Siberian Mathematical Journal, Vol. 46, *No.* 4, *pp.* 751–764, 2005
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ABSTRACT SUPERPOSITION OPERATORS ON MAPPINGS OF BOUNDED VARIATION OF TWO REAL VARIABLES. II

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UDC 517.98

Abstract: We define and study the metric semigroup $\mathrm{BV}_2(I_a^b; M)$ of mappings of two real variables of bounded total variation in the Vitali–Hardy–Krause sense on a rectangle I_a^b with values in a metric semigroup or abstract convex cone M. We give a complete description for the Lipschitzian Nemytskii superposition operators from $\mathrm{BV}_2(I_a^b; M)$ to a similar semigroup $\mathrm{BV}_2(I_a^b; N)$ and, as a consequence, characterize set-valued superposition operators. We establish a connection between the mappings in $\mathrm{BV}_2(I_a^b; M)$ and the mappings of bounded iterated variation and study the iterated superposition operators on the mappings of bounded iterated variation. The results of this article develop and generalize the recent results by Matkowski and Miś (1984), Zawadzka (1990), and the author (2002, 2003) to the case of (set-valued) superposition operators on the mappings of two real variables.

Keywords: mappings of two variables, total variation, metric semigroup, Nemytskii superposition operator, set-valued operator, Banach algebra type property, Lipschitz condition

§ 4. Lipschitzian Superposition Operators. A Sufficient Condition

This article is a continuation of the author's research [1] devoted to the complete description of Lipschitzian Nemytskii superposition operators $\mathscr{H}$ from a metric semigroup $\mathrm{BV}_2(I_a^b; N)$ of mappings of bounded variation of two real variables to a similar semigroup $\mathrm{BV}_2(I_a^b; M)$, where N and M are abstract metric semigroups. In [1] we obtained a necessary condition for an operator $\mathscr{H}$ to be Lipschitzian. The goal of this article is to establish a sufficient condition for the Lipschitz continuity of $\mathscr{H}$ (Theorems 2 and 3 in § 4) and characterize the iterated superposition operators on $\mathrm{BV}_2(I_a^b; N)$ (Theorem 4 in § 5). The results of this article were announced in [2, 3].

We adhere below to the terminology and notations of [1] wherein a detailed motivation, bibliography, and history of the problem are also given. However, for the reader's convenience we stand with briefly recalling the basic definitions of [1] we need for this part. Observe that the numeration of sections and assertions of this article continues that of [1].

Let I, M, and N be nonempty sets and let M^I be the family of all mappings from I to M. Given a mapping $h : I \times N \to M$, the operator $\mathscr{H} : N^I \to M^I$ defined by the rule $(\mathscr{H}g)(x) = h(x, g(x))$ for $x \in I$ and $g \in N^I$ is called an (*abstract Nemytskii*) *superposition operator with generator* h.

A *metric semigroup* is a triple $(M, d, +)$, where (M, d) is a metric space with metric d, while $(M, +)$ is an abelian semigroup with addition operation $+$, and d is translation-invariant: $d(u+w, v+w) = d(u, v)$ for all $u, v, w \in M$. The following inequality holds in a metric semigroup M:

$$d(u + \bar{u}, v + \bar{v}) \le d(u, v) + d(\bar{u}, \bar{v}), \quad u, v, \bar{u}, \bar{v} \in M; \tag{1}$$

in particular, the addition operation $M \times M \ni (u, v) \mapsto u + v \in M$ is continuous. If M contains the *zero* element $0 \in M$ (so that $u + 0 = 0 + u = u$ for all $u \in M$) then we put $|u|_d = d(u, 0)$ for $u \in M$.

An *abstract convex cone* is a quadruple $(M, d, +, \cdot)$, where $(M, d, +)$ is a metric semigroup with zero $0 \in M$ and the operation $\cdot : \mathbb{R}^+ \times M \to M$ of multiplication of elements of M by nonnegative numbers acting by the rule $(\lambda, u) \mapsto \lambda u$ possesses the following properties for all $\lambda, \mu \in \mathbb{R}^+$ and $u, v \in M$: $\lambda(u + v) = \lambda u + \lambda v$, $(\lambda + \mu)u = \lambda u + \mu u$, $\lambda(\mu u) = (\lambda\mu)u$, $1 \cdot u = u$, and $d(\lambda u, \lambda v) = \lambda d(u, v)$.

The author was supported by the Russian Foundation for Basic Research (Grant 03–01–00473).

Nizhniĭ Novgorod. Translated from *Sibirskiĭ Matematicheskiĭ Zhurnal*, Vol. 46, No. 4, pp. 942–957, July–August, 2005. Original article submitted March 13, 2004.

Numerous examples of metric semigroups and abstract convex cones are given in [1]. Here we are mainly interested in the semigroups and cones of mappings of bounded variation of one and two variables.

Let (M, d) be a metric space and let $[a, b] \subset \mathbb{R}$ be a closed interval. The classical (*Jordan*) *variation of a mapping* $\varphi : [a, b] \to M$ is the quantity

$$V_a^b(\varphi) = \sup_{\xi} \sum_{i=1}^{m} d(\varphi(t_i), \varphi(t_{i-1})),$$

where the supremum is taken over all partitions $\xi = \{t_i\}_{i=0}^m$ of the interval $[a, b]$ (i.e., $m \in \mathbb{N}$ and $a = t_0 < t_1 < \dots < t_{m-1} < t_m = b$). If $V_a^b(\varphi) < \infty$ then we write $\varphi \in \mathrm{BV}_1([a, b]; M)$ and say that φ is a *mapping of bounded variation* on $[a, b]$. If $(M, d, +)$ is a (complete) metric semigroup (or an abstract convex cone) then $\mathrm{BV}_1([a, b]; M)$ as well is a (complete) metric semigroup (or an abstract convex cone), where the addition operation (as well as multiplication by nonnegative numbers) is defined pointwise and the translation-invariant metric d_1 is given by the rule

$$d_1(\varphi, \psi) = d(\varphi(a), \psi(a)) + W_a^b(\varphi, \psi), \quad \varphi, \psi \in \mathrm{BV}_1([a, b]; M),$$

and the semimetric $W_a^b(\varphi, \psi)$ is defined as

$$W_a^b(\varphi, \psi) = \sup_{\xi} \sum_{i=1}^{m} d(\varphi(t_i) + \psi(t_{i-1}), \psi(t_i) + \varphi(t_{i-1})). \tag{2}$$

Below we need the following inequality [1, Lemma 1(b)]:

$$d(\varphi(t), \psi(t)) \le d_1(\varphi, \psi), \quad t \in [a, b]. \tag{3}$$

The corresponding definitions for the mappings of two variables with values in a semigroup M are as follows:

We write the coordinate representations of $x, y \in \mathbb{R}^2$ in the form $x = (x_1, x_2)$ and $y = (y_1, y_2)$ and assume that $x \le y$ or $x < y$ (in $\mathbb{R}^2$) if these inequalities hold coordinatewise. Suppose that $a = (a_1, a_2) < b = (b_1, b_2)$ in $\mathbb{R}^2$ and $I_a^b = I_{a_1,a_2}^{b_1,b_2} = [a_1, b_1] \times [a_2, b_2]$ is a *basic* rectangle on the plane (the domain of most mappings). Given a mapping $f : I_a^b \to M$ and points $x_1 \in [a_1, b_1]$ and $x_2 \in [a_2, b_2]$, define the two mappings $f(\cdot, x_2) : [a_1, b_1] \to M$ and $f(x_1, \cdot) : [a_2, b_2] \to M$ of a single variable by the rules: $f(\cdot, x_2)(t) = f(t, x_2)$ for $t \in [a_1, b_1]$ and $f(x_1, \cdot)(s) = f(x_1, s)$ for $s \in [a_2, b_2]$.

Suppose that $(M, d, +)$ is a metric semigroup and I_a^b is the basic rectangle.

