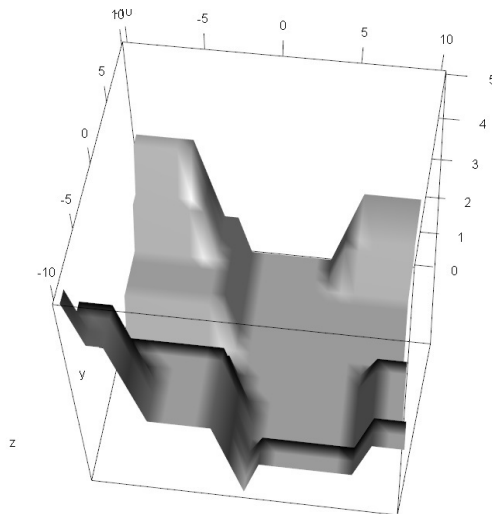


FIG 7. Nonparametric estimation of the copula for simulated process $Y(s)$