SYMBOLS OF TRUNCATED TOEPLITZ OPERATORS

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ABSTRACT. We consider three problems connected with coinvariant subspaces of the backward shift operator in Hardy spaces H^p:

– properties of truncated Toeplitz operators;

– Carleson-type embedding theorems for the coinvariant subspaces;

– factorizations of pseudocontinuable functions from H^1.

These problems turn out to be closely connected and even, in a sense, equivalent. The new approach based on the factorizations allows us to answer a number of challenging questions about truncated Toeplitz operators posed by Donald Sarason.

1. Introduction

Let H^p, $1 \leqslant p \leqslant \infty$, denote the Hardy space in the unit disk $\mathbb{D}$, and let $H^p_- = \overline{zH^p}$. As usual, we identify the functions in H^p in the disk and their nontangential boundary values on the unit circle $\mathbb{T}$.

A function θ which is analytic and bounded in $\mathbb{D}$ is said to be *inner* if $|\theta| = 1$ m-a.e. on $\mathbb{T}$ in the sense of nontangential boundary values; by m we denote the normalized Lebesgue measure on $\mathbb{T}$. With each inner function θ we associate the subspace

$$K^p_\theta = H^p \cap \theta H^p_-$$

of the space H^p. Equivalently, one can define K^p_θ as the set of all functions in H^p such that $\langle f, \theta g\rangle = \int_{\mathbb{T}} f\,\overline{\theta g}\,dm = 0$ for any $g \in H^q$, $1/p + 1/q = 1$. In particular,

$$K_\theta := K^2_\theta = H^2 \ominus \theta H^2$$

(in what follows we usually omit the exponent 2). It is well known that, for $1 \leqslant p \leqslant \infty$, any closed subspace of H^p invariant with respect to the backward shift $(S^*f)(z) = \frac{f(z)-f(0)}{z}$ is of the form K^p_θ for some inner function θ (see [22, Chapter II] or [14]). Subspaces K^p_θ are often called *star-invariant subspaces*. These subspaces play an outstanding role both in function and operator theory (see [14, 27, 28]) and, in particular, in the Sz.-Nagy–Foias model for contractions in a Hilbert space (therefore they are sometimes referred to as *model subspaces*).

A characteristic property of the elements of the spaces K^p_θ is the existence of a *pseudocontinuation* outside the unit disk: if $f \in K^p_\theta$, then there exists

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a function g which is meromorphic and of Nevanlinna class in $\{z : |z| > 1\}$ such that $g = f$ a.e. on $\mathbb{T}$ in the sense of nontangential boundary values.

Now we discuss in detail the three main themes of the paper as indicated in the abstract.

1.1. **Truncated Toeplitz operators on K_θ.** Recall that the classical Toeplitz operator on H^2 with a symbol $\varphi \in L^\infty(\mathbb{T})$ is defined by $T_\varphi f = P_+(\varphi f)$, $f \in H^2$, where P_+ stands for the orthogonal projection from $L^2 := L^2(\mathbb{T}, m)$ onto H^2.

Now let $\varphi \in L^2$. We define the *truncated Toeplitz operator* A_φ on bounded functions from K_θ by the formula

$$A_\varphi f = P_\theta(\varphi f), \qquad f \in K_\theta \cap L^\infty(\mathbb{T}).$$

Here $P_\theta f = P_+ f - \theta P_+(\overline{\theta} f)$ is the orthogonal projection onto K_θ. In contrast to the Toeplitz operators on H^2 (which satisfy $\|T_\varphi\| = \|\varphi\|_\infty$), the operator A_φ may be extended to a bounded operator on K_θ even for some unbounded symbols φ. The class of all bounded truncated Toeplitz operators on K_θ will be denoted by $\mathcal{T}(\theta)$.

Certain special cases of truncated Toeplitz operators are well known and play a prominent role in the operator theory. If $\varphi(z) = z$, then $A_\varphi = S_\theta$ is the so-called restricted shift operator, the scalar model operator from the Sz.-Nagy–Foias theory. If $\varphi \in H^\infty$, then $A_\varphi = \varphi(S_\theta)$. Truncated Toeplitz operators include all finite Toeplitz matrices (corresponding to the case $\theta(z) = z^n$) and the Wiener–Hopf convolution operators on an interval which are unitary equivalent to the truncated Toeplitz operators on the space generated by the singular inner function associated with a point mass on the circle,

$$\theta_a(z) = \exp\left(a\frac{z+1}{z-1}\right) \tag{1}$$

(for a more detailed discussion see [9]). Bercovici, Foias and Tannenbaum studied truncated (or *skew*) Toeplitz operators (mainly, with symbols which are rational functions with pole at zero) in connection with control theory (see [11, 12]). However, a systematic study of truncated Toeplitz operators with L^2 symbols was started recently by Sarason in [31]. This paper laid the basis of the theory and inspired much of the subsequent activity in the field [9, 15, 16, 21].

Unlike standard Toeplitz operators on H^2, the symbol of a truncated Toeplitz operator is not unique. The set of all symbols of an arbitrary operator A_φ is exactly the set $\varphi + \theta H^2 + \overline{\theta H^2}$, see [31]. Clearly, any bounded function $\varphi \in L^\infty$ determines the bounded operator A_φ with the norm $\|A_\varphi\| \leqslant \|\varphi\|_\infty$. The first basic question on truncated Toeplitz operators posed in [31] is *whether every bounded operator A_φ has a bounded symbol*, i.e., is a restriction of a bounded Toeplitz operator on H^2. Note that if a truncated Toeplitz operator with a symbol $\varphi \in H^2$ is bounded, then, as a consequence of the commutant lifting theorem, it admits an H^∞ symbol (see [31, Section 4]). On the other hand, by the results of Rochberg [30] (proved in the context of the Wiener–Hopf operators and the Paley–Wiener spaces) any operator in $\mathcal{T}(\theta_a)$ has a bounded symbol. However, in general the answer to this question is negative: in [9] inner functions θ are constructed, for which there exist operators in $\mathcal{T}(\theta)$ (even of rank one) that have no bounded symbols.

Thus, the following question seems to be of interest: *in which spaces K_θ does any bounded truncated Toeplitz operator admit a bounded symbol?* In the present paper we obtain a description of such inner functions. In particular, we show that this is true for an interesting class of *one-component inner functions* introduced by Cohn in [18]: these are functions θ such that the sublevel set

$$\{z \in \mathbb{D} : |\theta(z)| < \varepsilon\}$$

is connected for some $\varepsilon \in (0, 1)$. This statement was conjectured in [9]. A basic example of a one-component inner function is the function θ_a given by (1).

1.2. **Embeddings of the spaces K^p_θ.** Let μ be a finite positive Borel measure in the closed unit disk $\overline{\mathbb{D}}$. We are interested in the class of measures such that Carleson-type embedding $K^p_\theta \hookrightarrow L^p(\mu)$ is bounded. Since the functions in K^p_θ are well-defined only m-almost everywhere on $\mathbb{T}$, one should be careful when dealing with the restriction of μ to $\mathbb{T}$. Recall that, by a theorem of Aleksandrov, functions in K^p_θ which are continuous in the closed disk $\overline{\mathbb{D}}$ are dense in K^p_θ, see [2] or [13]. (While this statement is trivial for the Blaschke products, there is no constructive way to prove the statement in the general case.) This allows one to define the embedding on the dense set of all continuous functions from K_θ in a natural way and then ask if it admits a bounded continuation to the whole space K^p_θ. However, this extension may always be viewed as an embedding operator due to the following theorem by Aleksandrov.

