Derived sections and categorical resolutions in homotopical context

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Introduction

In (non-commutative) geometry, a categorical resolution of a (potentially singular) variety X is a full and faithful embedding of its derived category $\mathcal{D}^b(X)$ into a smooth and proper triangulated category $\mathcal{T}$ [11, 12]. The notion generalises the situation of rational singularities, where the geometric resolution functor $F : \tilde{X} \to X$ induces the full and faithful functor $F^* : \mathcal{D}^b(X) \to \mathcal{D}^b(\tilde{X})$ to the smooth and proper category $\mathcal{D}^b(\tilde{X})$.

Another example of a categorical resolution comes from considering a finite CW-complex Y of homotopy type $K(G, 1)$. The derived category of local systems over Y, which we denote as $Loc(Y, k)$, is the derived category of complexes of locally constant sheaves over some field k. Denote by BG the fundamental groupoid of Y. We then see that $Loc(Y, k)$ is exactly the derived category $\mathcal{D}(\mathrm{B}G, k)$ of functors from BG to the category of complexes of vector spaces

$\mathbf{DVect}_k$, which is nothing else than the derived category of representations of G. Since Y is of type $K(G,1)$, $Loc(Y,k)$ is a full subcategory of the derived category of constructible sheaves on Y.

Take I to be the partially ordered set associated to a chosen regular cellular decomposition of Y and consider the derived category $\mathcal{D}(I,k)$ of functors from I to $\mathbf{DVect}_k$. Choosing a point (say, the centre) in each cell of I and connecting them by paths when one cell is included in the other defines a functor

$$F : I \to \mathrm{B}G. \tag{i}$$

On the level of triangulated categories, F induces the pullback functor $F^* : Loc(Y,k) \to \mathcal{D}(I,k)$. One can then prove[1] that this functor is full and faithful, with its image consisting of those functors $I \to \mathbf{DVect}_k$ which are 'locally constant', in the sense that they send all morphisms of I to quasiisomorphisms. It is also seen that $\mathcal{D}(I,k)$ is indeed a good object: it is the category of modules over a finite-dimensional algebra generated by I.

The functor F satisfies a technical condition (up to an equivalence, it is a Grothendieck opfibration with contractible fibers, see Definition 3.21 for details). The functors with this property are called *resolutions* in this paper and play an important role.

The examples considered above are all additive: the categories $\mathcal{D}^b(X)$ and $Loc(Y,k)$ carry a triangulated structure. In this paper, we would like to develop the formalism of categorical resolutions in the setting of homotopical algebra, that is, the setting of non-linear algebraic structures, for example algebras that are associative, commutative or satisfy any other identities only up to (coherent) homotopy. Because of this, one has to abandon homological, linear methods in favour of homotopy-theoretic techniques. By the latter, we mean the techniques of categorical homotopy theory [5, 16], so that the basic objects of study are pairs $(\mathcal{M},\mathcal{W})$ of a category $\mathcal{M}$ and a class of weak equivalences $\mathcal{W}$, the pairs which should be viewed as enhancements of the categories $\mathrm{Ho}\,\mathcal{M}$ obtained by localising $\mathcal{M}$ along $\mathcal{W}$. In the linear case, the derived categories like $\mathcal{D}(I,k)$ are localisations of the functor categories $Fun(I,\mathbf{DVect}_k)$ along the class $\mathcal{W}$ of natural transformations which are quasiisomorphisms for each object i of I.

One of the approaches to homotopical algebraic structures goes back to Segal and consists, in the case of commutative structures, of the following. Define $\mathbf{Fin}_*$ to be the category of finite sets and partially defined maps, i.e. a morphism from a finite set S to another finite set T in $\mathbf{Fin}_*$ is a map $U \to T$ defined on some subset $U \hookrightarrow S$. Take a symmetric monoidal category $\mathcal{M}$ with the monoidal product operation denoted by $\otimes$. Define then $\mathcal{M}^{\otimes}$ to be the category with objects $(S,\{X_s\}_{s\in S})$ where $S \in \mathbf{Fin}_*$ and each X_s is an object of $\mathcal{M}$. A morphism $(S,\{X_s\}_{s\in S}) \to (T,\{Y_t\}_{t\in T})$ consists of a partially defined map $f : S \to T$, and for each $t \in T$, of a morphism $\otimes_{s\in f^{-1}(t)} X_s \to Y_t$; when $f^{-1}(t)$ is empty, the monoidal product over the empty set is the unit object. The compositions can be then defined with the help of the coherence isomorphisms for the product $\otimes$.

There is an evident forgetful functor $p : \mathcal{M}^{\otimes} \to \mathbf{Fin}_*$; one can see that a monoid object $A \in \mathcal{M}$ gives a *section* of p, defined as $S \mapsto (S,\{X_s\})$ with each $X_s = A$. Changing $\mathbf{Fin}_*$ to Δ^{op}, the opposite of the category of simplexes, one can, in a similar way, describe associative monoids in (not necessarily symmetric) monoidal categories.