The (Vitali) *mixed difference* of a mapping $f : I_a^b \to M$ on a subrectangle $I_x^y = [x_1, y_1] \times [x_2, y_2] \subset I_a^b$, where $x, y \in I_a^b$, $x \le y$, is defined by [1, 4]

$$\mathrm{md}(f, I_x^y) = \mathrm{md}(f, I_{x_1,x_2}^{y_1,y_2}) = d(f(x_1, x_2) + f(y_1, y_2), f(x_1, y_2) + f(y_1, x_2)).$$

A pair (ξ, η) is called a (net) *partition* of I_a^b if there exist $m, n \in \mathbb{N}$ such that $\xi = \{t_i\}_{i=0}^m$ is a partition of $[a_1, b_1]$ and $\eta = \{s_j\}_{j=0}^n$ is a partition of $[a_2, b_2]$. Then the mixed difference $\mathrm{md}(f, I_{ij})$ on the rectangles

$$I_{ij} = I_{t_{i-1},s_{j-1}}^{t_i,s_j} = [t_{i-1}, t_i] \times [s_{j-1}, s_j], \quad i = 1, \dots, m, \ j = 1, \dots, n, \tag{4}$$

constituting this partition, is calculated according to the equality

$$\mathrm{md}(f, I_{t_{i-1},s_{j-1}}^{t_i,s_j}) = d(f(t_{i-1}, s_{j-1}) + f(t_i, s_j), f(t_{i-1}, s_j) + f(t_i, s_{j-1})).$$

The *double variation* of a mapping $f : I_a^b \to M$ is defined by the rule (Vitali [4] for $M = \mathbb{R}$)

$$V_2(f, I_a^b) = \sup_{(\xi,\eta)} \sum_{i=1}^{m} \sum_{j=1}^{n} \mathrm{md}(f, I_{ij}),$$

where the supremum is taken over all partitions (ξ, η) of the rectangle I_a^b of the above form. The *total variation* (in the modification of Hardy and Krause, see [5, 6] if $M = \mathbb{R}$) of a mapping f is the quantity

$$TV_d\big(f, I_a^b\big) = V_{a_1}^{b_1}(f(\cdot, a_2)) + V_{a_2}^{b_2}(f(a_1, \cdot)) + V_2\big(f, I_a^b\big), \tag{5}$$

and the class of all mappings of finite total variation is called the *space of mappings of bounded variation* (in the Vitali–Hardy–Krause sense) and denoted by $\mathrm{BV}_2\big(I_a^b; M\big)$. The following inequality is valid for $f \in \mathrm{BV}_2\big(I_a^b; M\big)$ [7, 8]:

$$d(f(y), f(x)) \le TV_d\big(f, I_x^y\big) \le TV_d\big(f, I_a^y\big) - TV_d\big(f, I_a^x\big), \quad x, y \in I_a^b,\ x \le y. \tag{6}$$

If a metric semigroup $(M, d, +)$ contains zero then we also put

$$\|f\|_d = |f(a)|_d + TV_d\big(f, I_a^b\big), \quad f \in \mathrm{BV}_2\big(I_a^b; M\big).$$

The main property of V_2 is *additivity*: for every above-indicated partition (ξ, η) of the rectangle I_a^b into some subrectangles $\{I_{ij}\}_{i,j=1}^{m,n}$ as in (4) we obtain

$$V_2\big(f, I_a^b\big) = \sum_{i=1}^{m} \sum_{j=1}^{n} V_2(f, I_{ij}). \tag{7}$$

In the case when $(M, d, +)$ is a (complete) metric semigroup (abstract convex cone) then the structure of a (complete) metric semigroup (abstract convex cone) on $\mathrm{BV}_2\big(I_a^b; M\big)$ is defined as follows [1]: Let $f, g \in \mathrm{BV}_2\big(I_a^b; M\big)$. The addition operation $+$ (multiplication by nonnegative numbers) in $\mathrm{BV}_2\big(I_a^b; M\big)$ is introduced pointwise and the *translation-invariant metric* d_2 is defined by the rule

$$d_2(f, g) = d(f(a), g(a)) + TW_d\big(f, g, I_a^b\big),$$

where

$$TW_d\big(f, g, I_a^b\big) = W_{a_1}^{b_1}(f(\cdot, a_2), g(\cdot, a_2)) + W_{a_2}^{b_2}(f(a_1, \cdot), g(a_1, \cdot)) + W_2\big(f, g, I_a^b\big).$$

Here the first summand on the right-hand side is the quantity (2) calculated in the metric d for the mappings $t \mapsto f(t, a_2)$ and $t \mapsto g(t, a_2)$ on the interval $[a_1, b_1]$, the second summand has a similar meaning, and $W_2\big(f, g, I_a^b\big)$ is defined in the notations of (4) by the rule

$$W_2\big(f, g, I_a^b\big) = \sup_{(\xi,\eta)} \sum_{i=1}^{m} \sum_{j=1}^{n} \mathrm{md}_2\big(f, g, I_{ij}\big),$$

where the supremum is taken over all partitions $\xi = \{t_i\}_{i=0}^m$ and $\eta = \{s_j\}_{j=0}^n$ of the respective intervals $[a_1, b_1]$ and $[a_2, b_2]$ $(m, n \in \mathbb{N})$ and the *joint mixed difference* $\mathrm{md}_2\big(f, g, I_x^y\big)$ on the subrectangle $I_x^y = [x_1, y_1] \times [x_2, y_2] \subset I_a^b$ is

$$\begin{aligned} \mathrm{md}_2\big(f, g, I_{x_1,x_2}^{y_1,y_2}\big) = d(&f(x_1, x_2) + f(y_1, y_2) + g(x_1, y_2) + g(y_1, x_2), \\ &g(x_1, x_2) + g(y_1, y_2) + f(x_1, y_2) + f(y_1, x_2)). \end{aligned}$$

Observe that for $f, g \in \mathrm{BV}_2\big(I_a^b; M\big)$ we obtain [1, Lemma 2(b)]

$$\big|TV_d\big(f, I_a^b\big) - TV_d\big(g, I_a^b\big)\big| \le TW_d\big(f, g, I_a^b\big) \le TV_d\big(f, I_a^b\big) + TV_d\big(g, I_a^b\big). \tag{8}$$

Let $(N, \rho, +)$ and $(M, d, +)$ be two metric semigroups (two abstract convex cones). An operator $T : N \to M$ is called *Lipschitzian* if its (least) *Lipschitz constant* is finite:

$$L(T) = \sup\{d(Tu, Tv)/\rho(u, v) \mid u, v \in N, u \ne v\},$$

and the set of all these operators is denoted by $\mathrm{Lip}(N; M)$. An operator $T : N \to M$ is called *additive* if it satisfies the Cauchy equation: $T(u + v) = Tu + Tv$ for all $u, v \in N$. Denote by $\mathrm{L}(N; M)$ the set of all Lipschitzian additive operators from N to M.

Henceforth we consider only the case when N and M contain zeros (denoted by the same symbol 0). In this case if $T \in \mathrm{L}(N;M)$ then $T(0)=0$, for $T(0)=T(0+0)=T(0)+T(0)$ and $d(0,T(0))=d(T(0),T(0)+T(0))=0$. The set $\mathrm{L}(N;M)$ is closed with respect to the pointwise addition (multiplication by nonnegative numbers) by (1). The translation-invariant metric d_L on $\mathrm{L}(N;M)$ is defined by the rule [9]

$$d_L(T,S)=\sup\{d(Tu+Sv,Su+Tv)/\rho(u,v)\mid u,v\in N,\ u\neq v\},\quad T,S\in \mathrm{L}(N;M).$$

Thus, $(\mathrm{L}(N;M),d_L,+)$ is a metric semigroup (abstract convex cone) which is complete if such is the metric semigroup $(X,d,+)$; moreover $L(T)=d_L(T,0)=|T|_{d_L}$. For future reference, observe that [1, Lemma 4(b)]

$$|L(T)-L(S)|\le d_L(T,S)\le L(T)+L(S),\quad T,S\in \mathrm{L}(N;M). \tag{9}$$

In [1, Theorem 1] we proved the following necessary condition for Lipschitz continuity of a superposition operator $\mathscr{H}$ (we cite it under some additional assumptions which do not change the result much). *Suppose that $(N,\rho,+,\cdot)$ and $(M,d,+,\cdot)$ are two abstract convex cones such that M is complete and a mapping $h: I_a^b\times N\to M$ which is continuous in the first argument is the generator of a superposition operator $\mathscr{H}$ for $I=I_a^b$. If $\mathscr{H}\in \mathrm{Lip}\big(\mathrm{BV}_2(I_a^b;N);\mathrm{BV}_2(I_a^b;M)\big)$ then $h(x,\cdot)\in \mathrm{Lip}(N;M)$ for all $x\in I_a^b$ and there exist two mappings $f: I_a^b\to \mathrm{L}(N;M)$ and $h_0: I_a^b\to M$ such that $f(\cdot)u,\ h_0\in \mathrm{BV}_2(I_a^b;M)$ for all $u\in N$ and the representation $h(x,u)=f(x)u+h_0(x)$ holds for all $x\in I_a^b$ and $u\in N$, where $f(\cdot)u$ acts by the rule $x\mapsto f(x)u$.*

The main results of this section are Theorem 2 in which we establish the Banach algebra type property of the spaces $\mathrm{BV}_2\big(I_a^b;M\big)$ (cf. [7]) and Theorem 3 which gives a sufficient condition for the Lipschitz continuity of the superposition operator $\mathscr{H}$ which acts between metric semigroups $\mathrm{BV}_2\big(I_a^b;M\big)$.

