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# Optimizing Insurance and Reinsurance in the Dynamic Cramér–Lundberg Model

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**Abstract**—We find optimal (from the insurer's point of view) strategies for insurance and reinsurance in a controllable Cramér–Lundberg risk process that describes the capital dynamics of an insurance company over an extended time interval. As the optimality criterion being minimized, we use the stationary variation coefficient, taking into account additional constraints on residual risks for both insurers and reinsurer. We establish that it is best to use stop-loss reinsurance with an upper limit and insurance which is a combination of a stop-loss strategy and franchise. We derive equations that define optimal strategy parameters.

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## 1. INTRODUCTION

The original uncontrollable Cramér–Lundberg risk process (see, e.g., [1]) that describes the capital dynamics of an insurance company (insurer) looks like

$$X_t = x_0 + c(t) - \sum_{i=1}^{N(t)} Y_i,$$

where $x_0 > 0$ is the initial capital, $\{Y_i\}$ are independent identically distributed insurance premiums with distribution function $F(x)$ and finite second moment $E\,Y^2 < \infty$. The payment claims process, i.e., the number of insured events $\{N(t)\}$, is a non-uniform Poisson process with intensity $\lambda(t)$ such that the integral intensity $\Lambda(t) \overset{def}{=} \int_0^t \lambda(x)dx \to \infty$ for $t \to \infty$. The nonuniformity of $N(t)$ lets us take into account seasonal effects, while the condition on $\lambda(t)$ is not restrictive since it holds, for instance, if $\lambda(t) \geq a$ for some $a > 0$. The amount of premium obtained by the insurer on $[0, t]$ is determined by the mean value principle (see, e.g., [1, 2]), the total premium on $[0, t]$ is $c(t) = (1+\alpha)E\,X_t$, where the total loss $X_t = \sum_{i=1}^{N(t)} Y_i$ and $\alpha > 0$, a given load coefficient, is the markup percentage over the mean risk $E\,X_t$. Since $E\,N(t) = \Lambda(t)$, we get that $c(t) = (1+\alpha)\Lambda(t)E\,Y$.

Optimal control problems for the Cramér–Lundberg risk process have been studied in [3–9]. In [3], the problem of minimizing ruin probability for an insurance company was considered, solved with choosing reinsurance in the stop-loss class, divisions applied to the risk of every individual insurer. The work [4] was devoted to solving the same problem in case when the insurer has proportional reinsurance and risky asset investments. In a similar setting, but without reinsurance control, minimizing ruin probability has been studied in [5] with an asymptotics of the target functional. The risk process arises as a diffuse approximation of the Cramér–Lundberg process and has been studied in [6], where optimal investment and proportional reinsurance strategies have been found both without budget constraints on investment (with short sales) and with these constraints. Problems with a different optimality criterion, namely expected insurer consumption utility, have been solved in [7], where optimal strategies for investment and insurance have been found.

The basic differences of this work from previous studies are the simultaneous choice of insurance and reinsurance strategies by the insurer (we do not consider investment control) and an (upper) constraint with probability one on the risk left for the client after insurance. These constraints appear natural from the insurer's point of view, who would like to avoid "large" losses. It is interesting to note that while the basic instrument for constructing an optimal strategy in these works is the Hamilton–Jacobi–Bellman equation, which very rarely admits an analytic solution and requires numerical methods, in the present work we have managed to reduce the optimal control problem to the static case and get an analytic solution for this optimality criterion. Similar problems have been considered in [8–10], where the insurer is able to optimize insurance and/or reinsurance. But in [10] only the static model has been studied, and in [8], unlike this work, the minimization criterion for the process was chosen as average maximal insurer losses with a constraint on the insurer's risk, discarding the additional constraint with probability one. A significant difference of the model in [9] from the case considered by the authors is that it uses only one risk management instrument, namely insurance, and reinsurance has not been considered.

