ON APPLYING APPROXIMATIONS TO FIND OPTIMAL EXCESS OF LOSS REINSURANCE

Mikhail Batsyn
candidate of physical-mathematical sciences, associate professor of «Applied mathematics and informatics» department, National Research University Higher School of Economics, Nizhny Novgorod

Ekaterina Batsyna
assistant professor of «Applied mathematics and informatics» department, National Research University Higher School of Economics, Nizhny Novgorod

E-mail: mbatsyn@hse.ru, batcyna@hse.ru
Address: 25/12, B. Pecherskaya street, Nizhny Novgorod, Russia

In this paper we consider the problem of finding an optimal excess of loss reinsurance which maximizes the reliability (probability of no ruin) of the insurance company. We apply two approximate approaches to calculate the distribution of total payments. The first approach is based on normal approximation of the payments distribution. Using this approximation we have derived an integral equation on the optimal retention limit. The second approach is based on simulation techniques. To test the precision of our approaches we use an exact formula for the distribution of total payments known for the case when losses in one insured event are distributed uniformly.

Key words: excess of loss reinsurance, ruin probability, maximum reliability, normal approximation, simulation, claim payments distribution.

1. Introduction

Reinsurance is one of the most efficient and widely-used methods of increasing a reliability of an insurance company. It secures the company against losses related to high payments in different insurance contracts. When reinsurance is used the risks of high payments are shared between insurance and reinsurance companies. Reinsurance contracts differ in the methods of the risk sharing. The problem of analyzing and optimizing of insurance and reinsurance parameters is an important and difficult problem of mathematical modeling. Different aspects of this problem have been actively analyzed in recent decades.

Many works deal with the problem of optimal reinsurance form, or function of sharing losses between insurance and reinsurance companies (Borch (1960), Heerwarden et al. (1989), Gajek & Zagrodny (2004), Kaluszka (2005), Guerra & Centeno (2008), Balbas et al. (2009)). The following criteria are used as optimization criteria: minimization of claim payments variance, maximization of expected utility, minimization of ruin probability, minimization of a linear combination of claim payments expectation and variance, minimization of a general objective function depending on all moments of total claim payments distribution. In contrast to the enumerated papers in the work of Denuit & Vermandele (1998) along with a global reinsurance (reinsurance of total losses over all insurance contracts) a local, or individual, reinsurance (reinsurance of losses individually for every insurance contract) is considered. It is shown that an optimal form of individual reinsurance is the excess of loss reinsurance. Golubin (2008) considers the problem of minimization of the expected maximum losses within the long-term collective risk model and proves that the excess of loss reinsurance gives the optimal solution to this problem. Golubin & Gridin (2012) show that under certain restrictions on insurer's and reinsurer's risks the excess of loss reinsurance minimizes the «stationary coefficient of variation» equal to an upper limit of a ratio of the risk variance to the risk expectation.

A number of papers are devoted to optimization of parameters of standard types of reinsurance: stop-loss, excess of loss, quota-share, and surplus. In works of Sholomitsky & Rachkova (1998), Verlaak & Beirlant (2003), Krvavych & Sherris (2006) different types of global reinsurance are analyzed. In Centeno (2002) the problem of minimizing of an upper bound for ruin probability is considered for excess of loss reinsurance. An improved Gerber's bound (Grandell, 1991) is applied for this purpose. The same problem is solved for two dependent risks in Centeno (2005). In Kaishev & Dimitrova (2006) a numerical solution by means of Appell polynomials is suggested for the problem of optimal excess of loss reinsurance. The joint survival probability of insurance and reinsurance companies is used as an optimization criterion.