Theorem 1.1. [4, Theorem 2] *Let θ be an inner function, let μ be a positive Borel measure on $\mathbb{T}$, and let $1 \leqslant p < \infty$. Assume that for any continuous function $f \in K^p_\theta$ we have*

$$\|f\|_{L^p(\mu)} \leqslant C\|f\|_p. \tag{2}$$

Then all functions from K^p_θ possess angular boundary values μ-almost everywhere and for any $f \in K^p_\theta$, (2) holds for its boundary values.

The angular convergence μ-almost everywhere gives us a nice illustration of how the embedding acts. This approach, essentially based on results of Poltoratski's paper [29], uses deep analytic techniques. For our purposes we will need the L^2-convergence, which can be established much simpler.

To make the exposition more self-contained we present the corresponding arguments in Section 3.

Denote by $\mathcal{D}_p(\theta)$ the class of finite complex Borel measures μ on $\overline{\mathbb{D}}$ for which the embedding $K^p_\theta \subset L^p(|\mu|)$ holds; by $|\mu|$ we denote the total variation of the complex measure μ. The class of positive measures in $\mathcal{D}_p(\theta)$ is denoted by $\mathcal{D}^+_p(\theta)$. The classes $\mathcal{D}^+_p(\theta)$ contain all Carleson measures, i.e., measures for which the embedding $H^p \hookrightarrow L^p(\mu)$ is a bounded operator (for some, and hence for all $p > 0$). However, the class $\mathcal{D}_p(\theta)$ is usually much wider due to additional analyticity (pseudocontinuability) of the elements of K^p_θ on the boundary. The problem of the description of the class $\mathcal{D}_p(\theta)$ for general θ was posed by Cohn in 1982; it is still open. Many partial results may be found in [18, 19, 33, 6, 26, 7, 8]. In particular, the classes $\mathcal{D}_p(\theta)$ are described if θ is a one-component inner function; in this case there exists a nice geometric description analogous to the classical Carleson embedding theorem [33, 3, 6]. Moreover, Aleksandrov [6] has shown that θ is one-component if and only if all classes $\mathcal{D}_p(\theta)$ for $p > 0$ coincide.

In what follows we denote by $\mathcal{C}_p(\theta)$ the set of finite complex Borel measures μ on the unit circle $\mathbb{T}$ such that $|\mu|$ is in $\mathcal{D}_p(\theta)$; the class of positive measures in $\mathcal{C}_p(\theta)$ will be denoted by $\mathcal{C}^+_p(\theta)$.

If $\mu \in \mathcal{C}_2(\theta)$, we may define the bounded operator A_μ on K_θ by the formula

$$(A_\mu f, g) = \int f\bar{g}\, d\mu. \tag{3}$$

It is shown in [31] that $A_\mu \in \mathcal{T}(\theta)$. This follows immediately from the following characteristic property of truncated Toeplitz operators.

Theorem 1.2. [31, Theorem 8.1] *A bounded operator A on K_θ is a truncated Toeplitz operator if and only if the condition $f, zf \in K_\theta$ yields $(Af, f) = (Azf, zf)$.*

A complex measure μ on $\mathbb{T}$ with finite total variation (but not necessarily from $\mathcal{C}_2(\theta)$) will be called a *quasisymbol* for a truncated Toeplitz operator A if $(Af, g) = \int f\bar{g}\, d\mu$ holds for all continuous functions $f, g \in K_\theta$. The symbol φ of $A = A_\varphi$ can be regarded as a quasisymbol if we identify it with the measure φm.

The following conjecture was formulated by Sarason in [31]: *every bounded truncated Toeplitz operator A coincides with A_μ for some $\mu \in \mathcal{C}_2(\theta)$*. Below we prove this conjecture. Moreover, we show that nonnegative bounded truncated Toeplitz operators are of the form A_μ with $\mu \in \mathcal{C}^+_2(\theta)$. We also prove that truncated Toeplitz operators with bounded symbols correspond to complex measures from the subclass $\mathcal{C}_1(\theta^2)$ of $\mathcal{C}_2(\theta^2) = \mathcal{C}_2(\theta)$, and, finally, that every bounded truncated Toeplitz operator has a bounded symbol if and only if $\mathcal{C}_1(\theta^2) = \mathcal{C}_2(\theta^2)$.

1.3. **Factorizations.** Now we consider a factorization problem for pseudocontinuable functions in H^1, which proves to have an equivalent reformulation in terms of truncated Toeplitz operators.

It is well known that any function $f \in H^1$ can be represented as the product of two functions $g, h \in H^2$ with $\|f\|_1 = \|g\|_2 \cdot \|h\|_2$. By the definition of the spaces K_θ^p, there is a natural involution on K_θ:

$$f \mapsto \tilde{f} = \bar{z}\theta \bar{f} \in K_\theta, \qquad f \in K_\theta. \tag{4}$$

Hence, if $f, g \in K_\theta$, then $fg \in H^1$ and $\bar{z}^2\theta^2\bar{f}\bar{g} \in H^1$. Thus,

$$fg \in H^1 \cap \bar{z}^2\theta^2\overline{H^1} = H^1 \cap \bar{z}\theta^2 H^1_-.$$

If $\theta(0) = 0$, then θ^2/z is an inner function and $H^1 \cap \bar{z}\theta^2 H^1_- = K^1_{\theta^2/z}$.

It is not difficult to show that linear combinations of products of pairs of functions from K_θ form a dense subset of $H^1 \cap \bar{z}\theta^2 H^1_-$. We are interested in a stronger property:

For which θ may any function $f \in H^1 \cap \bar{z}\theta^2 H^1_-$ be represented in the form

$$f = \sum_k g_k h_k, \qquad g_k, h_k \in K_\theta, \qquad \sum_k \|g_k\|_2 \cdot \|h_k\|_2 < \infty? \tag{5}$$

We still use the term *factorization* for the representations of the form (5), by analogy with the usual row-column product.

Below we will see that, for functions $f \in H^1 \cap \bar{z}\theta^2 H^1_-$, not only a usual factorization $f = g \cdot h$, $g, h \in K_\theta$, but even a weaker factorization (5) may be impossible. We prove that this problem is equivalent to the problem of existence of bounded symbols for all bounded truncated Toeplitz operators on K_θ.

Let us consider two special cases of the problem. Take $\theta(z) = z^{n+1}$. The spaces K_θ and $K^1_{\theta^2/z}$ consist of polynomials of degree at most n and $2n$, respectively, and then, obviously, $K^1_{\theta^2/z} = K_\theta \cdot K_\theta$. However, it is not known if a norm controlled factorization is possible, i.e., if for a polynomial p of degree at most $2n$ there exist polynomials q, r of degree at most n such that $p = q \cdot r$ and $\|q\|_2 \cdot \|r\|_2 \leqslant C\|p\|_1$, where C is an absolute constant independent on n. On the other hand, it is shown in [32] that there exists a representation $p = \sum_{k=1}^4 q_k r_k$ with $\sum_{k=1}^4 \|q_k\|_2 \cdot \|r_k\|_2 \leqslant C\|p\|_1$.