## 2. MODEL DESCRIPTION

The following is a formal description of the controllable risk process that we consider here. At the payment moment $t = t_i$ ($i \geq 0$, $t_0 = 0$), the insurer makes a decision, that is, chooses division functions for the insurance risk $I_t(\cdot)$ and reinsurance risk $A_t(\cdot)$. Then $I_t(Y_{i+1})$ is the share of the next payment reimbursed to the client, $A_t(I_t(Y_{i+1}))$ is the insurer's risk share after reinsurance. Taking reinsurance into account, the accumulation rate of the premium left at the insurer becomes equal to

$$c_t = \lambda(t)\{(1+\alpha)E\,I_t(Y) - (1+\alpha_1)E\,[I_t(Y) - A_t(I_t(Y))]\}, \tag{1}$$

where $\alpha_1 > \alpha$ is the reinsurer's load coefficient. The controllable risk process is then

$$X_t = x_0 + \int_0^t c_s ds - \sum_{j=1}^{N(t)} A_{t_{j-1}}(I_{t_{j-1}}(Y_j)), \tag{2}$$

where we consider as admissible strategies $I = \{I_t\}$ and $A = \{A_t\}$, nonanticipating controls measurable with respect to a natural filtration $\{\mathcal{F}_t\}$ and satisfying standard inequalities $0 \leq I_t(x) \leq x$ and $0 \leq A_t(x) \leq x$ on $[0, \infty)$ (in other words, the reimbursement is nonnegative and does not exceed the damages).

Let us also assume that for each (fixed) $t$ the following inequality on the client's residual risk must hold: $Y - I_t(Y) \leq q$, where $q > 0$ is a constant specified by the insurer. This means that in any contract admissible for the client the amount of damages left after insurance should never exceed the value $q$. An equivalent constraint on the risk assumed by the insurer can be written as $I_t(x) \geq (x-q)_+$, $x \in [0, \infty)$, where $(y)_+$ denotes $\max\{y, 0\}$. We impose a similar constraint on the reinsurer's risk. Suppose that his requirement is that the maximal reinsurer's reimbursement does not exceed a given sum $Q > 0$ with probability one, $I_t(Y) - A_t(I_t(Y)) \leq Q$. This restriction can be equivalently rewritten as $A_t(x) \geq (x-Q)_+$. In particular, we can formally let $Q = \infty$ and/or $q = \infty$, which would mean that there are no additional constraints on the reinsurer's and/or insurer's strategies.

Suppose that the minimized functional is an upper limit

$$J[I, A] = \varlimsup_{t\to\infty} D\,X_t / E\,X_t \tag{3}$$

or, similar to a well-known term in insurance theory, the "stationary variation coefficient."[1] In actuary mathematics (see, e.g., [2]), the variation coefficient is often used as a criterion for the

[1] Usually this means the ratio of the standard deviation $\sqrt{D\,\xi}$ rather than variance to the expectation $E\,\xi$. In this case, as we will show, the variance and mean $X_t$ have the same growth order, so as the criterion here we have selected a modified variation coefficient (3).

insurer's financial stability: the less it is, the better balanced is the insurance portfolio. The optimal control problem in question looks like

$$\min J[I,A], \tag{4}$$

where the minimum is taken over the set of admissible strategies $\{I, A\}$ defined above.

The claims process $N(t)$ in a controllable risk process (2) is a nonuniform Poisson process and, therefore, $N(t) = N^0(\Lambda(t))$ almost surely (a.s.), where $N^0(t)$ is a standard Poisson process with intensity $\lambda(t) \equiv 1$. Denoting by $X_t^0$ the risk process corresponding to the claims process $N^0(t)$, we get $X_t = X^0_{\Lambda(t)}$ a.s. Then

$$J[I,A] = \overline{\lim_{t\to\infty}} D\,X_t/E\,X_t = \overline{\lim_{t\to\infty}} D\,X^0_{\Lambda(t)}/E\,X^0_{\Lambda(t)}.$$

Taking into account that $\Lambda(t) \to \infty$ as $t \to \infty$, we have

$$J[I,A] = \overline{\lim_{t\to\infty}} D\,X^0_t/E\,X^0_t.$$

Thus, the original problem (4) has been reduced to an equivalent problem with process $X_t^0$, which, by construction, is a uniform controllable Markov process with infinite horizon (see, e.g., [11]). However, the minimized "fractional" objective functional is not a Bellman type functional because the expression under the limit is not an expectation of some function of $X_t^0$. The standard theory does not contain a result that would guarantee that Markov strategies suffice to minimize it. Therefore, in what follows we will a priori consider only Markov strategies $\{I, A\}$ that satisfy constraints introduced above and will ignore other nonanticipating strategies.

