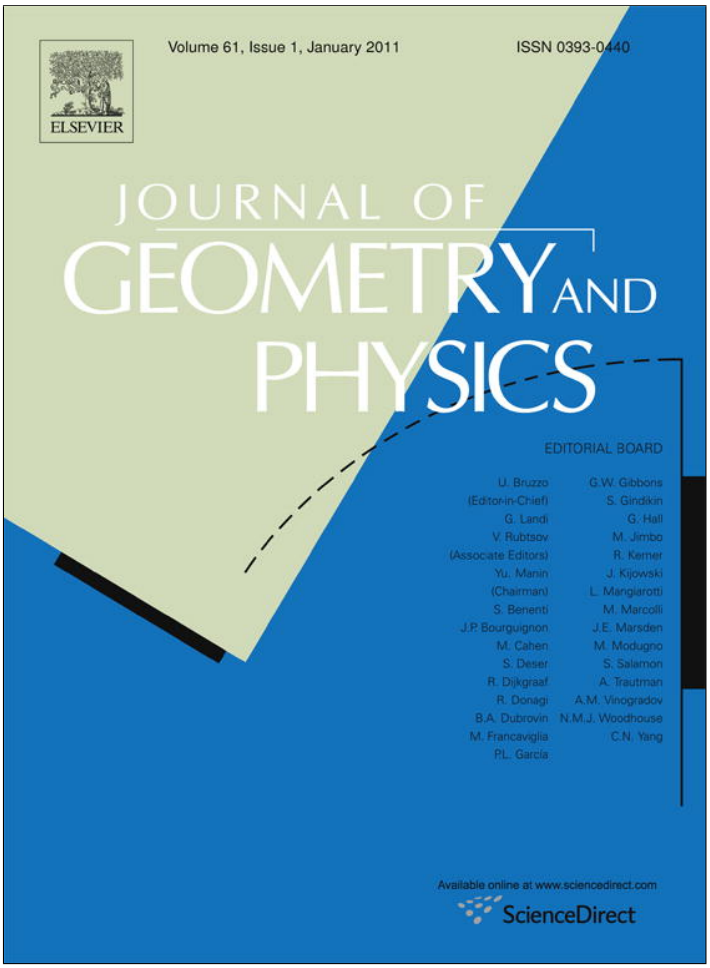

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Apparent singularities of Fuchsian equations and the Painlevé property for Garnier systems

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ABSTRACT

We study movable singularities of Garnier systems using the connection of the latter with Schlesinger isomonodromic deformations of Fuchsian systems. Questions on the existence of solutions for some inverse monodromy problems are also considered.

1. Introduction

In the middle of the XIXth century, B. Riemann considered the problem of the construction of a linear differential equation

$$\frac{d^p u}{dz^p} + b_1(z)\frac{d^{p-1}u}{dz^{p-1}} + \cdots + b_p(z)u = 0 \tag{1}$$

with the prescribed regular singularities $a_1, \ldots, a_n \in \overline{\mathbb{C}}$ (which are the poles of the coefficients) and prescribed monodromy.

Recall that a singular point a_i of Eq. (1) is said to be *regular* if any solution of the equation is of no more than a polynomial (with respect to $1/|z - a_i|$) growth in any sectorial neighbourhood of the point a_i.

By Fuchs's theorem [1] (see also [2, Th. 12.1]), a singular point a_i is regular if and only if the coefficient $b_j(z)$ has at this point a pole of order j or lower ($j = 1, \ldots, p$). Linear differential equations with regular singular points only are called *Fuchsian*.

Poincaré [3] has established that the number of parameters determining a Fuchsian equation of order p with n singular points is less than the dimension of the space $\mathcal{M}$ of monodromy representations, if $p > 2, n > 2$ or $p = 2, n > 3$ (see also [4, pp. 158–159]). Hence in the construction of a Fuchsian equation with the given monodromy there arise (besides $a_1, \ldots, a_n$) the so-called *apparent* singularities at which the coefficients of the equation have poles but the solutions are single-valued meromorphic functions, i. e., the monodromy matrices at these points are identity matrices. Below by apparent

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singular points of an equation we mean these very singularities. Thus, in general case the Riemann problem has a negative solution.

A similar problem for systems of linear differential equations is called the *Riemann–Hilbert problem*. This is the problem of the construction of a *Fuchsian system*

$$\frac{\mathrm{d}y}{\mathrm{d}z} = \left(\sum_{i=1}^{n} \frac{B_i}{z-a_i}\right) y, \quad y(z) \in \mathbb{C}^p,\ B_i \in \mathrm{Mat}(p, \mathbb{C}), \tag{2}$$

with the given singularities $a_1, \ldots, a_n$ (if ∞ is not a singular point of the system, then $\sum_{i=1}^{n} B_i = 0$) and monodromy

$$\chi : \pi_1(\overline{\mathbb{C}} \setminus \{a_1, \ldots, a_n\}, z_0) \longrightarrow \mathrm{GL}(p, \mathbb{C}). \tag{3}$$

A counterexample to the Riemann–Hilbert problem was obtained by Bolibrukh (see [4, Ch. 5]). The solution of this problem has a more complicated history than that of the Riemann problem for scalar Fuchsian equations (before Bolibrukh it had long been wrongly regarded as solved in the affirmative; for details see [5]).

Alongside Fuchsian equations consider the famous non-linear differential equations—the *Painlevé* VI *equation* ($\mathrm{P_{VI}}$) and *Garnier systems*.

The equation $\mathrm{P_{VI}}(\alpha, \beta, \gamma, \delta)$ is the non-linear differential equation

$$\begin{aligned}\frac{\mathrm{d}^2u}{\mathrm{d}t^2} &= \frac{1}{2}\left(\frac{1}{u} + \frac{1}{u-1} + \frac{1}{u-t}\right)\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{u-t}\right)\frac{\mathrm{d}u}{\mathrm{d}t} \\ &\quad + \frac{u(u-1)(u-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{u^2} + \gamma\frac{t-1}{(u-1)^2} + \delta\frac{t(t-1)}{(u-t)^2}\right)\end{aligned} \tag{4}$$

of second order with respect to the unknown function $u(t)$, where $\alpha, \beta, \gamma, \delta$ are complex parameters. This equation has three fixed singular points, $0, 1, \infty$. Its movable singularities (which depend on the initial conditions) can be poles only. In such a case one says that an equation satisfies the *Painlevé property*. The general $\mathrm{P_{VI}}$ equation (4) was first written down by Fuchs [6] and was added to the list of the equations now known as the *Painlevé* I–VI *equations* by Painlevé's student Gambier [7]. Among the non-linear differential equations of second order satisfying the Painlevé property, only the equations of this list in general case cannot be reduced to the known differential equations for elementary and classical special functions. The $\mathrm{P_{VI}}$ equation is the most general because all the other $\mathrm{P_{I-V}}$ equations can be derived from it by certain limit processes after the substitution of the independent variable t and parameters (see [8]).

The Garnier system $\mathcal{G}_n(\theta)$ depending on $n+3$ complex parameters $\theta_1, \ldots, \theta_{n+2}, \theta_\infty$ is a completely integrable system of non-linear partial differential equations of second order obtained by Garnier [9]. It was written down by Okamoto [8] in an equivalent Hamiltonian form

$$\frac{\partial u_i}{\partial a_j} = \frac{\partial H_j}{\partial v_i}, \qquad \frac{\partial v_i}{\partial a_j} = -\frac{\partial H_j}{\partial u_i}, \quad i, j = 1, \ldots, n, \tag{5}$$

with certain Hamiltonians $H_i = H_i(a, u, v, \theta)$ rationally depending on $a = (a_1, \ldots, a_n)$, $u = (u_1, \ldots, u_n)$, $v = (v_1, \ldots, v_n)$, $\theta = (\theta_1, \ldots, \theta_{n+2}, \theta_\infty)$. In the case $n = 1$ the Garnier system $\mathcal{G}_1(\theta_1, \theta_2, \theta_3, \theta_\infty)$ is an equivalent (Hamiltonian) form of $\mathrm{P_{VI}}(\alpha, \beta, \gamma, \delta)$, where

$$\alpha = \frac{1}{2}\theta_\infty^2, \qquad \beta = -\frac{1}{2}\theta_2^2, \qquad \gamma = \frac{1}{2}\theta_3^2, \qquad \delta = \frac{1}{2}(1-\theta_1^2).$$

There exist classical results [6,9] on the connection of scalar Fuchsian equations of second order with $\mathrm{P_{VI}}$ equations and Garnier systems. Let us consider a scalar Fuchsian equation of second order with singular points $a_1, \ldots, a_n, a_{n+1} = 0, a_{n+2} = 1, a_{n+3} = \infty$ and apparent singularities $u_1, \ldots, u_n$ whose Riemann scheme has the form

$$\begin{pmatrix} a_i & \infty & u_k \\ 0 & \alpha & 0 \\ \theta_i & \alpha + \theta_\infty & 2 \end{pmatrix}, \quad i = 1, \ldots, n+2,\ k = 1, \ldots, n,\ \theta_i \notin \mathbb{Z}$$

(α depends on the parameters θ_i according to the classical Fuchs relation $\sum_{i=1}^{n+2} \theta_i + \theta_\infty + 2\alpha + 2n = 2n+1$). There is freedom of choice of such an equation. Its coefficients $b_1(z), b_2(z)$ depend on a, u, θ and n arbitrary parameters $v_1, \ldots, v_n$ ($v_i = \mathrm{res}_{u_i} b_2(z)$).

Fix a set θ ($\theta_i \notin \mathbb{Z}$) and consider an ($n$-dimensional) integral manifold M of the system $\mathcal{G}_n(\theta)$. Due to Okamoto's theorem [8], *Fuchsian equations corresponding to points* $(a, u, v) \in M$ *have the same monodromy.*[1] *Inversely, points* (a, u, v) *corresponding to Fuchsian equations with the same monodromy lie on the integral manifold of the system* $\mathcal{G}_n(\theta)$.

[1] This property is defined precisely in Section 3.

Using the above relationship between Fuchsian and non-linear differential equations one can deduce the known properties of the latter as well as some new ones. In particular, we study movable singularities of Garnier systems and give estimates for pole orders of the elementary symmetric polynomials $\sigma_i(u_1, \dots, u_n)$ depending on solutions to these systems (see Theorem 3). For this end Bolibrukh's estimate for orders of movable poles of solutions to the Schlesinger equation are essentially used. We also consider the Riemann problem for some class of $\mathrm{GL}(2, \mathbb{C})$-representations proving that any element of this class for almost all locations of points $a_1, \dots, a_n$ ($n \geqslant 4$) is realized by a scalar Fuchsian equation of second order that has (besides $a_1, \dots, a_n$) exactly $n-3$ apparent singularities (see Propositions 3 and 4) (earlier Ohtsuki [10] has established that the number of apparent singularities is *at most* $n-3$). This fact is based on Bolibrukh's formula for the number of apparent singularities arising in the construction of a scalar Fuchsian equation with the prescribed *irreducible* monodromy.

2. Method of solution of the Riemann–Hilbert problem

In the study of problems related to the Riemann–Hilbert problem a very useful tool is provided by linear gauge transformations of the form

$$y' = \Gamma(z)y \tag{6}$$

of the unknown function $y(z)$. The transformation (6) is said to be *holomorphically (meromorphically) invertible* at some point $z = a$, if the matrix $\Gamma(z)$ is holomorphic (meromorphic) at this point and $\det \Gamma(a) \neq 0$ ($\det \Gamma(z) \not\equiv 0$). This transformation transforms system (2) into the system

$$\frac{\mathrm{d}y'}{\mathrm{d}z} = B'(z)y', \quad B'(z) = \frac{\mathrm{d}\Gamma}{\mathrm{d}z}\Gamma^{-1} + \Gamma\left(\sum_{i=1}^{n} \frac{B_i}{z-a_i}\right)\Gamma^{-1}, \tag{7}$$

which is said to be, respectively, *holomorphically* or *meromorphically equivalent* to the original system in a neighbourhood of the point a.