For $\theta = \theta_a$ defined by (1), the corresponding model subspaces K_θ^p are the natural analogs of the Paley–Wiener spaces $\mathcal{P}W_a^p$ of entire functions. The space $\mathcal{P}W_a^p$ consists of all entire functions of exponential type at most a, whose restrictions to $\mathbb{R}$ are in L^p. It follows from our results (and may be proved directly, see Section 7) that every entire function $f \in \mathcal{P}W^1_{2a}$ of exponential type at most $2a$ and summable on the real line $\mathbb{R}$ admits a representation $f = \sum_{k=1}^4 g_k h_k$ with $f_k, g_k \in \mathcal{P}W_a^2$, $\sum_{k=1}^4 \|g_k\|_2 \cdot \|h_k\|_2 \leqslant C\|f\|_1$.

2. Main results

Our first theorem answers Sarason's question about representability of bounded truncated Toeplitz operators via Carleson measures for K_θ.

Theorem 2.1. 1) *Any nonnegative bounded truncated Toeplitz operator on K_θ admits a quasisymbol which is a nonnegative measure from $\mathcal{C}_2^+(\theta)$.*

2) *Any bounded truncated Toeplitz operator on K_θ admits a quasisymbol from $\mathcal{C}_2(\theta)$.*

In general, in assertion 1) of the theorem, μ cannot be chosen absolutely continuous, i.e., bounded nonnegative truncated Toeplitz operators may have no nonnegative symbols. Let δ be the Dirac measure at a point of $\mathbb{T}$, for which the reproducing kernel belongs to K_θ. Then the operator A_δ cannot be realized by a nonnegative symbol unless the dimension of K_θ is 1. Indeed, if μ is a positive absolutely continuous measure, then the embedding $K_\theta \hookrightarrow L^2(\mu)$ must have trivial kernel, while in this example it is a rank-one operator.

The next theorem characterizes operators from $\mathcal{T}(\theta)$ that have bounded symbols.

Theorem 2.2. *A bounded truncated Toeplitz operator A admits a bounded symbol if and only if $A = A_\mu$ for some $\mu \in \mathcal{C}_1(\theta^2)$.*

In the proofs of these results the key role is played by the following Banach space X defined by

$$X = \Big\{ \sum_k x_k \bar{y}_k : \quad x_k, y_k \in K_\theta, \qquad \sum_k \|x_k\|_2 \cdot \|y_k\|_2 < \infty \Big\}. \tag{6}$$

The norm in X is defined as the infimum of $\sum \|x_k\|_2 \cdot \|y_k\|_2$ over all representations of the element in the form $\sum x_k \bar{y}_k$.

Theorem 2.3. 1) *The space dual to X can be naturally identified with $\mathcal{T}(\theta)$. Namely, continuous linear functionals over X are of the form*

$$\Phi_A(f) = \sum_k (Ax_k, y_k), \qquad f = \sum_k x_k \bar{y}_k \in X, \tag{7}$$

with $A \in \mathcal{T}(\theta)$, and the correspondence between the functionals over X and the space $\mathcal{T}(\theta)$ is one-to-one and isometric.

2) *With respect to the duality* (7) *the space X is dual to the class of all compact truncated Toeplitz operators.*

The next theorem establishes a connection between the factorization problem, Carleson-type embeddings, and the existence of a bounded symbol for every bounded truncated Toeplitz operator on K_θ.

Theorem 2.4. *The following are equivalent:*

1) *any bounded truncated Toeplitz operator on K_θ admits a bounded symbol;*

2) $\mathcal{C}_1(\theta^2) = \mathcal{C}_2(\theta^2)$;

3) *for any* $f \in H^1 \cap \bar{z}\theta^2 H^1_-$ *there exist* $x_k, y_k \in K_\theta$ *with* $\sum_k \|x_k\|_2 \cdot \|y_k\|_2 < \infty$ *such that* $f = \sum_k x_k y_k$.

In the proof it will be shown that condition 2) can be replaced by the stronger condition

2′) $\mathcal{D}_1(\theta^2) = \mathcal{D}_2(\theta^2)$.

Condition 3) also admits formally stronger, but equivalent reformulations. If 3) is fulfilled, then, by the Closed Graph theorem, one can always find x_k, y_k such that $\sum_k \|x_k\|_2 \cdot \|y_k\|_2 \leqslant C\|f\|_1$ for some constant C independent from f. Thus, 3) means that $X = H^1 \cap \bar{z}\theta^2 H^1_-$ and the norm of X is equivalent to L^1 norm. Moreover, it follows from Proposition 4.1 that one can require that the sum contain at most four summands.

If θ is a one-component inner function, then all classes $\mathcal{C}_p(\theta)$ coincide, see [6, Theorem 1.4]. If θ is one-component, then θ^2 is, too, hence $\mathcal{C}_1(\theta^2) = \mathcal{C}_2(\theta^2)$. As an immediate consequence of Theorem 2.4 we obtain the following result conjectured in [9]:

Corollary 2.5. *If θ is a one-component inner function, then the equivalent conditions of Theorem 2.4 are fulfilled.*

We do not know if the converse is true, that is, whether the equality $\mathcal{C}_1(\theta^2) = \mathcal{C}_2(\theta^2)$ implies that θ is one-component. If this is true, it would give us a nice geometrical description of inner functions θ satisfying the equivalent conditions of Theorem 2.4:

Conjecture 2.6. *The equivalent conditions of Theorem 2.4 are fulfilled if and only if θ is one-component.*

Theorem 2.4 also allows to extend considerably the class of counterexamples to the existence of a bounded symbol. Let us recall the definition of the Clark measures σ_α [17]. For each $\alpha \in \mathbb{T}$ there exists a finite (singular) positive measure σ_α on $\mathbb{T}$ such that

$$\operatorname{Re} \frac{\alpha + \theta(z)}{\alpha - \theta(z)} = \int_{\mathbb{T}} \frac{1 - |z|^2}{|1 - \bar{\tau} z|^2} \, d\sigma_\alpha(\tau), \qquad z \in \mathbb{D}. \tag{8}$$

If σ_α is purely atomic, i.e., if $\sigma_\alpha = \sum_n a_n \delta_{t_n}$, then the system $\{k_{t_n}\}$ is an orthogonal basis in K^2_θ; in particular, $k_{t_n} \in K^2_\theta$ and $\|k_{t_n}\|_2^2 = |\theta'(t_n)|/(2\pi)$.

It is shown in [4, Theorem 8] that the condition $\mathcal{C}_1(\theta^2) = \mathcal{C}_2(\theta^2)$ implies that all measures σ_α are discrete.

Corollary 2.7. *If, for some $\alpha \in \mathbb{T}$, the Clark measure σ_α is not discrete, then the conditions of Theorem 2.4 do not hold and, in particular, there exist operators from $\mathcal{T}(\theta)$ that do not admit a bounded symbol.*

3. Embeddings $K_\theta \hookrightarrow L^2(\mu)$: the radial L^2-convergence

In this section we present a more elementary approach to embedding theorems which is different from that of Theorem 1.1. Sometimes it may be more convenient to work in the L^2-convergence setting than with continuous functions from K_θ. Here we impose an extra assumption $\theta(0) = 0$, or, equivalently, $1 \in K_\theta$, to which the general case can easily be reduced (via transform (11) defined below), but we omit the details of the reduction.