We define the function $V(t, x) = \inf_{I,A} J[I, A]$ for the process on the interval $[t, \infty)$ with initial state $X_t^0 = x$, where the infimum is taken over the set of Markov admissible strategies. Due to the specifics of the considered criterion $J[I, A]$, we have $V(t, x) = V(0, x)$ and $V(t, x) = V(t, 0)$. Thus, we can look for the minimum in problem (4) in the class of constant strategies $I_t(\cdot) = I(\cdot)$ and $A_t(\cdot) = A(\cdot)$ that depend neither on the decision making moment $t$ nor on the current state $x$.

In the class of these strategies, the total damage for the insurer (see (2)) is a compound Poisson random variable $\sum_{j=1}^{N^0(t)} A(I(Y_j))$, so

$$E\,X_t^0 = x_0 + t\alpha_1\{E\,A(I(Y)) - \delta E\,I(Y)]\} \text{ and } D\,X_t^0 = tE\,A^2(I(Y)),$$

where the parameter $\delta \overset{def}{=} 1-\alpha/\alpha_1 \in (0,1)$. Then (3) implies that the original problem (4) assumes the form

$$J[I,A] \equiv \frac{E\,A^2(I(Y))}{\alpha_1\{E\,A(I(Y)) - \delta E\,I(Y)\}} \to \min. \tag{5}$$

The condition that the denominator is positive, $E\,A(I(Y)) > \delta E\,I(Y)$, is known in actuary mathematics as safety loading condition and means that the risk process on average increases because $E\,X_t^0 = x_0 + t\alpha_1\{E\,A(I(Y)) - \delta E\,I(Y)]\}$. The set of admissible divisions $\{I, A\}$ here is the set of Borel functions of $I(\cdot)$ and $A(\cdot)$ that satisfy inequalities $(x-q)_+ \le I(x) \le x$, $(x-Q)_+ \le A(x) \le x$, $E\,A(I(Y)) > \delta E\,I(Y)$.

Everywhere below the equality of divisions $(I', A') = (I'', A'')$ is understood as $I'(Y) = I''(Y)$ and $A'(I'(Y)) = A''(I''(Y))$ a.s.

Section 3.1 shows the solution of a version of problem (5) in case when the insurer can only choose reinsurance strategies satisfying a predefined upper bound; in Section 3.2, problem (5) is solved for the case when the insurer chooses insurance strategy $I$ and reinsurance strategy $A$, and a constraint limits the residual insurer's risk from above. In the first case, the optimal strategy turns out to be stop-loss reinsurance with an upper limit, where the reimbursement is bounded by $Q$. In the second problem, stop-loss reinsurance is optimal (or reinsurance for the damage excess) while the optimal insurance strategy is a combination of stop-loss strategy and franchise. The results are illustrated with a numerical example for the case of uniform payment distribution.

## 3. MINIMIZING THE STATIONARY VARIATION COEFFICIENT

We begin our studies of problem (5) with a lemma that proves the existence of optimal strategies.

**Lemma.** *Problem* (5) *is feasible.*

Proof of lemma is given in the Appendix.

### *3.1. Optimal Reinsurance*

Let us consider a special case of the optimal control problem shown above, where the stationary variation coefficient in (5) is minimized only by choosing a reinsurance division function $(x-Q)_+ \le A(x) \le x$, while the function $I(x) \equiv x$, i.e., the damages of each client are insured in full:

$$J[A] \equiv \frac{E\,A^2(Y)}{\alpha_1\{E\,A(Y) - \delta E\,Y\}} \to \min, \quad E\,A(Y) > \delta E\,Y. \tag{6}$$

Below we will need a special kind of reinsurance division functions (see Fig. 1), namely stop loss reinsurance with an upper limit $A_{a,Q}(x) = (x \wedge a) \vee (x - Q)$ (here and in what follows $x \wedge y = \min\{x, y\}$ and $x \vee y = \max\{x, y\}$), which represents a contract with an upper limit on the reinsurer's responsibility, well known in the practice of reinsurance: if the client's damages $Y$ do not exceed the insurer's stop loss point $a$, it is paid by the insurer in full; if $a + Q > Y > a$ then the insurer's reimbursement is $a$ while the remaining $Y - a$ is paid by the reinsurer; if $Y \ge a + Q$ then the reinsurer only reimburses the previously agreed sum of $Q$, and the insurer reimburses the rest.