An important property of meromorphic gauge transformations is the fact that they do not change the monodromy (being meromorphic, the matrix $\Gamma(z)$ is single-valued, therefore the ramification of the fundamental matrix $\Gamma(z)Y(z)$ of the new system coincides with the ramification of the matrix $Y(z)$).

Locally, in a neighbourhood of each point a_k, it is not difficult to produce a system for which a_k is a Fuchsian singularity and the monodromy matrix at this point coincides with the corresponding generator $G_k = \chi([\gamma_k])$ of the representation (3). This system is

$$\frac{\mathrm{d}y}{\mathrm{d}z} = \frac{E_k}{z-a_k}y, \quad E_k = \frac{1}{2\pi \mathrm{i}} \ln G_k, \tag{8}$$

with fundamental matrix $(z-a_k)^{E_k} := \mathrm{e}^{E_k \ln(z-a_k)}$. The branch of the logarithm of the matrix G_k is chosen such that the eigenvalues ρ_k^α of the matrix E_k satisfy the condition

$$0 \leqslant \operatorname{Re} \rho_k^\alpha < 1. \tag{9}$$

Indeed,

$$\frac{\mathrm{d}}{\mathrm{d}z}(z-a_k)^{E_k} = \frac{E_k}{z-a_k}(z-a_k)^{E_k},$$

and a single circuit around the point a_k counterclockwise transforms the matrix $(z-a_k)^{E_k}$ into the matrix

$$\mathrm{e}^{E_k(\ln(z-a_k)+2\pi \mathrm{i})} = \mathrm{e}^{E_k \ln(z-a_k)}\mathrm{e}^{2\pi \mathrm{i} E_k} = (z-a_k)^{E_k}G_k.$$

Of course, not any system with the Fuchsian singularity a_k and the local monodromy matrix G_k is holomorphically equivalent to the system (8) in a neighbourhood of this point.

Let S_k be a non-singular matrix reducing the matrix E_k to a block-diagonal form $E_k' = S_k E_k S_k^{-1} = \operatorname{diag}(E_k^1, \dots, E_k^m)$, where each block E_k^j is an upper-triangular matrix with the unique eigenvalue ρ_k^j. Consider a diagonal integer-valued matrix $\Lambda_k = \operatorname{diag}(\Lambda_k^1, \dots, \Lambda_k^m)$ with the same block structure and such that the diagonal elements of each block Λ_k^j form a non-increasing sequence. Then according to (7) the transformation

$$y' = \Gamma(z)y, \quad \Gamma(z) = (z-a_k)^{\Lambda_k}S_k,$$

transforms the system (8) into the system

$$\frac{\mathrm{d}y'}{\mathrm{d}z} = \left(\frac{\Lambda_k}{z-a_k} + (z-a_k)^{\Lambda_k}\frac{E_k'}{z-a_k}(z-a_k)^{-\Lambda_k}\right)y', \tag{10}$$

for which the point a_k is also a Fuchsian singularity[2] and the matrix G_k is the monodromy matrix. We call a set $\{\Lambda_1, \dots, \Lambda_n, S_1, \dots, S_n\}$ of matrices having the properties described above, a set of *admissible matrices*.

According to Levelt's theorem [11], the Fuchsian system (2) is *holomorphically* equivalent to a system of form (10) (with some matrix Λ_k) in a neighbourhood of the singular point a_k, i.e., the system has a fundamental matrix

$$Y_k(z) = U_k(z)(z-a_k)^{\Lambda_k}(z-a_k)^{E'_k},$$

where the matrix $U_k(z)$ is holomorphically invertible at $z = a_k$. The matrix $Y_k(z)$ is called the *Levelt fundamental matrix* (its columns form the *Levelt basis*).

The eigenvalues β_k^j of the residue matrix B_k are said to be the *exponents* of the Fuchsian system (2) at the point a_k. They are invariants of the *holomorphic* equivalence class of this system. From (10) it follows that the exponents coincide with the eigenvalues of the matrix $\Lambda_k + E'_k$. The matrix Λ_k is said to be the *valuation matrix* of the Fuchsian system (2) at the singularity a_k. According to (9), its diagonal elements coincide with the integer parts of the numbers $\operatorname{Re}\beta_k^j$.

The Riemann–Hilbert problem has a positive solution if one can pass from the local systems (10) to a global Fuchsian system defined on the whole Riemann sphere. The use of holomorphic vector bundles and meromorphic connections proves to be effective in the study of this question.

From the representation (3) one constructs over the Riemann sphere a family $\mathcal{F}$ of holomorphic vector bundles of rank p with logarithmic (Fuchsian) connections having the prescribed singular points $a_1, \dots, a_n$ and monodromy (3). The Riemann–Hilbert problem for the fixed representation (3) is solved in the affirmative if some bundle in the family $\mathcal{F}$ turns out to be holomorphically trivial (then the corresponding logarithmic connection defines a Fuchsian system with the given singularities $a_1, \dots, a_n$ and monodromy (3) on the whole Riemann sphere). We now briefly present the construction of the family $\mathcal{F}$ (see details in [4, Sect. 3.1, 3.2 and 5.1]).

1. First, from the representation (3) over the punctured Riemann sphere $B = \overline{\mathbb{C}} \setminus \{a_1, \dots, a_n\}$ one constructs a holomorphic vector bundle F of rank p with a holomorphic connection ∇ that has the given monodromy (3). The bundle F over B is obtained from the holomorphically trivial bundle $\widetilde{B} \times \mathbb{C}^p$ over the universal cover $\widetilde{B}$ of the punctured Riemann sphere after identifications of the form $(\tilde{z}, y) \sim (\sigma\tilde{z}, \chi(\sigma)y)$, where $\tilde{z} \in \widetilde{B}$, $y \in \mathbb{C}^p$ and σ is an element of the group of deck transformations of $\widetilde{B}$ which is identified with the fundamental group $\pi_1(B)$. Thus, $F = \widetilde{B} \times \mathbb{C}^p/\sim$ and $\pi : F \longrightarrow B$ is the natural projection. It is not difficult to show that a gluing cocycle $\{g_{\alpha\beta}\}$ of the bundle F is defined by constant matrices $g_{\alpha\beta}$ after some choice of a covering $\{U_\alpha\}$ of the punctured Riemann sphere.

The holomorphic connection ∇ can now be given by the set $\{\omega_\alpha\}$ of matrix differential 1-forms $\omega_\alpha \equiv 0$, which obviously satisfy the gluing conditions

$$\omega_\alpha = (\mathrm{d}g_{\alpha\beta})g_{\alpha\beta}^{-1} + g_{\alpha\beta}\omega_\beta g_{\alpha\beta}^{-1} \tag{11}$$

on the intersections $U_\alpha \cap U_\beta \neq \varnothing$. Furthermore, it follows from the construction of the bundle F that the monodromy of the connection ∇ coincides with χ.

2. Next, the pair (F, ∇) is extended to a bundle F^0 with a logarithmic connection ∇^0 over the whole Riemann sphere. For this, the set $\{U_\alpha\}$ should be supplemented by small neighbourhoods $O_1, \dots, O_n$ of the points $a_1, \dots, a_n$, respectively. An extension of the bundle F to each point a_i looks as follows. For some non-empty intersection $O_i \cap U_\alpha$ one takes $g_{i\alpha}(z) = (z-a_i)^{E_i}$ on this intersection. For any other neighbourhood U_β that intersects O_i one defines $g_{i\beta}(z)$ as the analytic continuation of the matrix function $g_{i\alpha}(z)$ into $O_i \cap U_\beta$ along a suitable path (so that the set $\{g_{\alpha\beta}, g_{i\alpha}(z)\}$ defines a cocycle for the covering $\{U_\alpha, O_i\}$ of the Riemann sphere). An extension of the connection ∇ to each point a_i is given by the matrix differential 1-form $\omega_i = E_i\mathrm{d}z/(z-a_i)$, which has a simple pole at this point. Then the set $\{\omega_\alpha, \omega_i\}$ defines a logarithmic connection ∇^0 in the bundle F^0, since along with the conditions (11) for non-empty $U_\alpha \cap U_\beta$, the conditions

$$(\mathrm{d}g_{i\alpha})g_{i\alpha}^{-1} + g_{i\alpha}\omega_\alpha g_{i\alpha}^{-1} = \frac{E_i}{z-a_i}\mathrm{d}z = \omega_i, \qquad O_i \cap U_\alpha \neq \varnothing,$$

also hold (see (8)). The pair (F^0, ∇^0) is called the *canonical extension* of the pair (F, ∇).

3. In a way similar to that for the construction of the pair (F^0, ∇^0), one can construct the family $\mathcal{F}$ of bundles F^Λ with logarithmic connections ∇^Λ having the given singularities $a_1, \dots, a_n$ and monodromy (3). For this, the matrices $g_{i\alpha}(z)$ in the construction of the pair (F^0, ∇^0) should be replaced by the matrices

$$g_{i\alpha}^\Lambda(z) = (z-a_i)^{\Lambda_i}S_i(z-a_i)^{E_i},$$

and the forms ω_i by the forms

$$\omega_i^\Lambda = \left(\Lambda_i + (z-a_i)^{\Lambda_i}E'_i(z-a_i)^{-\Lambda_i}\right)\frac{\mathrm{d}z}{z-a_i},$$

where $\{\Lambda_1, \dots, \Lambda_n, S_1, \dots, S_n\}$ are all possible sets of admissible matrices. Then the conditions

$$(\mathrm{d}g_{i\alpha}^\Lambda)(g_{i\alpha}^\Lambda)^{-1} + g_{i\alpha}^\Lambda\omega_\alpha(g_{i\alpha}^\Lambda)^{-1} = \omega_i^\Lambda \tag{12}$$

again hold on the non-empty intersections $O_i \cap U_\alpha$ (see (10)).

[2] As follows from the form of the matrices Λ_k and E'_k, the matrix $(z-a_k)^{\Lambda_k}E'_k(z-a_k)^{-\Lambda_k}$ is holomorphic.

Remark 1. Strictly speaking, the bundle F^{Λ} also depends on the set $S = \{S_1, \dots, S_n\}$ of the matrices S_i reducing the monodromy matrices G_i to an upper-triangular form. In view of this dependence the bundles of the family $\mathcal{F}$ should be denoted by $F^{\Lambda,S}$. But in the following two cases all bundles $F^{\Lambda,S}$ with a fixed Λ are holomorphically equivalent.

(i) All points a_i are *non-resonant*, i. e., for each valuation matrix Λ_i all its blocks Λ_i^j are scalar matrices.
(ii) Resonant points exist, but each resonant point a_i has the following property: for its monodromy matrix G_i and any $\lambda \in \mathbb{C}$ one has the inequality $\operatorname{rank}(G_i - \lambda I) \geqslant p - 1$.

In particular, in the two-dimensional case ($p = 2$) if all monodromy matrices G_i are non-scalar, then bundles of the family $\mathcal{F}$ depend on sets Λ only.

The exponents β_i^j of the local Fuchsian system $\mathrm{d}y = \omega_i^{\Lambda} y$ are called the *exponents of the logarithmic connection* ∇^{Λ} at the point $z = a_i$.

According to the Birkhoff–Grothendieck theorem, every holomorphic vector bundle E of rank p over the Riemann sphere is equivalent to a direct sum

$$E \cong \mathcal{O}(k_1) \oplus \cdots \oplus \mathcal{O}(k_p)$$

of line bundles which has a coordinate description of the form

$$\left(U_0 = \mathbb{C}, U_\infty = \overline{\mathbb{C}} \setminus \{0\}, g_{0\infty} = z^K\right), \quad K = \operatorname{diag}(k_1, \dots, k_p),$$

where $k_1 \geqslant \cdots \geqslant k_p$ is a set of integers which is called the *splitting type* of the bundle E. The bundle E is holomorphically trivial if and only if it has the zero splitting type.