We will show that the condition $\mu \in \mathcal{C}_2^+(\theta)$ is equivalent to the existence of an operator $J : K_\theta \to L^2(\mu)$ such that

(i) if $f, zf \in K_\theta$ then $Jzf = zJf$,

(ii) $J1 = 1$.

Moreover, these properties uniquely determine the operator, which turns out to coincide with the embedding operator $K_\theta \hookrightarrow L^2(\mu)$ defined by Theorem 1.1. The proofs are based on the following result of Poltoratski, see also [23]. For $g \in K_\theta$, g_r denotes the function $g_r(z) = g(rz)$.

Proposition 3.1. *(cf.* [29, Theorem 1.1]*) Let $\theta(0) = 0$. If a bounded operator $J : K_\theta \to L^2(\mu)$ satisfies the properties $(i), (ii)$, then for any $g \in K_\theta$ we have $\|g_r\|_{L^2(\mu)} \leqslant 2 \cdot \|Jg\|_{L^2(\mu)}$ and $g_r \to Jg$ in $L^2(\mu)$ as $r \nearrow 1$.*

Proof. Consider the Taylor expansion of $g \in K_\theta$,

$$g(z) = \sum_{k=0}^{\infty} a_k z^k,$$

and introduce the functions $g_n \in K_\theta$,

$$g_n(z) = \sum_{k=0}^{\infty} a_{k+n} z^k.$$

By induction from the relation $Jg_n = a_n + zJg_{n+1}$ we obtain the formula

$$Jg = \sum_{k=0}^{n-1} a_k z^k + z^n J g_n.$$

We have $\|g_n\|_2 \leqslant \|g\|_2$ and $\|g_n\|_2 \to 0$ as $n \to +\infty$. Therefore, $\sum_{k=0}^{n-1} a_k z^k \to Jg$ in $L^2(\mu)$. Since g_r are the Abel means of the sequence $\left(\sum_{k=0}^{n-1} a_k z^k\right)_{n\geqslant 1}$, we conclude that $g_r \to Jg$ as well. $\square$

Since for a continuous function $g \in K_\theta$, Jg coincides with g μ-almost everywhere, J is the same operator as the embedding from Theorem 1.1.

By Proposition 3.1 the function $z^{-1}\theta(z)$ (or $\frac{\theta(z)-\theta(0)}{z}$ in the general case, if $\theta(0) \neq 0$) has the boundary function defined by the limit in $L^2(\mu)$ of θ_r, where $\theta_r(z) = \theta(rz)$. This allows us to define the boundary values of θ μ-almost everywhere.

Proposition 3.2. *If $\mu \in \mathcal{C}_2(\theta)$, then $|\theta| = 1$ $|\mu|$-almost everywhere.*

This fact is mentioned in [4] and its proof there seems to use the techniques of convergence μ-almost everywhere. A more elementary proof is given below for the reader's convenience.

Proof. We may assume that $\mu \in \mathcal{C}_2^+(\theta)$. It is easy to check the relation

$$M_z J - J A_z = (\cdot, \bar{z}\theta)\theta,$$

where M_z is the operator of multiplication by z on $L^2(\mu)$, J is the embedding $K_\theta \hookrightarrow L^2(\mu)$, A_z is the truncated Toeplitz operator with symbol z, i.e., the model contraction S_θ. Indeed, on vectors orthogonal to $\bar{z}\theta$ both sides equal 0, and for $\bar{z}\theta$ the formula can be verified by a simple straightforward calculation. Similarly, $M_{\bar{z}} J - J A_{\bar{z}} = (\cdot, 1)\bar{z}$, hence

$$J^* M_z - A_z J^* = (\cdot, \bar{z})1.$$

We obtain

$$\begin{aligned} JJ^* M_z - M_z JJ^* =& J(J^* M_z - A_z J^*) - (M_z J - J A_z)J^* \\ =& (\cdot, \bar{z})J1 - (\cdot, J\bar{z}\theta)\theta = (\cdot, \bar{z})1 - (\cdot, \bar{z}\theta)\theta. \end{aligned}$$

Theorem 6.1 of [24] says that if $K = \sum(\cdot, \bar{u}_k)v_k$ is a finite rank (or even trace class) operator on $L^2(\mu)$, where μ is a singular measure on $\mathbb{T}$, and if $K = XM_z - M_z X$ for some bounded linear operator X on $L^2(\mu)$, then $\sum u_k v_k = 0$ μ-almost everywhere. By this theorem $z - z|\theta|^2 = 0$, hence $|\theta| = 1$ μ-almost everywhere, as required. $\square$

4. The space X

As above, the space X is defined by formula (6),

$$X = \left\{ \sum x_k \bar{y}_k : \quad x_k, y_k \in K_\theta, \quad \sum \|x_k\|_2 \cdot \|y_k\|_2 < \infty \right\}.$$

We also consider the analytic analog X_a of the space X,

$$X_a = \left\{ \sum x_k y_k : \quad x_k, y_k \in K_\theta, \quad \sum \|x_k\|_2 \cdot \|y_k\|_2 < \infty \right\}. \tag{9}$$

By (4),

$$X \subset \bar{\theta} z H^1 \cap \theta \overline{z H^1}$$

and

$$X_a = \{\bar{z}\theta f : f \in X\} \subset H^1 \cap \bar{z}\theta^2 H^1_- \subset K^1_{\theta^2}.$$

The norms in these spaces are defined as infimum of $\sum \|x_k\|_2 \cdot \|y_k\|_2$ over all possible representations, thus X, X_a are Banach spaces.

Proposition 4.1. 1) *Any nonnegative element of X can be written as $|g|^2$, $g \in K_\theta$.*

2) *Any element of X can be represented as a linear combination of four nonnegative elements of X.*

3) *Every element of X, X_a admits a representation as a sum containing only four summands in the definition of these spaces, and the norm of each*

summand in the space does not exceed the norm of the initial element of the space.

Proof. 1) Let $f = \sum x_k \bar{y}_k \in X$, $f \geqslant 0$. Since $\bar{z}\theta\bar{y}_k \in K_\theta$, we have $\bar{z}\theta f \in H^1$. Then, by Dyakonov's result [20], $f = |g|^2$ for some $g \in K_\theta$ (proof: take the outer function with modulus $f^{1/2}$ on $\mathbb{T}$ as g; then $\bar{z}\theta\bar{g} \in H^2$ and hence $g \in K_\theta$).

2) Since X is symmetric with respect to complex conjugation, it suffices to show that real functions from X may be represented as a difference of two nonnegative functions from X. The real part of a function from X of the form $\sum x_k \bar{y}_k$ with $x_k, y_k \in K_\theta$, $\sum \|x_k\|_2 \cdot \|y_k\|_2 < \infty$, is

$$\frac{1}{2}\sum (x_k\bar{y}_k + \bar{x}_k y_k) = \sum \left|\frac{x_k + y_k}{2}\right|^2 - \sum \left|\frac{x_k - y_k}{2}\right|^2,$$

which is the desired representation. We may suppose that $\|x_k\| = \|y_k\|$ for every k, then each of the norms $\left\|\sum \left|\frac{x_k+y_k}{2}\right|^2\right\|_X$, $\left\|\sum \left|\frac{x_k-y_k}{2}\right|^2\right\|_X$ obviously does not exceed $\sum \|x_k\|_2 \cdot \|y_k\|_2$.

