

Pseudoholomorphic curves on nearly Kähler manifolds

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Abstract

Let M be an almost complex manifold equipped with a Hermitian form such that its de Rham differential has Hodge type $(3,0)+(0,3)$, for example a nearly Kähler manifold. We prove that any connected component of the moduli space of pseudoholomorphic curves on M is compact. This can be used to study pseudoholomorphic curves on a 6-dimensional sphere with the standard (G_2 -invariant) almost complex structure.

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1 Introduction

Nearly Kähler manifolds were defined and studied by Alfred Gray, [Gra], in a general context of intrinsic torsion of $U(n)$ -structures and weak holonomies. An almost complex Hermitian manifold (M, I) is called **nearly Kähler**, in this sense, if $\nabla_X(I)X = 0$, for any vector fields X (∇ denotes the Levi-Civita connection). In other words, the tensor $\nabla\omega$ must be totally skew-symmetric, for ω the Hermitian form on M . If $\nabla_X(\omega) \neq 0$ for any non-zero vector field X , M is called **strictly nearly Kähler**.

For the last 10 years, the term “nearly Kähler” most often denotes strictly nearly Kähler 6-manifolds. In sequel we shall follow this usage, omitting “strictly” and “6-dimensional”.

In dimension 6, a manifold is (strictly) nearly Kähler if and only if it admits a Killing spinor ([Gru]). Therefore, such a manifold is Einstein, with positive Einstein constant.

As one can easily show (see e.g. [V]), strictly nearly Kähler 6-manifolds can be defined as 6-manifolds with structure group $SU(3)$ and fundamental forms

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$\omega \in \Lambda_{\mathbb{R}}^{1,1}(M)$, $\Omega \in \Lambda^{3,0}(M)$, satisfying $d\omega = 3\lambda \operatorname{Re} \Omega$, $d\operatorname{Im} \Omega = -2\lambda\omega^2$. An excellent introduction to nearly Kähler geometry is found in [MNS].

Only 4 compact examples of nearly Kähler manifolds are known, all of them homogeneous. In [Bu] it was shown that any homogeneous nearly Kähler 6-manifold belongs to this list. Existence of non-homogeneous compact examples is conjectured, but not proven yet.

1. The 6-dimensional sphere S^6 . The almost complex structure on S^6 is reconstructed from the octonion action, and the metric is standard.
2. $S^3 \times S^3$, with the complex structure mapping ξ_i to ξ'_i , ξ'_i to $-\xi_i$, where $\xi_i, \xi'_i, i = 1, 2, 3$ is a basis of left invariant 1-forms on the first and the second component.
3. Given a self-dual Einstein Riemannian 4-manifold M with positive Einstein constant, one defines its **twistor space** $\operatorname{Tw}(M)$ as a total space of a bundle of unit spheres in $\Lambda_-^2(M)$ of anti-self-dual 2-forms. Then $\operatorname{Tw}(M)$ has a natural Kähler-Einstein structure (I_+, g) , obtained by interpreting unit vectors in $\Lambda_-^2(M)$ as complex structure operators on TM . Changing the sign of I_+ on TM , we obtain an almost complex structure I_- which is also compatible with the metric g ([ES]). A straightforward computation insures that $(\operatorname{Tw}(M), I_-, g)$ is nearly Kähler ([M]).

As N. Hitchin proved, there are only two compact self-dual Einstein 4-manifolds: S^4 and $\mathbb{C}P^2$. The corresponding twistor spaces are $\mathbb{C}P^3$ and the flag space $F(1, 2)$. The almost complex structure operator I_- induces a nearly Kähler structure on these two symmetric spaces.

2 Pseudoholomorphic curves on nearly Kähler manifolds

2.1 Pseudoholomorphic curves on nearly Kähler manifolds

The study of pseudoholomorphic curves on almost complex manifolds is a big subject, spurred by advances in physics and symplectic topology. Even before the advent of the theory of pseudoholomorphic curves, complex curves in a 6-sphere and a nearly Kähler $\mathbb{C}P^3$ were used to study the minimal surfaces.