[1] We state this as a folklore result here in the introduction, although it can be obtained from the main result of this paper

The functor $p : \mathcal{M}^{\otimes} \to \mathbf{Fin}_*$ is a *Grothendieck opfibration* and also is the output of the following procedure. Up to an equivalence, the symmetric monoidal structure on $\mathcal{M}$ can be presented as a functor $M : \mathbf{Fin}_* \to \mathbf{Cat}$ with $M(1) \cong \mathcal{M}$, and so that M satisfies a set of special conditions. We then apply the Grothendieck construction (cf. Construction 2.6) to M to obtain a functor $\int M \to \mathbf{Fin}_*$ with domain $\int M \cong \mathcal{M}^{\otimes}$.

To make this discussion homotopical, assume that the symmetric monoidal category $\mathcal{M}$ has weak equivalences preserved by the monoidal product. Then in terms of the functor $M : \mathbf{Fin}_* \to \mathbf{Cat}$, one would equip each value $M(S)$ with weak equivalences preserved by each $M(f) : M(S) \to M(T)$; in terms of $\mathcal{M}^{\otimes} \to \mathbf{Fin}_*$, see Definition 2.15. One would then take the category of sections $\mathsf{Sect}(\mathbf{Fin}_*, \mathcal{M})$ and attempt to induce weak equivalences from $\mathcal{M}^{\otimes}$. However, while always possible, this may be not the right way to proceed. For example, over a field k of positive characteristic, the category of commutative DG-algebras $\mathbf{CDGA}_k$ is a subcategory of $\mathsf{Sect}(\mathbf{Fin}_*, \mathbf{DVect}_k^{\otimes})$. The homotopical structure induced on $\mathbf{CDGA}_k$ from the sections then matches the one induced by the forgetful functor $U : \mathbf{CDGA}_k \to \mathbf{DVect}_k$, and the latter homotopical structure is known to behave badly.

Sometimes [9] one can work with the category of sections and get reasonable results. In those cases however, given a functor $p : \mathcal{E} \to \mathcal{C}$ which is a Grothendieck opfibration, we assume that it can be reproduced as the Grothendieck construction of a functor $E : \mathcal{C} \to \mathbf{Cat}$, with each $E(c)$ being a (cofibrantly generated) model category, and for each $f : c \to c'$, we require that $E(f)$ is a part of a Quillen adjunction. However, when $\mathcal{E} \to \mathcal{C}$ does not have that much structure, the methods of [9] are inapplicable. In the case of a symmetric monoidal category, the functors $M(f) : M(S) \to M(T)$ are given, in essence, by tensor products, which at best can be expected to preserve weak equivalences (what happens e.g. for $\mathbf{DVect}_k$), but not to admit any adjoints. They do not even preserve (co)products.

This is where we propose our approach, which we call the formalism of *derived sections*. Its roots can be traced back to [3]. For any small category $\mathcal{C}$, there is another category, called the *simplicial replacement of* $\mathcal{C}$ (Definition 3.1), which we denote by $\mathbb{C}$. The objects of $\mathbb{C}$ are sequences of morphisms $c_0 \xrightarrow{f_1} ... \xrightarrow{f_n} c_n$ of $\mathcal{C}$, a morphism between $c_0 \xrightarrow{f_1} ... \xrightarrow{f_n} c_n$ and $c'_0 \xrightarrow{f'_1} ... \xrightarrow{f'_n} c'_m$ consists of a map $i : [m] \to [n]$ in the category of simplexes Δ (cf. Notation 1.1), such that $c_{i(0)} \to ... \to c_{i(m)}$ equals $c'_0 \to ... \to c'_m$. A functor $A : \mathbb{C} \to \mathcal{N}$ valued in any category $\mathcal{N}$ supplies us in particular with spans of the form

$$A(c_0) \longleftarrow A(c_0 \xrightarrow{f} c_1) \longrightarrow A(c_1), \tag{ii}$$

for any morphism $f : c_0 \to c_1$. If one requests that the morphism $A(c_0) \longleftarrow A(c_0 \xrightarrow{f} c_1)$ is an isomorphism, we then can view this span as a morphism $A(f) : A(c_0) \to A(c_1)$ in $\mathcal{N}$. The functor A supplies other span diagrams for longer sequences of morphisms as well; and one can see that imposing conditions like above on those spans permits to define a genuine functor $\mathcal{C} \to \mathcal{N}$.