The number $\deg E = \sum_{i=1}^p k_i$ equals the *degree* of the bundle E. For the pair $(F^{\Lambda}, \nabla^{\Lambda})$ the degree of the bundle F^{Λ} coincides with the sum $\sum_{i=1}^n \sum_{j=1}^p \beta_i^j = \sum_{i=1}^n \operatorname{tr}(\Lambda_i + E_i)$ of the exponents of the connection ∇^{Λ}.

If some bundle F^{Λ} in the family $\mathcal{F}$ is holomorphically trivial then the corresponding logarithmic connection ∇^{Λ} defines a global Fuchsian system (2) that solves the Riemann–Hilbert problem. On the other hand, in view of Levelt's theorem mentioned above, the existence of a Fuchsian system with the given singular points $a_1, \dots, a_n$ and monodromy (3) implies the triviality of some bundle in the family $\mathcal{F}$.

Thus, *the Riemann–Hilbert problem is soluble if and only if at least one of the bundles of the family $\mathcal{F}$ is holomorphically trivial* (see [4, Th. 5.1.1]).

3. Isomonodromic deformations of Fuchsian systems

Let us include a Fuchsian system

$$\frac{\mathrm{d}y}{\mathrm{d}z} = \left(\sum_{i=1}^n \frac{B_i^0}{z - a_i^0}\right) y, \quad \sum_{i=1}^n B_i^0 = 0, \tag{13}$$

of p equations into a family

$$\frac{\mathrm{d}y}{\mathrm{d}z} = \left(\sum_{i=1}^n \frac{B_i(a)}{z - a_i}\right) y, \quad \sum_{i=1}^n B_i(a) = 0, \ B_i(a^0) = B_i^0, \tag{14}$$

of Fuchsian systems holomorphically depending on the parameter $a = (a_1, \dots, a_n) \in D(a^0)$, where $D(a^0)$ is a disc of small radius centred at the point $a^0 = (a_1^0, \dots, a_n^0)$ of the space $\mathbb{C}^n \setminus \bigcup_{i \neq j} \{a_i = a_j\}$.

One says that the family (14) is *isomonodromic* (or it is an *isomonodromic deformation* of the system (13)), if for all $a \in D(a^0)$ the monodromies

$$\chi : \pi_1(\overline{\mathbb{C}} \setminus \{a_1, \dots, a_n\}) \longrightarrow \mathrm{GL}(p, \mathbb{C})$$

of the corresponding systems are the same. (Under small variations of the parameter a there exist canonical isomorphisms of the fundamental groups $\pi_1(\overline{\mathbb{C}} \setminus \{a_1, \dots, a_n\})$ and $\pi_1(\overline{\mathbb{C}} \setminus \{a_1^0, \dots, a_n^0\})$ generating canonical isomorphisms

$$\operatorname{Hom}\left(\pi_1(\overline{\mathbb{C}} \setminus \{a_1, \dots, a_n\}), \mathrm{GL}(p, \mathbb{C})\right) / \mathrm{GL}(p, \mathbb{C}) \cong \operatorname{Hom}\left(\pi_1(\overline{\mathbb{C}} \setminus \{a_1^0, \dots, a_n^0\}), \mathrm{GL}(p, \mathbb{C})\right) / \mathrm{GL}(p, \mathbb{C})$$

of the spaces of conjugacy classes of representations for the above fundamental groups; this allows one to compare χ for various $a \in D(a^0)$.) This means that for every value of a from $D(a^0)$ there exists a fundamental matrix $Y(z, a)$ of the system (14) that has the same monodromy matrices for all $a \in D(a^0)$. This matrix $Y(z, a)$ is called an *isomonodromic fundamental matrix*.

For any isomonodromic family (14) there exists an isomonodromic fundamental matrix that analytically depends on both variables z and a. An isomonodromic deformation preserves not only the monodromy but also the exponents of the initial system (thus, the eigenvalues of the residue matrices $B_i(a)$ of the family (14) do not depend on the parameter a; see [12] on the two latter statements).

Is it always possible to include the system (13) into an isomonodromic family of Fuchsian systems? The answer is positive. For instance, if the matrices $B_i(a)$ satisfy the *Schlesinger equation* [13]

$$\mathrm{d}B_i(a) = -\sum_{j=1, j\neq i}^{n} \frac{[B_i(a), B_j(a)]}{a_i - a_j} d(a_i - a_j),$$

then the family (14) is isomonodromic (in this case it is called the *Schlesinger isomonodromic family*).

A Schlesinger isomonodromic family has the following property: connection matrices between some fixed isomonodromic fundamental matrix $Y(z, a)$ and local Levelt's bases at singular points do not depend on a. Among all isomonodromic deformations of Fuchsian systems with this property, the Schlesinger ones are distinguished by the condition $(d_a Y(z, a))\, Y^{-1}(z, a)|_{z=\infty} \equiv 0$ (see [12]).

It is well known that *for arbitrary initial conditions* $B_i(a^0) = B_i^0$ *the Schlesinger equation has a unique solution* $\{B_1(a), \ldots, B_n(a)\}$ *in some disc* $D(a^0)$, *and the matrices* $B_i(a)$ *can be extended to the universal cover* Z *of the space* $\mathbb{C}^n \setminus \bigcup_{i\neq j}\{a_i = a_j\}$ *as meromorphic functions* (Malgrange's theorem [14]). Thus, the Schlesinger equation satisfies the Painlevé property.

Recall, that a function $f(a)$ is *meromorphic* on Z, if it is holomorphic on $Z \setminus P$, it cannot be extended to P holomorphically and is presented as a quotient $f(a) = \varphi(a)/\psi(a)$ of holomorphic functions in a neighbourhood of every point $a^0 \in P$ (hence, $\psi(a^0) = 0$). Thus, $P \subset Z$ is an analytic set of codimension one (it is defined locally by the equation $\psi(a) = 0$), which is called the *polar locus* of the meromorphic function f. The points of this set is divided into *poles* (at which the function φ does not vanish) and *ambiguous points* (at which $\varphi = 0$).

One can also define a divisor of a meromorphic function. Denote by $A = N \cup P$ the union of the set N of zeros and polar locus P of the function f. Any regular point a^0 of the set A can belong to only one irreducible component of N or P. Thus, one can define the *order* of this component as the degree (taken with "+", if $a^0 \in N$, and with "−", if $a^0 \in P$) of the corresponding factor in the decomposition of the function φ or ψ into irreducible factors. Then the *divisor* of the meromorphic function f is the pair (A, κ), where $\kappa = \kappa(a)$ is an integer-valued function on the set of regular points of A (which takes a constant value on each its irreducible component, this value is equal to the order of a component). The pair (P, κ) is called the *polar divisor* of the meromorphic function f. By $(f)_\infty$ we will mean the restriction of κ on regular points of P.

Notation. For the polar locus P of the function f, and $a^0 \in P$, let us denote by $\Sigma_{a^0}(f)$ the sum of orders of all irreducible components of $P \cap D(a^0)$.

Example. (a) The function $f(a) = 1/a_1a_2$ is meromorphic on $\mathbb{C}^2$. Its polar locus is $P = \{a_1a_2 = 0\}$ (all points are poles), the order of each component $\{a_i = 0\}$ is equal to -1 (thus, $(f)_\infty \equiv -1$), and $\Sigma_0(f) = -2$.

(b) The function $g(a) = a_1/a_2$ is meromorphic on $\mathbb{C}^2$. Its polar locus is $P = \{a_2 = 0\}$ (0 is an ambiguous point, all the others are poles), the set of zeros is $N = \{a_1 = 0\} \setminus \{0\}$. The order of the component $\{a_1 = 0\}$ is equal to 1, the order of the component $\{a_2 = 0\}$ is equal to -1 (thus, $(g)_\infty \equiv -1$), and $\Sigma_0(g) = -1$.

Let us return to the Schlesinger equation. The polar locus $\Theta \subset Z$ of the extended matrix functions $B_1(a), \ldots, B_n(a)$ is called the *Malgrange Θ-divisor*[3] (Θ depends on the initial conditions $B_i(a^0) = B_i^0$). Near a point $a^* \in \Theta$ it is defined by the equation $\tau^*(a) = 0$, where $\tau^*(a)$ is a holomorphic function in a neighbourhood of the point a^* called a *local τ-function* of the Schlesinger equation. According to Miwa's theorem [15] (see also [16]) there exists a function $\tau(a)$ holomorphic on the whole space Z whose set of zeros coincides with Θ. In a neighbourhood of the point $a^* \in \Theta$ the global τ-function differs from the local one by a holomorphic non-zero multiplier, and

$$\mathrm{d}\ln\tau(a) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1, j\neq i}^{n} \frac{\mathrm{tr}(B_i(a)B_j(a))}{a_i - a_j} d(a_i - a_j).$$

If we consider system (13) as an equation for horizontal sections of the logarithmic connection $\nabla_{a^0}^{\Lambda}$ (with singularities $a_1^0, \ldots, a_n^0$) in the trivial bundle $F_{a^0}^{\Lambda}$ (where Λ is a set of valuation matrices of the system), then the set Θ corresponds to those points, where the bundle F_a^{Λ} associated to the parameter a in the isomonodromic deformation of $(F_{a^0}^{\Lambda}, \nabla_{a^0}^{\Lambda})$ is not holomorphically trivial.

In what follows, we will use the theorem describing a general solution of the Schlesinger equation near the Θ-divisor in the case $p = 2$.

Theorem 1 (*Bolibrukh [17,5]*). *If the monodromy of the two-dimensional family (14) is irreducible, then* $\Sigma_{a^*}(B_i) \geqslant 2 - n$ *for any* $a^* \in \Theta$ $(i = 1, \ldots, n)$.

Further we present a simplified proof of this theorem based on the technique of the paper [16], but first we recall this technique in the proof of Proposition 1.

[3] In view of the above definition of a divisor, here the term "divisor" is not precise enough.

Consider an *irreducible* two-dimensional representation

$$\chi_a : \pi_1(\overline{\mathbb{C}} \setminus \{a_1, \dots, a_n\}) \longrightarrow \mathrm{GL}(2, \mathbb{C}),$$

$a \in D(a^0) \subset \mathbb{C}^n \setminus \bigcup_{i \neq j}\{a_i = a_j\}$, and the family $(F_a^\Lambda, \nabla_a^\Lambda)_{a \in D(a^0)}$ of holomorphic vector bundles with logarithmic connections constructed by the corresponding representations χ_a and a set Λ of admissible matrices.

Proposition 1. *If* $\deg F_a^\Lambda = 0$ *(recall that the degree does not change along the isomonodromic deformation* $(F_a^\Lambda, \nabla_a^\Lambda)_{a \in D(a^0)}$*), then for all* $a \in D(a^0)$*, may be, with the exception of an analytic subset of codimension one, the bundle* F_a^Λ *is holomorphically trivial (i.e., for almost all* $a \in D(a^0)$ *there exists a Fuchsian system with the given singular points* $a_1, \dots, a_n$*, monodromy* χ_a *and set* Λ *of valuation matrices).*

We should note that this proposition for connections with $sl(2, \mathbb{C})$-residues contains in Corollary 1 from the paper of Heu [18]. She considers universal isomonodromic deformations of irreducible tracefree meromorphic rank 2 connections over compact Riemann surfaces using a geometrical approach. Here we give an analytical proof.

