ISSN 0027-1322, Moscow University Mathematics Bulletin, 2012, Vol. 67, No. 4, pp. 182–184 . © Allerton Press, Inc., 2012.
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BRIEF COMMUNICATIONS

Behavior of the Shannon Function for Some Families of Classes of Three-Valued Logic Functions

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Received December 12, 2011

Abstract—A certain countable set of families of classes of three-valued logic functions taking values from the set {0,1} is considered. For each class from these families and for each its finite generating system, the order of growth of the corresponding Shannon depth function is obtained.

DOI: 10.3103/S0027132212040092

It is known [1, 2] that for an arbitrary finite system of Boolean functions, any function from the closed class generated by this system can be realized by a formula whose depth has not more than a linear order of growth with respect to the number of variables. In this paper we consider some countable set of families of classes of functions from $P_{3,2}$ which is the set of all functions of three-valued logic taking the values 0 or 1. For each class from these families and for each its finite generating system we obtain a linear by order estimate for the corresponding Shannon function of depth.

Present necessary definitions (see also [3–5]). The set of all function of k-valued logic is denoted by P_k, $k \geq 2$. Let $F \subseteq P_k$, $k \geq 2$. By $[F]$ we denote the closure of F with respect to the superposition operation, and by $F(n)$ we denote the set of all functions from F dependent on the variables $x_1, \ldots, x_n$, $n \geq 1$. Let $f(x_1, \ldots, x_n) \in [F]$, Φ be a formula over F realizing the function f, and $H \subseteq [F]$. By $D(\Phi)$ we denote the depth of the formula Φ, by $D_F(f)$ we denote the minimum of $D(\widehat{\Phi})$ over all formulas $\widehat{\Phi}$ over the system F realizing the function f, by $D_F(H(n))$ we denote the Shannon function of depth for the set H.

Let $f \in P_2$ and $A \subseteq P_2$. By f^* we denote the function dual to f and by A^* we denote the set $\bigcup\{f^*\}$, where the union is taken over all functions $f \in A$. We say that a function f satisfies the condition $< 0^m >$ ($< 1^m >$, respectively), $2 \leq m < \infty$, if any m collections where the function f equals zero (one, respectively) have a common zero (one, respectively) component. We follow the notations for closed classes of Boolean functions from [3], namely, P_2 is the set of all Boolean function; S is the set of all self-dual functions; T_i is the set of all function preserving the constant i, $i = 0, 1$; M is the set of all monotone functions; L is the set of all linear functions; O^m is the set of all functions satisfying the condition $< 0^m >$; I^m is the set of all functions satisfying the condition $< 1^m >$, $2 \leq m < \infty$. Assume

$$M_0 = M \cap T_0, M_1 = M \cap T_1, T_{01} = T_0 \cap T_1, M_{01} = M_0 \cap M_1, S_{01} = S \cap T_{01}, SM = S \cap M$$

and for each $m \geq 2$ assume

$$O_0^m = T_0 \cap O^m, I_1^m = T_1 \cap I^m, MO^m = M \cap O^m, MI^m = M \cap I^m, MO_0^m = M \cap O_0^m, MI_1^m = M \cap I_1^m.$$

By $\mathcal{Q}$ we denote the set of closed classes of Boolean functions

$$\{P_2, T_0, T_1, T_{01}, M, M_0, M_1, M_{01}, S, S_{01}, SM, O^m, O_0^m, MO^m, MO_0^m, I^m, I_1^m, MI^m, MI_1^m, \quad 2 \leq m < \infty\},$$

and by $\mathcal{R}$ we denote the set

$$\{P_2, T_1, T_{01}, M, M_1, M_{01}, O^m, O_0^m, MO^m, MO_0^m,\ 2 \leq m < \infty\}.$$

It is easy to see that for any class $A \in \mathcal{Q} \setminus \{S, S_{01}, SM\}$ at least one of the two following conditions holds:

1) $A \in \mathcal{R}$;

2) there exists a class $B \in \mathcal{R}$ such that $A = B^*$.

Let $f(x_1, \ldots, x_n) \in P_{3,2}$ and $F \subseteq P_{3,2}$. The projection of the function $f(x_1, \ldots, x_n)$ is said to be the Boolean function $\mathrm{pr} f(x_1, \ldots, x_n)$ whose value at each collection $\widetilde{\alpha} \in E_2^n$ is determined by the equality $\mathrm{pr} f(\widetilde{\alpha}) = f(\widetilde{\alpha})$. The projection $\mathrm{pr} F$ of the set of functions F is said to be the set $\bigcup\{\mathrm{pr} f\}$, where the union is taken over all functions $f \in F$. Let B be an arbitrary closed class of Boolean functions. Define the set $\mathrm{pr}^{-1}B$ in the following way. Assume

$$\mathrm{pr}^{-1}B = \{f \in P_{3,2} | \mathrm{pr} f \in B\}.$$

Obviously, the set $\mathrm{pr}^{-1}B$ is a closed class. In this case for any closed class $F \subseteq P_{3,2}$ such that $\mathrm{pr}F = B$ the relation $F \subseteq \mathrm{pr}^{-1}B$ is valid. The closed class $\mathrm{pr}^{-1}B$ is called maximal. Assume

$$\mathfrak{N}(B) = \{A \subseteq P_{3,2} | A = [A],\ \mathrm{pr}A = B\}.$$

It is known (see, e.g., [5]) that $|\mathfrak{N}(B)| < \infty$ if and only if $B \in \mathcal{Q}$.

