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Integrability of Hurwitz partition functions

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Abstract

Partition functions often become τ -functions of integrable hierarchies, if they are considered dependent on infinite sets of parameters called time variables. The Hurwitz partition functions $Z = \sum_R d_R^{2-k} \chi_R(t^{(1)}) \dots \chi_R(t^{(k)}) \exp(\sum_n \xi_n C_R(n))$ depend on two types of such time variables, t and ξ . KP/Toda integrability in t requires that $k \leq 2$ and also that $C_R(n)$ are selected in a rather special way, in particular the naive cut-and-join operators are not allowed for $n > 2$. Integrability in ξ further restricts the choice of $C_R(n)$, forbidding, for example, the free cumulants. It also requires that $k \leq 1$. The quasi-classical integrability (the WDVV equations) is naturally present in ξ variables, but also requires a careful definition of the generating function.

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1. Introduction

Since the early papers by M Sato school on the KP hierarchy [1], it is known that linear combinations of the Schur functions $\chi_R(t)$ (or characters of the linear groups, here R is the Young diagram)

$$\tau(t) = \sum_R w_R \chi_R(t) \quad (1)$$

are KP τ -functions (that depends on the infinite set of times t_k) satisfying the bilinear Hirota identities if and only if the coefficients w_R in these combinations satisfy the bilinear Plücker relations and, in this way, describe a point of the infinite-dimensional Grassmannian. Since then, this fact and its generalizations were exploited in many concrete examples.

One of the possible generalizations is parameterizing the coefficients w_R by some (infinite) set of new continuous variables \bar{t} . One can do this in such a way that $\tau(t|\bar{t})$ satisfies the KP hierarchy in both t and \bar{t} variables and becomes the Toda-lattice τ -function [2].

The generic condition for these coefficients in this Toda lattice case is quite involved; however, one may naturally consider simpler sub-classes of the \bar{t} -dependent coefficients in (1), that is,

$$w_R(\bar{t}) = v_R \chi_R(\bar{t}), \tag{2}$$

where v_R are the coefficients yet to be determined in order to guarantee that $\tau(t|\bar{t})$ is the Toda-lattice τ -function.

It was earlier demonstrated [3] that this is the case if v_R are given by an (natural) exponential parameterization

$$v_R \sim \exp\left(\sum_n \xi_n C_R(n)\right), \tag{3}$$

where $C_R(n)$ are eigenvalues of the corresponding Casimir operators in representation R and ξ_n are arbitrary constants.

A new interest to integrable properties of combinations of the kind

$$\tau(t|\bar{t}) = \sum_R \chi_R(\bar{t}) \chi_R(t) \exp\left(\sum_n \xi_n C_R(n)\right) \tag{4}$$

emerged after demonstration in [4] that this kind of sum provides natural generating functions of the Hurwitz numbers.

In this paper, we study the integrable properties of (4) and generalize the results of [3] in two directions. First of all, instead of infinite sums over representations R , one can consider in (4) finite sums. It turns out that this can preserve integrability; however, the corresponding hierarchy is the forced KP/Toda hierarchy [5]. Second, one can consider $\tau(t|\bar{t})$ as a function of the coefficients ξ_n . We study in this paper when (4) is a τ -function also with respect to the ξ_n variables.

This paper contains only a description of the results and brief comments; the detailed proofs will be presented elsewhere.

2. Hurwitz numbers and the Frobenius formula

Hurwitz numbers [6] count the ramified coverings of a Riemann surface. Their calculation is obviously important in string theory and from time to time it attracts certain attention. The current interest is due to the possibility of expressing the Hurwitz numbers through group characters, which implies close relation to matrix models, integrable systems, Virasoro constraints, AGT relation and other basic chapters of modern theory. This expression is given by the Frobenius formula [7]

$$\text{Cover}_n(\Delta_1, \dots, \Delta_k) = \sum_R d_R^2 \varphi_R(\Delta_1) \dots \varphi_R(\Delta_k) \delta_{|R|,n} \tag{5}$$

for the number of n -sheet coverings of a Riemann sphere with k ramification points of given types. The type of ramification at a point i characterizes the way in which the sheets are glued, and is labeled by a Young diagram (integer partition of n) Δ_i of weight $|\Delta_i| = n$. The sum in (5) goes over all the Young diagrams $R = \{r_1 \geq r_2 \geq \dots\}$ of the same weight $|R| = r_1 + r_2 + \dots = n$, and $\varphi_R(\Delta)$ are the symmetric group characters, to appear in equation (8) below.