Proof. Choose an arbitrary point $a^* = (a_1^*, \dots, a_n^*) \in D(a^0)$. Suppose the corresponding bundle $F_{a^*}^\Lambda$ is not holomorphically trivial:

$$F_{a^*}^\Lambda \cong \mathcal{O}(-k) \oplus \mathcal{O}(k), \quad k \geqslant 1.$$

Let us show that the set of points a, for which the corresponding bundle F_a^Λ is not holomorphically trivial, is given by an equation $\tau^*(a) = 0$ in a neighbourhood of the point a^*, where $\tau^*(a) \not\equiv 0$ is a holomorphic function.

Consider an auxiliary system

$$\frac{\mathrm{d}y}{\mathrm{d}z} = \left(\sum_{i=1}^n \frac{B_i^*}{z - a_i^*}\right) y$$

with the monodromy χ_{a^*}, valuation matrices $\Lambda_1, \dots, \Lambda_n$ at the points $a_1^*, \dots, a_n^*$ respectively but also with the apparent Fuchsian singularity at the infinity.

As follows from Bolibrukh's permutation lemma (Lemma 2 from [19]), a fundamental matrix of the constructed system has the form $Y(z) = U(z)z^K$ near the infinity, where

$$U(z) = I + U_1\frac{1}{z} + U_2\frac{1}{z^2} + \cdots, \qquad K = \mathrm{diag}(-k, k).$$

Therefore, the residue matrix at the infinity is equal to $-K$, and $\sum_{i=1}^n B_i^* = K$.

We need the following proposition which will be also used further.

Proposition 2 (*Bolibrukh [19]*)**.** *Consider the Fuchsian system (2) with the singularities* $a_1, \dots, a_n$*, apparent singularity* ∞*, monodromy (3) and set* $\Lambda = \{\Lambda_1, \dots, \Lambda_n\}$ *of valuation matrices; furthermore* $\sum_{i=1}^n B_i = K' = \mathrm{diag}(k_1, \dots, k_p)$*, where* $k_1 \leqslant \cdots \leqslant k_p$ *are integers.*

The matrix K' *defines the splitting type of the bundle* F^Λ *if and only if the transformation* $y' = z^{-K'}y$ *transforms this system into the system that is holomorphic at the infinity.*

Due to this proposition the transformation $y' = z^{-K}y$ transforms our auxiliary system into the system that is holomorphic at the infinity, hence

$$U(z)z^K = z^K V(z)$$

for some matrix $V(z)$ holomorphically invertible at the infinity. The latter relation implies that the upper-right element u_1^{12} of the matrix U_1 equals zero.

Using the theorem of existence and uniqueness for the Schlesinger equation, the constructed Fuchsian system was included into the Schlesinger isomonodromic family

$$\frac{\mathrm{d}y}{\mathrm{d}z} = \left(\sum_{i=1}^n \frac{B_i(a)}{z - a_i}\right) y, \qquad B_i(a^*) = B_i^*, \qquad \sum_{i=1}^n B_i(a) = K. \tag{15}$$

As shown in [16], there exists an isomonodromic fundamental matrix $Y(z, a)$ of this family of the form

$$Y(z, a) = U(z, a)z^K, \quad U(z, a) = I + U_1(a)\frac{1}{z} + U_2(a)\frac{1}{z^2} + \cdots, \tag{16}$$

at the infinity, $U(z, a^*) = U(z)$ and

$$\frac{\partial U_1(a)}{\partial a_i} = -B_i(a), \quad i = 1, \dots, n. \tag{17}$$

Since the monodromy χ_a is irreducible, among the upper-right elements $b_i^{12}(a)$ of the corresponding matrices $B_i(a)$ there exists at least one that is not identically zero. Hence, in view of (17), the similar element $u_1^{12}(a)$ of the matrix $U_1(a)$ does not equal zero identically (while $u_1^{12}(a^*) = u_1^{12} = 0$).

Further, whereas

$$\frac{\mathrm{d}Y(z,a)}{\mathrm{d}z}Y^{-1}(z,a) = \sum_{i=1}^{n}\frac{B_i(a)}{z-a_i} = \frac{1}{z}\sum_{i=1}^{n}\frac{B_i(a)}{1-\frac{a_i}{z}},$$

from (16) one gets the relation

$$-U_1(a)\frac{1}{z^2} + o(z^{-2}) + \left(I + U_1(a)\frac{1}{z} + o(z^{-1})\right)\frac{K}{z} = \left(\frac{K}{z} + \left(\sum_{i=1}^{n}B_i(a)a_i\right)\frac{1}{z^2} + o(z^{-2})\right)\left(I + U_1(a)\frac{1}{z} + o(z^{-1})\right).$$

Hence,

$$-U_1(a) + [U_1(a), K] = \sum_{i=1}^{n}B_i(a)a_i.$$

Therefore,

$$(2k-1)u_1^{12}(a) = \sum_{i=1}^{n}b_i^{12}(a)a_i.$$

Denote by $b_1(a)$ the sum $\sum_{i=1}^{n}b_i^{12}(a)a_i$. Then

$$b_1(a) = (2k-1)u_1^{12}(a) \not\equiv 0, \qquad b_1(a^*) = 0.$$

Consider the matrix

$$\Gamma_1'(z,a) = \begin{pmatrix} 1 & 0 \\ \dfrac{1-2k}{b_1(a)}z & 1 \end{pmatrix},$$

holomorphically invertible (in z) off the infinity. One can directly check that the matrix $U'(z,a) = \Gamma_1'U(z,a)$ has the form

$$U'(z,a) = \left(U_0'(a) + U_1'(a)\frac{1}{z} + \cdots\right)z^{\mathrm{diag}(1,-1)}, \quad U_0'(a) = \begin{pmatrix} 0 & \dfrac{b_1(a)}{2k-1} \\ \dfrac{1-2k}{b_1(a)} & \dfrac{f(a)}{b_1(a)} \end{pmatrix},$$

where $f(a)$ is a holomorphic function at the point a^*. Thus, the gauge transformation $y_1 = \Gamma_1(z,a)y$, $\Gamma_1(z,a) = U_0'(a)^{-1}\Gamma_1'(z,a)$, transforms a system of the family (15) into the Fuchsian system with the fundamental matrix $Y^1(z,a) = \Gamma_1(z,a)Y(z,a)$ of the form (16) at the infinity (and does not change valuations at the points $a_1,\dots,a_n$), where all involved matrices are equipped with the upper index 1, and $K^1 = \mathrm{diag}(-k+1, k-1)$. This expansion is valid only in the exterior of some analytic subset of codimension one which is the set of zeros of the function $b_1(a) \not\equiv 0$.

Note also that the transformed family is a Schlesinger isomonodromic family. Indeed, its connection matrices do not depend on a (the transformation does not change those of the Schlesinger family (15)), and $(d_aY^1(z,a))Y^1(z,a)^{-1}|_{z=\infty} = (d_aU^1(z,a))U^1(z,a)^{-1}|_{z=\infty} \equiv 0$ according to the form of the matrix $U^1(z,a)$.

After k steps of the above procedure of Bolibrukh we will get a Fuchsian family holomorphic at the infinity. It is defined in a neighbourhood of the point a^* outside of the analytic subset $\{\tau^*(a) = 0\}$, $\tau^*(a) = b_1(a)\dots b_k(a)$, where $b_j(a)$ appears at the j-th step of the Bolibrukh procedure in the same way as $b_1(a)$ does. This means that for all $a \notin \{\tau^*(a) = 0\}$ from the neighbourhood of the point a^* there exists a Fuchsian system with the singularities $a_1,\dots,a_n$, monodromy χ_a and set Λ of valuation matrices. □

Definition. Recall that if all generators G_i of the *two-dimensional* representation χ are non-scalar matrices, then bundles of the family $\mathcal{F}$ depend on sets Λ only (see Remark 1). One calls such representations *non-smaller*. In the opposite case, if l monodromy matrices are scalar, χ is called *l-smaller*.

Corollary 1. *If χ_a is an irreducible non-smaller* $\mathrm{SL}(2,\mathbb{C})$*-representation with generators $G_1,\dots,G_n$, then for almost all $a \in D(a^0)$ there exists a family (depending on the parameter $\mathbf{m} = (m_1,\dots,m_n) \in \mathbb{Z}_+^n$)*

$$\frac{\mathrm{d}y}{\mathrm{d}z} = \left(\sum_{i=1}^{n}\frac{B_i^{\mathbf{m}}(a)}{z-a_i}\right)y, \quad \sum_{i=1}^{n}B_i^{\mathbf{m}}(a) = 0,$$

of Fuchsian systems with the singularities $a_1, \dots, a_n$, monodromy χ and exponents $\pm(m_k+\rho_k)$, where ρ_k is one of the eigenvalues of the matrix $E_k = (1/2\pi \mathrm{i}) \ln G_k (k = 1, \dots, n)$. Furthermore, $B_n^{\mathbf{m}}(a) = \mathrm{diag}(m_n + \rho_n, -m_n - \rho_n)$ are diagonal matrices.

Proof. If a set $\Lambda = \{\Lambda_1, \dots, \Lambda_n\}$ of admissible matrices satisfies the conditions $\mathrm{tr}(\Lambda_k + E_k) = 0,\ k = 1, \dots, n$, then by Proposition 1 for all $a \in D(a^0)$, may be, with the exception of an analytic subset Θ_Λ of codimension one, the corresponding bundle F_a^Λ is holomorphically trivial and the logarithmic connection ∇_a^Λ defines a Fuchsian system with the singularities $a_1, \dots, a_n$, monodromy χ_a and set Λ of valuation matrices.

By the relations $\mathrm{e}^{2\pi \mathrm{i} E_k} = G_k$, $\det G_k = 1$, the sum $\rho_k^1 + \rho_k^2$ of the eigenvalues of the matrix E_k is an integer, and it equals 0 or 1 by condition (9). Fix an order of the eigenvalues ρ_k^1, ρ_k^2 and put $\rho_k = \rho_k^1$.

(1) If $\rho_k^1 + \rho_k^2 = 0$, then one can take $\Lambda_k = \mathrm{diag}(m_k, -m_k)$, $m_k \in \mathbb{Z}_+$ (but if $\rho_n = 0$, then $m_n \in \mathbb{N}$).
(2) If $\rho_k^1 + \rho_k^2 = 1$, then one can take $\Lambda_k = \mathrm{diag}(m_k, -m_k - 1)$, $m_k \in \mathbb{Z}_+$.

Thus, for all $a \in D(a^0) \setminus \Theta_{\mathbf{m}}$ the representation χ_a can be realized by a Fuchsian system with the singular points $a_1, \dots, a_n$ and exponents $\pm(m_1 + \rho_1), \dots, \pm(m_n + \rho_n)$. Moreover, the residue matrix at the point a_n is diagonalizable (because its eigenvalues $\pm(m_n + \rho_n)$ do not equal zero by the construction). Then the statement of the corollary is valid for all $a \in D(a^0) \setminus \bigcup_{\mathbf{m}} \Theta_{\mathbf{m}}$. □

Proof of Theorem 1. For $a^* \in \Theta$ the corresponding vector bundle $F_{a^*}^\Lambda \cong \mathcal{O}(-k) \oplus \mathcal{O}(k)$ is not holomorphically trivial and, as shown in the proof of Proposition 1, the Θ-divisor of the family (14) in a neighbourhood of the point a^* is the set of zeros of the function $\tau^*(a) = b_1(a) \dots b_k(a)$ constructed by the auxiliary family (15). Let us denote by $B_i^*(a)$ the residue matrices of the latter (to tell them from those $B_i(a)$ of the initial family (14)). They are holomorphic in a neighbourhood of the point a^*.

The functions $b_j(a)$ are irreducible at a^*, since $\mathrm{d}b_j(a^*) \not\equiv 0$. For instance,

$$\mathrm{d}b_1(a) = (2k-1)\mathrm{d}u_1^{12}(a) = (1-2k)\sum_{i=1}^n b_i^{12}(a)\mathrm{d}a_i$$

in view of (17), and the equality $\mathrm{d}b_1(a^*) \equiv 0$ implies $b_1^{12}(a^*) = \dots = b_n^{12}(a^*) = 0$, which contradicts the irreducibility of the monodromy.