Many authors (Borch (1960), Heerwarden et al. (1989), Gajek & Zagrodny (2004), Kaluszka (2005), Guerra & Centeno (2008), Balbas et al. (2009), Sholomitsky & Rachkova (1998), Verlaak & Beirlant (2003), Krvavych & Sherris (2006) and others) consider a global reinsurance for which the total losses over all insured events are shared between the insurance and reinsurance companies according to some function of losses sharing. Some authors (Denuit & Vermandele (1998), Golubin (2008), Golubin & Gridin (2012), Centeno (2002), Centeno (2005), Kaishev & Dimitrova (2006) and others) consider an individual reinsurance which is more complicated from the mathematical point of view than a global one. When individual reinsurance is used, payments in every insured event which exceed a certain limit r, called retention limit, are passed to the reinsurance company. The difficulty is that the distribution of an insurance company payment in an insured event has a discrete component due to the individual reinsurance (the distribution function becomes discontinuous). This peculiarity considerably complicates the problem of obtaining the distribution of the sum of payments over all insured events. Some authors (Denuit & Vermandele (1998), Golubin (2008), Golubin & Gridin (2012), Centeno (2002), Centeno (2005) and others) avoid this difficulty by using optimization criteria which do not require calculating the distribution of total payments of the insurance company. Other works (Krvavych & Sherris (2006), for example) apply different approximations and numerical methods.

In this paper we suggest two approaches to the problem of finding the optimal retention limit in the excess of loss reinsurance which maximizes the reliability of an insurance company. This problem is considered within the short-term collective risk model. In the first approach we approximate the distribution function of the total claim payments of an insurance company by the normal distribution. This allows us to derive an analytical equation on the optimal value of the retention limit. In the second approach we apply simulation techniques to build an approximate distribution function of the total claim payments. To test the precision of the approximate approaches we use an exact formula for the distribution function of total claim payments known for the case when losses in one insured event are distributed uniformly (Batsyn & Kalyagin, 2012). Our computational experiments allow us to determine when either approximation gives results accurate enough and when it is not applicable. We show that the simulation approach can be applied for any parameters values, but it requires a long computational time to get the necessary precision. The applicability of the normal approximation is limited by the distribution of the number of insured events. The expected number of insured events should be high to guarantee high precision.

The paper is organized as follows. In the next section we briefly describe the short-term collective risk model used in this work. The third section is devoted to our first approach (based on normal approximation) to the considered excess of loss reinsurance optimization problem. The second approach based on simulation techniques is presented in the fourth section.

2. Short-term collective risk model

The main goal of the current work is to study the behavior of the insurance company reliability depending on the parameters of the excess of loss reinsurance. In the short-term collective risk model an insurance company has a portfolio of insurance contracts and insured events for these contracts are modeled as a Poisson random process. An average number λ of events in this process is known from statistics. The number N of insured events has a Poisson distribution,

$$N \sim \frac{e^{-\lambda}\lambda^n}{n!}.$$

Losses in every insured event i are represented as a random variable X_i with distribution function $F(x)$ and probability density $f(x)$. If the losses X_i are less than the retention limit r in the excess of loss reinsurance then the insurance company payment Y_i is equal to X_i. Otherwise the insurer pays only r and the excess $X_i - r$ is paid by the reinsurer:

$$Y_i = \begin{cases} X_i, \ X_i \le r \\ r, \ X_i > r \end{cases}.$$

The retention limit r is chosen by the insurance company on its own. The total claim payments of the insurance company are equal to a random sum of random variables:

$$Y = \sum_{i=1}^{N} Y_i .$$

The distribution function of Y is represented by the following formula:

$$F^Y(x) = P\{Y < x\} =$$
$$= \sum_{n=0}^{\infty}\left(P\{N=n\}P\left\{\sum_{i=1}^{n} Y_i < x\right\}\right) = \sum_{n=0}^{\infty}\left(\frac{\lambda^n}{n!}e^{-\lambda}F_n(x)\right), \quad (1)$$

where $F_n(x)$ is the distribution function of the sum of n claim payments (for $n=0$ we set $F_0(x)=1$). Calculation of this function is a difficult mathematical problem due to the retention limit r which makes this distribution a mixed one (a combination of discrete and continuous distributions). An analytical expression of $F_n(x)$ has been obtained in Batsyn & Kalyagin (2012) for the case when losses X_i have uniform distribution. In this work we use approximate approaches to calculate this function.