For $k = 1$, one can extend the sum in (5) to all R [8]. Remarkably, though the sum is now infinite, this produces only a factor of $e = 2.718\dots$,

$$\sum_R d_R^2 \varphi_R(\Delta) \delta_{|R|,|\Delta|} = \frac{1}{e} \sum_R d_R^2 \varphi_R(\Delta). \tag{6}$$

This motivates the change of the definition (5). Instead of (5) one can write

$$\text{Cover}(\Delta_1, \dots, \Delta_k) = \frac{1}{e} \sum_R d_R^2 \varphi_R(\Delta_1) \dots \varphi_R(\Delta_k) \tag{7}$$

without the $\delta_{|R|,n}$ projector. In this form, the rhs is defined for an arbitrary Δ_i and the index n can be omitted. These generalized Hurwitz numbers [9] are much more interesting, but for $k > 1$ they are different from (5), even when all the diagrams have the same sizes, $|\Delta_1| = \dots = |\Delta_k|$.

3. Hurwitz partition functions and two ways to introduce time variables

To put the problem into a string theory context, it remains to substitute particular Hurwitz numbers by generating functions. This can be done in two ways.

First, one can keep k fixed and sum over all types of ramification at a given point. The clever way to do this is to use the crucial property of $\varphi_R(\Delta)$ as expansion coefficients of the Schur functions ($GL(\infty)$ characters) $\chi_R(t)$:

$$\chi_R(t) = \sum_{\Delta} d_R \varphi_R(\Delta) p(\Delta) \delta_{|\Delta|,|R|}, \tag{8}$$

where

$$\begin{aligned} \chi_R(t) &= \det_{ij} S_{r_i-i+j}(t), \\ \exp\left(\sum_k t_k z^k\right) &= \sum_k z^k S_k(t), \end{aligned} \tag{9}$$

$p_k = kt_k$, and for $\Delta = \{\delta_1 \geq \delta_2 \geq \dots\}$ we define $p(\Delta) = p_{\delta_1} p_{\delta_2} \dots = p_1^{m_1} p_2^{m_2} \dots$ and, for the future use, $\tilde{p}(\Delta) = p(\Delta)/z(\Delta)$, where $z(\Delta) = \prod_k m_k! k^{m_k}$.⁷

In (8) one can remove the projector $\delta_{|\Delta|,|R|}$ from the sum. Since [9]

$$\varphi_R(\Delta) = \varphi_R(\tilde{\Delta}, 1^{|\Delta|-|\tilde{\Delta}|}) = \frac{(|R| - |\tilde{\Delta}|)!}{(|R| - |\Delta|)! (|\Delta| - |\tilde{\Delta}|)!} \varphi_R(\tilde{\Delta}, 1^{|\Delta|-|\tilde{\Delta}|}), \tag{10}$$

where $\tilde{\Delta}$ is the sub-diagram of Δ which does not contain unit lines, in particular, $\varphi_R(\Delta) = 0$ for $|R| < |\Delta|$, one has

$$\chi_R(t+1) = \chi_R(t_k + \delta_{k,1}) = \sum_{\Delta} d_R \varphi_R(\Delta) p(\Delta), \tag{11}$$

i.e. the projector is removed at the expense of shifting the first time t_1 , for which we introduce a condensed notation $t \rightarrow t + 1$. For example, in this formula one can put all $t_k = 0$; then only the term with $\Delta = \emptyset$ contributes, and one obtains $d_R = \chi_R(\delta_{k,1})$.