One can assume that $\tau^*(a) = b_1^{m_1}(a) \dots b_r^{m_r}(a)$, $m_1 + \dots + m_r = k$ (some factors are equal). Now let us show that the order of each component $\{b_j(a) = 0\}$ is not less than $-2m_j$. It is sufficient to consider the first step of the Bolibrukh procedure. The transformation $y_1 = \Gamma_1(z, a)y$ transforms the auxiliary family into the family with the coefficient matrix of the form

$$\frac{\mathrm{d}\Gamma_1}{\mathrm{d}z}\Gamma_1^{-1} + \Gamma_1\left(\sum_{i=1}^n \frac{B_i^*(a)}{z - a_i}\right)\Gamma_1^{-1},$$

where

$$\Gamma_1(z, a) = U_0'(a)^{-1}\Gamma_1'(z, a) = \begin{pmatrix} \dfrac{f(a)}{b_1(a)} & \dfrac{b_1(a)}{1-2k} \\ \dfrac{2k-1}{b_1(a)} & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \dfrac{1-2k}{b_1(a)}z & 1 \end{pmatrix} = \begin{pmatrix} \dfrac{f(a)}{b_1(a)} + z & \dfrac{b_1(a)}{1-2k} \\ \dfrac{2k-1}{b_1(a)} & 0 \end{pmatrix}.$$

Thus, the residue matrices $B_i^1(a)$ of the transformed family have the form

$$B_i^1(a) = \begin{pmatrix} \dfrac{f(a)}{b_1(a)} + a_i & \dfrac{b_1(a)}{1-2k} \\ \dfrac{2k-1}{b_1(a)} & 0 \end{pmatrix} B_i^*(a) \begin{pmatrix} \dfrac{f(a)}{b_1(a)} + a_i & \dfrac{b_1(a)}{1-2k} \\ \dfrac{2k-1}{b_1(a)} & 0 \end{pmatrix}^{-1},$$

i.e., the matrices $b_1(a)^2 B_i^1(a)$ are holomorphic in $D(a^*)$.

After the final (k-th) step of the procedure we get the Schlesinger isomonodromic family with the residue matrices $B_i^k(a)$ which are simultaneously conjugated to the corresponding $B_i(a)$ of the initial family (14) by some constant matrix S (this follows from the uniqueness of a solution to the Schlesinger equation). Therefore, $\Sigma_{a^*}(B_i) \geqslant -2m_1 - \dots - 2m_r = -2k \geqslant 2-n$ (see (18)). □

In the case of dimension $p > 2$ one can also apply a similar procedure to find a local τ-function $\tau^*(a) = b_1(a) \dots b_s(a)$. We cannot assert that the functions $b_j(a)$ are irreducible at the point a^*. But if for each $b_j(a)$ all its irreducible factors are distinct (this is the case when the discriminant of the Weierstrass polynomial of each $b_j(a)$ is not identically zero), then one can estimate the order of each irreducible component of the Θ-divisor as follows (see [20]):

$$(B_i)_\infty \geqslant -\frac{(n-2)p(p-1)}{2},$$

if the monodromy of the family is irreducible, and

$$(B_i)_\infty \geqslant -\sum_{i=1}^{n}(M_i-\mu_i)\frac{p(p-1)}{2}$$

in the case of reducible monodromy, where $\mu_i < M_i$ are integers that bound real parts of the eigenvalues of the residue matrix $B_i(a)$.

Conclude this section by the following auxiliary lemma.

Lemma 1. *Consider a two-dimensional Schlesinger isomonodromic family of the form*

$$\frac{\mathrm{d}y}{\mathrm{d}z}=\left(\sum_{i=1}^{n}\frac{B_i(a)}{z-a_i}\right)y,\quad \sum_{i=1}^{n}B_i(a)=K=\mathrm{diag}(\theta,-\theta),\ \theta\in\mathbb{C},$$

and the function $b(a)=\sum_{i=1}^{n}b_i^{12}(a)a_i$, where $b_i^{12}(a)$ are the upper-right elements of the matrices $B_i(a)$, respectively. Then the differential of the function $b(a)$ is given by the formula

$$\mathrm{d}b(a)=(2\theta+1)\sum_{i=1}^{n}b_i^{12}(a)\mathrm{d}a_i.$$

Note that we cannot directly apply calculations of Proposition 1, because an isomonodromic fundamental matrix of the family not necessary has the form (16) (the monodromy at the infinity can be non-diagonal).

Proof. The differential $\mathrm{d}b(a)$ has the form

$$\mathrm{d}b(a)=\sum_{i=1}^{n}a_i\mathrm{d}b_i^{12}(a)+\sum_{i=1}^{n}b_i^{12}(a)\mathrm{d}a_i.$$

To find the first of the two latter summands, let us use the Schlesinger equation

$$\mathrm{d}B_i(a)=-\sum_{j=1,j\neq i}^{n}\frac{[B_i(a),B_j(a)]}{a_i-a_j}d(a_i-a_j)$$

for the matrices $B_i(a)$. Then we have

$$\begin{aligned}\sum_{i=1}^{n}a_i\,\mathrm{d}B_i(a)&=-\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}a_i\frac{[B_i(a),B_j(a)]}{a_i-a_j}d(a_i-a_j)=-\sum_{i=1}^{n}\sum_{j>i}^{n}[B_i(a),B_j(a)]d(a_i-a_j)\\&=-\sum_{i=1}^{n}\left[B_i(a),\sum_{j=1,j\neq i}^{n}B_j(a)\right]\mathrm{d}a_i=-\sum_{i=1}^{n}[B_i(a),K]\mathrm{d}a_i.\end{aligned}$$

The upper-right element of the latter matrix 1-form is equal to $\sum_{i=1}^{n}2\theta b_i^{12}(a)\mathrm{d}a_i$, hence $\sum_{i=1}^{n}a_i\mathrm{d}b_i^{12}(a)=2\theta\sum_{i=1}^{n}b_i^{12}(a)\mathrm{d}a_i$, and $\mathrm{d}b(a)=(2\theta+1)\sum_{i=1}^{n}b_i^{12}(a)\mathrm{d}a_i$. □

4. The Riemann–Hilbert problem and the Painlevé VI equation

As mentioned earlier, the problem of constructing a Fuchsian differential equation (1) with the given singularities $a_1,\dots,a_n$ and monodromy (3) has a negative solution in general case. In the construction there arise apparent singular points. In the case of *irreducible* representation Bolibrukh [19] obtained the formula for the minimal number of such singularities. It is given below.

We consider the family $\mathcal{F}$ of holomorphic vector bundles F^Λ with logarithmic connections ∇^Λ constructed from the representation (3). The *Fuchsian weight* of the bundle F^Λ is defined as the quantity

$$\gamma(F^\Lambda)=\sum_{i=1}^{p}(k_1-k_i),$$

where $(k_1,\dots,k_p)$ is the splitting type of F^Λ.

If the representation (3) is *irreducible*, then the splitting type of the bundle F^Λ satisfies the inequalities

$$k_i-k_{i+1}\leqslant n-2,\quad i=1,\dots,p-1 \tag{18}$$

(see [19, Cor. 3]). Therefore, the quantity

$$\gamma_{\max}(\chi) = \max_{F^\Lambda \in \mathcal{F}} \gamma(F^\Lambda) \leqslant \frac{(n-2)p(p-1)}{2}$$

is defined for such a representation, and is called the *maximal Fuchsian weight* of the irreducible representation χ.

The minimal possible number m_0 of apparent singular points emerging in the construction of a Fuchsian equation (1) with the *irreducible* monodromy (3), is given by the formula

$$m_0 = \frac{(n-2)p(p-1)}{2} - \gamma_{\max}(\chi). \tag{19}$$

In the case of reducible representation there exists the estimate $m_0 \leqslant 1 + (n+1)p(p-1)/2$ obtained in [21].

In particular, it follows from the formula (19) that a set of singular points $a_1, a_2, a_3 (n = 3)$ and irreducible two-dimensional representation ($p = 2$) can always be realized by a Fuchsian differential equation of second order, since in this case $\gamma(F^\Lambda) = 1$ for any bundle F^Λ of odd degree.

A P_{VI} equation appears when one solves the problem of constructing a Fuchsian differential equation of *second* order with *four* given singularities and an *irreducible* monodromy. Further we recall this fact.

Let us consider the four points $t, 0, 1, \infty$ ($t \in D(t^*)$, where $D(t^*) \subset \mathbb{C} \setminus \{0, 1\}$ is a disc of small radius centred at the point t^*) and an *irreducible non-smaller* representation

$$\chi^* : \pi_1(\mathbb{C} \setminus \{t, 0, 1\}) \longrightarrow \mathrm{GL}(2, \mathbb{C}) \tag{20}$$

generated by matrices G_1, G_2, G_3 corresponding to the points $t, 0, 1$ (recall that in this case bundles of the family $\mathcal{F}$ depend on sets Λ of valuation matrices only).

Depending on the location of the point t, there are two possible cases.

(1) *Every vector bundle F^Λ in the family $\mathcal{F}$ constructed with respect to the given four points and representation χ^*, such that* $\deg F^\Lambda = 0$, *is holomorphically trivial* (as follows from Proposition 1, this is the case for almost all values $t \in D(t^*)$).
(2) *Among the elements of the family $\mathcal{F}$ there exists a non-trivial holomorphic vector bundle F^Λ of degree zero.* (Denote by $\widetilde{\Theta}$ the set of values of the parameter t that correspond to this case.)

It follows from the inequalities (18) that $\gamma_{\max}(\chi^*) \leqslant 2$; therefore, in the first case the splitting type of a non-trivial holomorphic vector bundle F^Λ (of non-zero degree) can be $(k, k-1)$ or (k, k) only. The case $(k+1, k-1)$ is impossible, since then the bundle $F^\Lambda \otimes \mathcal{O}(-k)$ constructed with respect to the set of valuation matrices $\Lambda_1 - kI, \Lambda_2, \Lambda_3, \Lambda_\infty$ has degree zero, i.e., it is holomorphically trivial, but at the same time its splitting type is $(1, -1)$. Consequently, $\gamma_{\max}(\chi^*) = 1$ in the first case.

In the second case, the splitting type of the non-trivial holomorphic vector bundle of degree zero equals $(1, -1)$, and $\gamma_{\max}(\chi^*) = 2$ in this case.

Thus, in view of formula (19), for almost all values $t \in D(t^*)$ the set of points $t, 0, 1, \infty$ and representation χ^* can be realized by a Fuchsian differential equation of second order with one apparent singularity. We denote this singularity by $u(t)$ regarding it as a function of the parameter t. It turns out that the function $u(t)$ satisfies Eq. (4) for some values of the constants $\alpha, \beta, \gamma, \delta$, if χ^* is an $\mathrm{SL}(2, \mathbb{C})$-representation.[4] Let us explain this interesting fact by using isomonodromic deformations of Fuchsian systems.

By Corollary 1, we can choose a value $t = t^0 \in D(t^*)$ for which the representation χ^* is realized by Fuchsian systems

$$\frac{\mathrm{d}y}{\mathrm{d}z} = \left(\frac{B_1^{\mathbf{m}}}{z - t^0} + \frac{B_2^{\mathbf{m}}}{z} + \frac{B_3^{\mathbf{m}}}{z-1} \right) y, \quad \mathbf{m} = (m_1, m_2, m_3, m_\infty) \in \mathbb{Z}_+^4, \tag{21}$$

with the singular points $t^0, 0, 1, \infty$ (the eigenvalues of the matrices $B_k^{\mathbf{m}}$ are $\pm(m_k + \rho_k)$, and the matrices $B_\infty^{\mathbf{m}} = -B_1^{\mathbf{m}} - B_2^{\mathbf{m}} - B_3^{\mathbf{m}}$ are diagonal).