The insurance company takes from every contract an insurance premium which depends on risk premium Π, risk loading θ, and expenses loading η. A risk loading is a loading of a risk premium paid by an insured and needed to increase reliability of the insurance company. Expenses loading η is necessary for different administrative needs and cannot be used for claim payments. That is why we will not take it into account further. So the insurance company takes $\Pi(1+\theta)$ from every contract. A risk premium Π is determined from risk equivalence principle, so that the total sum of risk premiums over all contracts is equal to the expected total losses in all insured events, $\lambda\mu$ (where μ denotes the expectation of losses X_i). A risk loading θ is set by the insurance company on its own. The total sum S of insurance premiums taken from all contracts is $S = \lambda\mu(1+\theta)$. The reinsurance company also calculates its risk premium $\tilde{\Pi}$ according to the risks equivalence principle. So total risk premium is equal to $\lambda\tilde{\mu}$, where $\tilde{\mu}$ denotes the expectation of the reinsurance company payment $\tilde{Y}_i$.

Proposition 1. The expectation $\tilde{\mu}$ of a reinsurance company payment $\tilde{Y}_i$ in one insured event i is equal to $\tilde{\mu} = \mu - I(r)$, where

$$I(r) = \int_0^r (1 - F(x))dx.$$

Proof. A payment of the reinsurance company depends on losses in an insured event in the following way:

$$\tilde{Y}_i = \begin{cases} 0, \ X_i \le r \\ X_i - r, \ X_i > r. \end{cases}$$

The distribution function of this random variable is equal to $\tilde{F}(x) = P\{\tilde{Y}_i < x\} = F(x+r)$. It is not difficult to calculate the expectation of payment $\tilde{Y}_i$ applying integration by parts:

$$\tilde{\mu} = E(Y_i) = \int_0^{\infty} xf(x+r)dx = \mu - \int_0^r (1 - F(x))dx. \ \square$$

The insurance company pays an insurance premium $\tilde{\Pi}(1+\tilde{\theta}+\tilde{\eta})$ for reinsurance of every contract. Here $\tilde{\theta}$ is a risk loading of the reinsurance company which is always greater than θ, and $\tilde{\eta}$ is its expenses loading. So the total cost of reinsurance for the insurance company is

$$\tilde{S} = \lambda(\mu - I(r))(1+\tilde{\theta}+\tilde{\eta}).$$

The funds which can be used to cover claim payments Y are $S - \tilde{S}$. The reliability of the insurance company Rl is measured as the probability of its survival, or the probability that all the claim payments will be covered by the company funds:

$$Rl = P\{Y < S - \tilde{S}\} = F^Y(S - \tilde{S}).$$

The reliability depends on the chosen value of retention limit r in a sophisticated way, because both the distribution of Y and the value of $\tilde{S}$ depend on r.

Proposition 2. The expectation of the total payments Y is $E(Y) = \lambda I(r)$, the variance of Y is $var(Y) = \lambda I_2(r)$, where

$$I(r) = \int_0^r (1 - F(x))dx, \text{ and}$$

$$I_2(r) = \int_0^r 2x(1 - F(x))dx.$$

Proof. These expressions are derived using the laws of total expectation and variance (via conditional expectation and variance) and integration by parts, same as for proposition 1. □

3. Normal approximation approach

To calculate the reliability Rl of the insurance company a normal approximation Φ to the total payments distribution can be applied.

Theorem 1. The reliability function $Rl(r)$ calculated via normal approximation has its maximum value in point r^*, where r^* is the unique solution of equation

$$I_2(r) - rI(r) + \left(1 - \frac{\theta}{\tilde{\theta} + \tilde{\eta}}\right) r\mu = 0. \qquad (2)$$

Proof. The reliability function is calculated as follows:

$$Rl(r) = P\{Y < S - \tilde{S}\} \approx \Phi_{(0,1)}\left(\frac{S - \tilde{S} - E(Y)}{\sqrt{var(Y)}}\right) =$$

$$= \Phi_{(0,1)}\left(\sqrt{\frac{\lambda}{I_2(r)}}\left(\tilde{\theta} + \tilde{\eta}\right)\left[I(r) - \left(1 - \frac{\theta}{\tilde{\theta} + \tilde{\eta}}\right)\mu\right]\right) \qquad (3)$$

Since $\Phi_{(0,1)}$ is an increasing function then behavior (intervals of increasing/decreasing) of $Rl(r)$ coincides with behavior of

$$\delta(r) = \sqrt{\frac{\lambda}{I_2(r)}}\left(\tilde{\theta} + \tilde{\eta}\right)\left[I(r) - \left(1 - \frac{\theta}{\tilde{\theta} + \tilde{\eta}}\right)\mu\right].$$