By $Z_{2,i}$ we denote the set of all functions from $P_{3,2}$ possessing the following property: if the collection $\widetilde{\alpha} \in \{0,1\}^n$ is obtained from the collection $\widetilde{\beta} \in \{0,1,2\}^n$ by the change of all twos by i, then $f(\widetilde{\alpha}) = f(\widetilde{\beta})$, $i = 0,1$.

Define some functions from $P_{3,2}$. By $j_i(x)$ we denote the function equal to one for $x = i$ and to zero in other cases, $i = 0,1,2$. By $x \vee y$ and $x \cdot y$ we denote the functions from $P_{3,2}$ whose projections are the Boolean disjunction and conjunction functions, respectively, and which are equal to zero on collections containing at least one argument two.

The main result of the paper is the following theorem (see also [6]).

Theorem. *Let B be an arbitrary closed class of Boolean functions from the set $\mathcal{Q}$, H be an arbitrary closed class of functions from $P_{3,2}$ such that $\mathrm{pr}H = B$, and G be an arbitrary finite generating system of the class H. Then the relation $D_G(H(n)) \asymp n$ holds.*

In the proof of the upper estimate we use well-known synthesis methods for formulas over incomplete bases realizing logic algebra functions (see [1, 2, 7]) and we also use some properties of functions from $P_{3,2}$. Describe principal stages of the proof.

The set $\mathcal{Q}$ is represented in the form of a union of three nonintersecting subsets, and the proof of the upper estimate is performed separately for each of them.

First we consider classes B from the set $\mathcal{Q} \setminus \{S, S_{01}, SM\}$. Let H be an arbitrary closed class of functions from $P_{3,2}$ such that $\mathrm{pr}H = B$ and G be an arbitrary finite generating system of the class H. Let $B \in \mathcal{R}$. It is not difficult to show that there exist a natural number $r \geq 2$ and a function $\Delta_r(x_1, \ldots, x_{r(r-1)})$ such that for $n \geq 10$ any function $f(x_1, \ldots, x_n) \in H$ can be represented in the form $f = \Delta_r(\widetilde{Y}^r)$, where $\widetilde{Y}^r$ is the collection of $r(r-1)$ functions each of which is obtained by identification of variables in the function f and depends on not more than $n-1$ variables (see Lemma 3 from [7]). Then for any finite generating system G_0 of the class H containing the function Δ_r and all functions from the set $H(9)$ the relation $D_{G_0}(H(n)) \leq c_0 n$ holds, where c_0 is some constant. This implies that for the generating system G of the class H the following relation holds: $D_G(H(n)) \leq cn$, where c is some constant (because the transition from one basis to another causes an increase in the depth of formulas realizing the function f not more than by a constant factor). If B is a class such that $B^* \in \mathcal{R}$, then the upper estimate for the function $D_G(H(n))$ follows from duality reasons.

After that we consider classes $B \in \{S, S_{01}\}$. Let H be an arbitrary closed class of functions from $P_{3,2}$ such that $\mathrm{pr}H = B$ and G be an arbitrary finite generating system of the class H. By $\varepsilon_1^{(2)}(x_1, x_2)$ we denote the function from $P_{3,2}$ satisfying the condition $\mathrm{pr}\varepsilon_1^{(2)}(x_1, x_2) = x_1$ and equal to zero on all collections containing at least one two. The description of the set of all classes whose projection coincides with the class S, or with S_{01} (see [5]) implies that for each such class at least one of the following three conditions holds:

1) the relation $H \subseteq Z_{2,i}$ is valid for a certain $i \in \{0,1\}$;

2) $\varepsilon_1^{(2)}(x_1, x_2) \in H$;

3) the class H is dual to one of the classes satisfying condition 2 with respect to the substitution (01)(2).

The definition of the sets $Z_{2,i}$ implies that for the classes H satisfying condition 1 the upper estimate for the function $D_G(H(n))$ follows directly from the upper estimate for the function $D_{\mathrm{pr}G}(B(n))$. If the class H satisfies condition 2, then the upper estimate for the Shannon function is obtained with the use of an analogue of the constant simulation method (see [2]) for functions from $P_{3,2}$, in this case the function $\varepsilon_1^{(2)}(x_1, x_2)$ is used instead of the constant 0. For a class satisfying condition 3 the corresponding upper estimate is valid because of duality reasons.