3) For the space X this directly follows from 1) and 2), for X_a it remains to use the relation $X_a = \bar{z}\theta X$, which is a consequence of (4). $\square$

Given a function f in the unit disk, define functions f_r, $0 < r < 1$, by $f_r(z) = f(rz)$. We may think of functions $f \in X_a$ as analytic functions in $\mathbb{D}$. For $f \in X_a$ write $f = \sum x_k y_k$ with $x_k, y_k \in K_\theta$ and $\sum \|x_k\|_2 \cdot \|y_k\|_2 < \infty$. We have $f_r = \sum (x_k)_r (y_k)_r$. Now it follows from Proposition 3.1 that, for any $\mu \in \mathcal{C}_2^+(\theta)$, the embedding of the space X_a into $L^1(\mu)$ is a well-defined bounded map realized by the limit of f_r in $L^1(\mu)$ as $r \nearrow 1$.

We will need the following important lemma.

Lemma 4.2. *Let $\mu \in \mathcal{C}_2(\theta)$ and let $x_k, y_k \in K_\theta$, $\sum \|x_k\|_2 \cdot \|y_k\|_2 < \infty$. If $\sum x_k \bar{y}_k = 0$ in the space X, then also $\sum x_k \bar{y}_k = 0$ $|\mu|$-almost everywhere.*

In other words, the embedding $X \hookrightarrow L^1(|\mu|)$ is well defined.

Proof. There is no loss of generality if we assume that $\mu \in \mathcal{C}_2^+(\theta)$. As above, let J stand for the embedding $K_\theta \hookrightarrow L^2(\mu)$. If $g \in K_\theta$, then $\tilde{g} \in K_\theta$, where $\tilde{g} = \bar{z}\theta\bar{g}$. By Proposition 3.1 the functions $\tilde{g}_r$ have a limit as $r \nearrow 1$, and we want to show that

$$\lim_{r \to 1-} \tilde{g}_r = \bar{z}\theta\bar{g} \qquad \text{in } L^2(\mu). \tag{10}$$

It suffices to check this relation on a dense set. It is easily seen that for reproducing kernels $\frac{1-\overline{\theta(\lambda)}\theta}{1-\bar{\lambda}z}$ this property is equivalent to the fact that $|\theta_r|^2 \to 1$ proved in Proposition 3.2.

Take $x_k, y_k \in K_\theta$ such that $\sum \|x_k\|_2 \cdot \|y_k\|_2 < \infty$ and $\sum x_k \bar{y}_k = 0$. Consider the functions $\tilde{y}_k \in K_\theta$, $\tilde{y}_k = \bar{z}\theta\bar{y}_k$. By (10) we have $(\tilde{y}_k)_r \to \bar{z}\theta\bar{y}_k$

in $L^2(\mu)$. The formula $\sum x_k \tilde{y}_k$ determines the zero element of X_a, hence $\sum (x_k)_r (\tilde{y}_k)_r = (\sum x_k \tilde{y}_k)_r = 0$. We obtain

$$\sum x_k \cdot \bar{z}\theta \bar{y}_k = \lim_{r\to 1-} \sum (x_k)_r (\tilde{y}_k)_r = 0$$

in the norm of the space $L^1(\mu)$. Since $\theta \neq 0$ μ-a.e. (e.g., by Proposition 3.2), we conclude that $\sum x_k \bar{y}_k = 0$ μ-almost everywhere. □

5. Proofs of Theorems 2.1–2.3

Proof of Theorem 2.1. 1) Let A be a nonnegative bounded truncated Toeplitz operator with symbol φ. Denote by X_c the set of all continuous functions from X and define the functional l on X_c by $l : f \mapsto \int \varphi f\, dm$. If $f \in X_c$, $f \geqslant 0$, then $lf \geqslant 0$. Indeed, by item 1) of Proposition 4.1, there exists a function $g \in K_\theta$ such that $|g|^2 = f$ (g turns out to be bounded), and hence

$$lf = \int \varphi f\, dm = \int \varphi g \bar{g}\, dm = (Ag, g) \geqslant 0.$$

Assume first that $\theta(0) = 0$, then $1 \in X$. Extend the functional l to the space $C(\mathbb{T})$ of all continuous functions on $\mathbb{T}$ by the Hahn–Banach theorem so that the norm of the extended functional equal the norm of l. Since $1 \in X$, it will be nonnegative automatically, hence $lf = \int f\, d\mu$, $f \in X_c$, for some nonnegative Borel measure μ on $\mathbb{T}$. The map $K_\theta \to L^2(\mu)$, which takes continuous functions to their traces on the support of μ, is bounded. Indeed, if $g \in K_\theta$ is continuous, then

$$\int |g|^2 d\mu = (Ag, g) \leqslant \|A\| \cdot \|g\|_2^2.$$

This proves that $\mu \in \mathcal{C}_2^+(\theta)$. By linearity and continuity the relation $\int |g|^2 d\mu = (Ag, g)$, $g \in K_\theta$, implies $\int x\bar{y}\, d\mu = (Ax, y)$ for all $x, y \in K_\theta$, hence $A = A_\mu$.

If $w = \theta(0) \neq 0$, consider so-called Crofoot's transform

$$U : f \mapsto \sqrt{1 - |w|^2}\, \frac{f}{1 - \overline{w}\theta} \tag{11}$$

which is a unitary map of K_θ onto K_Θ, where $\Theta = \frac{\theta - w}{1 - \overline{w}\theta}$ is the Frostman shift of θ. Take a bounded truncated Toeplitz operator $A \geq 0$ acting on the space K_θ. By [31, Theorem 13.2] the operator $UAU^* \geq 0$ is a bounded truncated Toeplitz operator on K_Θ. Note that $\Theta(0) = 0$. Let $\mu \in \mathcal{C}_2^+(\Theta)$ be a quasisymbol of UAU^*. Then the measure $\nu = \frac{1 - |w|^2}{|1 - \overline{w}\theta|^2}\mu$ is a quasisymbol of A. Indeed, $\nu \in \mathcal{C}_2^+(\theta)$, and from (11) it follows that for any $f, g \in K_\theta$, we have

$$(Af, g) = (UAU^*Uf, Ug) = \int Uf \cdot \overline{Ug}\, d\mu = \int f\bar{g}\, d\nu.$$

Thus, each nonnegative bounded truncated Toeplitz operator admits a quasisymbol from $\mathcal{C}_2^+(\theta)$.

2) Let A be a bounded truncated Toeplitz operator. It may be represented in the form $A = A_1 - A_2 + iA_3 - iA_4$, where all A_i, $i = 1, 2, 3, 4$, are nonnegative truncated Toeplitz operators. Indeed, A^* is a truncated Toeplitz operators as well, which allows us to consider only selfadjoint operators. The identity operator I is trivially a truncated Toeplitz operator (with symbol 1), and A is the difference of two nonnegative operators $\|A\| \cdot I$ and $\|A\| \cdot I - A$. For each A_i construct μ_i as above. It remains to take $\mu = \mu_1 - \mu_2 + i\mu_3 - i\mu_4$. $\square$

Proof of Theorem 2.2. If A has a bounded symbol φ, then $A = A_\mu$ with $d\mu = \varphi dm$, and $\mu \in \mathcal{C}_1(\theta^2)$.