Coming back to the generating function, one can use (8) to convert (5) into

$$\begin{aligned} Z\{t^{(1)}, \dots, t^{(k)}|q\} &= \sum_n q^n \sum_{\Delta_1, \dots, \Delta_k} \text{Cover}_n(\Delta_1, \dots, \Delta_k) p^{(1)}(\Delta_1) \dots p^{(k)}(\Delta_k) \delta_{|\Delta_1|,n} \dots \delta_{|\Delta_k|,n} \\ &= \sum_n q^n \sum_R d_R^{2-k} \chi_R(t^{(1)}) \dots \chi_R(t^{(k)}) \delta_{|R|,n}. \end{aligned} \tag{12}$$

⁷ This combinatorial coefficient that counts the order of the automorphism group of the Young diagram, appears everywhere in the theory of symmetric functions and symmetric group $S(\infty)$. In particular, the standardly normalized symmetric group characters $\hat{\chi}_R(\Delta)$

$$\hat{\chi}_R(\Delta) = d_R z(\Delta) \varphi_R(\Delta).$$

These $\hat{\chi}_R(\Delta)$ are generated by the command *Chi*(R, Δ) in MAPLE in the package COMBINAT, and we use the hat to distinguish them from the linear group characters, i.e. the Schur functions $\chi_R(t)$ differ by this factor from our $\varphi_R(\Delta)$.

Remarkably, the generating function of the *generalized* Hurwitz numbers (7) is given by the same formula (!), with the projector $\delta_{|R|,n}$ removed and substituted by the factor of $1/e$ [8]:

$$\sum_{\Delta_1, \dots, \Delta_k} \text{Cover}(\Delta_1, \dots, \Delta_k) p^{(1)}(\Delta_1) \dots p^{(k)}(\Delta_k) = \frac{1}{e} \sum_R d_R^{2-k} \chi_R(t^{(1)} + 1) \dots \chi_R(t^{(k)} + 1) = \frac{1}{e} Z\{t^{(1)} + 1, \dots, t^{(k)} + 1 | q = 1\}. \tag{13}$$

Non-unit q can be introduced into this sum by the substitution $p_k \rightarrow q^k p_k$, which leads to the factor of $q^{|R|}$ in the sum. Note that neither the constant shift of t variables, $t \rightarrow t + 1$ nor the normalization factor of $1/e$ is essential for integrability properties below; therefore, we are not concerned with them in what follows. We emphasize that though (7) is different from (5), the generating functions in (12) and (13) are the same function; only the arguments are shifted.

The second way to make a partition function is to exponentiate $\varphi_R(\Delta)$:

$$Z_{\text{excessive}}\{\xi\} = \sum_R d_R^2 \exp\left(\sum_{\Delta} \xi_{\Delta} \varphi_R(\Delta)\right). \tag{14}$$

This, however, introduces an excessive set of time variables ξ , labeled by all Young diagrams Δ . In the matrix model case, this would correspond to exponentiating all multi-trace operators with independent couplings, $\int dM \exp(\sum_{\Delta} \xi_{\Delta} \prod_k (\text{tr } M^k)^{m_k})$ (and, from the integrable point of view to non-Cartanian hierarchies [10]). However, this is not a clever choice, leading to anything nice: instead one should better consider just $\int dM \exp(\sum_k \xi_k \text{tr } M^k)$. Similarly, instead of (14), one should better consider

$$Z_{C(n)}\{\xi\} = \sum_R d_R^2 \exp\left(\sum_n \xi_n C_R(n)\right), \tag{15}$$

where $C_R(n)$ is *some* linear combination of $\varphi_R(\Delta)$, one for each n . The question is, however, what combination to choose, and this is the main subject of our consideration below. For historical reasons, $C_R(n)$ are often called (eigenvalues of) Casimir operators.

Finally, one can consider the mixed partition function, with both t and ξ variables:

$$Z(t|\xi) = \frac{1}{e} \sum_R d_R^{2-k} \chi_R(t^{(1)} + 1) \dots \chi_R(t^{(k)} + 1) \exp\left(\sum_n \xi_n C_R(n)\right). \tag{16}$$

In particular, q in (13) is just $q = e^{\xi_1}$, and $C_R(1) = \varphi_R(1)$ is defined unambiguously because there is just one Young diagram of weight 1.