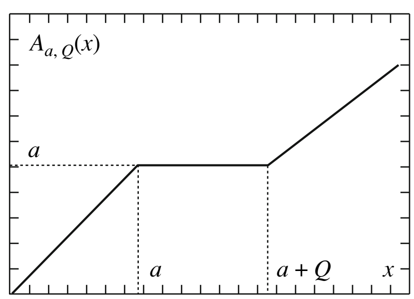


**Fig. 1.** The stop-loss division function for reinsurance with an upper limit.

We denote by $a' \in (0, \infty)$ the root of equation $EA_{a,Q}(Y) = \delta EY$ or $\int_0^a \bar{F}(x)dx + \int_a^\infty \bar{F}(x+Q)dx = \delta EY$, where, recall, $\delta = 1 - \alpha/\alpha_1 \in (0, 1)$ and $\bar{F}(x)$ denotes $1 - F(x)$. In other words, $a'$ is the stop-loss point in stop-loss reinsurance with an upper limit for which the denominator in (6) becomes zero. We let $a_0 = a' \vee 0$, taking into account a possible case when $E\,A_{0,Q}(Y) = E\,(Y - Q)_+ > \delta E\,Y$.

**Theorem 1.** *Problem* (6) *has a unique solution, a stop-loss reinsurance with upper limit* $A^*(x) = (x \wedge a^*) \vee (x - Q)$, *where* $a^*$ *is the unique root on* $(a_0, \infty)$ *for the equation*

$$\int_0^a (a - x)\bar{F}(x)dx + \int_a^\infty (a - x)\bar{F}(x + Q)dx - a\delta E\,Y = 0. \tag{7}$$

**Proof.** By lemma, problem (6) has a solution $A^*$. The set of admissible divisions is convex because if $A_1(x)$ and $A_2(x)$ are admissible divisions, i.e., they satisfy constraints $(x-Q)_+ \le A_i(x) \le x$, $EA_i(Y) > \delta EY$, then, obviously, $\rho A_1(x) + (1-\rho)A_2(x)$ also satisfies these constraints for every $\rho \in [0,1]$. Then the necessary optimality condition $A^*$ looks like

$$\frac{d}{d\rho} J[\rho A^* + (1-\rho)A]\Big|_{\rho=1} \le 0$$

for every (admissible) division function $A$. Substituting into the expression for $J$ (see (6)) and formally differentiating by $\rho$, we get that the left-hand side of this inequality equals

$$\int_0^\infty \eta(x)(A^*(x) - A(x))dF(x).$$

The function $\eta(x) = 2\beta A^*(x) - \gamma$, where $\beta > 0$ and $-\gamma < 0$ are partial derivatives of $J$ with respect to $E\,[A_\rho(Y)]^2$, and $E\,[A_\rho(Y)]$ respectively. The derivative by direction $dJ[\rho A^* + (1-\rho)A]/d\rho|_{\rho=1}$ exists because the expectation $E\,A^*(Y)A(Y) \le E\,Y^2 < \infty$ is finite. Thus, $A^*(x)$ is a solution for the minimization problem for integral

$$\min_A \int_0^\infty \eta(x)A(x)dF(x) \tag{8}$$

on the set of measurable functions $\{A : (x-Q)_+ \le A(x) \le x\}$. Optimality conditions for this kind of problems are given by the generalized Neyman–Pearson lemma, well-known in the theory of moments (see, e.g., [12]): $A^*(x)$ is optimal in (8) if and only if

$$A^*(x) = \begin{cases} (x-Q)_+, & \eta(x) > 0 \\ x, & \eta(x) < 0 \end{cases} \tag{9}$$