Now let $\mu \in \mathcal{C}_1(\theta^2)$. We need to prove that A_μ coincides with a truncated Toeplitz operator with a bounded symbol. Define the functional $l : f \mapsto \int f\, d\mu$ on functions from X which are finite sums of the functions of the form $x_k \bar{y}_k$ with $x_k, y_k \in K_\theta$. For $f = \sum x_k \bar{y}_k$ we have

$$\left| \int f\, d\mu \right| \leqslant \int |f| d|\mu| \leqslant C\|f\|_1,$$

since $\theta \bar{z} f \in K^1_{\theta^2}$ and $\mu \in \mathcal{C}_1(\theta^2)$. Hence, the functional l can be continuously extended to L^1, and so there exists a function $\varphi \in L^\infty$ such that $l(f) = \int \varphi f\, dm$, $f \in X$. Hence, for any $x, y \in K^2_\theta$, we have

$$\int x\bar{y}\, d\mu = l(x\bar{y}) = \int \varphi\, x\bar{y}\, dm = (A_\varphi x, y),$$

and thus $A_\mu = A_\varphi$. $\square$

Proof of Theorem 2.3. 1) First, we verify that the functional (7) is well defined for any operator $A \in \mathcal{T}(\theta)$. We need to prove that $\sum (Ax_k, y_k) = 0$ if $x_k, y_k \in K_\theta$, $\sum \|x_k\|_2 \cdot \|y_k\|_2 < \infty$, and $\sum x_k \bar{y}_k = 0$ almost everywhere with respect to the Lebesgue measure. To this end, apply Theorem 2.1 and find a measure $\mu \in \mathcal{C}_2(\theta)$ such that $(Ax, y) = \int x\bar{y}\, d\mu$ for all $x, y \in K_\theta$. Lemma 4.2 holds for all complex measures from $\mathcal{C}_2(\theta)$; we conclude that $\sum x_k \bar{y}_k = 0$ μ-almost everywhere. By the definition of A_μ we have

$$\sum (Ax_k, y_k) = \int \left(\sum x_k \bar{y}_k \right) d\mu.$$

Thus $\sum (Ax_k, y_k) = 0$, and the functional is defined correctly.

Now prove the equality $\|\Phi_A\| = \|A\|$. Indeed, for any function $\sum x_k \bar{y}_k \in X$ we have

$$\left| \Phi_A \left(\sum x_k \bar{y}_k \right) \right| = \left| \sum (Ax_k, y_k) \right| \leqslant \|A\| \sum \|x_k\|_2 \cdot \|y_k\|_2.$$

Hence $\|\Phi_A\| \leqslant \|A\|$. On the other hand, for any unit norm vectors $x, y \in K^2_\theta$ we have $\|x\bar{y}\|_X \leqslant 1$ and

$$\|A\| = \sup_{\|x\|_2, \|y\|_2 \leqslant 1} |(Ax, y)| = \sup_{\|x\|_2, \|y\|_2 \leqslant 1} |\Phi_A(x\bar{y})| \leqslant \|\Phi_A\|.$$

This proves the inverse inequality.

It remains to show that any linear continuous functional Φ on X may be represented in the form $\Phi = \Phi_A$ for some (unique) truncated Toeplitz operator A. Take a continuous functional Φ on X and define the operator A_Φ by its bilinear form: $(A_\Phi x, y) \stackrel{\text{def}}{=} \Phi(x\bar{y})$. If $f, zf \in K_\theta$, we have

$$(A_\Phi f, f) = \Phi(|f|^2) = \Phi(|zf|^2) = (A_\Phi zf, zf)$$

Now, applying Theorem 1.2, we obtain $A \in \mathcal{T}(\theta)$. The uniqueness of A is a consequence of the relation $\|A_\Phi\| = \|\Phi\|$.

2) Consider the duality $\langle f, A\rangle = \sum (Ax_k, y_k)$ where $A \in \mathcal{T}(\theta)$, $f = \sum x_k \bar{y}_k \in X$ (by Lemma 4.2 $\langle f, A\rangle$ does not depend on the choice of factorization). We need to prove that every continuous functional Φ on the space $\mathcal{T}_0(\theta)$ of all compact truncated Toeplitz operators is realized by an element of X. Extend Φ by Hahn–Banach theorem to the space of all compact operators in K_θ. The trace class is the dual space to the class of all compact operators, hence the functional may be written in the form $\Phi(A) = \sum (Ax_k, y_k)$, $A \in \mathcal{T}_0(\theta)$, for some $x_k, y_k \in K_\theta$ with $\|\Phi\| = \sum \|x_k\|_2 \cdot \|y_k\|_2$. Then $f = \sum x_k \bar{y}_k \in X$ and $\Phi(A) = \langle f, A\rangle$. Repeating the arguments from 1) we conclude that $\|\Phi\| \leqslant \|f\|_X$. On the other hand, $\|f\|_X \leqslant \sum \|x_k\|_2 \cdot \|y_k\|_2 = \|\Phi\|$. ☐

Now we prove two corollaries which give us additional information on the structure of the space of truncated Toeplitz operators.

Corollary 5.1. *The closed (in the norm) linear span of rank-one truncated Toeplitz operators coincides with the set of all compact truncated Toeplitz operators.*

Proof. For $\lambda \in \mathbb{D}$, denote by $k_\lambda, \tilde{k}_\lambda$ the functions from K_θ,

$$k_\lambda(z) = \frac{1 - \overline{\theta(\lambda)}\theta(z)}{1 - \bar{\lambda} z}, \qquad \tilde{k}_\lambda(z) = \frac{\theta(z) - \theta(\lambda)}{z - \lambda}$$

(recall that k_λ is the reproducing kernel for the space K_θ). If $x, y \in K_\theta$, then $(x, k_\lambda) = x(\lambda)$, $(\tilde{y}, \tilde{k}_\lambda) = \overline{y(\lambda)}$. It is shown in [31] that the operators $T_\lambda = (\cdot, k_\lambda)\tilde{k}_\lambda$ are rank-one truncated Toeplitz operators. Take $f \in X$ as an element of the dual space to the class of all compact truncated Toeplitz operators. Set $g = \bar{z}\theta f \in X_a$, let $g = \sum x_k y_k$ with $x_k, y_k \in K_\theta$. The following formula illustrates the duality on rank-one truncated Toeplitz operators:

$$\begin{aligned}\langle f, T_\lambda\rangle &= \Big\langle \sum x_k \cdot \overline{\tilde{y}_k}, T_\lambda \Big\rangle = \sum (T_\lambda x_k, \tilde{y}_k) \\ &= \sum x_k(\lambda) \cdot (\tilde{k}_\lambda, \tilde{y}_k) = \sum x_k(\lambda) \cdot y_k(\lambda) = g(\lambda).\end{aligned}$$

Suppose that f annihilates all operators of the form T_λ with $|\lambda| < 1$. Then $g \equiv 0$ in $\mathbb{D}$, hence f is the zero element of X. ☐

The space of all bounded linear operators on a Hilbert space is dual to the space of trace class operators. This duality generates the ultraweak topology on the former space. Formally, the ultraweak topology is stronger than the weak operator topology, but on the subspace of all truncated Toeplitz operators they coincide.