Its first derivative is:

$$\delta'(r) = \sqrt{\frac{\lambda}{(I_2(r))^3}}\left(\tilde{\theta} + \tilde{\eta}\right)(1 - F(r))\left[I_2(r) - rI(r) + \left(1 - \frac{\theta}{\tilde{\theta} + \tilde{\eta}}\right) r\mu\right]$$

The intervals of positive/negative values for $\delta'(r)$ coincide with these intervals for the expression in square brackets

$$\gamma(r) = I_2(r) - rI(r) + \left(1 - \frac{\theta}{\tilde{\theta} + \tilde{\eta}}\right) r\mu.$$

Its first derivative is

$$\gamma'(r) = \left(1 - \frac{\theta}{\tilde{\theta} + \tilde{\eta}}\right)\mu + r(1 - F(r)) - I(r),$$

and the second derivative is $\gamma''(r) = -rf(r) \le 0$.

So $\gamma'(r)$ decreases from

$$\gamma'(0) = \left(1 - \frac{\theta}{\tilde{\theta} + \tilde{\eta}}\right)\mu > 0$$

to $\gamma'(r) = 0$, and goes to $-\infty$. This means that $\gamma(r)$ first increases from $\gamma(0) = 0$ to some positive value, where $\gamma'(r) = 0$, and then decreases to $-\infty$. Since $\gamma(r)$ is a continuous function then there exists exactly one point r^* in which it is equal to 0. The same is true for $\delta'(r)$: it is positive for $r \in (0, r^*)$, equal to 0 in $r = r^*$, and negative for $r \in (r^*, \infty)$. Consequently, $\delta(r)$ and $Rl(r)$ has its maximum value in r^*. □

Theorem 1 allows us to find an optimal retention limit by solving equation (2) which can be easily solved numerically in a general case or analytically for some concrete distributions $F(x)$ like uniform, for example. Using expression (3) we calculate the reliability function. The precision of the normal approximation can be estimated using the Berry-Esseen inequality (see Bening & Korolev, 2002; Korolev & Shevtsova, 2010, 2012). The precision increases with increasing of an average insured events number λ. To determine the precision of our approach empirically we compare the approximate reliability function with the precise reliability calculated via expression (1). To compute the expression (1) we use the analytical formula for $F_n(x)$ derived in Batsyn & Kalyagin (2012) for the case of uniform losses distribution.

The graphs showing examples of comparison for different values of λ are presented on *fig. 1*. The optimal value r^* of the retention limit found from equation (2) is also shown on these graphs. In average for $\lambda = 1$ the precision is about 3%, for $\lambda = 10$ – about 1%, for $\lambda = 50$ – about 0.5%, for $\lambda = 100$ – 0.3%, for $\lambda = 1000$ – 0.1%, and for $\lambda = 100000$ – 0.01%. It confirms the dependency of the precision ε on λ known from the Berry-Esseen inequality:

$$\varepsilon \sim \frac{1}{\sqrt{\lambda}}.$$

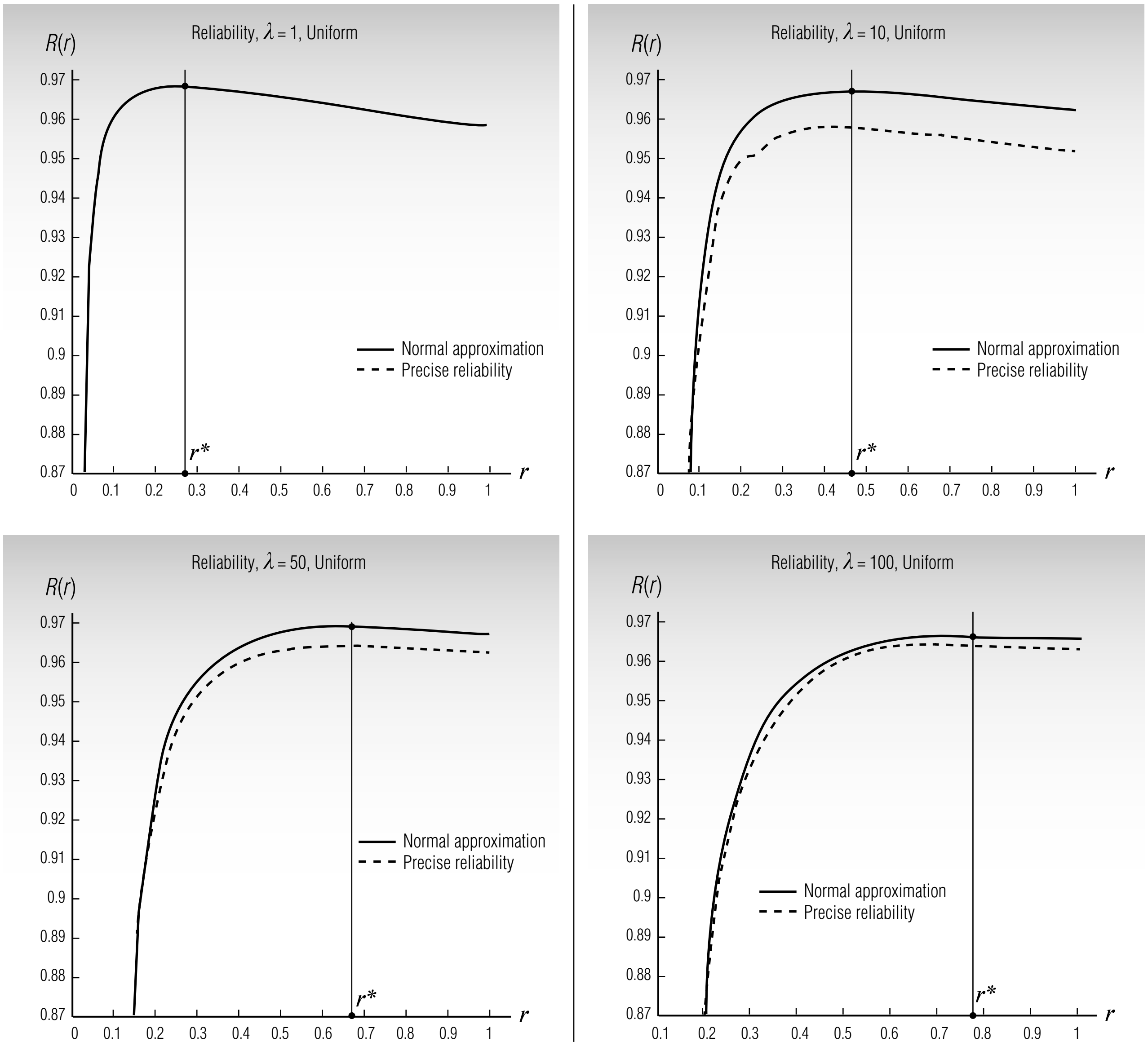

Fig. 1. Comparison of approximate and precise *Rl*(*r*) for λ = 1, 10, 50, 100

4. Simulation approach

The simulation approach is based on random generation of the number of insured events and the losses in each of these events. We generate these random variables many times for every value of the retention limit *r*. The number of insured events *N* has a Poisson distribution with parameter λ and the losses have distribution *F*(*x*). For each event $i=\overline{1,N}$ if losses X_i are greater than *r* then the payment Y_i is taken equal to *r*, otherwise it is taken equal to X_i. If the total payments

$$Y=\sum_{i=1}^{N} Y_i$$

are less than the total funds $S-\tilde{S}$ then no ruin occurs. After a large number of such iterations we calculate the empirical probability (the frequency) of «no ruin» events. According to the law of large numbers this probability can be used as an approximation for the reliability of the insurance company. To measure the precision of the simulation approach we compare the obtained reliability function with the precise reliability known for the case of uniform losses distribution.