Theorem 2. *Suppose that $(N,\rho,+)$ and $(M,d,+)$ are two metric semigroups with zeros. If $f\in \mathrm{BV}_2\big(I_a^b;\mathrm{L}(N;M)\big)$ and $g\in \mathrm{BV}_2\big(I_a^b;N\big)$ then the mapping $fg: I_a^b\to M$ acting by the rule $(fg)(x)=f(x)g(x)$ for all $x\in I_a^b$ lies in $\mathrm{BV}_2(I_a^b;M)$ and the inequality $\|fg\|_d\le 4\|f\|_{d_L}\|g\|_\rho$ is valid.*

Proof. Since $fg: I_a^b\to M$, by (5), we have

$$\|fg\|_d=|(fg)(a)|_d+V_{a_1}^{b_1}((fg)(\cdot,a_2))+V_{a_2}^{b_2}((fg)(a_1,\cdot))+V_2\big(fg,I_a^b\big). \tag{10}$$

For the first summand from the definitions of the Lipschitz constant of the operator $f(a)$ we obtain

$$\begin{aligned}|(fg)(a)|_d&=d((fg)(a),0)=d(f(a)g(a),f(a)(0))\\&\le L(f(a))\rho(g(a),0)=|f(a)|_{d_L}\cdot|g(a)|_\rho.\end{aligned} \tag{11}$$

Let us estimate the remaining three terms in (10). To estimate the second summand, we use the definition of the Lipschitz constant $L(\cdot)$ and the metric d_L, so that if $t,s\in[a_1,b_1]$ then

$$\begin{aligned}d((fg)(t,a_2),(fg)(s,a_2))&\le d(f(t,a_2)g(t,a_2),f(t,a_2)g(s,a_2))\\&\quad+d(f(t,a_2)g(s,a_2),f(s,a_2)g(s,a_2))\\&\le L(f(t,a_2))\rho(g(t,a_2),g(s,a_2))+d_L(f(t,a_2),f(s,a_2))\rho(g(s,a_2),0),\end{aligned}$$

whence

$$V_{a_1}^{b_1}((fg)(\cdot,a_2))\le(\sup_{[a_1,b_1]}L(f(\cdot,a_2)))V_{a_1}^{b_1}(g(\cdot,a_2))+V_{a_1}^{b_1}(f(\cdot,a_2))(\sup_{[a_1,b_1]}\rho(g(\cdot,a_2),0)).$$

Observing that (see, in particular, (9))

$$\sup_{t\in[a_1,b_1]}L(f(t,a_2))\le L(f(a))+V_{a_1}^{b_1}(f(\cdot,a_2)),$$

$$\sup_{s\in[a_1,b_1]}\rho(g(s,a_2),0)\le\rho(g(a),0)+V_{a_1}^{b_1}(g(\cdot,a_2)),$$

we find that

$$\begin{aligned}V_{a_1}^{b_1}((fg)(\cdot,a_2))&\le|f(a)|_{d_L}V_{a_1}^{b_1}(g(\cdot,a_2))+V_{a_1}^{b_1}(f(\cdot,a_2))|g(a)|_\rho\\&\quad+2V_{a_1}^{b_1}(f(\cdot,a_2))V_{a_1}^{b_1}(g(\cdot,a_2)).\end{aligned} \tag{12}$$

A similar estimate holds also for the third summand in (10):

$$V_{a_2}^{b_2}((fg)(a_1,\cdot)) \le |f(a)|_{d_L} V_{a_2}^{b_2}(g(a_1,\cdot)) + V_{a_2}^{b_2}(f(a_1,\cdot))|g(a)|_\rho$$
$$+2V_{a_2}^{b_2}(f(a_1,\cdot))V_{a_2}^{b_2}(g(a_1,\cdot)). \tag{13}$$

To estimate the fourth summand $V_2\big(fg, I_a^b\big)$ in (10), we use the following observation concerning the elements of the metric semigroup $(M, d, +)$:

$$\text{if } n \in \mathbb{N},\ \{l_k, r_k\}_{k=0}^n \subset M, \text{ and } \sum_{k=0}^n l_k = \sum_{k=0}^n r_k \text{ then } d(l_0, r_0) \le \sum_{k=1}^n d(r_k, l_k). \tag{14}$$

Indeed, from the translation invariance of d and (1) we obtain

$$d(l_0, r_0) = d\bigg(l_0 + \sum_{k=1}^n l_k, r_0 + \sum_{k=1}^n l_k\bigg) = d\bigg(r_0 + \sum_{k=1}^n r_k, r_0 + \sum_{k=1}^n l_k\bigg)$$
$$= d\bigg(\sum_{k=1}^n r_k, \sum_{k=1}^n l_k\bigg) \le \sum_{k=1}^n d(r_k, l_k).$$

Let $\{t_i\}_{i=0}^m$ and $\{s_j\}_{j=0}^n$ be respective partitions of the intervals $[a_1, b_1]$ and $[a_2, b_2]$. Note that, by additivity of the operator $f(x)$ for all $x \in I_a^b$, for $i = 1, \dots, m$ and $j = 1, \dots, n$ the following equality holds (the subscripts of brackets in this equality only establish enumeration and indicate the correspondence between summands on the left- and right-hand sides to be used below):

$$[(fg)(t_{i-1}, s_{j-1}) + (fg)(t_i, s_j)]_0 + [(f(t_{i-1}, s_j) + f(t_i, s_{j-1}))g(t_{i-1}, s_{j-1})]_1$$
$$+[f(t_i, s_j)(g(t_{i-1}, s_j) + g(t_i, s_{j-1}))]_2 + [f(a_1, s_{j-1})g(t_i, a_2) + f(a_1, s_j)g(t_{i-1}, a_2)]_3$$
$$+[f(a_1, s_{j-1})(g(t_{i-1}, a_2) + g(t_i, s_{j-1})) + f(a_1, s_j)(g(t_{i-1}, s_{j-1}) + g(t_i, a_2))]_4$$
$$+[(f(a_1, s_j) + f(t_i, s_{j-1}))g(t_i, a_2) + (f(a_1, s_{j-1}) + f(t_i, s_j))g(t_{i-1}, a_2)]_5$$
$$+[(f(a_1, s_j) + f(t_i, s_{j-1}))(g(t_{i-1}, a_2) + g(t_i, s_{j-1}))$$
$$+(f(a_1, s_{j-1}) + f(t_i, s_j))(g(t_{i-1}, s_{j-1}) + g(t_i, a_2))]_6$$
$$+[f(t_{i-1}, a_2)g(a_1, s_j) + f(t_i, a_2)g(a_1, s_{j-1})]_7$$
$$+[f(t_{i-1}, a_2)(g(a_1, s_{j-1}) + g(t_{i-1}, s_j)) + f(t_i, a_2)(g(a_1, s_j) + g(t_{i-1}, s_{j-1}))]_8$$
$$+[(f(t_{i-1}, s_j) + f(t_i, a_2))g(a_1, s_j) + (f(t_{i-1}, a_2) + f(t_i, s_j))g(a_1, s_{j-1})]_9$$
$$+[(f(t_{i-1}, s_j) + f(t_i, a_2))(g(a_1, s_{j-1}) + g(t_{i-1}, s_j))$$
$$+(f(t_{i-1}, a_2) + f(t_i, s_j))(g(a_1, s_j) + g(t_{i-1}, s_{j-1}))]_{10}$$
$$= [(fg)(t_{i-1}, s_j) + (fg)(t_i, s_{j-1})]_0 + [(f(t_{i-1}, s_{j-1}) + f(t_i, s_j))g(t_{i-1}, s_{j-1})]_1$$
$$+[f(t_i, s_j)(g(t_{i-1}, s_{j-1}) + g(t_i, s_j))]_2 + [f(a_1, s_j)g(t_i, a_2) + f(a_1, s_{j-1})g(t_{i-1}, a_2)]_3$$
$$+[f(a_1, s_j)(g(t_{i-1}, a_2) + g(t_i, s_{j-1})) + f(a_1, s_{j-1})(g(t_{i-1}, s_{j-1}) + g(t_i, a_2))]_4$$
$$+[(f(a_1, s_{j-1}) + f(t_i, s_j))g(t_i, a_2) + (f(a_1, s_j) + f(t_i, s_{j-1}))g(t_{i-1}, a_2)]_5$$
$$+[(f(a_1, s_{j-1}) + f(t_i, s_j))(g(t_{i-1}, a_2) + g(t_i, s_{j-1}))$$
$$+(f(a_1, s_j) + f(t_i, s_{j-1}))(g(t_{i-1}, s_{j-1}) + g(t_i, a_2))]_6$$
$$+[f(t_i, a_2)g(a_1, s_j) + f(t_{i-1}, a_2)g(a_1, s_{j-1})]_7$$
$$+[f(t_i, a_2)(g(a_1, s_{j-1}) + g(t_{i-1}, s_j)) + f(t_{i-1}, a_2)(g(a_1, s_j) + g(t_{i-1}, s_{j-1}))]_8$$
$$+[(f(t_{i-1}, a_2) + f(t_i, s_j))g(a_1, s_j) + (f(t_{i-1}, s_j) + f(t_i, a_2))g(a_1, s_{j-1})]_9$$
$$+[(f(t_{i-1}, a_2) + f(t_i, s_j))(g(a_1, s_{j-1}) + g(t_{i-1}, s_j))$$
$$+(f(t_{i-1}, s_j) + f(t_i, a_2))(g(a_1, s_j) + g(t_{i-1}, s_{j-1}))]_{10}.$$