Corollary 5.2. *The weak operator topology on $\mathcal{T}$ coincides with the ultraweak topology.*

Proof. Any ultraweakly continuous functional Φ on $\mathcal{T}$ is generated by some trace class operator $\sum_k(\cdot, y_k)x_k$, where $x_k, y_k \in K_\theta$, $\sum_k \|x_k\|_2 \cdot \|y_k\|_2 < \infty$, and is of the form

$$\Phi(A) = \sum_k (Ax_k, y_k).$$

The function $f = \sum_k x_k \bar{y}_k$ belongs to the space X. It follows from Proposition 4.1 that there exist $f_1, g_1 \dots f_4, g_4 \in K_\theta$ such that $f = \sum_{k=1}^4 f_k \bar{g}_k$. Therefore, by the duality from Theorem 2.3,

$$\Phi(A) = \langle f, A\rangle = \Big\langle \sum_{k=1}^4 f_k \bar{g}_k, A\Big\rangle = (Af_1, g_1) + \ldots + (Af_4, g_4).$$

Now the statement of the corollary is obvious. □

6. Proof of Theorem 2.4

Throughout this section we will assume, for simplicity, that $\theta(0) = 0$. The general case follows immediately by means of the transform (11) (note that in this case $\mathcal{C}_p(\Theta) = \mathcal{C}_p(\theta)$ for any p, see, e.g., [5, Theorem 1.1]).

For the proof of Theorem 2.4 we need the following obvious lemma (see [5]) based on the relations $K_{\theta^2} = K_\theta \oplus \theta K_\theta$ and $K_\theta \cdot K_\theta \subset K^1_{\theta^2}$.

Lemma 6.1. *For any inner function θ we have $\mathcal{C}_2(\theta) = \mathcal{C}_2(\theta^2)$ and $\mathcal{C}_1(\theta^2) \subset \mathcal{C}_2(\theta^2)$. If $\theta(0) = 0$, we also have $\mathcal{C}_p(\theta^2) = \mathcal{C}_p(\theta^2/z)$ for any p. The same equalities or inclusions hold for the classes $\mathcal{D}_p(\theta)$.*

Proof of Theorem 2.4. 3) $\Rightarrow$ 2). We will establish condition 2$'$), which is formally stronger than 2). By Lemma 6.1 it suffices to prove the inclusion $\mathcal{D}_2(\theta) \subset \mathcal{D}_1(\theta^2/z)$. Take a complex measure $\mu \in \mathcal{D}_2(\theta)$. We must check that the embedding $K^1_{\theta^2/z} \hookrightarrow L^1(|\mu|)$ is a bounded operator. By condition 3), there is a positive constant c_1 such that any function $f \in K^1_{\theta^2/z}$ can be represented in the form $f = \sum_{k=1}^\infty f_k g_k$, where the functions f_k, g_k are in K_θ and $\sum_{k=1}^\infty \|f_k\|_2 \cdot \|g_k\|_2 \leqslant c_1 \|f\|_1$. Since $\mu \in \mathcal{D}_2(\theta)$, we have $\sum_{k=1}^\infty \|f_k\|_{L^2(|\mu|)} \cdot \|g_k\|_{L^2(|\mu|)} < \infty$, so the series converges in $L^1(|\mu|)$ and $f \in L^1(|\mu|)$. Moreover,

$$\int |f|\, d|\mu| = \int \left|\sum_{k=1}^\infty f_k g_k\right| d|\mu| \leqslant \sum_{k=1}^\infty \|f_k\|_{L^2(|\mu|)} \cdot \|g_k\|_{L^2(|\mu|)}$$

$$\leqslant c_2 \sum_{k=1}^\infty \|f_k\|_2 \cdot \|g_k\|_2 \leqslant c_1 c_2 \|f\|_1,$$

and, thus, $\mu \in \mathcal{D}_1(\theta^2/z)$. Therefore, we have $\mathcal{D}_2(\theta) \subset \mathcal{D}_1(\theta^2/z)$ which implies $\mathcal{D}_2(\theta^2) = \mathcal{D}_1(\theta^2)$ (and, in particular, $\mathcal{C}_2(\theta^2) = \mathcal{C}_1(\theta^2)$).

The implication 2) $\Rightarrow$ 1) follows directly from Theorems 2.1 and 2.2 (even with the weaker assumption $\mathcal{C}_1(\theta^2) = \mathcal{C}_2(\theta^2)$).

1) $\Rightarrow$ 3). Condition 3) can be written in the form $X_a = K^1_{\theta^2/z}$ or, equivalently, as $X = z\bar\theta K^1_{\theta^2/z}$ (see Section 4). In the general case we have that X is a dense subset of $z\bar\theta K^1_{\theta^2/z}$. By the Closed Graph theorem, $X = z\bar\theta K^1_{\theta^2/z}$ if and only if the norms in the spaces X and $K^1_{\theta^2/z}$ are equivalent. Take an arbitrary function $h = \sum x_k \bar y_k \in X$. Clearly, we have $\|h\|_1 \leqslant \|h\|_X$. On the other hand, it follows from Theorem 2.3 that

$$\|h\|_X = \sup\left\{\left|\sum (Ax_k, y_k)\right| : A \in \mathcal{T}(\theta), \|A\| \leqslant 1\right\}. \tag{12}$$

The Closed Graph theorem and condition 1) guarantee the existence of a bounded symbol $f_A \in L^\infty$ for any operator $A \in \mathcal{T}(\theta)$ with $\|f_A\|_\infty \leqslant c\|A\|$. Therefore, the supremum in (12) does not exceed

$$\sup\left\{\left|\sum (fx_k, y_k)\right| : f \in L^\infty, \|f\|_\infty \leqslant c\right\} = c \sup_{\|f\|_\infty \leqslant 1} \left|\int f \cdot \sum x_k \bar y_k \, dm\right|.$$

Thus, $\|h\|_X \leqslant c\|h\|_1$, $h \in X$, which proves the theorem. $\square$

7. One-component inner functions

As we have noted in Section 2, one-component inner functions satisfy the condition 2) of Theorem 2.4: $\mathcal{C}_2(\theta) = \mathcal{C}_1(\theta^2)$. It is also possible to show that one-component inner functions satisfy the factorization condition 1) in Theorem 2.4. We will start with a particular case of the Paley–Wiener spaces.

Example 7.1. Let $\Theta_a(z) = \exp(iaz)$, $a > 0$, be an inner function in the upper half-plane. Then for the corresponding model subspace we have $K^p_{\Theta_a} = \mathcal{P}W^p_a \cap H^p$. Note, that the model subspaces in the half-plane case are defined as $K^p_\Theta = H^p \cap \Theta\overline{H^p}$, the involution is given by $f \mapsto \Theta\bar f$, and so $fg \in K^1_{\Theta^2}$ for any $f, g \in K^2_\Theta$. Thus, in view of Proposition 4.1, the factorization for the corresponding space X is equivalent to the following property: for any $f \in \mathcal{P}W^1_{2a}$ which takes real values there exist $g \in \mathcal{P}W^1_{2a}$ such that $|f| \leqslant g$. This can be easily achieved. Let $a = \pi/2$. Put

$$g(z) = \sum_{n\in\mathbb{Z}} c_n \frac{\sin^2 \frac{\pi}{2}(t-n)}{(t-n)^2},$$

where $c_n = \max_{[n,n+1]} |f|$. By the Plancherel–Pólya inequality (see, e.g., [25, Lecture 20]), $\sum_n c_n \leqslant C\|f\|_1$, and so $g \in \mathcal{P}W^1_\pi$. Also, if $t \in [n, n+1]$, then

$$|f(t)| \leqslant c_n \leqslant c_n \frac{\sin^2 \frac{\pi}{2}(t-n)}{(t-n)^2} \leqslant g(t).$$

$\square$

An analogous argument works for general one-component inner function. Let θ be an inner function in the unit disk. In view of Proposition 4.1, the property 1) in Theorem 2.4 will be obtained as soon as we prove the following theorem:

Theorem 7.2. *For any real-valued element f of X there exist $t_n \in \mathbb{T}$ and $c_n > 0$ such that*

$$g = \sum c_n |k_{t_n}|^2 \in X,$$

$\|g\|_1 \leqslant C\|f\|_1$, *and* $|f| \leqslant g$ *on* $\mathbb{R}$.