4. Integrability properties: a summary

A cleverly defined partition function should be a τ -function of some integrable hierarchy [10]. Although this is not necessary, in quite many cases the hierarchies are ‘Cartanian’, belonging to the Toda/KP family associated with the Kac–Moody algebra $\widehat{U}(1)$ (for more general τ -functions see [11]). This turns out to be possible also for Hurwitz partition functions, but imposes certain restrictions. What is true is the following set of statements:

- *Quasi-classical integrability.* Quasi-classical integrability (WDVV equations) in ξ variables is most natural for the Hurwitz partition function because of its clear algebraic topological nature. In fact, this is the most difficult kind of integrability; it actually appears when the set of ξ is *excessive* and when the partition function is defined with the help of the $*$ -product. This is a separate story to be considered in [12].

- *The basic example of t-integrability.* $Z\{t^{(1)}, \dots, t^{(k)}|q\}$ is the KP τ -function in $t^{(1)}$ only for $k = 1$ and $k = 2$:

$$Z(t, \bar{t}|q) = \sum_R q^{|R|} \chi_R(t) \chi_R(\bar{t}) = \exp\left(\sum_k kq^k t_k \bar{t}_k\right) \tag{17}$$

is a KP τ -function w.r.t. both sets of times, t_k and \bar{t}_k .

This is a particular case of a more general statement:

$$\tau(t) = \sum_R w_R \chi_R(t) \tag{18}$$

is a KP τ -function provided w_R satisfy the bilinear Plucker relations

$$\begin{aligned} w_{22}w_0 - w_{21}w_1 + w_2w_{11} &= 0 \\ w_{32}w_0 - w_{31}w_1 + w_3w_{11} &= 0 \\ w_{221}w_0 - w_{211}w_1 + w_2w_{111} &= 0 \\ &\dots \end{aligned} \tag{19}$$

which possess a solution $w_R = \det_{ij} A_{i,r_j-j}$ with any matrix A_{ij} such that $A_{ij} = 0$ for $i < 0$ or $j < 0$ (i.e. non-zero only in the positive quadrant). In particular, $w_R = \chi_R(\bar{t}) = \det_{ij} S_{i+r_j-j}(\bar{t})$ for the Schur functions, $\sum_r z^r S_r = e^{\sum z^k t_k}$. Moreover, one can restrict the sum over R in (18) to the Young diagrams with a finite number of lines, $l(R)$:

$$\tau_{\mathcal{N}}(t) = \sum_{R: l(R) \leq \mathcal{N}} w_R \chi_R(t). \tag{20}$$

It is still a KP τ -function. The parameter \mathcal{N} plays the role of an additional time variable, ‘zero time’.

- *KP τ -function w.r.t. (t, \bar{t}) variables.* ξ -deformation preserves the t -integrability, i.e.

$$\mathcal{Z}_{C(n)}\{t, \bar{t}|\xi\} = \sum_R \chi_R(t) \chi_R(\bar{t}) \exp\left(\sum_n \xi_n C_R(n)\right) \tag{21}$$

is a KP τ -function in t, \bar{t} variables [3] only if $C_R(n)$ is of the form

$$\sum_n \xi_n C_R(n) = \sum_{k,i} \zeta_k ((r_i - i)^k - (-i)^k) \tag{22}$$

with the arbitrary ζ_k .

This is a restrictive condition: in particular, the choice $C_R(n) = \varphi_R(n)$ with single line diagrams is *not* allowed beyond $n = 1, 2$. Indeed,

$$\varphi_R(1) = \sum_j r_j = |R|, \quad \varphi_R(2) = \frac{1}{2} \sum_j ((r_i - i + 1/2)^2 - (-i + 1/2)^2) \tag{23}$$

perfectly fits (22), but already

$$\varphi_R(3) = \frac{1}{3} \sum_j r_j (r_j^2 - 3jr_j + 3j^2 - 3j + 2) - \sum_{i < j} r_i r_j \tag{24}$$

is not of the form (22) due to the last ‘mixing’ term. Thus, $\sum_R \chi_R \bar{\chi}_R e^{\xi_1 \varphi_R(1) + \xi_2 \varphi_R(2)}$ is, but $\sum_R \chi_R \bar{\chi}_R e^{\xi_3 \varphi_R(3)}$ is *not* a KP τ -function in t, \bar{t} .