up to a set of zero measure $F$. Let us show that $\eta(x)$ first grows to zero, then stays zero on a certain interval, and then increases further. Since $\eta(0) < 0$, as $x$ rises from zero the value $\eta(x)$ increases (here $A^*(x) = x$ due to (9)). After touching the horizontal axis at some point $a$, the function $\eta(x)$ cannot take negative values because otherwise for such $x$ (see (9)) the value $A^*(x) = x$ and we would get a contradiction: $\eta(x) > 0$. $\eta(x)$ also cannot increase from zero since for such $x$ it holds that $A^*(x) = 0$, and, therefore, $\eta(x) < 0$. The value of $\eta(x)$ stays zero up to the point $x = a + Q$ when the lower bound of admissible divisions is achieved. By (9), $A^*(x) = x - Q$ for $x > a + Q$. As a result, $A^*(x)$ looks like $(x \wedge a) \vee (x - Q)$. Subsituting this division function, we get an equation to find the touching point on interval $(a_0, \infty)$: $2\beta a - \gamma = 0$. It is easy to show that $E\,[A^*(Y)]^2 = 2\left\{\int_0^a x\,\bar{F}(x)dx + \int_a^\infty x\bar{F}(x+Q)dx\right\}$ and $E\,A^*(Y) = \int_0^a \bar{F}(x)dx + \int_a^\infty \bar{F}(x+Q)dx$, so we see that the function $2\beta a - \gamma$ coincides, up to a positive factor, with the left-hand side of (7) which we denote by $\psi(a)$. Taking into account the form of $\psi(a)$ and the definition of $a_0$, we have $\psi(a_0) < 0$. Since $\psi'(a) = \int_0^a \bar{F}(x)dx + \int_a^\infty \bar{F}(x+Q)dx - \delta E\,Y > 0$ on $(a_0, \infty)$ and $\psi'(a) \to (1-\delta)E\,Y > 0$ for $a \to \infty$, we get that the touching point in question is the unique root of Eq. (7). □

*Remark 1.* If we denote by $T \le \infty$ the supremum of the support of distribution $F$, i.e., the maximal possible value of the payment $Y$, it is possible that for $T < \infty$ (bounded payment) the level $a^*$ found in Theorem 1 will exceed $T$. In this case $A^*(Y) = Y$ a.s., the insurer takes all risk of the client without any help from the reinsurer. For a distribution $F$ with unbounded support (e.g., for an exponential distribution) $a^* < T = \infty$, and the insurer always uses reinsurance.

### *3.2. Optimal Insurance and Reinsurance*

Let us assume that the choice of reinsurance is free of additional constraints, and admissible insurance divisions are only those functions that satisfy an upper bound on the risk of the client (see Section 2), $I(x) \geq (x-q)_+$. Then problem (5) assumes the form

$$J[I,A] \equiv \frac{E\,A^2(I(Y))}{\alpha_1\{E\,A(I(Y)) - \delta E\,I(Y)\}} \to \min, \qquad x \geq I(x) \geq (x-q)_+, \quad x \geq A(x) \geq 0, \quad E\,A(I(Y)) - \delta E\,I(Y) > 0. \tag{10}$$

Similar to Section 3.1, we introduce a two-level division function, this time for insurance rather than reinsurance $I_{k,q}(x) = (x \wedge k) \vee (x-q)$, together with stop-loss reinsurance $A_a(x) = x \wedge a$. Let $k = a(1-\delta)$ and recall that $\delta = 1 - \alpha/\alpha_1$. We denote by $a^0 \in (0,\infty)$ the root of the equation that results from equating the denominator of (10) to zero: $E\,A_a(I_{k,q}(Y)) = \delta E\,I_{k,q}(Y)$, or

$$z(a) \overset{def}{=} (1-\delta)\left[\int\limits_0^k \bar{F}(x)dx + \int\limits_k^a \bar{F}(x+q)dx\right] - \delta\int\limits_a^\infty \bar{F}(x+q)dx = 0.$$

Thus, $a^0$ is the hold level in "stop-loss" reinsurance which, together with insurance $I_{k,q}$ for $k = a^0(1-\delta)$, turns the denominator of (10) into zero. (Formally speaking, $z(a)$ may have more than one zero since $z(0) < 0$, $\lim\limits_{a\to\infty} z(a) > 0$ and $z'(a) = (1-\delta)^2[\bar{F}(k) - \bar{F}(k+q)] + \bar{F}(a+q) \geq 0$. In this case, we let $a^0$ be the maximal zero of $z(a)$.)

**Theorem 2.** *The optimal reinsurance in* (10) *is a stop-loss reinsurance* $A^*(x) = x \wedge a^*$. *The optimal insurance is a combination of stop-loss insurance and franchise,* $I^*(x) = (x \wedge k^*) \vee (x-q)$. *Here* $k^* = a^*(1-\delta)$, *the parameter* $a^*$ *is the unique root on* $(a^0,\infty)$ *for the equation*

$$(1-\delta)\left[\int\limits_0^k (k-x)\bar{F}(x)dx + \int\limits_k^a (k-x)\bar{F}(x+q)dx\right] - k\delta\int\limits_a^\infty \bar{F}(x+q)dx = 0, \tag{11}$$

*where* $k = a(1-\delta)$.