Finally, we consider the case $B = SM$. Present here additional definitions and a draft of proof of the theorem for this case. Let $p \geq 2$ and $1 \leq i \leq p$. Assume $x^i = (x_1^i, \ldots, x_{i-1}^i, x_{i+1}^i, \ldots, x_p^i)$. By $\widetilde{X}^p$ we denote the collection $(x^1, \ldots, x^p)$ consisting of $p(p-1)$ variables. Let $f(x_1, \ldots, x_n)$ be an arbitrary function from

$P_{3,2}$, $n \geq p$. By $f_j^i(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$ we denote the function $f(x_1, \ldots, x_{j-1}, x_i, x_{j+1}, \ldots, x_n)$. By $\widetilde{Y}^p$ we denote the collection of functions obtained from the collection $\widetilde{X}^p$ by the change of the variables x_j^i by the functions f_j^i, respectively, where $i, j = 1, \ldots, p$, $i \neq j$.

Let $f(x_1, \ldots, x_n) \in P_{3,2}$, $n \geq 5$. Define the functions $g_i^f(y_1, \ldots, y_i, x_1, \ldots, x_n) \in P_{3,2}$, $i = 2, 3, 4, 5, 6$, in the following way. Assume

$$g_i^f(y_1, \ldots, y_i, x_1, \ldots, x_n) = f(x_1, \ldots, x_n)(j_1(y_1) \vee \ldots \vee j_1(y_i)) \vee (j_1(y_1) \cdot \ldots \cdot j_1(y_i)).$$

The proof is based on a special decomposition of functions from $\mathrm{pr}^{-1}SM$. First, using recurrent relations, we construct the functions $\Omega_i \in \mathrm{pr}^{-1}SM$, $i = 2, 3, 4, 5, 6$, so that for any function $f(x_1, \ldots, x_n) \in P_{3,2}$, $n \geq 10$, and any collection $\widetilde{\beta} \in \{0, 1, 2\}^n$ the following equality holds:

$$\Omega_i(j_1(x_1), \ldots, j_1(x_i), \widetilde{Y}^{10}(\widetilde{\beta})) = g_i^f(x_1, \ldots, x_i, \widetilde{\beta}).$$

Substituting the functions $f_2^1, f_3^1, f_4^1, f_3^2, f_4^2, f_4^3$ instead of the variables $x_1, \ldots, x_6$, respectively, in the case $i = 6$, we get the following relation from this equality:

$$\Omega_6(f_2^1, f_3^1, f_4^1, f_3^2, f_4^2, f_4^3, \widetilde{Y}^{10}(\widetilde{\beta})) = g_i^f(f_2^1, f_3^1, f_4^1, f_3^2, f_4^2, f_4^3, \widetilde{\beta}).$$

Using the definition of the functions g_i^f, it is not difficult to prove that

$$g_i^f(f_2^1(\widetilde{\beta}), f_3^1(\widetilde{\beta}), f_4^1(\widetilde{\beta}), f_3^2(\widetilde{\beta}), f_4^2(\widetilde{\beta}), f_4^3(\widetilde{\beta}), \widetilde{\beta}) = f(\widetilde{\beta}).$$

Therefore,

$$f(\widetilde{\beta}) = \Omega_6(f_2^1(\widetilde{\beta}), f_3^1(\widetilde{\beta}), f_4^1(\widetilde{\beta}), f_3^2(\widetilde{\beta}), f_4^2(\widetilde{\beta}), f_4^3(\widetilde{\beta}), \widetilde{Y}^{10}(\widetilde{\beta})).$$

Let H be an arbitrary closed class of functions from $P_{3,2}$ such that $\mathrm{pr}H = SM$, G be an arbitrary finite generating system of the class H, $f(x_1, \ldots, x_n) \in SM$, $n \geq 10$. Then $f = \Omega_6(f_2^1, f_3^1, f_4^1, f_3^2, f_4^2, f_4^3, \widetilde{Y}^{10})$. Each function substituted into the function Ω_6 is obtained by identification of variables of the function f and depends on not more than $n-1$ variables. Then the set $G_0 = \{\Omega_6\} \cup H(9)$ is a generating system of the class H. Moreover, the following inequality holds:

$$D_{G_0}(f^{(n)}) \leq 1 + D_{G_0}(H(n-1)).$$

Therefore, there exists a constant c_0 such that $D_{G_0}(H(n)) \leq c_0 n$. Thus, for any finite basis G of the class H there exists a constant $c = c(G)$ such that $D_G(H(n)) \leq cn$.

The corresponding lover estimates follows from capacity arguments.

ACKNOWLEDGMENTS

The author is grateful to Prof. A. B. Ugolnikov for attention to the work.

The work was supported by the Russian Foundation for Basic Research (project no. 11–01–00508) and by the Program of fundamental research of Mathematical division of RAS “Algebraic and combinatorial methods of mathematical cybernetics and information systems of new generation,” project “Problems of optimal synthesis of control systems.”

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Translated by V. Valedinskii