- *Forced Toda-lattice τ -function w.r.t. (t, \bar{t}) variables.* In fact, (21) can be further promoted to a Toda-lattice τ -function, which depends on additional zero-time \mathcal{N} , and, in addition to being a KP τ -function both in t and \bar{t} variables, satisfies an extra equation

$$\tau_{\mathcal{N}} \frac{\partial^2 \tau_{\mathcal{N}}}{\partial t_1 \partial \bar{t}_1} - \frac{\partial \tau_{\mathcal{N}}}{\partial t_1} \frac{\partial \tau_{\mathcal{N}}}{\partial \bar{t}_1} = \tau_{\mathcal{N}+1} \tau_{\mathcal{N}-1}. \tag{25}$$

However, for (21) to satisfy this equation, the ζ variables should depend on \mathcal{N} in a rather peculiar way. Instead, one can say that the Casimir operators in (21) should be substituted by their peculiar \mathcal{N} -dependent combinations

$$\mathcal{Z}_{C(n)}\{t, \bar{t}, \mathcal{N}|\xi\} = e^{Q_{\mathcal{N}}} \sum_{R: l(R) \leq \mathcal{N}} \chi_R(t) \chi_R(\bar{t}) \exp\left(\sum_n \xi_n C_{R;\mathcal{N}}(n)\right), \quad (26)$$

which is a Toda-lattice τ -function *only* if

$$\begin{aligned} \sum_n \xi_n C_{R;\mathcal{N}}(n) &= \sum_{k,i} \zeta_k ((r_i + \mathcal{N} + \gamma - i)^k - (\mathcal{N} + \gamma - i)^k) = \sum_{n,k} \binom{n}{k} \zeta_n C_R(k) \mathcal{N}^{n-k} \\ Q_{\mathcal{N}} &= \sum_k \sum_{i=1}^{\mathcal{N}} \zeta_k (\mathcal{N} + \gamma - i)^k, \end{aligned} \quad (27)$$

where ζ_k and γ are arbitrary \mathcal{N} -independent constants.

Note that (26) can be rewritten in the form

$$\mathcal{Z}_{C(n)}\{t, \bar{t}, \mathcal{N}|\xi\} = \sum_{R: l(R) \leq \mathcal{N}} \chi_R(t) \chi_R(\bar{t}) \exp\left(\sum_n \xi_n \tilde{C}_{R;\mathcal{N}}(n)\right) \quad (28)$$

with

$$\sum_n \xi_n \tilde{C}_{R;\mathcal{N}}(n) = \sum_k \sum_{i=1}^{\mathcal{N}} \zeta_k (r_i + \mathcal{N} + \gamma - i)^k, \quad (29)$$

where the sum over i is now terminated at $i = \mathcal{N}$ not automatically, but ‘by hands’.

In fact, this is a τ -function of *forced* Toda-lattice hierarchy [5, 13], i.e. $\tau_0 = 1$ and $\tau_{\mathcal{N}} = 0$ for $\mathcal{N} < 0$.

- *Toda-lattice τ -function w.r.t. (t, \bar{t}) variables.* One can lift this restriction by shifting \mathcal{N} with constant \mathcal{M} and then taking the limit $\mathcal{N}, \mathcal{M} \rightarrow \infty$ in such a way that the new (shifted) zero-time N remains finite: $\mathcal{N} = \mathcal{M} + N$. With this procedure one is led to the τ -function of the generic (unforced) Toda-lattice τ -function:

$$\tau_N\{t, \bar{t}|\xi\} = e^{Q_N} \sum_R \chi_R(t) \chi_R(\bar{t}) \exp\left(\sum_n \xi_n C_{R;N}(n)\right), \quad (30)$$

where the sum is now over *all* diagrams R , independently of N , and

$$\begin{aligned} \sum_n \xi_n C_R(n) &= \sum_{k,i} \zeta_k ((r_i + N + \gamma - i)^k - (N + \gamma - i)^k) = \sum_{n,k} \binom{n}{k} \zeta_n C_R(k) N^{n-k} \\ Q_N &= \sum_k \sum_{i=1}^N \zeta_k (N + \gamma - i)^k. \end{aligned} \quad (31)$$