**Proof.** Let $(I^*, A^*)$ be a solution of (10) that exists by lemma. The function $A^*(x)$ is optimal in problem $\min_A J[I^*, A]$, so, repeating the derivations of Theorem 1, where no additional restriction on $A$ formally means $Q = \infty$, we get $A^*(x) = x \wedge a^*$. The value of $a^*$ is defined as the point where the function

$$\eta^*(x) = 2\beta x - \gamma, \tag{12}$$

touches the horizontal axis, where $\beta > 0$ and $-\gamma < 0$ are partial derivatives $J[I^*, A]$ with respect to $E\,[A(I^*(Y))]^2$ and $E\,[A(I^*(Y))]$ respectively, computed at $A = A^*$.

Let us now consider the problem $\min_I J[I, A^*]$. A sufficient optimality condition is

$$\frac{d}{d\rho}\,J[I_\rho, A^*]\big|_{\rho=1} \leq 0$$

for every admissible $I$, where $I_\rho = \rho I^* + (1-\rho)I$. After differentiating we get that $I^*$ is a solution of problem

$$\min_I \int\limits_0^\infty \theta(x)I(x)dF(x), \quad (x-q)_+ \leq I(x) \leq x,$$

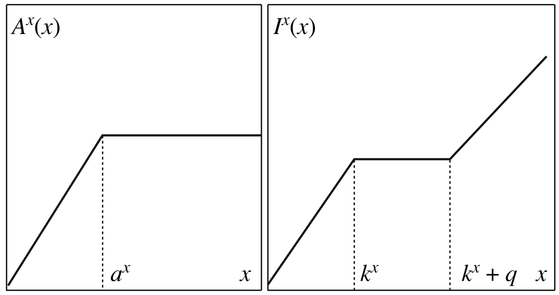


**Fig. 2.** Optimal division functions: reinsurance $A^*(x)$ and insurance $I^*(x)$.

where $\theta(x) = \mathbf{I}\{I^*(x) < a^*\}[2\beta I^*(x) - \gamma] + \delta\gamma$, $\mathbf{I}\{\cdot\}$ denotes the indicator function, $\delta\gamma > 0$ is a partial derivative of $J$ (see (10)) with respect to $E\, I(Y)$ for $I = I^*$. Now the Neyman–Pearson lemma gives necessary and sufficient optimality conditions for this problem:

$$I^*(x) = \begin{cases} (x-q)_+ & \text{for} \quad \theta(x) > 0 \\ x & \text{for} \quad \theta(x) < 0. \end{cases} \tag{13}$$

Since $\theta(0) < 0$, as $x$ grows from zero the function $I^*(x) = x$ due to (13), and the value $\theta(x)$ increases. The indicator $\mathbf{I}\{I^*(x) < a^*\} = \mathbf{I}\{x < a^*\}$ remains equal to one up to the point $k^*$ where the function $\theta(x)$ touches the horizontal axis. Indeed, the function $\theta^*(x) \overset{def}{=} 2\beta x - \gamma + \delta\gamma > \eta^*(x) = 2\beta x - \gamma$ (see (12)), so $k^* < a^*$. As $x$ moves out from $k^*$, the value of $\theta(x)$ remains zero up to the point $x = k^* + q$ where the lower bound of admissible insurance divisions is reached: $\theta(x)$ cannot take negative values since in the opposite case for such $x$ (see (13)) the value $I^*(x) = x$, so $\theta(x) > 0$ and we have reached a contradiction. It is also impossible for $\theta(x)$ to increase from 0 because for these $x$ it holds that $I^*(x) = 0$, which implies $\theta(x) < 0$. Due to (13), $I^*(x) = x - q$ for $x > k^* + q$. As a result, $I^*(x) = (x \wedge k^*) \vee (x - q)$. After substituting this division function and $A^*(x) = x \wedge a^*$ into the expressions for the moments $E\, A^2(I(Y))$, $E\, A(I(Y))$ and $E\, I(Y)$, we get a pair of optimality equations to determine the levels of $a^*$ and $k^*$: $\eta^*(a) \overset{def}{=} 2\beta a - \gamma = 0$ and $\theta^*(k) \overset{def}{=} 2\beta k - (1-\delta)\gamma = 0$. This immediately implies that it holds for their roots that $k = a(1-\delta)$. After substituting the expression for $k$ into the second equation, it is easy to see that this equation, up to a positive factor in the left-hand side, coincides with Eq. (11). Let us show that the solution of (11) is unique on the interval $(a^0, \infty)$. We denote by $\phi(a)$ the left-hand side of (11); then $\phi(a) = kz(a) - (1-\delta)\left[\int_0^k x\bar{F}(x)dx + \int_k^a x\bar{F}(x+q)dx\right]$, where $k = a(1-\delta)$, and $z(a)$ is the denominator in (10). Therefore, $\phi(a^0) < 0$, and since $\phi'(a) = (1-\delta)z(a) > 0$ on $(a^0, \infty)$ and $\phi(a) \to \infty$ for $a \to \infty$, we get that the solution (11) is unique. □