When $\mathcal{N}$ has weak equivalences $\mathcal{W}$, one can instead require than in the spans like (ii), the left map is a weak equivalence. This would provide us with the set of data which would constitute a 'weak' functor $\mathcal{C} \to \mathcal{N}$ in a rather natural sense. For example, from a functor $\mathbb{C} \to \mathcal{N}$ satisfying the prescribed conditions, we easily obtain a functor from $\mathcal{C}$ to the localisation $\operatorname{Ho}\mathcal{N}$ of $\mathcal{N}$ with respect to $\mathcal{W}$. In good cases, the morphisms in $\operatorname{Ho}\mathcal{N}$ can be described as fractions of the form $X \leftarrow Y \to Z$ with $Y \to X$ in $\mathcal{W}$, thus $\mathbb{C}$ can be also seen as the 'fraction replacement' of the source $\mathcal{C}$.

When one works with opfibrations $\mathcal{E} \to \mathcal{C}$, we show that one can extend them to $\mathbb{C}$ using a contravariant functor to $\mathcal{C}$, defined by $c_0 \xrightarrow{f_1} ... \xrightarrow{f_n} c_n \mapsto c_n$. The resulting lift $\mathbf{E} \to \mathbb{C}$ will then be a fibration (that is, it is equivalent to the Grothendieck construction of a contravariant functor from $\mathbb{C}$ to $\mathbf{Cat}$). Then a section $A : \mathbb{C} \to \mathbf{E}$ of this fibration, acting on $f : c_0 \to c_1$, will supply the following span in $\mathbf{E}(c_1) = \mathcal{E}(c_1)$

$$E(f)A(c_0) \longleftarrow A(c_0 \xrightarrow{f} c_1) \longrightarrow A(c_1), \tag{iii}$$

where $E(f) : \mathcal{E}(c_0) \to \mathcal{E}(c_1)$ is the transition functor along f induced by the opfibration structure. When there is a suitable homotopical structure on $\mathcal{E}$, we can again require the left map to be a weak equivalence. The category of such derived sections $\mathbb{R}\mathsf{Sect}(\mathcal{C}, \mathcal{E})$ is thus a subcategory of sections $\mathsf{Sect}(\mathbb{C}, \mathbf{E})$ singled out by such conditions, and it can be equipped with a homotopical structure.

When the opfibration in question is the symmetric monoidal homotopical category opfibration $\mathcal{M}^{\otimes} \to \mathbf{Fin}_*$, the span (iii) for a suitably normalised derived section A will, in effect, lead to the following diagram in $\mathcal{M}^{\otimes}(1) = \mathcal{M}$ for each fully defined map of finite sets $S \to 1$:

$$A(1)^{\otimes S} \longleftarrow A(S \to 1) \longrightarrow A(1).$$

The left map is moreover a weak equivalence; we thus see that, when inverting weak equivalences, there are operations $A(1)^{\otimes S} \to A(1)$, which can be thought of equipping $A(1)$ with a weakly associative and commutative multiplication. A similar pattern occurs when one studies factorisation algebras [1].

The authors of [3] introduced simplicial replacements to calculate homotopy colimits. For similar reasons, the simplicial replacement technique is also useful if one studies the base-change of derived sections along functors $F : \mathcal{D} \to \mathcal{C}$. There is an induced functor $\mathbb{F} : \mathbb{D} \to \mathbb{C}$ between the simplicial replacements, which induces a homotopical inverse image functor $\mathbb{F}^* : \mathbb{R}\mathsf{Sect}(\mathcal{C}, \mathcal{E}) \to \mathbb{R}\mathsf{Sect}(\mathcal{D}, F^*\mathcal{E})$, where $F^*\mathcal{E} \to \mathcal{D}$ is the pullback of $\mathcal{E} \to \mathcal{C}$. When F is a resolution, just like the functor (i) considered in the linear case, one would like to see that $\mathbb{F}^*$ is full and faithful on homotopy level, and ideally one would also like to characterise its essential image. It turns out that the technique of simplicial replacements is flexible enough as to allow to construct a homotopical 'direct image' functor[2]

$$\mathbb{F}_! : \operatorname{Ho}\mathsf{Sect}(\mathbb{D}, \mathbb{F}^*\mathbf{E}) \to \operatorname{Ho}\mathsf{Sect}(\mathbb{C}, \mathbf{E})$$

and a natural map $\epsilon : \mathbb{F}_!\mathbb{F}^* \to id$. While ϵ is not in general a counit map (so that $\mathbb{F}_!$ is not a left adjoint to $\mathbb{F}^*$), and while $\mathbb{F}_!$ will not in general map $\mathbb{R}\mathsf{Sect}(\mathcal{D}, F^*\mathcal{E})$ to derived sections over $\mathcal{C}$, in the case of a resolution $F : \mathcal{D} \to \mathcal{C}$, the functor $\mathbb{F}_!$ and ϵ will ensure the homotopical fullness and faithfulness of $\mathbb{F}^*$. It will furthermore preserve those derived sections which are locally constant (Definition 3.23), at least when $\mathcal{E} \to \mathcal{C}$ is good enough (Definition 5.11), ensuring that the essential image of $\mathbb{F}^*$ on homotopy level consists exactly of locally constant derived sections. Thus our main results, Theorems 3.24 and 3.25, allow us to homotopically embed $\mathbb{R}\mathsf{Sect}(\mathcal{C}, \mathcal{E})$ into the category $\mathbb{R}\mathsf{Sect}(\mathcal{D}, F^*\mathcal{E})$ by the means of the functor $\mathbb{F}^*$ and have enough control over the essential image of this embedding.