Any system of the form (21) can be included into the Schlesinger isomonodromic family

$$\frac{\mathrm{d}y}{\mathrm{d}z} = \left(\frac{B_1^{\mathbf{m}}(t)}{z - t} + \frac{B_2^{\mathbf{m}}(t)}{z} + \frac{B_3^{\mathbf{m}}(t)}{z-1} \right) y, \qquad B_k^{\mathbf{m}}(t^0) = B_k^{\mathbf{m}}, \tag{22}$$

of Fuchsian systems with the singularities $t, 0, 1, \infty$ which depends holomorphically on the parameter $t \in D(t^0)$. Furthermore, $B_1^{\mathbf{m}}(t) + B_2^{\mathbf{m}}(t) + B_3^{\mathbf{m}}(t) = -B_\infty^{\mathbf{m}} = \mathrm{diag}(-m_\infty - \rho_\infty, m_\infty + \rho_\infty)$.

Denote by $B_{\mathbf{m}}(z, t) = (b_{ij}^{\mathbf{m}}(z, t))$ the coefficient matrix of the family (22). Since the upper-right element of the matrix $B_1^{\mathbf{m}}(t) + B_2^{\mathbf{m}}(t) + B_3^{\mathbf{m}}(t) = -B_\infty^{\mathbf{m}}$ is equal to zero, for every fixed t the same element of the matrix $z(z-1)(z-t)B_{\mathbf{m}}(z, t)$ is a polynomial of first degree in z. We define $\tilde{u}_{\mathbf{m}}(t)$ as the unique root of this polynomial. Next we use the following theorem (see, for instance, [15]).

[4] The P_{VI} equation was obtained by Fuchs precisely as a differential equation that is satisfied by the apparent (fifth) singularity $\lambda(t)$ of some Fuchsian equation of second order with the singular points $0, 1, t, \infty$ and $\mathrm{SL}(2, \mathbb{C})$-monodromy independent of the parameter t.

Theorem 2. *The function $\tilde{u}_{\mathbf{m}}(t)$ satisfies Eq. (4), where the constants $\alpha, \beta, \gamma, \delta$ are connected with the parameter $\mathbf{m} = (m_1, m_2, m_3, m_\infty)$ by the relations*

$$\alpha = \frac{(2m_\infty + 2\rho_\infty - 1)^2}{2}, \qquad \beta = -2(m_2 + \rho_2)^2, \qquad \gamma = 2(m_3 + \rho_3)^2, \qquad \delta = \frac{1}{2} - 2(m_1 + \rho_1)^2.$$

Let us consider the row vectors

$$h_0^{\mathbf{m}} = (1, 0), \qquad h_1^{\mathbf{m}}(z, t) = \frac{\mathrm{d}h_0^{\mathbf{m}}}{\mathrm{d}z} + h_0^{\mathbf{m}} B_{\mathbf{m}}(z, t) = (b_{11}^{\mathbf{m}}, b_{12}^{\mathbf{m}})$$

and the matrix composed from them,

$$\Gamma_{\mathbf{m}}(z, t) = \begin{pmatrix} h_0^{\mathbf{m}} \\ h_1^{\mathbf{m}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ b_{11}^{\mathbf{m}} & b_{12}^{\mathbf{m}} \end{pmatrix},$$

which is meromorphically invertible on $\overline{\mathbb{C}} \times D(t^0)$, since $\det \Gamma_{\mathbf{m}}(z, t) = b_{12}^{\mathbf{m}}(z, t) \not\equiv 0$ by the irreducibility of the representation χ^*. We define functions $p_{\mathbf{m}}(z, t)$ and $q_{\mathbf{m}}(z, t)$, meromorphic on $\overline{\mathbb{C}} \times D(t^0)$, so that the relation

$$h_2^{\mathbf{m}}(z, t) := \frac{\mathrm{d}h_1^{\mathbf{m}}}{\mathrm{d}z} + h_1^{\mathbf{m}} B_{\mathbf{m}}(z, t) = (-q_{\mathbf{m}}, -p_{\mathbf{m}}) \Gamma_{\mathbf{m}}(z, t)$$

holds. Then

$$\frac{\mathrm{d}\Gamma_{\mathbf{m}}}{\mathrm{d}z} = \frac{\mathrm{d}}{\mathrm{d}z} \begin{pmatrix} h_0^{\mathbf{m}} \\ h_1^{\mathbf{m}} \end{pmatrix} = \begin{pmatrix} h_1^{\mathbf{m}} \\ h_2^{\mathbf{m}} \end{pmatrix} - \begin{pmatrix} h_0^{\mathbf{m}} \\ h_1^{\mathbf{m}} \end{pmatrix} B_{\mathbf{m}}(z, t) = \begin{pmatrix} 0 & 1 \\ -q_{\mathbf{m}} & -p_{\mathbf{m}} \end{pmatrix} \Gamma_{\mathbf{m}} - \Gamma_{\mathbf{m}} B_{\mathbf{m}}(z, t),$$

whence,

$$\begin{pmatrix} 0 & 1 \\ -q_{\mathbf{m}} & -p_{\mathbf{m}} \end{pmatrix} = \frac{\mathrm{d}\Gamma_{\mathbf{m}}}{\mathrm{d}z} \Gamma_{\mathbf{m}}^{-1} + \Gamma_{\mathbf{m}} B_{\mathbf{m}}(z, t) \Gamma_{\mathbf{m}}^{-1}.$$

The latter means that for every fixed $t \in D(t^0)$ the gauge transformation $y' = \Gamma_{\mathbf{m}}(z, t) y$ transforms the corresponding system of the family (22) into the system

$$\frac{\mathrm{d}y'}{\mathrm{d}z} = \begin{pmatrix} 0 & 1 \\ -q_{\mathbf{m}} & -p_{\mathbf{m}} \end{pmatrix} y',$$

the first coordinate of whose solution is the solution of the scalar equation

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} + p_{\mathbf{m}}(z, t) \frac{\mathrm{d}w}{\mathrm{d}z} + q_{\mathbf{m}}(z, t) w = 0. \tag{23}$$

This (Fuchsian) equation has the singular points $t, 0, 1, \infty$ and monodromy χ^*, but it also has the apparent singularity $u_{\mathbf{m}}(t)$ which is a zero of the function $\det \Gamma_{\mathbf{m}}(z, t) = b_{12}^{\mathbf{m}}(z, t)$, as follows from the construction of the functions $p_{\mathbf{m}}(z, t)$, $q_{\mathbf{m}}(z, t)$. By Theorem 2, the function $u_{\mathbf{m}}(t)$ satisfies an equation $\mathrm{P_{VI}}$.

Thus, we can formulate the following statement.

Proposition 3. (i) *The set of the points $t, 0, 1, \infty$ and any irreducible non-smaller $\mathrm{SL}(2, \mathbb{C})$-representation (20) can be realized by the family (depending on the parameter $\mathbf{m} \in \mathbb{Z}_+^4$) of scalar Fuchsian equations (23) with one apparent singularity.*[5]

(ii) *The set $\widetilde{\Theta} \supset \bigcup_{\mathbf{m}} \{t \in D(t^*) | u_{\mathbf{m}}(t) = t, 0, 1, \text{ or } \infty\}$ is a countable set of parameter values for which the Riemann–Hilbert problem for scalar Fuchsian equations under consideration is soluble without apparent singularities.*

Being solutions of $\mathrm{P_{VI}}$ equations, the functions $u_{\mathbf{m}}(t)$ have only poles as movable singularities (in other words, they can be extended to the universal covering H of the space $\mathbb{C} \setminus \{0, 1\}$ as meromorphic functions). What one can say about their pole orders?

Denote by $b_1^{\mathbf{m}}(t), b_2^{\mathbf{m}}(t), b_3^{\mathbf{m}}(t)$ the upper-right elements of the matrices $B_1^{\mathbf{m}}(t), B_2^{\mathbf{m}}(t), B_3^{\mathbf{m}}(t)$, respectively (recall that $b_1^{\mathbf{m}}(t) + b_2^{\mathbf{m}}(t) + b_3^{\mathbf{m}}(t) \equiv 0$). Since

$$b_{12}^{\mathbf{m}}(z, t) = \frac{(t b_1^{\mathbf{m}} + b_3^{\mathbf{m}}) z + t b_2^{\mathbf{m}}}{z(z-1)(z-t)},$$

the function $u_{\mathbf{m}}(t)$ is given by the relation

$$(t b_1^{\mathbf{m}} + b_3^{\mathbf{m}}) u_{\mathbf{m}} = -t b_2^{\mathbf{m}},$$

from which it follows that poles of the function $u_{\mathbf{m}}(t)$ are poles of the function $b_2^{\mathbf{m}}(t)$ or zeros of the function $t b_1^{\mathbf{m}}(t) + b_3^{\mathbf{m}}(t)$.

[5] The apparent singular point $u_{\mathbf{m}}(t)$ of every equation from this family, as a function of the parameter $t \in D(t^*)$, satisfies the equation $\mathrm{P_{VI}}$ with the constants $\alpha, \beta, \gamma, \delta$ given by Theorem 2.

By Theorem 1 (with $n = 4$), a pole order of the function $b_i^{\mathbf{m}}(t)$ does not exceed two. Applying Lemma 1 to the family (22), where $(a_1, a_2, a_3) = (t, 0, 1)$, one gets

$$\frac{\mathrm{d}}{\mathrm{d}t}(tb_1^{\mathbf{m}}(t) + b_3^{\mathbf{m}}(t)) = (-2m_\infty - 2\rho_\infty + 1)b_1^{\mathbf{m}}(t).$$

If $(m_\infty, \rho_\infty) \neq (0, 1/2)$, then $\theta = -2m_\infty - 2\rho_\infty + 1 \neq 0$. In this case, a pole of the function $b_2^{\mathbf{m}}(t)$ is also a pole for $tb_1^{\mathbf{m}}(t) + b_3^{\mathbf{m}}(t)$, since

$$tb_1^{\mathbf{m}}(t) + b_3^{\mathbf{m}}(t) = b_1^{\mathbf{m}}(t) + b_3^{\mathbf{m}}(t) + (t-1)b_1^{\mathbf{m}}(t) = -b_2^{\mathbf{m}}(t) + \frac{t-1}{\theta}\frac{\mathrm{d}}{\mathrm{d}t}(tb_1^{\mathbf{m}}(t) + b_3^{\mathbf{m}}(t)).$$

From this relation it also follows that any zero t_0 of the function $tb_1^{\mathbf{m}}(t) + b_3^{\mathbf{m}}(t)$ can be simple only. Indeed, if $t_0 b_1^{\mathbf{m}}(t_0) + b_3^{\mathbf{m}}(t_0) = 0$ and $\frac{\mathrm{d}}{\mathrm{d}t}(tb_1^{\mathbf{m}}(t) + b_3^{\mathbf{m}}(t))\big|_{t=t_0} = 0$, then $b_2^{\mathbf{m}}(t_0) = 0$ and $b_1^{\mathbf{m}}(t_0) = b_3^{\mathbf{m}}(t_0) = 0$, which contradicts the irreducibility of the representation (20).

If $(m_\infty, \rho_\infty) = (0, 1/2)$, then $-2m_\infty - 2\rho_\infty + 1 = 0$ and $tb_1^{\mathbf{m}}(t) + b_3^{\mathbf{m}}(t) \equiv c = \text{const}$. Hence $u_{\mathbf{m}}(t) = -tb_2^{\mathbf{m}}(t)/c$. Note that $c \neq 0$, since in the opposite case for all $t \in D(t^*)$ the function $b_{12}^{\mathbf{m}}(z, t)$ has no zeros and the Riemann–Hilbert problem for scalar Fuchsian equations under consideration is soluble without apparent singularities, and $\gamma_{\max}(\chi^*) = 2$ (but this contradicts the above construction).