For $k = 0, 1, \dots, 10$ we denote by l_k^{ij} (r_k^{ij}) the kth summand in the brackets on the left (right) side of the above equality, so that we can rewrite it in the form $\sum_{k=0}^{10} l_k^{ij} = \sum_{k=0}^{10} r_k^{ij}$. By (4) and (14), we find that

$$\operatorname{md}(fg, I_{ij}) = d\big(l_0^{ij}, r_0^{ij}\big) \le \sum_{k=1}^{10} d\big(r_k^{ij}, l_k^{ij}\big);$$

therefore,

$$\sum_{i=1}^{m}\sum_{j=1}^{n} \operatorname{md}(fg, I_{ij}) \le \sum_{k=1}^{10}\sum_{i=1}^{m}\sum_{j=1}^{n} d\big(r_k^{ij}, l_k^{ij}\big) = \sum_{k=1}^{10} S_k.$$

Estimate the expressions $S_k = \sum_{i=1}^{m}\sum_{j=1}^{n} d\big(r_k^{ij}, l_k^{ij}\big)$, $k = 1, \dots, 10$, separately. It follows from (6) that if $(t, s) \in I_a^b$ then

$$|g(t,s)|_\rho = \rho(g(t,s), 0) \le \rho(g(a), 0) + \rho(g(t,s), g(a)) \le |g(a)|_\rho + TV_\rho\big(g, I_a^b\big) = \|g\|_\rho;$$

similarly, it follows from (9) and (6) that

$$\begin{aligned} |f(t,s)|_{d_L} &= L(f(t,s)) \le L(f(a)) + d_L(f(t,s), f(a)) \\ &\le |f(a)|_{d_L} + TV_{d_L}\big(f, I_a^b\big) = \|f\|_{d_L}. \end{aligned}$$

By the definition of d_L and the estimate for $|g(t,s)|_\rho$, for S_1 we have

$$\begin{aligned} d(r_1^{ij}, l_1^{ij}) &\le d_L(f(t_{i-1}, s_{j-1}) + f(t_i, s_j), f(t_{i-1}, s_j) + f(t_i, s_{j-1}))|g(t_{i-1}, s_{j-1})|_\rho \\ &\le \operatorname{md}(f, I_{ij})\|g\|_\rho, \end{aligned}$$

whence

$$S_1 \le V_2\big(f, I_a^b\big)\|g\|_\rho.$$

Using the definition of the Lipschitz constant and the estimate for $|f(t,s)|_{d_L}$, for S_2 we find that

$$\begin{aligned} d\big(r_2^{ij}, l_2^{ij}\big) &\le L(f(t_i, s_j))\rho(g(t_{i-1}, s_{j-1}) + g(t_i, s_j), g(t_{i-1}, s_j) + g(t_i, s_{j-1})) \\ &= |f(t_i, s_j)|_{d_L} \operatorname{md}(g, I_{ij}) \le \|f\|_{d_L} \operatorname{md}(g, I_{ij}); \end{aligned}$$

consequently,

$$S_2 \le \|f\|_{d_L} V_2\big(g, I_a^b\big).$$

For the summand S_3 (using again the definition of d_L) we obtain

$$d\big(r_3^{ij}, l_3^{ij}\big) \le d_L(f(a_1, s_j), f(a_1, s_{j-1}))\rho(g(t_i, a_2), g(t_{i-1}, a_2))$$

and hence

$$S_3 \le V_{a_2}^{b_2}(f(a_1, \cdot))V_{a_1}^{b_1}(g(\cdot, a_2)).$$

By analogy with S_3 we estimate the expression S_7:

$$d\big(r_7^{ij}, l_7^{ij}\big) \le d_L(f(t_i, a_2), f(t_{i-1}, a_2))\rho(g(a_1, s_j), g(a_1, s_{j-1}));$$

$$S_7 \le V_{a_1}^{b_1}(f(\cdot, a_2))V_{a_2}^{b_2}(g(a_1, \cdot)).$$

For S_4 we obtain

$$\begin{aligned} d\big(r_4^{ij}, l_4^{ij}\big) &\le d_L(f(a_1, s_j), f(a_1, s_{j-1}))\rho(g(t_{i-1}, a_2) + g(t_i, s_{j-1}), g(t_{i-1}, s_{j-1}) + g(t_i, a_2)) \\ &= d_L(f(a_1, s_j), f(a_1, s_{j-1})) \operatorname{md}\big(g, I_{t_{i-1}, a_2}^{t_i, s_{j-1}}\big) \le d_L(f(a_1, s_j), f(a_1, s_{j-1}))V_2\big(g, I_{t_{i-1}, a_2}^{t_i, b_2}\big), \end{aligned}$$

whence, by (monotonicity and) additivity of V_2 (see (7)), we find that

$$S_4 \le V_{a_2}^{b_2}(f(a_1,\cdot))V_2\big(g, I_a^b\big).$$

By analogy with S_4, we obtain the following estimate for S_8:

$$\begin{aligned} d\big(r_8^{ij}, l_8^{ij}\big) &\le d_L(f(t_i,a_2), f(t_{i-1},a_2))\,\mathrm{md}\big(g, I_{a_1,s_{j-1}}^{t_{i-1},s_j}\big) \\ &\le d_L(f(t_i,a_2), f(t_{i-1},a_2))V_2\big(g, I_{a_1,s_{j-1}}^{b_1,s_j}\big); \\ S_8 &\le V_{a_1}^{b_1}(f(\cdot,a_2))V_2\big(g, I_a^b\big). \end{aligned}$$

To estimate S_5, observe that

$$\begin{aligned} d\big(r_5^{ij}, l_5^{ij}\big) &\le d_L(f(a_1,s_{j-1}) + f(t_i,s_j), f(a_1,s_j) + f(t_i,s_{j-1}))\rho(g(t_i,a_2), g(t_{i-1},a_2)) \\ &= \mathrm{md}\big(f, I_{a_1,s_{j-1}}^{t_i,s_j}\big)\rho(g(t_i,a_2), g(t_{i-1},a_2)) \le V_2\big(f, I_{a_1,s_{j-1}}^{b_1,s_j}\big)\rho(g(t_i,a_2), g(t_{i-1},a_2)), \end{aligned}$$

whence, by monotonicity and additivity of the double variation V_2,

$$S_5 \le V_2\big(f, I_a^b\big)V_{a_1}^{b_1}(g(\cdot,a_2)).$$

By analogy with S_5, we estimate the summand S_9:

$$\begin{aligned} d\big(r_9^{ij}, l_9^{ij}\big) &\le \mathrm{md}\big(f, I_{t_{i-1},a_2}^{t_i,s_j}\big)\rho(g(a_1,s_j), g(a_1,s_{j-1})) \\ &\le V_2\big(f, I_{t_{i-1},a_2}^{t_i,b_2}\big)\rho(g(a_1,s_j), g(a_1,s_{j-1})); \\ S_9 &\le V_2\big(f, I_a^b\big)V_{a_2}^{b_2}(g(a_1,\cdot)). \end{aligned}$$

From the inequalities

$$\begin{aligned} d\big(r_6^{ij}, l_6^{ij}\big) &\le d_L(f(a_1,s_{j-1}) + f(t_i,s_j), f(a_1,s_j) + f(t_i,s_{j-1})) \\ &\times\rho(g(t_{i-1},a_2) + g(t_i,s_{j-1}), g(t_{i-1},s_{j-1}) + g(t_i,a_2)) \\ &= \mathrm{md}\big(f, I_{a_1,s_{j-1}}^{t_i,s_j}\big)\,\mathrm{md}\big(g, I_{t_{i-1},a_2}^{t_i,s_{j-1}}\big) \le V_2\big(f, I_{a_1,s_{j-1}}^{b_1,s_j}\big)V_2\big(g, I_{t_{i-1},a_2}^{t_i,b_2}\big) \end{aligned}$$

based on the definition of d_L and from additivity of V_2 we obtain the following estimate for S_6:

$$S_6 \le V_2\big(f, I_a^b\big)V_2\big(g, I_a^b\big).$$

The summand S_{10} is estimated by analogy with S_6:

$$\begin{aligned} d\big(r_{10}^{ij}, l_{10}^{ij}\big) &\le \mathrm{md}\big(f, I_{t_{i-1},a_2}^{t_i,s_j}\big)\,\mathrm{md}\big(g, I_{a_1,s_{j-1}}^{t_{i-1},s_j}\big) \le V_2\big(f, I_{t_{i-1},a_2}^{t_i,b_2}\big)V_2\big(g, I_{a_1,s_{j-1}}^{b_1,s_j}\big); \\ S_{10} &\le V_2\big(f, I_a^b\big)V_2\big(g, I_a^b\big). \end{aligned}$$

Thus, we obtain the following estimate for $V_2\big(fg, I_a^b\big)$:

$$\begin{aligned} V_2\big(fg, I_a^b\big) &\le |f(a)|_{d_L}V_2\big(g, I_a^b\big) + 2V_{a_1}^{b_1}(f(\cdot,a_2))V_2\big(g, I_a^b\big) \\ &+2V_{a_2}^{b_2}(f(a_1,\cdot))V_2\big(g, I_a^b\big) + V_2\big(f, I_a^b\big)|g(a)|_\rho \\ &+2V_2\big(f, I_a^b\big)V_{a_1}^{b_1}(g(\cdot,a_2)) + 2V_2\big(f, I_a^b\big)V_{a_2}^{b_2}(g(a_1,\cdot)) \\ &+V_{a_1}^{b_1}(f(\cdot,a_2))V_{a_2}^{b_2}(g(a_1,\cdot)) + V_{a_2}^{b_2}(f(a_1,\cdot))V_{a_1}^{b_1}(g(\cdot,a_2)) + 4V_2\big(f, I_a^b\big)V_2\big(g, I_a^b\big). \end{aligned}$$

Recalling (10)–(13) and the last estimate, we obtain the desired inequality in Theorem 2. $\square$

REMARK 1. If in Theorem 2 we put $I_a^b = [a,b] \subset \mathbb{R}$ and replace BV_2 with BV_1 and $\mathrm{L}(N;M)$ with $\mathrm{Lip}_0(N;M) = \{T \in \mathrm{Lip}(N;M) \mid T(0) = 0\}$ then $fg \in \mathrm{BV}_1\big(I_a^b;M\big)$; moreover $\|fg\|_d \le 2\|f\|_{d_L}\|g\|_\rho$, where $\|fg\|_d = d((fg)(a),0) + V_a^b(fg)$, $\|f\|_{d_L} = L(f(a)) + V_a^b(f)$, and $\|g\|_\rho = \rho(g(a),0) + V_a^b(g)$.