For pseudoholomorphic curves in a 6-sphere, the basic reference is a paper of R. Bryant [Br]. Bryant observed that the complex curves in S^6 can be approached similarly to the real curves in \mathbb{R}^3 , by computing their first and second fundamental forms and the torsion form. These operators, as defined by Bryant, turn out to be holomorphic on a curve. Bryant classifies the curves which have vanishing torsion tensor, and constructs many examples of such curves, using the correspondence between pseudoholomorphic curves in S^6 and holomorphic curves in a 5-dimensional quadric in $\mathbb{C}P^6$.

Results of Bryant were extended to $\mathbb{C}P^3$ with the nearly Kähler almost complex structure by Feng Xu ([Xu]). The usual (Kähler) projective 3-space

$\mathbb{C}P^3$ can be obtained as a twistor space of S^4 fibered over S^4 with fibers isomorphic to $\mathbb{C}P^1$, $\mathbb{C}P^3 \xrightarrow{\pi} S^4$. This gives a direct sum decomposition $T\mathbb{C}P^3 = \pi^*TS^4 \oplus (\pi^*TS^4)^\perp$. From the twistor construction, it is clear that this decomposition is compatible with the complex structure. To obtain the nearly Kähler complex structure, we take the opposite complex structure on $(\pi^*TS^4)^\perp$, and the usual one on π^*TS^4 .

The pullback π^*TS^4 is identified with the holomorphic contact bundle on $\mathbb{C}P^3$ and the Legendrian curves (ones that are tangent to the contact distribution) are by the above construction pseudoholomorphic with respect to the nearly Kähler almost complex structure.

Feng Xu shows that a pseudoholomorphic curve is Legendrian if and only if its torsion vanishes, and classifies pseudoholomorphic curves of genus 0.

2.2 Compactness of the moduli spaces

Definition 2.1: An **almost complex Hermitian manifold** is a manifold (M, I, ω) equipped with an almost complex structure I and a **Hermitian form** ω , that is, a (1,1)-form $\omega := g(\cdot, I\cdot)$, where g is a **Hermitian** (I -invariant Riemannian) metric.

Definition 2.2: Let (M, I, ω) be an almost complex Hermitian manifold. A **smooth, open pseudoholomorphic curve** is a smooth 2-dimensional submanifold $S_0 \subset M$ (not necessarily closed) satisfying $I(T_s S_0) = T_s S_0$ for each point $s \in S_0$.

Remark 2.3: By Wirtinger's theorem, the Riemannian volume form on a smooth pseudoholomorphic curve S_0 is equal to $\omega|_{S_0}$.

Definition 2.4: Let $S \subset (M, I)$ be a compact subset, smooth and 2-dimensional outside of a subset $S_{\text{sing}} \subset S$ of Hausdorff dimension 0. Assume that the **smooth part** $S \setminus S_{\text{sing}}$ is pseudoholomorphic, and S is its closure. Then S is called a **pseudoholomorphic curve** in M .¹

Definition 2.5: Define **the volume** of a pseudoholomorphic curve as an integral $\int_{S \setminus S_{\text{sing}}} \omega$.

Remark 2.6: This integral equal to the Riemannian volume of $S \setminus S_{\text{sing}}$. Since S_{sing} has Hausdorff codimension 2 in S , it is always finite ([F]).

Definition 2.7: Let M be a metric space, and \mathfrak{C} the set of its compact subsets. Define **the Hausdorff distance** on \mathfrak{C} as

$$d_H(X, Y) := \max\left(\sup_{x \in X} \left(\inf_{y \in Y} (d(x, y))\right), \sup_{y \in Y} \left(\inf_{x \in X} (d(x, y))\right)\right).$$

¹Sometimes such curves are also called **holomorphic curves** and **complex curves**.

It is well known (see [G1]) that d_H defines a metric on \mathfrak{C} , which is complete and, when M is a manifold, locally compact.

The following fundamental theorem about pseudoholomorphic curves is due to M. Gromov ([G2], [AL]).

Theorem 2.8: Let (M, I, ω) be a compact almost complex Hermitian manifold, and \mathfrak{S} the set of pseudoholomorphic curves on (M, I, ω) , equipped with the Hausdorff distance. Consider the volume function $\text{Vol} : \mathfrak{S} \rightarrow \mathbb{R}$. Then Vol is continuous, and, moreover, $\text{Vol}^{-1}([0, C])$ is compact, for any $C \in \mathbb{R}$.