A restriction of such a Toda-lattice τ -function to just two non-vanishing ξ_n and $\gamma = 1/2$ appeared in [4]:

$$\tau_N\{t, \bar{t}|\xi\} = e^{\xi_1 \frac{N^2}{2} + \xi_2 \frac{N(4N^2-1)}{12}} \underbrace{\sum_R \chi_R(t) \chi_R(\bar{t}) e^{(\xi_1 + 2N\xi_2)C_R(1) + \xi_2 C_R(2)}}_{\tilde{\tau}(t, \bar{t}|q, \beta)}. \quad (32)$$

Rescaled τ -function $\tilde{\tau}(t, \bar{t}|q, \beta)$ with $q = e^{\xi_1 + 2N\xi_2}$ and $\beta = \xi_2/2$ satisfies the equation (see [4, equation (10)])

$$\tilde{\tau}(t, \bar{t}|q, \beta) \frac{\partial^2 \tilde{\tau}(t, \bar{t}|q, \beta)}{\partial t_1 \partial \bar{t}_1} - \frac{\tilde{\tau}(t, \bar{t}|q, \beta)}{\partial t_1} \frac{\tilde{\tau}(t, \bar{t}|q, \beta)}{\partial \bar{t}_1} = q \tilde{\tau}(t, \bar{t}|e^\beta q, \beta) \tilde{\tau}(t, \bar{t}|e^{-\beta} q, \beta) \quad (33)$$

which is a slight modification of (25), taking into account that $e^{Q_{N+1} + Q_{N-1} - 2Q_N} = q$.

- An example of ξ integrability. Integrability in ξ variables is even more restrictive:

$$\mathcal{Z}_{C_n}\{\xi\} = \sum_R d_R^2 \exp\left(\sum_n \xi_n C_R(n)\right), \tag{34}$$

which is a KP τ -function in ξ variables only if

$$\sum_n \xi_n C_R(n) = \sum_{n,i} \xi_n ((r_i - i + \gamma)^n - (-i + \gamma)^n) \tag{35}$$

with arbitrary γ , i.e. one cannot choose the function $\zeta(\xi)$ in (22) in an arbitrary way: only a very restricted class of linear triangular changes $\{\xi\} \rightarrow \{\zeta\}$ in (22) is allowed.

In particular, the expressions naturally emerging in the theory of Kerov polynomials [14]

$$\sum_n \xi_n C_R(n) = \sum_{k,i} \xi_k \frac{1}{k+1} ((r_i - i + 1)^{k+1} - (r_i - i)^{k+1} - (-i + 1)^{k+1} + (-i)^{k+1}). \tag{36}$$

do not provide a τ -function in ξ variables.

- Toda-chain τ -function in ξ variables. Again, with the sum restricted to the Young diagrams with \mathcal{N} lines, like in (20) or (26), one can consider instead of (34) the generating function

$$\mathcal{Z}_{C(n)}\{t, \bar{t}, \mathcal{N}|\xi\} = e^{\mathcal{Q}_{\mathcal{N}}} \sum_{R: l(R) \leq \mathcal{N}} d_R^2 \exp\left(\sum_n \xi_n C_{R,\mathcal{N}}(n)\right). \tag{37}$$

It is a Toda-chain τ -function in the ξ variables, with the zero-time \mathcal{N} and

$$\begin{aligned} \sum_n \xi_n C_{R,\mathcal{N}}(n) &= \sum_{k,i} \xi_k ((r_i + \mathcal{N} + \gamma - i)^k - (\mathcal{N} + \gamma - i)^k) = \sum_{n,k} \binom{\mathcal{N}}{k} \xi_n C_R(k) \mathcal{N}^{n-k} \\ \mathcal{Q}_{\mathcal{N}} &= \sum_k \sum_{i=1}^{\mathcal{N}} \xi_k (\mathcal{N} + \gamma - i)^k. \end{aligned} \tag{38}$$

The difference with (27) is that now on the rhs there should be ξ_k s instead of arbitrary coefficients ζ_k s.