Organisation of the paper. In the first section, we introduce the formalism of homotopical Δ-categories. The content of this section is not new and is somehow present in the

[2]The pullback fibration $\mathbb{F}^*\mathbf{E} \to \mathbb{D}$ coincides with the extension of $F^*\mathcal{E} \to \mathcal{D}$ to $\mathbb{D}$.

folklore. For instance, the geometric realisation functor for homotopical categories, in the setting which goes beyond simplicial model categories, has been considered in [2, Appendix].

In the second section, we also introduce some of the basic notions and constructions related to the theory of Grothendieck (op)fibrations. Since Grothendieck opfibrations are natural tools for encoding the notion of families of categories, we introduce a class of suitably structured opfibrations, called homotopical Δ-opfibrations, which formalise the notion of a family of Δ-categories.

In the third section, we introduce simplicial replacements and then use them to define derived sections. We then define resolutions and formulate our main results, Theorems 3.24 and 3.25. The fourth section deals with the construction of the pushforward functor $\mathbb{F}_!$ and the map $\epsilon : \mathbb{F}_!\mathbb{F}^* \to id$, the data which one can use to verify if the 'right adjoint' $\mathbb{F}^*$ is full and faithful.

Finally, the fifth section consists of the analysis of the case of a resolution, outlining the proof of Theorems 3.24 and 3.25. It is proven that in this case, the inverse image on the derived sections is full and faithful on the homotopy level. In addition, under mild assumptions we can characterise the essential image of the inverse image.

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1 Generalities on geometric realisation

1.1 Homotopy colimits

Notation 1.1. For any category $\mathcal{C}$, $x \in \mathcal{C}$ means that x is an object of $\mathcal{C}$. We also write $f \in \mathcal{C}$ for morphisms $f : x \to y$ of $\mathcal{C}$ if there is no confusion. The set of morphisms between two objects x, y of $\mathcal{C}$ is denoted $\mathcal{C}(x, y)$. The category of functors $Fun(I, \mathcal{M})$ between two categories I and $\mathcal{M}$ is often denoted as $\mathcal{M}^I$. Sometimes, given an object $x \in \mathcal{C}$, we denote again by x the functor from the terminal category to $\mathcal{C}$ which picks out x.

From now on, Δ denotes the usual category of simplexes, i.e. the full subcategory of the category of categories **Cat** (we ignore the size issues in this paper) spanned, for $n \geq 0$, by categories $[n]$ with $n+1$ objects $0, ..., n$ and exactly one morphism from i to j whenever $i \leq j$.

By $\mathbf{SSet} = Fun(\Delta^{\mathsf{op}}, \mathbf{Set})$ we denote the category of simplicial sets. We often identify Δ with its image in **SSet** by the Yoneda embedding

$$\Delta^\bullet : \Delta \to \mathbf{SSet} = Fun(\Delta^{\mathsf{op}}, \mathbf{Set}), \quad [n] \mapsto \Delta^n := \Delta^\bullet([n]) = \Delta(-, [n]). \qquad (1.1)$$

For a simplicial object $X : \Delta^{\mathsf{op}} \to \mathcal{M}$ in a category $\mathcal{M}$, denote $X_n := X([n])$ for any $[n] \in$

Δ, and similarly, for bisimplicial objects $Y : \Delta^{\mathsf{op}} \times \Delta^{\mathsf{op}} \to \mathcal{M}$, we write $Y_{nm} = Y(([n],[m]))$. We also write $\Delta^{\mathsf{op}}\mathcal{M} := Fun(\Delta^{\mathsf{op}}, \mathcal{M})$, $(\Delta \times \Delta^{\mathsf{op}})\mathcal{M} := Fun(\Delta \times \Delta^{\mathsf{op}}, \mathcal{M})$ and so on.

Definition 1.2. A *homotopical category* is a pair $(\mathcal{M}, \mathcal{W})$ where $\mathcal{M}$ is a category and $\mathcal{W}$ is a subcategory of $\mathcal{M}$ which contains all objects and isomorphisms. We moreover require that for a composable pair of morphisms f, g of $\mathcal{M}$, if any two elements of $\{f, g, gf\}$ are in $\mathcal{W}$, then the third one is in $\mathcal{W}$ as well.

We call $\mathcal{W}$ the category of *weak equivalences.* A morphism $f : x \to y$ of $\mathcal{M}$ is a *weak equivalence* if it belongs to $\mathcal{W}$.