Remark 2.9: The set \mathfrak{S} with the Hausdorff distance and the induced topology is called **the moduli space of pseudoholomorphic curves**. It is a finite-dimensional, locally compact topological space. The moduli of pseudoholomorphic curves is a real analytic variety if (M, I, ω) is real analytic.

The main result of this section is the following theorem, which is then applied to the nearly Kähler manifolds, such as S^6 .

Theorem 2.10: Let (M, I, ω) be an almost complex Hermitian manifold satisfying $d\omega \in \Lambda^{3,0}(M) \oplus \Lambda^{0,3}(M)$. Then the volume function Vol is constant on each connected component of the moduli space of pseudoholomorphic curves.

Proof: Let R be a connected component of the moduli space, and $\gamma : [0, 1] \rightarrow R$ a continuous path. Denote by $R_\gamma \subset [0, 1] \times M$ a set

$$R_\gamma := \{(t, m) \in [0, 1] \times M \mid m \in \gamma(t)\}.$$

Without restricting the generality, we may assume that R_γ is smooth outside of a Hausdorff codimension 2 subset (see [AL], where the local structure of the moduli of pseudoholomorphic curves is describe analytically). Therefore, we may integrate bounded differential forms over R_γ and use the Stokes' theorem as if R_γ were a smooth, compact manifold with boundary.

Denote by $\gamma : R_\gamma \rightarrow M$ the tautological projection. Then $\text{Vol}(\gamma(1)) - \text{Vol}(\gamma(0)) = \int_{\partial R_\gamma} \gamma^* \omega$. Therefore, Stokes' theorem gives

$$\text{Vol}(\gamma(1)) - \text{Vol}(\gamma(0)) = \int_{R_\gamma} \gamma^* d\omega.$$

Consider the projection map $\pi : R_\gamma \rightarrow [0, 1]$. Then $\int_{R_\gamma} \gamma^* d\omega = \int_{[0, 1]} \pi_* \gamma^* d\omega$, where π_* is a pushforward of differential forms.

However, $\pi_* d\omega = 0$, because $d\omega$ is a $(3, 0) + (0, 3)$ -form, and π_* is an integration over $(1, 1)$ -cycles. ■

Comparing this statement with Gromov's compactness theorem (Theorem 2.8) we obtain the following result (applicable to S^6 and other nearly Kähler varieties).

Corollary 2.11: Let (M, I, ω) be an almost complex Hermitian manifold satisfying $d\omega \in \Lambda^{3,0}(M) \oplus \Lambda^{0,3}(M)$. Then each connected component of the moduli of pseudoholomorphic curves on M is compact. Moreover, there is only a finite number of components with volume bounded from above by any $C \in \mathbb{R}$.

2.3 Open questions

From Corollary 2.11, it follows that the pseudoholomorphic curves on nearly Kähler manifolds behave in essentially the same way as on symplectic manifolds: their moduli space is a countable union of compact components. On Kähler manifolds the following questions are very easy to answer, but on nearly Kähler manifolds, they seem quite mysterious.

Question 2.12: With each component of the moduli of pseudoholomorphic curves, one associates two numbers: a genus of a generic curve in a family, and its volume, which is independent from the choice of a curve by Theorem 2.10. Is there any relation between those numbers? What real numbers can occur as volumes of pseudoholomorphic curves on a given nearly Kähler manifold?

Notice that in [Br], R. Bryant proved existence of complex curves of arbitrary genus on S^6 .

It is interesting that on any nearly Kähler manifold M the first Chern class $c_1(M)$ vanishes, but the curves behave in a way which is completely different from observed on the Calabi-Yau manifolds. In particular, the big-dimensional families of rational curves seem to occur quite often, while on Calabi-Yau this is impossible for topological reasons.

Question 2.13: The moduli space of pseudoholomorphic curves on a Kähler manifold is a complex manifold. What kind of geometric structure arises on the moduli space of pseudoholomorphic curves on a nearly Kähler manifold?

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