Thus, if $(m_\infty, \rho_\infty) \neq (0, 1/2)$, then the poles of the function $u_{\mathbf{m}}(t)$ can be simple only, and if $(m_\infty, \rho_\infty) = (0, 1/2)$, then pole orders of the function $u_{\mathbf{m}}(t)$ do not exceed two.

Remark 2. Alongside formulae for the transition from a two-dimensional Schlesinger isomonodromic family with $sl(2, \mathbb{C})$-residues to an equation $\mathrm{P_{VI}}$, there also exist formulae for the inverse transition (see [15] or [22]).

Hence, the latter reasonings prove the well known statement about movable poles of the equation $\mathrm{P_{VI}}(\alpha, \beta, \gamma, \delta)$. In the case $\alpha \neq 0$ they can be simple only, and in the case $\alpha = 0$ their orders do not exceed two or $u(t) \equiv \infty$.

Indeed, if a solution $u(t)$ of Eq. (4) corresponds to a two-dimensional Schlesinger isomonodromic family with *irreducible* monodromy, then the statement follows from the above construction ($\alpha \neq 0 \Longrightarrow (m_\infty, \rho_\infty) \neq (0, 1/2)$; $\alpha = 0 \Longrightarrow (m_\infty, \rho_\infty) = (0, 1/2)$, furthermore the case $\alpha = 0$, $u(t) \equiv \infty$ is possible, if the monodromy is 1-smaller). If the monodromy of the corresponding family is *reducible*, then $u(t)$ satisfies a Riccati equation (as shown by Mazzocco [23]), whose movable poles are simple.

5. The Riemann–Hilbert problem and Garnier systems

The arguments given above can be extended to a general case of $n + 3$ singular points $a_1, \ldots, a_n, a_{n+1} = 0, a_{n+2} = 1, a_{n+3} = \infty$ and an *irreducible non-smaller* representation

$$\chi_a^* : \pi_1(\mathbb{C} \setminus \{a_1, \ldots, a_n, 0, 1\}) \longrightarrow \mathrm{GL}(2, \mathbb{C}), \tag{24}$$

$a = (a_1, \ldots, a_n) \in D(a^*)$, where $D(a^*)$ is a disc of small radius centred at the point a^* of the space $(\mathbb{C}\setminus\{0, 1\})^n \setminus \bigcup_{i\neq j}\{a_i = a_j\}$.

(Continuing investigations of Fuchs) Garnier [9] obtained for $n > 1$ the system of non-linear partial differential equations of second order that must be satisfied by apparent singularities $\lambda_1(a), \ldots, \lambda_n(a)$ of some Fuchsian differential equation of second order with singular points $a_1, \ldots, a_n, 0, 1, \infty$ and $\mathrm{SL}(2, \mathbb{C})$-monodromy not depending on the parameter a. We supplement these results by the reasonings following after Lemma 2.

Lemma 2. *One has $\gamma_{\max}(\chi_a^*) = 1$ for almost all $a \in D(a^*)$.*

Proof. For an arbitrary set Λ of admissible matrices consider the family F_a^Λ of holomorphic vector bundles constructed by the corresponding representations χ_a^* and the set Λ. It is sufficient to prove that $\gamma(F_a^\Lambda) \leqslant 1$ for all $a \in D(a^*)$, may be, with the exception of an analytic subset of codimension one.

If $F_{a^0}^\Lambda \cong \mathcal{O}(k_1) \oplus \mathcal{O}(k_2)$, $k_1 - k_2 > 1$, for some $a^0 \in D(a^*)$, then we can apply Bolibrukh's procedure (which was used in the proof of Proposition 1) to get a Schlesinger isomonodromic family of the form (15) with an isomonodromic fundamental matrix $Y(z, a)$ of the form (16), where

$$K = \mathrm{diag}(k_1', k_2'), \qquad k_1' - k_2' \leqslant 1.$$

This family is defined in the exterior of some analytic subset $\Theta_\Lambda \subset D(a^*)$ of codimension one.

In view of the form of the matrix $Y(z, a)$, the transformation $y' = z^{-K}y$ transforms this family into the family that is holomorphic at the infinity. Hence, due to Proposition 2, the matrix K defines the splitting type of the bundles F_a^Λ for $a \in D(a^*) \setminus \Theta_\Lambda$ (and $\gamma(F_a^\Lambda) \leqslant 1$ for these values of a). □

Thus, in view of the formula (19), for almost all $a \in D(a^*)$ the set of points $a_1, \ldots, a_n, 0, 1, \infty$ and representation χ_a^* can be realized by a Fuchsian differential equation of second order with n apparent singularities $u_1(a), \ldots, u_n(a)$. Let us recall how they are connected with a Garnier system in the case when χ_a^* is an $\mathrm{SL}(2, \mathbb{C})$-representation.

Applying again Corollary 1, let us choose a value of the parameter $a = a^0 = (a_1^0, \dots, a_n^0) \in D(a^*)$ for which the representation $\chi_{a^0}^*$ is realized by Fuchsian systems

$$\frac{\mathrm{d}y}{\mathrm{d}z} = \left(\sum_{i=1}^{n+2} \frac{B_i^{\mathbf{m}}}{z - a_i^0}\right) y, \quad \mathbf{m} = (m_1, \dots, m_{n+2}, m_\infty) \in \mathbb{Z}_+^{n+3}, \tag{25}$$

with the singular points $a_1^0, \dots, a_n^0, a_{n+1}^0 = 0, a_{n+2}^0 = 1, a_{n+3}^0 = \infty$ (here the eigenvalues of the matrices $B_i^{\mathbf{m}}$ are $\pm(m_i + \rho_i)$, and the matrices $B_\infty^{\mathbf{m}} = -\sum_{i=1}^{n+2} B_i^{\mathbf{m}}$ are diagonal).

Every system of the form (25) can be included into the Schlesinger isomonodromic family

$$\frac{\mathrm{d}y}{\mathrm{d}z} = \left(\sum_{i=1}^{n+2} \frac{B_i^{\mathbf{m}}(a)}{z - a_i}\right) y, \qquad B_i^{\mathbf{m}}(a^0) = B_i^{\mathbf{m}}, \tag{26}$$

of Fuchsian systems with singularities $a_1, \dots, a_n, 0, 1, \infty$ which depends holomorphically on the parameter $a = (a_1, \dots, a_n) \in D(a^0)$; furthermore, $\sum_{i=1}^{n+2} B_i^{\mathbf{m}}(a) = -B_\infty^{\mathbf{m}} = \mathrm{diag}(-m_\infty - \rho_\infty, m_\infty + \rho_\infty)$.

By Malgrange's theorem the matrix functions

$$B_i^{\mathbf{m}}(a) = \begin{pmatrix} c_i^{\mathbf{m}}(a) & b_i^{\mathbf{m}}(a) \\ d_i^{\mathbf{m}}(a) & -c_i^{\mathbf{m}}(a) \end{pmatrix}$$

can be extended to the universal covering Z of the space $(\mathbb{C}\setminus\{0, 1\})^n \setminus \bigcup_{i\neq j}\{a_i = a_j\}$ as meromorphic functions (holomorphic off the analytic subset of codimension one).

Denote by $B_{\mathbf{m}}(z, a)$ the coefficient matrix of the family (26). Since the upper-right element of the matrix $B_\infty^{\mathbf{m}}$ equals zero, for every fixed a the same element of the matrix $z(z-1)(z-a_1)\dots(z-a_n)B_{\mathbf{m}}(z, a)$ is a polynomial $P_{\mathbf{m}}(z, a)$ of degree n in z. We denote by $u_1^{\mathbf{m}}(a), \dots, u_n^{\mathbf{m}}(a)$ the roots of this polynomial and define the functions $v_1^{\mathbf{m}}(a), \dots, v_n^{\mathbf{m}}(a)$:

$$v_j^{\mathbf{m}}(a) = \sum_{i=1}^{n+2} \frac{c_i^{\mathbf{m}}(a) + m_i + \rho_i}{u_j^{\mathbf{m}}(a) - a_i}, \quad j = 1, \dots, n.$$

Then the following statement takes place: *the pair* $(u^{\mathbf{m}}, v^{\mathbf{m}}) = (u_1^{\mathbf{m}}, \dots, u_n^{\mathbf{m}}, v_1^{\mathbf{m}}, \dots, v_n^{\mathbf{m}})$ *satisfies the Garnier system* (5) *with the parameters* $2m_1 + 2\rho_1, \dots, 2m_{n+2} + 2\rho_{n+2}, 2m_\infty + 2\rho_\infty - 1$ (see proof of Proposition 3.1 from [8], or [24, Cor. 6.2.2 (p. 207)]).

Thus, using arguments analogous to those given in the case $n = 1$, we get the following statement.

Proposition 4. *The set of the points* $a_1, \dots, a_n, 0, 1, \infty$ *and any irreducible non-smaller* $\mathrm{SL}(2, \mathbb{C})$*-representation* (24) *can be realized by the family (depending on the parameter* $\mathbf{m} \in \mathbb{Z}_+^{n+3}$*) of scalar Fuchsian equations*

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} + p_{\mathbf{m}}(z, a)\frac{\mathrm{d}w}{\mathrm{d}z} + q_{\mathbf{m}}(z, a)w = 0$$

with n apparent singularities.

Recall that, due to Okamoto's theorem, the apparent singular points $u_1^{\mathbf{m}}(a), \dots, u_n^{\mathbf{m}}(a)$ of every equation from the above family and the functions $v_1^{\mathbf{m}}(a) = \operatorname{res} q_{\mathbf{m}}(z, a)|_{z=u_1^{\mathbf{m}}}, \dots, v_n^{\mathbf{m}}(a) = \operatorname{res} q_{\mathbf{m}}(z, a)|_{z=u_n^{\mathbf{m}}}$, $a \in D(a^*)$, form a solution $(u^{\mathbf{m}}(a), v^{\mathbf{m}}(a))$ of the Garnier system (5) with the parameters $2m_1 + 2\rho_1, \dots, 2m_{n+2} + 2\rho_{n+2}, 2m_\infty + 2\rho_\infty - 1$.

One can express the coefficients of the polynomial $P_{\mathbf{m}}(z, a)$ in terms of the upper-right elements $b_i^{\mathbf{m}}(a)$ of the matrices $B_i^{\mathbf{m}}(a)$. Let

$$\sigma_1(a) = \sum_{i=1}^{n+2} a_i, \qquad \sigma_2(a) = \sum_{1\leqslant i<j\leqslant n+2} a_i a_j, \dots, \sigma_{n+1}(a) = a_1 \dots a_n$$

be the elementary symmetric polynomials in $a_1, \dots, a_n, a_{n+1} = 0, a_{n+2} = 1$, and $Q(z) = \prod_{i=1}^{n+2}(z - a_i)$. Then

$$P_{\mathbf{m}}(z, a) = \sum_{i=1}^{n+2} b_i^{\mathbf{m}}(a)\frac{Q(z)}{z - a_i} =: b_{\mathbf{m}}(a)z^n + f_1^{\mathbf{m}}(a)z^{n-1} + \dots + f_n^{\mathbf{m}}(a)$$

(recall that $\sum_{i=1}^{n+2} b_i^{\mathbf{m}}(a) = 0$). By the Viète theorem one has

$$b_{\mathbf{m}}(a) = \sum_{i=1}^{n+2} b_i^{\mathbf{m}}(a)(-\sigma_1(a) + a_i) = \sum_{i=1}^{n+2} b_i^{\mathbf{m}}(a)a_i = \sum_{i=1}^{n} b_i^{\mathbf{m}}(a)a_i + b_{n+2}^{\mathbf{m}}(a),$$

$$f_1^{\mathbf{m}}(a) = \sum_{i=1}^{n+2} b_i^{\mathbf{m}}(a)\left(\sigma_2(a) - \sum_{j=1, j\neq i}^{n+2} a_i a_j\right) = -\sum_{1\leqslant i<j\leqslant n+2} (b_i^{\mathbf{m}}(a) + b_j^{\mathbf{m}}(a))a_i a_j.$$

In the similar way,

$$f_k^{\mathbf{m}}(a) = (-1)^k \sum_{1\leqslant i_1<\cdots<i_{k+1}\leqslant n+2} (b_{i_1}^{\mathbf{m}}(a) + \cdots + b_{i_{k+1}}^{\mathbf{m}}(a))a_{i_1}\dots a_{i_{k+1}}$$

for each $k = 1, \dots, n$.