This is the forced Toda-chain τ -function which satisfies the equation

$$\tau_{\mathcal{N}} \frac{\partial^2 \tau_{\mathcal{N}}}{\partial \xi_1^2} - \left(\frac{\partial \tau_{\mathcal{N}}}{\partial \xi_1}\right)^2 = \tau_{\mathcal{N}+1} \tau_{\mathcal{N}-1}. \tag{39}$$

One can again repeat the procedure of removing the forced condition in order to obtain

$$\tau_{\mathcal{N}}\{t, \bar{t}|\xi\} = e^{\mathcal{Q}_{\mathcal{N}}} \sum_R d_R^2 \exp\left(\sum_n \xi_n C_{R,\mathcal{N}}(n)\right) \tag{40}$$

with

$$\begin{aligned} \sum_n \xi_n C_{R,\mathcal{N}}(n) &= \sum_{k,i} \xi_k ((r_i + \mathcal{N} + \gamma - i)^k - (\mathcal{N} + \gamma - i)^k) = \sum_{n,k} \binom{\mathcal{N}}{k} \xi_n C_R(k) \mathcal{N}^{n-k} \\ \mathcal{Q}_{\mathcal{N}} &= \sum_k \sum_{i=1}^{\mathcal{N}} \xi_k (\mathcal{N} + \gamma - i)^k \end{aligned} \tag{41}$$

and again the difference with (31) is that here all ζ_k s on the rhs are replaced with ξ_k s.

- KP τ -function w.r.t. (ξ, t) variables. ξ -integrability is preserved if d_R^2 is substituted by $d_R w_R$, where w_R is any solution to the Plucker relations; in particular, w_R can be a character:

$$\mathcal{Z}_{C_n}\{t|\xi\} = \sum_R d_R \chi_R(t) \exp\left(\sum_n \xi_n C_R(n)\right), \tag{42}$$

which is a KP τ -function both in t and ξ variables, provided Casimirs are chosen to satisfy (35).

However, $\sum_R \chi_R(t) \chi_R(\bar{t}) \exp(\sum_n \xi_n C_R(n))$, though still a KP τ -function in t and \bar{t} variables, is *not* a KP τ -function in ξ .

Also, there is no way to introduce an \mathcal{N} -dependence into (42) to make it a Toda-lattice τ -function. This is in accordance with the general fact that a Toda-chain τ -function can be promoted into a Toda-lattice τ -function only in a trivial way, so that it depends on t and \bar{t} only through differences $t_k - \bar{t}_k$.

5. Technical approaches

Technical details behind the checks and proofs of all these statements will be presented in a separate publication. They depend heavily on the theory of integrable hierarchies; however, these are relatively old results. A principle new piece is the interplay with the newer chapters of Hurwitz theory. They are based on the study of associative and commutative algebra (actually isomorphic to Kerov algebra [15]) of cut-and-join operators $\hat{W}(\Delta)$, which have linear group and symmetric group characters $\chi_R(t)$ and $\varphi_R(\Delta)$ as their common eigenvectors and eigenvalues, respectively [9]:

$$\hat{W}(\Delta) \chi_R(t) = \varphi_R(\Delta) \chi_R(t). \tag{43}$$

These operators can be represented as differential operators in t variables, for example,

$$\begin{aligned} \hat{W}[2] &= \sum_{a,b \geq 1} \left((a+b) p_a p_b \frac{\partial}{\partial p_{a+b}} + a b p_{a+b} \frac{\partial^2}{\partial p_a \partial p_b} \right) \\ &= \sum_{a,b \geq 1} \left(a b t_a t_b \frac{\partial}{\partial t_{a+b}} + (a+b) t_{a+b} \frac{\partial^2}{\partial t_a \partial t_b} \right) \end{aligned} \tag{44}$$