The graphs demonstrating this comparison are shown on *fig. 2*. As one can see the precision is higher for a greater iterations number and does not depend on λ. Even for $\lambda=1$, when the precise reliability has a considerable jump, the approximate function

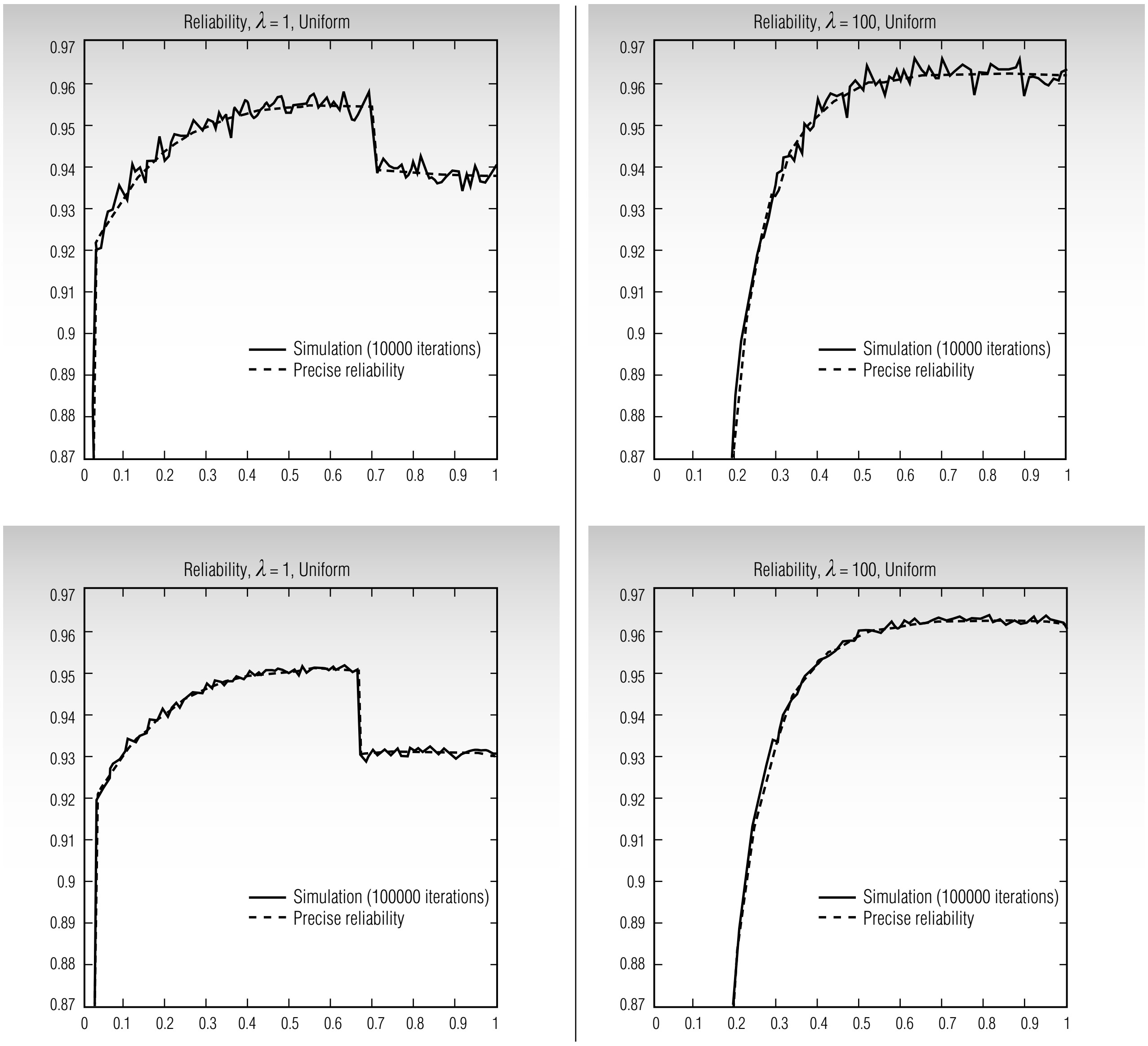

Fig. 2. Simulation-based approximation and precise *Rl*(*r*) for different λ and iterations number

precisely repeats this jump. In whole this approach gives much better results than the approach based on normal approximation. The only limitation of the simulation is the computational time which depends linearly on the average number λ of generated insured events and on the number of iterations. For example, for $\lambda = 100$ and 100000 iterations the program written in Matlab takes about 10 hours on a standard personal computer with 1.66 GHz CPU and 1.5 Gb of memory. ■

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