First we collect some known properties of one-component inner functions.

(i) Let $\rho(\theta)$ be the so-called *spectrum* of the inner function θ, that is, the set of all $\zeta \in \overline{\mathbb{D}}$ such that $\liminf\limits_{z\to\zeta,\, z\in\mathbb{D}} |\theta(z)| = 0$. Then θ, as well as any element of K^p_θ, has an analytic extension across any subarc of the set $\mathbb{T} \setminus \sigma(\theta)$.

It is shown in [6] that for a one-component inner function $\sigma_\alpha(\rho(\theta)) = 0$ for any Clark measure σ_α defined by (8). Thus, all Clark measures are purely atomic and supported on the set $\mathbb{T} \setminus \rho(\theta)$.

(ii) On each arc of the set $\mathbb{T} \setminus \rho(\theta)$, there exists a smooth increasing branch of the argument of θ (denote it by ψ) and the change of the argument between two neighboring points t_n and t_{n+1} from the support of one Clark measure is exactly 2π.

(iii) By (t_n, t_{n+1}) we denote the closed arc with endpoints t_n, t_{n+1}, which contains no other points from the Clark measure support. There exists a constant $A = A(\theta)$ such that for any two points t_n and t_{n+1} satisfying $|\psi(t_{n+1}) - \psi(t_n)| = 2\pi$ and for any s, t from the arc (t_n, t_{n+1}),

$$A^{-1} \leqslant \frac{|\theta'(s)|}{|\theta'(t)|} \leqslant A, \tag{13}$$

that is, $|\theta'|$ is almost constant, when the change of the argument is small. This follows from the results of [6], a detailed proof may be found in [10, Lemma 5.1].

(iv) If θ is one-component, then $\mathcal{C}_1(\theta) = \mathcal{C}_2(\theta)$. The same holds for the function θ^2 which is also one-component. By Lemma 6.1, $\mathcal{C}_1(\theta^2) = \mathcal{C}_2(\theta^2) = \mathcal{C}_2(\theta)$, and so there is a constant B such that for any measure in $\mathcal{C}_2^+(\theta)$,

$$\sup_{f\in K^1_{\theta^2}} \frac{\|f\|_{L^1(\mu)}}{\|f\|_1} \leqslant B \sup_{f\in K^2_\theta} \frac{\|f\|_{L^2(\mu)}}{\|f\|_2}. \tag{14}$$

(v) Let $\{t_n\}$ be the support of some Clark measure for θ and let $s_n \in (t_n, t_{n+1})$. There exists a constant $C = C(\theta)$ which does not depend on $\{t_n\}$ and $\{s_n\}$ such that for any $f \in K^2_\theta$,

$$\sum_n \frac{|f(s_n)|^2}{|\theta'(s_n)|} \leqslant C\|f\|_2^2.$$

This follows from the stability result due to Cohn [19, Theorem 3].

So (v) means that for the measures of the form $\sum_n |\theta'(s_n)|^{-1}\delta_{s_n}$ the supremum in the right-hand side of (14) is uniformly bounded. From this, (iii) and (iv) we have the following Plancherel–Polya type inequality:

Corollary 7.3. *Let $\{t_n\}$ be the support of some Clark measure for θ and let $s_n, u_n \in (t_n, t_{n+1})$. There exists a constant $C = C(\theta)$ which does not depend on $\{t_n\}$, $\{s_n\}$, $\{u_n\}$, such that for any $f \in X \subset K^1_{\theta^2}$,*

$$\sum_n \frac{|f(s_n)|}{|\theta'(u_n)|} \leqslant C\|f\|_1. \tag{15}$$

Proof of Theorem 7.2. Let $f \in X$. Take two Clark bases corresponding to 1 and -1, and let $\{t_n\}$ be the union of their supports. If t_n and t_{n+1} are two neighbor points from our set, then

$$\int_{(t_n, t_{n+1})} |\theta'(t)|dm(t) = \pi.$$

If we write $t_n = e^{ix_n}$ and take the branch of the argument ψ so that $\theta(e^{ix}) = e^{2i\psi(x)}$, then $|\psi(x_{n+1}) - \psi(x_n)| = \pi/2$.

Let $c_n = \sup_{t\in(t_n,t_{n+1})} |f(t)|$ and put, for some constant D whose value will be specified later,

$$g(z) = D\sum c_n \frac{|k_{t_n}(z)|^2}{|\theta'(t_n)|^2}.$$

Then $g \in L^1$ since the series converges in L^1-norm. Indeed, $c_n = |f(s_n)|$ for some $s_n \in (t_n, t_{n+1})$ and

$$\sum |c_n| \frac{\|k^2_{t_n}\|_1}{|\theta'(t_n)|^2} = \sum \frac{|f(s_n)|}{|\theta'(t_n)|} \leqslant C\|f\|_1$$

by Corollary 7.3. Also $g \in X$ and $g \geqslant 0$.

It remains to show that $g \geqslant |f|$. Let $t = e^{ix} \in (t_n, t_{n+1})$. We have

$$|k_{t_n}(t)| = \left|\frac{\theta(t) - \theta(t_n)}{t - t_n}\right| = \left|2\frac{\sin(\psi(x) - \psi(x_n))}{e^{ix} - e^{ix_n}}\right|. \tag{16}$$

Since $|\psi(x) - \psi(x_n)| \leqslant \pi/2$, we have $|\sin(\psi(x) - \psi(x_n))| \geqslant 2|\psi(x) - \psi(x_n)|/\pi$. Of course we have $|e^{ix} - e^{ix_n}| \leqslant |x - x_n|$. Hence, the last quantity in (16) is

$$\geqslant \frac{4}{\pi} \cdot \left|\frac{\psi(x) - \psi(x_n)}{x - x_n}\right| = \frac{4\psi'(y_n)}{\pi}$$

for some $y_n \in [x_n, x]$. If we put $u_n = e^{iy_n}$ we have $\psi'(y_n) = |\theta'(u_n)|/2$. Thus, we have shown that $|k_{t_n}(t)| \geqslant 2|\theta'(u_n)|/\pi$ for some $u_n \in (t_n, t_{n+1})$. Hence, if we take $D > \pi^2 A^2/4$, then

$$g(t) > Dc_n \frac{|k_{t_n}(t)|^2}{|\theta'(t_n)|^2} \geqslant D \frac{4|\theta'(u_n)|^2}{\pi^2|\theta'(t_n)|^2} c_n \geqslant \frac{4D}{\pi^2 A^2} c_n \geqslant c_n \geqslant |f(t)|.$$

The theorem is proved. □

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