By Theorem 2, Theorem 1 of [1] admits the following conversion:

Theorem 3. *Suppose that $(N, \rho, +)$ and $(M, d, +)$ are two metric semigroups with zeros and the mapping $h : I_a^b \times N \to M$ defined by the rule $h(x, u) = f(x)u + h_0(x)$, where $f \in \mathrm{BV}_2(I_a^b; \mathrm{L}(N; M))$ and $h_0 \in \mathrm{BV}_2(I_a^b; M)$, is the generator of a superposition operator $\mathscr{H}$. Then $\mathscr{H} \in \mathrm{Lip}(\mathrm{BV}_2(I_a^b; N); \mathrm{BV}_2(I_a^b; M))$ and the inequality $L(\mathscr{H}) \le 4\|f\|_{d_L}$ holds.*

Proof. First assume that $h_0 = 0$. Then the superposition operator $\mathscr{H}$ with such a generator acts by the rule: $(\mathscr{H}g)(x) = f(x)g(x) = (fg)(x)$ for $x \in I_a^b$ and $g : I_a^b \to N$. By Theorem 2, if $g \in \mathrm{BV}_2\big(I_a^b; N\big)$ then $\mathscr{H}g \in \mathrm{BV}_2\big(I_a^b; M\big)$, so that $\mathscr{H}$ acts from $\mathrm{BV}_2\big(I_a^b; N\big)$ to $\mathrm{BV}_2\big(I_a^b; M\big)$. Show that $\mathscr{H}$ is Lipschitzian.

Let $g_1, g_2 \in \mathrm{BV}_2\big(I_a^b; N\big)$. From the definition of d_2 we obtain

$$d_2(\mathscr{H}g_1, \mathscr{H}g_2) = d((\mathscr{H}g_1)(a), (\mathscr{H}g_2)(a)) + TW_d\big(\mathscr{H}g_1, \mathscr{H}g_2, I_a^b\big),$$

where the last summand is equal to

$$W_{a_1}^{b_1}((\mathscr{H}g_1)(\cdot, a_2), (\mathscr{H}g_2)(\cdot, a_2))$$
$$+W_{a_2}^{b_2}((\mathscr{H}g_1)(a_1, \cdot), (\mathscr{H}g_2)(a_1, \cdot)) + W_2\big(\mathscr{H}g_1, \mathscr{H}g_2, I_a^b\big).$$

Estimate each of the four summands in $d_2(\mathscr{H}g_1, \mathscr{H}g_2)$ separately. For the first summand we obtain

$$d((\mathscr{H}g_1)(a), (\mathscr{H}g_2)(a)) = d(f(a)g_1(a), f(a)g_2(a)) \le |f(a)|_{d_L}\rho(g_1(a), g_2(a)).$$

To estimate the second summand, note that, by additivity of $f(t, a_2)$, for all $t, s \in [a_1, b_1]$ we have

$$[(fg_1)(t, a_2) + (fg_2)(s, a_2)]_0 + [f(t, a_2)(g_2(t, a_2) + g_1(s, a_2))]_1$$
$$+[f(s, a_2)g_1(s, a_2) + f(t, a_2)g_2(s, a_2)]_2$$
$$= [(fg_2)(t, a_2) + (fg_1)(s, a_2)]_0 + [f(t, a_2)(g_1(t, a_2) + g_2(s, a_2))]_1$$
$$+[f(t, a_2)g_1(s, a_2) + f(s, a_2)g_2(s, a_2)]_2.$$

Hence, by (14) we find that

$$d((\mathscr{H}g_1)(t, a_2) + (\mathscr{H}g_2)(s, a_2), (\mathscr{H}g_2)(t, a_2) + (\mathscr{H}g_1)(s, a_2))$$
$$= d((fg_1)(t, a_2) + (fg_2)(s, a_2), (fg_2)(t, a_2) + (fg_1)(s, a_2))$$
$$\le d(f(t, a_2)(g_1(t, a_2) + g_2(s, a_2)), f(t, a_2)(g_2(t, a_2) + g_1(s, a_2)))$$
$$+d(f(t, a_2)g_1(s, a_2) + f(s, a_2)g_2(s, a_2), f(s, a_2)g_1(s, a_2) + f(t, a_2)g_2(s, a_2))$$
$$\le L(f(t, a_2))\rho(g_1(t, a_2) + g_2(s, a_2), g_2(t, a_2) + g_1(s, a_2))$$
$$+d_L(f(t, a_2), f(s, a_2))\rho(g_1(s, a_2), g_2(s, a_2))$$

and consequently

$$W_{a_1}^{b_1}((\mathscr{H}g_1)(\cdot, a_2), (\mathscr{H}g_2)(\cdot, a_2)) \le (\sup_{t \in [a_1, b_1]} L(f(t, a_2)))W_{a_1}^{b_1}(g_1(\cdot, a_2), g_2(\cdot, a_2))$$
$$+V_{a_1}^{b_1}(f(\cdot, a_2))(\sup_{s \in [a_2, b_2]} \rho(g_1(s, a_2), g_2(s, a_2))).$$

As observed in the proof of Theorem 2, in this inequality we have

$$\sup_{t \in [a_1, b_1]} L(f(t, a_2)) \le |f(a)|_{d_L} + V_{a_1}^{b_1}(f(\cdot, a_2))$$

and

$$\sup_{s \in [a_1, b_1]} \rho(g_1(s, a_2), g_2(s, a_2)) \le \rho(g_1(a), g_2(a)) + W_{a_1}^{b_1}(g_1(\cdot, a_2), g_2(\cdot, a_2)).$$

By analogy with (12), we thus obtain

$$W_{a_1}^{b_1}((\mathscr{H}g_1)(\cdot,a_2),(\mathscr{H}g_2)(\cdot,a_2)) \le |f(a)|_{d_L} W_{a_1}^{b_1}(g_1(\cdot,a_2),g_2(\cdot,a_2))$$
$$+V_{a_1}^{b_1}(f(\cdot,a_2))\rho(g_1(a),g_2(a)) + 2V_{a_1}^{b_1}(f(\cdot,a_2))W_{a_1}^{b_1}(g_1(\cdot,a_2),g_2(\cdot,a_2)).$$

A similar estimate holds also for the third summand:

$$W_{a_2}^{b_2}((\mathscr{H}g_1)(a_1,\cdot),(\mathscr{H}g_2)(a_1,\cdot)) \le |f(a)|_{d_L} W_{a_2}^{b_2}(g_1(a_1,\cdot),g_2(a_1,\cdot))$$
$$+V_{a_2}^{b_2}(f(a_1,\cdot))\rho(g_1(a),g_2(a)) + 2V_{a_2}^{b_2}(f(a_1,\cdot))W_{a_2}^{b_2}(g_1(a_1,\cdot),g_2(a_1,\cdot)).$$

To estimate the fourth summand $W_2\big(\mathscr{H}g_1,\mathscr{H}g_2,I_a^b\big)$ we proceed as follows: Let $\{t_i\}_{i=0}^m$ and $\{s_j\}_{j=0}^n$ be respective partitions of $[a_1,b_1]$ and $[a_2,b_2]$. Denote (a bit more exactly) by $l_k^{ij}(g)$ and $r_k^{ij}(g)$ the expressions in the brackets l_k^{ij} and r_k^{ij} in the proof of Theorem 2. Then we find that the following equality is valid in M:

$$\sum_{k=0}^{10}\big(l_k^{ij}(g_1)+r_k^{ij}(g_2)\big) = \sum_{k=0}^{10}\big(r_k^{ij}(g_1)+l_k^{ij}(g_2)\big)$$