Definition 1.3. Given two homotopical categories $(\mathcal{M}, \mathcal{W}_{\mathcal{M}})$, $(\mathcal{N}, \mathcal{W}_{\mathcal{N}})$ a functor $F : \mathcal{M} \to \mathcal{N}$ is *homotopical* iff $F(\mathcal{W}_M) \subset \mathcal{W}_N$. Equivalently, F takes weak equivalences of $\mathcal{M}$ to weak equivalences of $\mathcal{N}$.

Definition 1.4. A subcategory $\mathcal{W} \subset \mathcal{M}$ satisfies the *two-out-of-six* property, if given three maps in $\mathcal{M}$ denoted f, g, h, so that they are composable with compositions gf, hg, hgf, if any two maps of f, g, h, gf, hg, hgf are in $\mathcal{W}$, then all maps in this list are in $\mathcal{W}$.

The subcategory of isomorphisms in any category satisfies two-out-of-six. The subcategory of weak equivalences in any model category satisfies two-out-of-six as well [5].

Example 1.5. Some well known examples of homotopical categories are

- the category **SSet** of simplicial sets which can be equipped with a homotopical structure by defining $\mathcal{W}$ to be the subcategory of weak homotopy equivalences [6] of simplicial sets.
- the category $\mathbf{DVect}_k$ of unbounded chain complexes over a field k, with $\mathcal{W}$ being the subcategory of quasiisomorphisms [10].

For a homotopical category $(\mathcal{M}, \mathcal{W})$ its localisation[3] [5, 10] $\mathcal{W}^{-1}\mathcal{M}$ will be also denoted by $\operatorname{Ho}\mathcal{M}$. Any homotopical functor $F\mathcal{M} \to \mathcal{N}$ descends to a functor $\overline{F} : \operatorname{Ho}\mathcal{M} \to \operatorname{Ho}\mathcal{N}$.

Definition 1.6. For $I \in \mathbf{Cat}$ and a homotopical category $(\mathcal{M}, \mathcal{W})$, the standard homotopical structure $(\mathcal{M}^I, \mathcal{W}_I)$ on the category of functors $A : I \to \mathcal{M}$ consists of those natural transformations $\alpha : A \to B$ which are valued in the maps of $\mathcal{W}$. That is, for each $i \in I$, the map $\alpha(i) : A(i) \to B(i)$ is a weak equivalence.

1.2 Tensors and Δ-categories

Denote by $\delta : \Delta \to \Delta \times \Delta$ the diagonal functor for Δ.

Definition 1.7. A Δ-structure on a category $\mathcal{M}$ consists of

1. a functor
$$\otimes : \Delta \times \mathcal{M} \to \mathcal{M}, \quad ([n], x) \mapsto \Delta^n \otimes x,$$

[3] We ignore the size issues in the discussion.

2. a natural transformation *diag* depicted as a 2-square

$$\begin{tikzcd}
\Delta \times \mathcal{M} \arrow[rr, "\otimes"] \arrow[d, "\delta \times id"'] & \Downarrow diag & \mathcal{M} \\
\Delta \times \Delta \times \mathcal{M} \arrow[rr, "id \times \otimes"'] & & \Delta \times \mathcal{M} \arrow[u, "\otimes"']
\end{tikzcd}$$

3. a natural isomorphism of $\mathcal{M}$-endofunctors: $\Delta^0 \otimes - \xrightarrow{\sim} id_{\mathcal{M}}$.

These data should satisfy the obvious coassociativity and counitality identities. A category $\mathcal{M}$ with a Δ-structure is called a Δ-*category* if $\mathcal{M}$ is cocomplete and the functor $\otimes$ preserves colimits in the second argument.

Remark 1.8. It is immediate that a Δ-category $\mathcal{M}$ has a **SSet**-enrichment given by the mapping spaces

$$\mathrm{Map}_{\mathcal{M}}(x, y)_n := \mathcal{M}(\Delta^n \otimes x, y).$$

Example 1.9. The terminal category [0] can be equipped with a (trivial) Δ-structure.

Example 1.10. The category $\mathbf{DVect}_k$ is a Δ-category for $\Delta^n \otimes M := C_\bullet(\Delta^n) \otimes_k M$, where $C_\bullet$ is the chain complex functor. The natural transformation *diag* comes from the Alexander-Whitney map as follows:

$$diag : C_\bullet(\Delta^n) \xrightarrow{C_\bullet(\delta)} C_\bullet(\Delta^n \times \Delta^n) \to C_\bullet(\Delta^n) \otimes_k C_\bullet(\Delta^n).$$

Example 1.11. Any simplicial model category $\mathcal{M}$ is a Δ-category in the obvious way.