or, after the Miwa transform $p_k = k t_k = \text{tr} X^k$, as elements of the center of the universal enveloping of $GL(\infty)$:

$$\hat{W}(\Delta) = : \widehat{D}(\Delta) : = \frac{1}{z(\Delta)} : \prod_k (\text{tr} \hat{D}^k)^{m_k} : \tag{45}$$

with $\hat{D}_{ij} = \sum_k X_{ik} \frac{\partial}{\partial X_{jk}}$, for example,

$$\hat{W}[2] = \frac{1}{2!} : \text{tr} \hat{D}^2 : = \frac{1}{2} \sum_{i,j,k,l} X_{ik} X_{jl} \frac{\partial}{\partial X_{jk}} \frac{\partial}{\partial X_{il}}. \tag{46}$$

(If the factor of $z(\Delta)$ was omitted from the normalization of $\hat{W}(\Delta)$, then the eigenvalues would be $\frac{\hat{\chi}_R(\Delta)}{d_R}$.) The structure constants are the same for the multiplication of the \hat{W} -operators and of their eigenvalues:

$$\begin{aligned} \hat{W}(\Delta_1) \hat{W}(\Delta_2) &= \sum_{\Delta} C_{\Delta_1 \Delta_2}^{\Delta} \hat{W}(\Delta), \\ \varphi_R(\Delta_1) \varphi_R(\Delta_2) &= \sum_{\Delta} C_{\Delta_1 \Delta_2}^{\Delta} \varphi_R(\Delta) \quad \forall R \end{aligned} \tag{47}$$

and they are vanishing outside the interval $\max(|\Delta_1|, |\Delta_2|) \leq |\Delta| \leq |\Delta_1| + |\Delta_2|$. This algebra has various sets $\{\hat{C}(n)\}$ of *multiplicative* generators, with one $\hat{C}(n)$ at each level $|\Delta| = n$. $C_R(n)$ are their eigenvalues:

$$\hat{C}(n) \chi_R(t) = C_R(n) \chi_R(t). \tag{48}$$

An obvious choice is to take $\{\hat{W}[n]\}$ with single line diagrams for such a set, but, as explained in section 3, this is *not* the choice, preserving *any* kind of integrability. Actually, the t -integrability can be preserved if $\hat{C}(n)$ are chosen to be free cumulants (36), whose (nonlinear) relation to $\hat{W}[n]$ (represented by Kerov's polynomials) is known from [14]. However, even this set is *not* good for ξ -integrability. Fortunately, transformation to both ξ - and t -integrability preserving basis (35) from the basis of free cumulants is linear and elementary being given by the Newton binomial formulas.

In fact, equation (43) is equivalent to [9]:

$$\sum_{\Delta, R} d_R \varphi_R(\Delta) \chi_R(t) p'(\Delta) = e^t \sum_{\Delta, R} d_R \varphi_R(\Delta) \chi_R(t) p'(\Delta) \delta_{|R|, |\Delta|}, \quad (49)$$

which also implies (6). The difference between (5) and (7) for $k > 1$ arises because of the contribution of the structure constants $C_{\Delta_1, \Delta_2}^\Delta \neq 0$ for $|\Delta| \neq |\Delta_1|$ even if $|\Delta_1| = |\Delta_2|$.

The algebra of the cut-and-join operators is the Hurwitz theory part of the story. Many puzzles remain there, including matrix model realizations [16], mysterious form of the Virasoro constraints [17] and an open string generalization to the non-commutative algebra [18]. As to the integrability theory part, it includes relation to the character calculus and determinant representations of KP τ -functions. It summarizes many old developments, from the studies of Kontsevich matrix models in [19] to those of unitary models in [20]. The story of ξ -integrability and the difference between various choices of γ in (35) is intimately related to the theory of equivalent hierarchies [21], a rather sophisticated and not enough widely known, though important, chapter of integrability theory.

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