(formally, it is a consequence of the equality $\sum_{k=0}^{10} l_k^{ij}(g) = \sum_{k=0}^{10} r_k^{ij}(g)$ for $g=g_1-g_2$ used in the proof of Theorem 2), from which, by (4) and (14), we obtain

$$\mathrm{md}_2(g_1,g_2,I_{ij}) = d\big(l_0^{ij}(g_1)+r_0^{ij}(g_2), l_0^{ij}(g_2)+r_0^{ij}(g_1)\big)$$
$$\le \sum_{k=1}^{10} d\big(r_k^{ij}(g_1)+l_k^{ij}(g_2), r_k^{ij}(g_2)+l_k^{ij}(g_1)\big) \equiv \sum_{k=1}^{10} d_k^{ij}.$$

Put

$$S_k = \sum_{i=1}^m\sum_{j=1}^n d_k^{ij}, \quad k=1,\dots,10.$$

To estimate the quantities S_k, observe that from (8) and the definition of ρ_2 for all $(t,s)\in I_a^b$ we derive

$$\rho(g_1(t,s),g_2(t,s)) \le \rho(g_1(a),g_2(a)) + TW_\rho\big(g_1,g_2,I_a^b\big) = \rho_2(g_1,g_2).$$

As in the proof of Theorem 2, the estimate for S_1 follows from the definition of d_L:

$$d_1^{ij}=d((f(t_{i-1},s_{j-1})+f(t_i,s_j))g_1(t_{i-1},s_{j-1}) + (f(t_{i-1},s_j)+f(t_i,s_{j-1}))g_2(t_{i-1},s_{j-1}),$$
$$(f(t_{i-1},s_{j-1})+f(t_i,s_j))g_2(t_{i-1},s_{j-1}) + (f(t_{i-1},s_j)+f(t_i,s_{j-1}))g_1(t_{i-1},s_{j-1}))$$
$$\le d_L(f(t_{i-1},s_{j-1})+f(t_i,s_j), f(t_{i-1},s_j)+f(t_i,s_{j-1}))\rho(g_1(t_{i-1},s_{j-1}),g_2(t_{i-1},s_{j-1}))$$
$$\le \mathrm{md}(f,I_{ij})\rho_2(g_1,g_2),$$

whence

$$S_1 \le V_2\big(f,I_a^b\big)\rho_2(g_1,g_2).$$

By analogy, we obtain estimates for S_k similar to those in the proof of Theorem 2 in which we should replace $V_{a_1}^{b_1}(g(\cdot,a_2))$ with $W_{a_1}^{b_1}(g_1(\cdot,a_2),g_2(\cdot,a_2))$, $V_{a_2}^{b_2}(g(a_1,\cdot))$ with $W_{a_2}^{b_2}(g_1(a_1,\cdot),g_2(a_1,\cdot))$, and $V_2\big(g,I_a^b\big)$ with $W_2\big(g_1,g_2,I_a^b\big)$.

Consequently, combining these estimates, we find that

$$d_2(\mathscr{H}g_1,\mathscr{H}g_2) \le 4\|f\|_{d_L}\rho_2(g_1,g_2).$$

The general case for $h_0 \in \mathrm{BV}_2\big(I_a^b; M\big)$ follows from that above by the translation invariance of d_2 on $\mathrm{BV}_2\big(I_a^b; M\big)$. $\square$

Remark 2. Suppose that N and M are the same as in Theorem 3 and $g \in \mathrm{BV}_2\big(I_a^b; N\big)$. Then the operator $H : \mathrm{BV}_2\big(I_a^b; \mathrm{L}(N;M)\big) \to \mathrm{BV}_2\big(I_a^b; M\big)$ acting by the rule $H(f) = fg$ is Lipschitzian with a Lipschitz constant $L(H) \le 4\|g\|_\rho$.

Remark 3. It is immediate from the Banach Fixed Point Theorem and Theorem 3 that if M is a complete metric semigroup with zero, $h_0 \in \mathrm{BV}_2\big(I_a^b; M\big)$, $f \in \mathrm{BV}_2\big(I_a^b; \mathrm{L}(N;M)\big)$, and $\|f\|_{d_L} < 1/4$ then there is a unique mapping $g \in \mathrm{BV}_2\big(I_a^b; M\big)$ such that $g(x) = f(x)g(x) + h_0(x)$ for all $x \in I_a^b$.

Remark 4. In view of Remark 1 (see also Remark 6 in [1]), an analog of Theorem 3 holds also for mappings of a single variable.

§ 5. Lipschitzian Iterated Superposition Operators

Consider another approach to defining the space $\mathrm{BV}_2\big(I_a^b; M\big)$, when $(M, d, +)$ is a metric semigroup. Take $f \in \mathrm{BV}_2\big(I_a^b; M\big)$. Then $f(\cdot, s) \in \mathrm{BV}_1([a_1, b_1]; M)$ for all $s \in [a_2, b_2]$ and similarly $f(t, \cdot) \in \mathrm{BV}_1([a_2, b_2]; M)$ for all $t \in [a_1, b_1]$; moreover, the following inequalities hold [7, 8]:

$$V_{x_1}^{y_1}(f(\cdot, s)) \le V_{x_1}^{y_1}(f(\cdot, a_2)) + V_2\big(f, I_{x_1,a_2}^{y_1,s}\big), \quad x_1, y_1 \in [a_1, b_1],\ x_1 \le y_1, \tag{15}$$

$$V_{x_2}^{y_2}(f(t, \cdot)) \le V_{x_2}^{y_2}(f(a_1, \cdot)) + V_2\big(f, I_{a_1,x_2}^{t,y_2}\big), \quad x_2, y_2 \in [a_2, b_2],\ x_2 \le y_2. \tag{16}$$

Put $I_k = [a_k, b_k]$, $k = 1, 2$, so that $I_a^b = I_1 \times I_2$. By (16), $f(t, \cdot) \in \mathrm{BV}_1(I_2; M)$ for every $t \in I_1$; therefore, if $\mathscr{F}(t) = f(t, \cdot)$ for $t \in I_1$ then the mapping $\mathscr{F} : I_1 \to \mathrm{BV}_1(I_2; M)$ acts by the rule $\mathscr{F}(t)(s) = f(t, s)$, $t \in I_1$, $s \in I_2$. As observed above, the space $\mathrm{BV}_1(I_2; M)$ is a metric semigroup with the metric $d_1(\varphi, \psi) = d(\varphi(a_2), \psi(a_2)) + W_{a_2}^{b_2}(\varphi, \psi)$ and hence we can compute the variation of $\mathscr{F}$ on the interval I_1. To this end, let $\xi = \{t_i\}_{i=0}^m$ be a partition of I_1. Consider the expression

$$d_1(\mathscr{F}(t_i), \mathscr{F}(t_{i-1})) = d(\mathscr{F}(t_i)(a_2), \mathscr{F}(t_{i-1})(a_2)) + W_{a_2}^{b_2}(\mathscr{F}(t_i), \mathscr{F}(t_{i-1})). \tag{17}$$

It is clear that the first summand on the right-hand side is equal to $d(f(t_i, a_2), f(t_{i-1}, a_2))$. To estimate the second summand, suppose that $\eta = \{s_j\}_{j=0}^n$ is a partition of I_2. Then (see (2) and (4))

$$d(\mathscr{F}(t_i)(s_j) + \mathscr{F}(t_{i-1})(s_{j-1}), \mathscr{F}(t_{i-1})(s_j) + \mathscr{F}(t_i)(s_{j-1})) = \mathrm{md}(f, I_{ij}) \tag{18}$$

and from additivity of V_2 we find that

$$\sum_{j=1}^n d(\mathscr{F}(t_i)(s_j) + \mathscr{F}(t_{i-1})(s_{j-1}), \mathscr{F}(t_{i-1})(s_j) + \mathscr{F}(t_i)(s_{j-1}))$$
$$= \sum_{j=1}^n \mathrm{md}(f, I_{ij}) \le \sum_{j=1}^n V_2(f, I_{ij}) = V_2\big(f, I_{t_{i-1},a_2}^{t_i,b_2}\big).$$

Consequently, in view of the arbitrariness of η,

$$W_{a_2}^{b_2}(\mathscr{F}(t_i), \mathscr{F}(t_{i-1})) \le V_2\big(f, I_{t_{i-1},a_2}^{t_i,b_2}\big), \quad i = 1, \dots, m. \tag{19}$$

Then from (17) we obtain

$$\sum_{i=1}^m d_1(\mathscr{F}(t_i), \mathscr{F}(t_{i-1})) \le \sum_{i=1}^m d(f(t_i, a_2), f(t_{i-1}, a_2)) + \sum_{i=1}^m V_2\big(f, I_{t_{i-1},a_2}^{t_i,b_2}\big)$$
$$\le V_{a_1}^{b_1}(f(\cdot, a_2)) + V_2\big(f, I_a^b\big),$$

whence (in view of the arbitrariness of the partition ξ)

$$V_{a_1}^{b_1}(\mathscr{F}) \le V_{a_1}^{b_1}(f(\cdot, a_2)) + V_2\bigl(f, I_a^b\bigr).$$