Proposition 1.12. *If $\mathcal{M}$ is a Δ-category then $\otimes : \Delta \times \mathcal{M} \to \mathcal{M}$ can be extended uniquely to a functor $\otimes :$ **SSet** $\times \mathcal{M} \to \mathcal{M}$ such that*

1. *$\otimes$ preserves colimits in each argument,*

2. *there is a family of maps*

$$a(S, T, x) : (S \times T) \otimes x \to S \otimes (T \otimes x) \tag{1.2}$$

*natural in $S, T \in$ **SSet** and $x \in \mathcal{M}$, associative in a suitable sense and so that for each $[n]$, the composition*

$$\Delta^n \otimes x \to (\Delta^n \times \Delta^n) \otimes x \to \Delta^n \otimes (\Delta^n \otimes x)$$

equals $diag(n, x)$ of Definition 1.7. Moreover, $a(S, T, x)$ is an isomorphism whenever S or T is discrete.

We sometimes call the natural map $a(S, T, x)$ the *action map.*

Proof. Recall that to each simplicial set S we can associate its category of simplexes Δ/S. Its objects are all simplexes of S, represented as maps $\Delta^n \to S$, and a morphism between two such objects is given by a map $[n] \to [m]$ in Δ compatible with morphisms to S. Let $s : \Delta/S \to \Delta$ denote the functor $(\Delta^n \to S) \mapsto [n]$, and define $S \otimes x := \varinjlim_{\Delta/S} s \otimes x$. $\square$

Example 1.13. For any cocomplete category $\mathcal{M}$, there is canonical Δ-structure on $\Delta^{\mathsf{op}}\mathcal{M} = Fun(\Delta^{\mathsf{op}}, \mathcal{M})$, which produces a strict associative action of simplicial sets. Given a simplicial set K and a simplicial object $X \in \Delta^{\mathsf{op}}\mathcal{M}$, we define

$$(K \otimes X)_n = K_n \otimes X_n = \coprod_{K_n} X_n.$$

Definition 1.14. Given two Δ-categories $\mathcal{M}, \mathcal{N}$, a *Δ-functor* $F : \mathcal{M} \to \mathcal{N}$ is a functor between underlying categories together with a family of morphisms

$$m_F([n], x) : \Delta^n \otimes F(x) \to F(\Delta^n \otimes x)$$

natural in both $[n]$ and x. It is required to be compatible with the diagonal maps and unit isomorphisms.

Remark 1.15. Equivalently, a Δ-functor $F : \mathcal{M} \to \mathcal{N}$ is a simplical functor for the simplicial enrichment mentioned in Remark 1.8. It is evident that the composition of Δ-functors is naturally a Δ-functor.

Example 1.16. The tensor product of chain complexes

$$\otimes_k : \mathbf{DVect}_k \times \mathbf{DVect}_k \to \mathbf{DVect}_k,$$

or more generally, for any finite[4] set S, the S-fold tensor product

$$\otimes_k : \mathbf{DVect}_k^S \to \mathbf{DVect}_k$$

can be naturally equipped with the structure of a Δ-functor.

Proposition 1.17. *A Δ-functor on $F : \mathcal{M} \to \mathcal{N}$ between Δ-categories determines a family of maps $m_F(S, x) : S \otimes F(x) \to F(S \otimes x)$ natural in $S \in$ **SSet** and $x \in \mathcal{M}$, which restricts to $m_F([n], x)$ for $S = \Delta^n$ and respects the action maps of Proposition 1.12.*

Proof. Define $m_F(S, x)$ as

$$S \otimes F(x) \cong \varinjlim_{\Delta/S} s \otimes F(x) \xrightarrow{m_F} \varinjlim_{\Delta/S} F(s \otimes x) \to F(\varinjlim_{\Delta/S} s \otimes x) \cong F(S \otimes x).$$

Then the result follows. $\square$

Recall that for a functor $F : I^{\mathsf{op}} \times I \to \mathcal{M}$, its *coend* [14, IX.6] is defined as the universal object $\int^I F$ in $\mathcal{M}$ together with maps $F(i, i) \to \int^I F$ for each $i \in I$, such that for any morphism $i \to i'$, the induced diagram commutes:

$$\begin{tikzcd} F(i', i) \arrow[r] \arrow[d] & F(i, i) \arrow[d] \\ F(i', i') \arrow[r] & \int^I F. \end{tikzcd}$$

Coends exist in $\mathcal{M}$ when $\mathcal{M}$ is cocomplete.

[4] $S = *$ corresponds to the identity functor, $S = \emptyset$ corresponds to the inclusion of k in $\mathbf{DVect}_k$.

Definition 1.18. Let $\mathcal{M}$ be a Δ-category, and $X : \Delta^{\mathsf{op}} \to \mathcal{M}$ a simplicial object in $\mathcal{M}$. Its *geometric realisation* is defined as

$$|X| := \int^{\Delta^{\mathsf{op}}} \Delta^\bullet \otimes X$$

Where $\Delta^\bullet$ is the Yoneda functor (*1.1*). Varying X, we get a functor $|-| : \Delta^{\mathsf{op}}\mathcal{M} \to \mathcal{M}$.