The function $I^*(x)$ found in Theorem 2 is a sort of generalization for the franchise $I(x) = (x-q)_+$ since the "tail" of the damages distribution is left for the insurer; small damages are divided according to stop-loss insurance, $I(x) = x \wedge k^*$, and the client's damage is fully paid by the insurer if it does not exceed $k^*$. The form of $I^*(x)$, similar to the division of stop-loss reinsurance $A^*(x)$, is shown on Fig. 2.

The client is left with the "middle" part of his original risk, $Y - I^*(Y) = (Y - k^*)_+ \wedge q$. After the reinsurance deal is closed, the risk share paid by the insurer is a piecewise linear function $A^*(I^*(Y)) = I^*(Y) \wedge a^*$ "cut off" by the insurer's stop-loss level $a^*\,(> k^*)$, the maximal sum he agrees to reimburse to a client.

*Remark 2.* If we assume in addition that the reinsurer, similar to Section 3.1, sets an upper bound $Q$ on his admissible risk, i.e., the insurer's division function $A(x) \geq (x-Q)_+$, then, repeating the considerations from Theorem 1, it is easy to show that the optimal division $A^*$ in (10) will be a stop-loss reinsurance with upper limit $A^*(x) = (x \wedge a^*) \vee (x - Q)$. Here the form of the optimal insurance division remains the same, $I^*(x) = (x \wedge k^*) \vee (x - q)$, while parameters $k^*$ and $a^*$ are connected by the same relation $k^* = a^*(1 - \delta)$.

## 4. EXAMPLE

Let us consider a numerical example that illustrates Theorem 2 for the case when payments have a uniform distribution on the interval $[0, 10]$. Suppose that load coefficients $\alpha$ and $\alpha_1$ for the insurer and reinsurer are given, and the upper bound $q$ on the client risk left after insurance is also given, $Y - I(Y) \leq q$ almost surely. Setting the optimal insurance and reinsurance parameters $k^*$ and $a^*$ in problem (10) reduces to solving Eq. (11), where all integrals are now easy to express via elementary functions. Getting a solution $a^*$ of this equation (after substituting $k = (1 - \delta)a$, where $\delta = 1 - \alpha/\alpha_1$ and $\bar{F}(x) = 1 - 0.1x$, $x \in [0, 10]$), we find the insurer's stop-loss level $k^* = (1 - \delta)a^*$. According to Theorem 2, the variation coefficient is minimized for the stop-loss reinsurance $A^*(x) = x \wedge a^*$ and a combination of stop-loss insurance and franchise $I^*(x) = (x \wedge k^*) \vee (x - q)$. Numerical results for $\alpha_1 = 1$ and different values of $q$ and $\alpha$ are given in the table.

**Table**

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $\alpha = 0.6$ | | | | | | | |
| $q$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| $k^*$ | 3.409 | 2.932 | 2.465 | 2.010 | 1.570 | 1.151 | 0.760 |
| $a^*$ | 5.681 | 4.887 | 4.108 | 3.350 | 2.617 | 1.918 | 1.268 |
| $J^*$ | 8.632 | 7.298 | 6.028 | 4.824 | 3.695 | 2.651 | 1.709 |
| $\alpha = 0.7$ | | | | | | | |
| $q$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| $k^*$ | 2.624 | 2.196 | 1.789 | 1.407 | 1.053 | 0.733 | 0.453 |
| $a^*$ | 3.749 | 3.137 | 2.556 | 2.010 | 1.504 | 1.046 | 0.647 |
| $J^*$ | 5.991 | 4.966 | 4.007 | 3.117 | 2.306 | 1.584 | 0.966 |

The table indicates that as the upper bound on the client's risk left after insurance $q$ increases, the value $J^*$ of the optimal (i.e., minimal) variation coefficient decreases. Indeed, as $q$ grows the set of admissible insurance divisions $\{I(x) : (x - q)_+ \leq I(x) \leq x\}$ grows and, therefore, the optimal $J^*$ decreases.