Returning again to the partitions ξ and η, from (18) we also find that

$$\sum_{i=1}^{m}\sum_{j=1}^{n} \mathrm{md}(f, I_{ij}) \le \sum_{i=1}^{m} W_{a_2}^{b_2}(\mathscr{F}(t_i), \mathscr{F}(t_{i-1})) \tag{20}$$

and therefore $V_2\bigl(f, I_a^b\bigr) \le V_{a_1}^{b_1}\bigl(\mathscr{F}; W_{a_2}^{b_2}\bigr)$, where $V_{a_1}^{b_1}\bigl(\mathscr{F}; W_{a_2}^{b_2}\bigr)$ is the variation of $\mathscr{F}$ over the interval I_1 calculated in the semimetric $W_{a_2}^{b_2}$. The last inequality together with (19) gives

$$V_2\bigl(f, I_a^b\bigr) = V_{a_1}^{b_1}\bigl(\mathscr{F}; W_{a_2}^{b_2}\bigr).$$

Moreover, using the first summand of (17), from (20) we find that

$$\sum_{i=1}^{m} d(f(t_i, a_2), f(t_{i-1}, a_2)) + \sum_{i=1}^{m}\sum_{j=1}^{n} \mathrm{md}\bigl(f, I_{t_{i-1}, s_{j-1}}^{t_i, s_j}\bigr)$$
$$\le \sum_{i=1}^{m} d_1(\mathscr{F}(t_i), \mathscr{F}(t_{i-1})) \le V_{a_1}^{b_1}(\mathscr{F}),$$

and since the sums on the left-hand side do not decrease upon refining of the partition ξ, we have

$$V_{a_1}^{b_1}(f(\cdot, a_2)) + V_2\bigl(f, I_a^b\bigr) \le V_{a_1}^{b_1}(\mathscr{F}).$$

We have thus shown that $\mathscr{F} \in \mathrm{BV}_1(I_1; \mathrm{BV}_1(I_2; M))$ and

$$V_{a_1}^{b_1}(\mathscr{F}) = V_{a_1}^{b_1}(f(\cdot, a_2)) + V_2\bigl(f, I_a^b\bigr).$$

Similarly, if $\mathscr{G}(s)(t) = f(t, s)$, $t \in I_1$, $s \in I_2$, then $\mathscr{G} \in \mathrm{BV}_1(I_2; \mathrm{BV}_1(I_1; M))$,

$$V_{a_2}^{b_2}(\mathscr{G}) = V_{a_2}^{b_2}(f(a_1, \cdot)) + V_2\bigl(f, I_a^b\bigr) \quad \text{and} \quad V_2\bigl(f, I_a^b\bigr) = V_{a_2}^{b_2}\bigl(\mathscr{G}; W_{a_1}^{b_1}\bigr).$$

Indicating the dependence of the mappings $\mathscr{F}$ and $\mathscr{G}$ on f, i.e., writing them in the form $\mathscr{F}_f$ and $\mathscr{G}_f$, we arrive at the equality

$$\mathrm{BV}_2\bigl(I_a^b; M\bigr) = \bigl\{f : I_a^b \to M \mid \mathscr{F}_f \in \mathrm{BV}_1(I_1; \mathrm{BV}_1(I_2; M)) \text{ and } \mathscr{G}_f \in \mathrm{BV}_1(I_2; \mathrm{BV}_1(I_1; M))\bigr\},$$

which can be written in the following *symbolic* form:

$$\mathrm{BV}_2\bigl(I_a^b; M\bigr) = \mathrm{BV}_1(I_1; \mathrm{BV}_1(I_2; M)) \cap \mathrm{BV}_1(I_2; \mathrm{BV}_1(I_1; M)).$$

Henceforth we put $f(t, s) = f(t)(s)$, $t \in I_1$, $s \in I_2$, for $f \in \mathrm{BV}_1(I_1; \mathrm{BV}_1(I_2; M))$.

To study iterated superposition operators (see below), we need the following lemma:

Lemma 1. *If $(M, d, +)$ is a metric semigroup then the metric $d_{11} = (d_1)_1$ on the metric semigroup $\mathrm{BV}_1(I_1; \mathrm{BV}_1(I_2; M))$ is given by the equality $d_{11} = d_2$.*

Proof. Let d_1 be the above-introduced metric on $\mathrm{BV}_1\bigl(I_2; M\bigr)$. Suppose that mappings f and g lie in $\mathrm{BV}_1(I_1; \mathrm{BV}_1(I_2; M))$. Then

$$d_{11}(f, g) = d_1(f(a_1), g(a_1)) + \sup_{\xi} \sum_{i=1}^{m} d_1(f(t_i) + g(t_{i-1}), g(t_i) + f(t_{i-1})),$$

where the supremum is taken over all partitions $\xi = \{t_i\}_{i=0}^m$ of the interval $[a_1, b_1]$. We have

$$d_1(f(a_1), g(a_1)) = d(f(a_1)(a_2), g(a_1)(a_2))$$
$$+ \sup_{\eta} \sum_{j=1}^{n} d(f(a_1)(s_j) + g(a_1)(s_{j-1}), g(a_1)(s_j) + f(a_1)(s_{j-1}))$$
$$= d(f(a), g(a)) + W_{a_2}^{b_2}(f(a_1, \cdot), g(a_1, \cdot)),$$

where the supremum is taken over all partitions $\eta = \{s_j\}_{j=0}^n$ of the interval $[a_2, b_2]$. Now,

$$d_1(f(t_i) + g(t_{i-1}), g(t_i) + f(t_{i-1})) = d(f(t_i)(a_2) + g(t_{i-1})(a_2), g(t_i)(a_2) + f(t_{i-1})(a_2))$$
$$+ \sup_{\eta} \sum_{j=1}^{n} d(f(t_i)(s_j) + g(t_{i-1})(s_j) + g(t_i)(s_{j-1}) + f(t_{i-1})(s_{j-1}),$$
$$g(t_i)(s_j) + f(t_{i-1})(s_j) + f(t_i)(s_{j-1}) + g(t_{i-1})(s_{j-1}))$$
$$= d(f(t_i)(a_2) + g(t_{i-1})(a_2), g(t_i)(a_2) + f(t_{i-1})(a_2)) + \sup_{\eta} \sum_{j=1}^{n} \mathrm{md}_2(f, g, I_{ij}),$$

where (4) is used. It follows from additivity of W_2 that

$$\sum_{j=1}^{n} \mathrm{md}_2(f, g, I_{ij}) \le \sum_{j=1}^{n} W_2(f, g, I_{ij}) = W_2\big(f, g, I_{t_{i-1}, a_2}^{t_i, b_2}\big),$$

and therefore

$$\sup_{\xi} \sum_{i=1}^{m} d_1(f(t_i) + g(t_{i-1}), g(t_i) + f(t_{i-1})) \le W_{a_1}^{b_1}(f(\cdot, a_2), g(\cdot, a_2)) + W_2\big(f, g, I_a^b\big). \tag{21}$$

Observing that

$$d(f(t_i)(a_2) + g(t_{i-1})(a_2), g(t_i)(a_2) + f(t_{i-1})(a_2)) + \sum_{j=1}^{n} \mathrm{md}_2(f, g, I_{ij})$$
$$\le d_1(f(t_i) + g(t_{i-1}), g(t_i) + f(t_{i-1}))$$

and that the summands on the left-hand side of the above inequality do not decrease as we add points to the partition $\xi = \{t_i\}_{i=0}^m$, we arrive at the reverse inequality in (21). We are left with using the expression for $d_2(f, g)$. $\square$

Although, as demonstrated above, the mappings in $\mathrm{BV}_2\big(I_a^b; M\big)$ are of bounded iterated variation and the equality $d_{11} = d_2$ holds for the metrics, the Lipschitzian superposition operators $\mathscr{H}$ on the latter have a somewhat different structure (see Theorem 4 below). The point here is that the left-left regularization may fail to exist for mappings of bounded iterated variation.

Given $g \in (N^{I_2})^{I_1}$ (i.e., $g : I_1 \to N^{I_2}$) or $g \in N^{I_a^b}$ (i.e., $g : I_a^b \to N$), we put $g(t, s) = g(t)(s)$ for all $t \in I_1$ and $s \in I_2$. Given a mapping $h : I_a^b \times N \to M$, the operator $\mathscr{H} : (N^{I_2})^{I_1} \to (M^{I_2})^{I_1}$ acting by the rule

$$(\mathscr{H} g)(t)(s) \equiv (\mathscr{H} g)(t, s) = h(t, s, g(t, s)) \equiv h(t, s, g(t)(s)) \tag{22}$$

for $(t, s) \in I_1 \times I_2$ and $g \in (N^{I_2})^{I_1}$ is called the *Nemytskii iterated superposition operator with generator h*.

In the theorem below we use the following notation for the left regularization h^- (in the one-dimensional sense, see [1, Remark 6]) of a mapping $h \in \mathrm{BV}_1([a, b]; M)$, when $(M, d, +)$ is a complete metric semigroup: $h^-(t) = \lim_{s \to t-0} h(s)$ for $a < t \le b$ and $h^-(a) = \lim_{t \to a+0} h^-(t)$ in M. If we denote by $\mathrm{BV}_1^-([a, b]; M)$ the set of mappings in $\mathrm{BV}_1([a, b]; M)$ that are left continuous on $(a, b]$ then $h^- \in \mathrm{BV}_1^-([a, b]; M)$ and $V_a^b(h^-) \le V_a^b(h)$.