For $S \in \mathbf{SSet}$ and $A \in \mathcal{M}$, it is evident that the realisation of the simplicial object $[n] \mapsto S_n \otimes A$ is canonically isomorphic to $S \otimes A$.

Proposition 1.19. *For a Δ-functor $f : \mathcal{M} \to \mathcal{N}$ we have a canonical natural transformation*

$$s_f : |f(-)|_{\mathcal{N}} \to f|-|_{\mathcal{M}}$$

between the corresponding geometric realisations, where $f : \Delta^{\mathsf{op}}\mathcal{M} \to \Delta^{\mathsf{op}}\mathcal{N}$ is the induced functor. It is compatible with the composition in the following sense: the pasting of

$$\begin{tikzcd}
\Delta^{\mathsf{op}}\mathcal{M} \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\overset{s_f}{\Downarrow}"] & \Delta^{\mathsf{op}}\mathcal{N} \arrow[r, "g"] \arrow[d] \arrow[dr, phantom, "\overset{s_g}{\Downarrow}"] & \Delta^{\mathsf{op}}\mathcal{K} \arrow[d] \\
\mathcal{M} \arrow[r, "f"'] & \mathcal{N} \arrow[r, "g"'] & \mathcal{K}
\end{tikzcd}$$

with vertical functors given by realisations, is equal to s_{gf}.

Proof. A tedious but straightforward check. $\square$

1.3 Homotopical Δ-categories

For any bisimplicial object $X \in (\Delta^{\mathsf{op}} \times \Delta^{\mathsf{op}})\mathcal{M}$ denote by $\delta^* X \in \Delta^{\mathsf{op}}\mathcal{M}$ the diagonal simplicial object, that is, the pullback of X along the diagonal map $\delta : \Delta^{\mathsf{op}} \to \Delta^{\mathsf{op}} \times \Delta^{\mathsf{op}}$.

Definition 1.20. A *homotopical Δ-structure* on a category $\mathcal{M}$ consists of

- a homotopical structure given by the subcategory $\mathcal{W} \subset \mathcal{M}$,
- a Δ-structure with the functor $\otimes : \Delta \times \mathcal{M} \to \mathcal{M}$,

so that the following conditions are satisfied:

1. the subcategory $\mathcal{W}$ satisfies two-out-of-six (Definition 1.4),
2. $\mathcal{M}$ is a Δ-category and $\mathcal{W}$ is preserved by small coproducts,
3. the induced functor $\otimes : \mathbf{SSet} \times \mathcal{M} \to \mathcal{M}$ respects weak equivalences in each variable,
4. the induced action map (*1.2*) $a(S, T, x) : (S \times T) \otimes x \to S \otimes (T \otimes x)$ is a weak equivalence for each $x \in \mathcal{M}$ and $S, T \in \mathbf{SSet}$,

5. the geometric realisation functor $|-|:\Delta^{\mathsf{op}}\mathcal{M}\to\mathcal{M}$ preserves pointwise weak equivalences and for each bisimplicial object $X\in(\Delta^{\mathsf{op}}\times\Delta^{\mathsf{op}})\mathcal{M}$, the natural composite map

$$\int^{\Delta^{\mathsf{op}}}\Delta^\bullet\otimes\delta^*X\to\int^{\Delta^{\mathsf{op}}}\Delta^\bullet\otimes(\Delta^\bullet\otimes\delta^*X)\to\int^{\Delta^{\mathsf{op}}\times\Delta^{\mathsf{op}}}\Delta^\bullet\otimes(\Delta^\bullet\otimes X)$$

is a weak equivalence.

A category together with a homotopical Δ-structure is called a *homotopical Δ-category.*

Example 1.21. Some simplicial model categories $\mathcal{M}$, for instance simplicial presheaves with injective model structure or simplicial vector spaces, produce examples of homotopical Δ-categories. The nontrivial point here is that the realisation functor $\Delta^{\mathsf{op}}\mathcal{M}\to\mathcal{M}$ only preserves weak equivalences between Reedy cofibrant objects [6, VII.3.6], but for the model categories just mentioned, all objects of $\Delta^{\mathsf{op}}\mathcal{M}$ are automatically cofibrant.

Example 1.22. The category $\mathbf{DVect}_k$ is a homotopical Δ-category for the Δ-structure of Example 1.10 and $\mathcal{W}$ being the class of quasiisomorphisms. In this case, all simplicial objects are Reedy-cofibrant, and the functor of geometric realisation is known to be left Quillen for the Reedy model structure on simplicial objects [2, Lemma 9.8].

We assemble together some of the properties of geometric realisation. Define the category Δ_* as a subcategory of Δ consisting of all objects and maps $f:[m]\to[n]$ such that $f(m)=n$. One has the adjunction

$$j:\Delta\rightleftharpoons\Delta_*:i$$

where $j([n])=[n+1]$ should be thought as the inclusion of $[n]$ as first $n+1$ elements of $[n+1]$.