It immediately follows from the above formulae and Malgrange's theorem that *the elementary symmetric polynomials* $\sigma_k(u_1^{\mathbf{m}}, \dots, u_n^{\mathbf{m}}) = (-1)^k f_k^{\mathbf{m}}(a)/b_{\mathbf{m}}(a)$, *depending on solutions of the Garnier system extended to* Z, *are meromorphic functions.*

For $n > 1$, a Garnier system generically does not satisfy the Painlevé property (coordinates $(u_1, \dots, u_n)$ are defined as roots of a polynomial of degree n), but it can be transformed by a certain (symplectic) transformation $(u, v, a, H) \mapsto (q, p, s, K)$, $\sum_{i=1}^n (p_i dq_i - K_i ds_i) = \sum_{i=1}^n (v_i du_i - H_i da_i)$, into a Hamiltonian system satisfying the Painlevé property (see [24, Ch. III, Section 7]).

By Theorem 1, for each function $f_k^{\mathbf{m}}(a)$ extended to Z and any point a^* of the Θ-divisor of the family (26) one has $\Sigma_{a^*}(f_k^{\mathbf{m}}) \geqslant -n-1$. Similarly to the case $n = 1$, here we can tell something about the behaviour of the function $b_{\mathbf{m}}(a)$ along Θ.

Lemma 3. *Consider the family (26) with the irreducible non-smaller monodromy (24), and the function* $b_{\mathbf{m}}(a)$ *constructed by the residue matrices* $B_i^{\mathbf{m}}(a)$. *Then*

(i) *in the case* $(m_\infty, \rho_\infty) = (0, 1/2)$ *one has* $b_{\mathbf{m}}(a) \equiv \text{const} \neq 0$;
(ii) *in the case* $(m_\infty, \rho_\infty) \neq (0, 1/2)$ *the set* $\{a \in Z \mid b_{\mathbf{m}}(a) = 0\}$ *is an analytic submanifold of codimension one in* Z, *and if the function* $b_{\mathbf{m}}(a)$ *is holomorphic at a point* $a^0 \in Z$, *so are the functions* $f_k^{\mathbf{m}}(a)$.

Proof. By Lemma 1, we have $\mathrm{d}b_{\mathbf{m}}(a) = (-2m_\infty - 2\rho_\infty + 1)\sum_{i=1}^n b_i^{\mathbf{m}}(a)\mathrm{d}a_i$.

(i) In the case $(m_\infty, \rho_\infty) = (0, 1/2)$ one has $\mathrm{d}b_{\mathbf{m}}(a) \equiv 0$ for all $a \in D(a^*)$, hence $b_{\mathbf{m}}(a) \equiv \text{const} \neq 0$. Indeed, if $b_{\mathbf{m}}(a) \equiv 0$, then $P_{\mathbf{m}}(z, a)$ is a polynomial of degree $n-1$ in z. Therefore, for every $a \in D(a^*)$ the representation χ_a^* is realized by a scalar Fuchsian equation with at most $n-1$ apparent singularities (which are the roots of $P_{\mathbf{m}}(z, a)$) and $\gamma_{\max}(\chi_a^*) > 1$, which contradicts Lemma 2.

(ii) In the case $(m_\infty, \rho_\infty) \neq (0, 1/2)$ one has $\theta = -2m_\infty - 2\rho_\infty + 1 \neq 0$, and

$$b_i^{\mathbf{m}}(a) = \frac{1}{\theta}\frac{\partial b_{\mathbf{m}}(a)}{\partial a_i}, \quad i = 1, \dots, n;$$

$$b_{n+2}^{\mathbf{m}}(a) = b_{\mathbf{m}}(a) - \sum_{i=1}^n b_i^{\mathbf{m}}(a)a_i, \qquad b_{n+1}^{\mathbf{m}}(a) = -b_{n+2}^{\mathbf{m}}(a) - \sum_{i=1}^n b_i^{\mathbf{m}}(a). \tag{27}$$

Thus, if the function $b_{\mathbf{m}}(a)$ is holomorphic at some point $a^0 \in Z$, so are the functions $b_i^{\mathbf{m}}(a)$, $i = 1, \dots, n+2$, and hence, the functions $f_k^{\mathbf{m}}(a)$.

If for some $a^0 \in \{b_{\mathbf{m}}(a) = 0\}$ one has $\mathrm{d}b_{\mathbf{m}}(a^0) \equiv 0$, then $\sum_{i=1}^n b_i^{\mathbf{m}}(a^0)\mathrm{d}a_i \equiv 0$ and $b_1^{\mathbf{m}}(a^0) = \cdots = b_n^{\mathbf{m}}(a^0) = 0$. Taking into consideration the relations (27), one gets also $b_{n+2}^{\mathbf{m}}(a^0) = 0$ and $b_{n+1}^{\mathbf{m}}(a^0) = 0$. This contradicts the irreducibility of the representation $\chi_{a^0}^*$. □

As a consequence of Theorem 1 and Lemma 3, one gets the following statement.

Proposition 5. *Denote by* Δ_i *the polar loci of the functions* $\sigma_i(u_1^{\mathbf{m}}(a), \dots, u_n^{\mathbf{m}}(a))$ *extended to* Z, *respectively (in the conditions of Proposition 4). Then*

(a) *in the case* $(m_\infty, \rho_\infty) = (0, 1/2)$ *one has* $\Sigma_{a^*}(\sigma_i) \geqslant -n-1$ *for any point* $a^* \in \Delta_i$;
(b) *in the case* $(m_\infty, \rho_\infty) \neq (0, 1/2)$ *one has* $\Sigma_{a^*}(\sigma_i) \geqslant -n$ *for any point* $a^* \in \Delta_i \setminus \Delta^0$, *where* $\Delta^0 \subset \Delta_i$ *is some subset of positive codimension (or the empty set);*
(c) *in the case* $(m_\infty, \rho_\infty) \neq (0, 1/2)$, $a^* \in \Delta^0$, *one can estimate the order* κ *of each irreducible component of* $\Delta_i \cap D(a^*)$ *as follows:* $\kappa \geqslant -n$.

Proof. Recall that $\sigma_i(u_1^{\mathbf{m}}, \dots, u_n^{\mathbf{m}}) = (-1)^i f_i^{\mathbf{m}}(a)/b_{\mathbf{m}}(a)$ and $\Sigma_{a^*}(f_i^{\mathbf{m}}) \geqslant -n-1$ for any point a^* of the Θ-divisor of the family (26).

Therefore, the statement (a) of the proposition is a consequence of Lemma 3, (i).

(b) As follows from Lemma 3, (ii), the points $a^* \in \Delta_i$ can be of two types: such that $b_{\mathbf{m}}(a^*) = 0$ (then $\Sigma_{a^*}(\sigma_i) \geqslant -1$) or that belong to the polar locus $\Delta \subset \Theta$ of the function $b_{\mathbf{m}}(a)$.

Denote by $\Delta^0 \subset \Delta$ the set of ambiguous points of $b_{\mathbf{m}}(a)$. Then in a neighbourhood of any point $a^* \in \Delta \setminus \Delta^0$ each function $f_i^{\mathbf{m}}(a)$ can be presented in the form

$$f_i^{\mathbf{m}}(a) = \frac{g(a)}{\tau_1^{k_1}(a)\dots\tau_r^{k_r}(a)}, \quad k_1+\dots+k_r \leqslant n+1, \tag{28}$$

where $\tau_i(a)$, $g(a)$ are holomorphic near a^*; furthermore, $\tau_i(a)$ are irreducible at a^*, just as

$$b_{\mathbf{m}}(a) = \frac{h(a)}{\tau_1^{j_1}(a)\dots\tau_r^{j_r}(a)}, \quad j_1+\dots+j_r \geqslant 1, \tag{29}$$

where $h(a)$ is holomorphic near a^*, $h(a^*) \neq 0$. Thus,

$$\frac{f_i^{\mathbf{m}}(a)}{b_{\mathbf{m}}(a)} = \frac{g(a)}{\tau_1^{k_1}(a)\dots\tau_r^{k_r}(a)} : \frac{h(a)}{\tau_1^{j_1}(a)\dots\tau_r^{j_r}(a)} = \frac{g(a)/h(a)}{\tau_1^{k_1-j_1}(a)\dots\tau_r^{k_r-j_r}(a)},$$

therefore,

$$\Sigma_{a^*}(\sigma_i) = -(k_1-j_1)-\dots-(k_r-j_r) \geqslant -n.$$

(c) In a neighbourhood of a point $a^* \in \Delta^0$ the decompositions (28), (29) take place for the functions $f_i^{\mathbf{m}}(a)$, $b_{\mathbf{m}}(a)$, respectively, but $h(a^*) = 0$. However, due to Lemma 3, (ii), all irreducible factors of $h(a)$ in its decomposition $h(a) = h_1(a)\dots h_s(a)$ near a^* are distinct (we can assume also that none of h_i coincides with some of τ_j). One also has $k_i = 0$, if $j_i = 0$ ($b_{\mathbf{m}}(a)$ is holomorphic along $\{\tau_i(a) = 0\} \Longrightarrow f_i^{\mathbf{m}}(a)$ is holomorphic along $\{\tau_i(a) = 0\}$). Therefore, $k_i - j_i \leqslant n$, and the statement (c) follows from the decomposition

$$\frac{f_i^{\mathbf{m}}(a)}{b_{\mathbf{m}}(a)} = \frac{g(a)}{h_1(a)\dots h_s(a)\tau_1^{k_1-j_1}(a)\dots\tau_r^{k_r-j_r}(a)}. \quad \square$$

Alongside formulae for the transition from a two-dimensional Schlesinger isomonodromic family with $sl(2,\mathbb{C})$-residues to a Garnier system, there also exist formulae for the inverse transition (see [8, Prop. 3.2]). Hence, the latter proposition implies some addition to Garnier's theorem [9] (which claims that the elementary symmetric polynomials of solutions of a Garnier system are meromorphic on Z).

Theorem 3. *Consider a solution $(u(a), v(a))$ of the Garnier system (5), that corresponds to a two-dimensional Schlesinger isomonodromic family with irreducible monodromy, and the polar loci Δ_i of the functions $\sigma_i(u_1(a),\dots,u_n(a))$ meromorphic on Z. Then*

(a) *in the case $\theta_\infty = 0$ and the non-smaller monodromy one has $\Sigma_{a^*}(\sigma_i) \geqslant -n-1$ for any point $a^* \in \Delta_i$;*
(b) *in the case $\theta_\infty \neq 0$ one has $(\sigma_i)_\infty \geqslant -n$; moreover, $\Sigma_{a^*}(\sigma_i) \geqslant -n$ for any point $a^* \in \Delta_i$, may be, with the exception of some subset $\Delta^0 \subset \Delta_i$ of positive codimension.*

Remark 3. Mazzocco [25] has shown that the solutions of the Garnier system (5), that correspond to two-dimensional Schlesinger isomonodromic families with *reducible* monodromy, are classical functions (in each variable, in sense of Umemura [26]) and can be expressed via Lauricella hypergeometric equations (see [24, Ch. III, Section 9]). Thus, Theorem 3 can be applied, for example, to non-classical solutions of Garnier systems.

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