Theorem 4. *Suppose that $(N, \rho, +, \cdot)$ and $(M, d, +, \cdot)$ are two abstract convex cones, where M is complete and $h : I_a^b \times N \to M$ is the generator of an iterated superposition operator $\mathscr{H}$ in (22). If $\mathscr{H}$ takes $\mathrm{BV}_1(I_1; \mathrm{BV}_1(I_2; N))$ to $\mathrm{BV}_1(I_1; \mathrm{BV}_1(I_2; M))$ and is Lipschitzian then $h(x, \cdot) \in \mathrm{Lip}(N; M)$ for all $x \in I_a^b$ and there exist two mappings $f : I_a^b \to \mathrm{L}(N; M)$ and $h_0 : I_a^b \to M$ such that $f(\cdot)(\cdot)u, h_0 \in \mathrm{BV}_1^-(I_1; \mathrm{BV}_1(I_2; M))$ for all $u \in N$ and $h^-(t, s, u) = f(t, s)u + h_0(t, s)$ in M for all $(t, s) \in I_a^b$ and $u \in N$, where $h^-(t, s, u)$ is the left regularization (in the one-dimensional sense) of the mapping $\tau \mapsto h(\tau, s, u)$ at $t \in [a_1, b_1]$ for all fixed $s \in I_2$ and $u \in N$ (observe that the mappings $\tau \mapsto f(\tau, s)u$ and $\tau \mapsto h_0(\tau, s)$ are left continuous on $(a_1, b_1]$ for all $s \in I_2$ and $u \in N$).*

Proof. Put $N_1 = \mathrm{BV}_1(I_2; N)$ and $M_1 = \mathrm{BV}_1(I_2; M)$. Then the quadruples $(N_1, \rho_1, +, \cdot)$ and $(M_1, d_1, +, \cdot)$ are also abstract convex cones; moreover, M_1 is complete. Define the mapping $h_1 : I_1 \times N^{I_2} \to M^{I_2}$ by the rule

$$h_1(t, u_1)(s) = h(t, s, u_1(s)), \quad t \in I_1,\ s \in I_2,\ u_1 \in N^{I_2}. \tag{23}$$

With this in mind, we define the superposition operator $\mathscr{H}_1 : (N^{I_2})^{I_1} \to (M^{I_2})^{I_1}$ as follows:

$$\mathscr{H}_1(g)(t) = h_1(t, g(t)), \quad t \in I_1,\ g \in (N^{I_2})^{I_1}. \tag{24}$$

Observe that if $t \in I_1$ and $s \in I_2$ then

$$\mathscr{H}_1(g)(t)(s) = h_1(t, g(t))(s) = h(t, s, g(t)(s)) = (\mathscr{H} g)(t, s). \tag{25}$$

Show that $h_1 : I_1 \times N_1 \to M_1$. Indeed, let $t \in I_1$ and $u_1 \in N_1$. Put $g(t)(s) = u_1(s)$ for $t \in I_1$ and $s \in I_2$, so that $g \in \mathrm{BV}_1(I_1; N_1)$. By assumption, $\mathscr{H} g$ lies in $\mathrm{BV}_1(I_1; M_1)$; therefore, $(\mathscr{H} g)(t) \in M_1$, however

$$h_1(t, u_1)(s) = h(t, s, u_1(s)) = h(t, s, g(t)(s)) = (\mathscr{H} g)(t)(s), \quad s \in I_2,$$

whence $h_1(t, u_1) = (\mathscr{H} g)(t) \in M_1$. Hence, $\mathscr{H}_1 : (N_1)^{I_1} \to (M_1)^{I_1}$ and (24) is valid for $t \in I_1$ and $g \in (N_1)^{I_1}$; i.e., the mapping $h_1 : I_1 \times N_1 \to M_1$ is the generator of the superposition operator $\mathscr{H}_1 : (N_1)^{I_1} \to (M_1)^{I_1}$. Moreover, $\mathscr{H}_1 : \mathrm{BV}_1(I_1; N_1) \to \mathrm{BV}_1(I_1; M_1)$, since, by (25) and the conditions of the theorem, $g \in \mathrm{BV}_1(I_1; N_1)$ implies $\mathscr{H}_1(g) = \mathscr{H} g \in \mathrm{BV}_1(I_1; M_1)$. From the Lipschitz continuity of $\mathscr{H}$ we obtain $d_2(\mathscr{H} g_1, \mathscr{H} g_2) \le L(\mathscr{H})\rho_2(g_1, g_2)$ for all $g_1, g_2 \in \mathrm{BV}_1(I_1; N_1)$, however $d_2 = (d_1)_1$ and $\rho_2 = (\rho_1)_1$ by Lemma 1; therefore, by (25), we find that $\mathscr{H}_1 \in \mathrm{Lip}(\mathrm{BV}_1(I_1; N_1); \mathrm{BV}_1(I_1; M_1))$. By Remark 6 of [1],

$$h_1(t, \cdot) \in \mathrm{Lip}(N_1; M_1) \quad \text{for all } t \in I_1 \tag{26}$$

and there exist two mappings $f_1 : I_1 \to \mathrm{L}(N_1; M_1)$ and $h_0 : I_1 \to M_1$ such that the mappings $f_1(\cdot)u_1$ and h_0 lie in $\mathrm{BV}_1^-(I_1; M_1)$ for all $u_1 \in N_1$; moreover,

$$h_1^-(t, u_1) = f_1(t)u_1 + h_0(t) \quad \text{in } M,\ t \in I_1,\ u_1 \in N_1, \tag{27}$$

where $h_1^-(\cdot, u_1)$ is the left regularization of $h_1(\cdot, u_1)$, $u_1 \in N_1$.

In the proof below, given $u, v \in N$, we put $u_1(s) = u$ and $v_1(s) = v$ for all $s \in I_2$. Using (23), (3), and (26), for $u, v \in N$ we find that

$$\begin{aligned} d(h(t, s, u), h(t, s, v)) &= d(h_1(t, u_1)(s), h_1(t, v_1)(s)) \le d_1(h_1(t, u_1), h_1(t, v_1)) \\ &\le L(h_1(t, \cdot))\rho_1(u_1, v_1) = L(h_1(t, \cdot))\rho(u, v), \end{aligned}$$

whence $h(x, \cdot) \in \mathrm{Lip}(N; M)$ for all $x = (t, s) \in I_a^b$.

Given $(t, s) \in I_a^b$, we define the mapping $f(t, s) = f(t)(s) : N \to M$ by the rule

$$f(t, s)u = [f_1(t)u_1](s), \quad u \in N.$$

Then from (27) we derive the following equality in M:

$$h_1^-(t, u_1)(s) = [f_1(t)u_1](s) + h_0(t)(s) = f(t, s)u + h_0(t, s). \tag{28}$$

Show that, in fact, the mapping $f(t, s)$ is additive and Lipschitzian; i.e., $f(t, s) \in \mathrm{L}(N; M)$. By additivity of $f_1(t)$, for $u, v \in N$ we obtain

$$\begin{aligned} f(t, s)(u + v) &= [f_1(t)(u + v)_1](s) = [f_1(t)(u_1 + v_1)](s) = [f_1(t)u_1 + f_1(t)v_1](s) \\ &= [f_1(t)u_1](s) + [f_1(t)v_1](s) = f(t, s)u + f(t, s)v; \end{aligned}$$

moreover, (3) and the Lipschitz continuity of $f_1(t)$ imply that

$$\begin{aligned} d(f(t, s)u, f(t, s)v) &= d([f_1(t)u_1](s), [f_1(t)v_1](s)) \le d_1(f_1(t)u_1, f_1(t)v_1) \\ &\le L(f_1(t))\rho_1(u_1, v_1) = L(f_1(t))\rho(u, v). \end{aligned}$$

Since $f(t)(\cdot) = f_1(t)u_1 \in M_1$, we have $f : I_1 \to M_1$. Moreover, $f(\cdot)(\cdot)u = f_1(\cdot)u_1$ belongs to $\mathrm{BV}_1(I_1; M_1)$ and

$$d_1(f(\tau)(\cdot)u, f(t)(\cdot)u) = d_1(f_1(\tau)u_1, f_1(t)u_1) \to 0 \quad \text{as } \tau \to t - 0;$$

therefore, $\tau \mapsto f(\tau)(\cdot)u$ is left continuous, so that $f(\cdot)(\cdot)u \in \mathrm{BV}_1^-(I_1; M_1)$. It remains to calculate the left-hand side of (28). From (23) and (3) we obtain

$$\begin{aligned} d(h(\tau, s, u), h_1^-(t, u_1)(s)) &= d(h_1(\tau, u_1)(s), h_1^-(t, u_1)(s)) \\ &\le d_1(h_1(\tau, u_1), h_1^-(t, u_1)) \to 0 \quad \text{as } \tau \to t - 0, \end{aligned}$$

and it remains to put $h^-(t, s, u) = h_1^-(t, u_1)(s) = \lim_{\tau \to t-0} h(\tau, s, u)$. $\square$

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