As the insurer's load coefficient $\alpha$ increases, the denominator in expression (10) for the objective functional $J[I, A]$ increases or, in other words, the mean value of the insurer's capital per time unit increases. This lets us reduce the optimal value $J^*$ by decreasing the levels of $k^*$ and $a^*$, thus reducing both client's risk paid by the insurer $I^*(Y) = (Y \wedge k^*) \vee (Y - q)$ and residual insurer's risk $A^*(I^*(Y)) = I^*(Y) \wedge a^*$ after reinsurance.

*APPENDIX*

**Proof of Lemma.** Let $\{I_n, A_n\}$ be the minimizing sequence of admissible divisions in problem (5), i.e.,

$$\lim_{n\to\infty} J[I_n, A_n] = J^* \stackrel{def}{=} \inf_{(I,A)} J[I, A].$$

Applying the Helly–Bray theorem separately to the numerator and denominator in the objective functional, we get that there exists a subsequence of pairs of random values $\{I_m(Y), A_m(I_m(Y))\}$

that weakly converges to a certain limit $(\rho, \xi)$. To prove that $\rho$ and $\xi$ are eigen random variables, in other words, $P\{\rho < \infty, \xi < \infty\} = 1$, it suffices to note that by definition of division functions, $I_m(Y) \le Y$ and $A_m(I_m(Y)) \le Y$ almost surely (a.s.). Since $\rho$ is measurable with respect to the sigma-algebra $\sigma(Y)$, and $(Y-q)_+ \le \rho \le Y$, this random variable can be represented as $\rho = I^*(Y)$ for some Borel function $(x-q)_+ \le I^*(x) \le x$. Similarly, another limit can be represented as $\xi = A^*(I^*(Y))$, where $(x-Q)_+ \le A^*(x) \le x$ is the reinsurance division.

Let us show that these divisions are admissible in the sense that the denominator in (5) is positive. Assume the opposite, $E\,A^*(Y^*) - M = 0$, where $Y^* = I^*(Y)$ and $M = del E\,Y^* > 0$. Let us find the solution of problem

$$\inf_{A:\,E\,A(Y^*)=M} E\,A^2(Y^*). \tag{A.1}$$

Repeating the derivation of Theorem 1 and taking into account that $E\,A^2(Y^*)$ is convex in $A$, we see that the division $A_M(x)$ optimal in (A.1) coincides with the solution of problem

$$\min_I \int_0^\infty A_M(x)A(x)dF^*(x) \quad \text{under the constraint} \quad \int_0^\infty A(x)dF^*(x) = M,$$

where $F^*(x) \stackrel{def}{=} P\{Y^* \le x\}$. By the generalized Neyman–Pearson lemma (see [12]), an admissible division function $A_M$ is optimal in this problem if and only if there exists a constant $a$ such that

$$A_M(x) = \begin{cases} (x-Q)_+, & \text{if} \quad A_M(x) - a > 0 \\ x, & \text{if} \quad A_M(x) - a < 0, \end{cases}$$

up to a set of $F^*$-measure zero. Similar to the derivation of Theorem 1, it is easy to show that the only function satisfying this condition is $A_M(x) = (x \wedge a) \vee (x-Q)_+$, where $a$ is given by equation $E\,[(Y^* \wedge a) \vee (Y^* - Q)_+] = M$. Thus, $E\,[A^*(Y^*)]^2 \ge E\,[A_M(Y^*)]^2 > 0$ and $E\,A_m^2(I_m(Y)) \ge \varepsilon$ for some $\varepsilon > 0$ uniformly in $m$. But then the assumption $E\,A_m(I_m(Y)) - \delta E\,I_m(Y) \downarrow 0$ contradicts the fact that $\{I_m, A_m\}$ is a minimizing subsequence for $J[I, A]$.

Now the equality $J^* = J[I^*, A^*]$ follows from weak convergence $\{I_m(Y), A_m(I_m(Y))\}$ and finiteness of expectations $E\,[A^*(I^*(Y))^2]$, $E\,A^*(I^*(Y))$, and $E\,I^*(Y)$. $\square$

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