Definition 1.23. An *augmented*[5] simplicial object is a functor $\bar{X}:\Delta_*^{\mathsf{op}}\to\mathcal{M}$. A simplicial object $X:\Delta^{\mathsf{op}}\to\mathcal{M}$ admits an augmentation iff $X\cong j^*\bar{X}$ for some $\bar{X}:\Delta_*^{\mathsf{op}}\to\mathcal{M}$.

For a *bisimplicial* object $X:\Delta^{\mathsf{op}}\times\Delta^{\mathsf{op}}\to\mathcal{M}$, we denote by $||X|_2|_1$ its repeated realisation, that is the coend of the functor

$$([i],[j],[k],[l])\mapsto\Delta^i\otimes(\Delta^j\otimes X_{kl})$$

and by $||X|_1|_2|$ its transpose realisation, which is just a repeated realisation of a transposed bisimplicial object $([n],[m])\mapsto X_{mn}$.

Proposition 1.24. *For a homotopical Δ-category $\mathcal{M}$, the following is true:*

1. *For any simplicial object X admitting an augmentation $\bar{X}$, its realisation is weakly equivalent to $X_{-1}:=\bar{X}_0$. Precisely, there are weak equivalences*

$$X_{-1}\to|X|\to X_{-1}$$

with composition identity that come from the extra maps $X_{-1}\to X_n$ and $X_n\to X_{-1}$.

[5]In effect, that corresponds to what is usually called an augmented simplicial object together with a choice of a contracting homotopy.

2. Given a morphism $X \to Y$ of bisimplicial objects, we have

$$(||X|_2|_1 \to ||Y|_2|_1) \in \mathcal{W} \Leftrightarrow (||X|_1|_2 \to ||Y|_1|_2) \in \mathcal{W}$$

To distinguish simplicial and bisimplicial objects, we write $X_\bullet$ ($[n] \mapsto X_n$) and $X_{\bullet\bullet}$ ($([n],[m]) \mapsto X_{nm}$) for a simplicial and a bisimplicial object, correspondingly.

Proof. The first statement is proven in a few steps. Denote the tensoring of Example 1.13 by $\circledast : \mathbf{SSet} \times \Delta^{\mathsf{op}}\mathcal{M} \to \Delta^{\mathsf{op}}\mathcal{M}$. We now prove that the structure of a homotopical Δ-category on $\mathcal{M}$ gives rise to a family of weak equivalences $|K \circledast X_\bullet| \to K \otimes |X_\bullet|$ natural in $K \in \mathbf{SSet}$ and $X_\bullet \in \Delta^{\mathsf{op}}\mathcal{M}$. The maps are constructed as the following sequence:

$$K \otimes |X_\bullet| = \left(\int^{\Delta^{\mathsf{op}}} \Delta^\bullet \otimes K_\bullet\right) \otimes \left(\int^{\Delta^{\mathsf{op}}} \Delta^\bullet \otimes X_\bullet\right) \cong \int^{\Delta^{\mathsf{op}} \times \Delta^{\mathsf{op}}} \Delta^\bullet \otimes (\Delta^\bullet \otimes \coprod_{K_\bullet} X_\bullet)$$

$$\leftarrow \int^{\Delta^{\mathsf{op}}} \Delta^\bullet \otimes \delta^*(\coprod_{K_\bullet} X_\bullet) \cong \int^{\Delta^{\mathsf{op}}} \Delta^\bullet \otimes (K \circledast X_\bullet) = |K \circledast X_\bullet|.$$

The only non-invertible map in the chain above,

$$\int^{\Delta^{\mathsf{op}} \times \Delta^{\mathsf{op}}} \Delta^\bullet \otimes (\Delta^\bullet \otimes \coprod_{K_\bullet} X_\bullet) \leftarrow \int^{\Delta^{\mathsf{op}}} \Delta^\bullet \otimes \delta^*(\coprod_{K_\bullet} X_\bullet)$$

is a weak equivalence by Definition 1.20.

By definition, a simplicial homotopy equivalence in $\Delta^{\mathsf{op}}\mathcal{M}$ consists of two maps $f : X_\bullet \to Y_\bullet$ and $g : Y_\bullet \to X_\bullet$, and two diagrams

$$\begin{tikzcd} X_\bullet \arrow[d] \arrow[dr, "gf"] & \\ \Delta^1 \circledast X_\bullet \arrow[r, "h"] & X_\bullet, \\ X_\bullet \arrow[u] \arrow[ur, "id"'] & \end{tikzcd} \qquad \begin{tikzcd} Y_\bullet \arrow[d] \arrow[dr, "fg"] & \\ \Delta^1 \circledast Y_\bullet \arrow[r, "h'"] & Y_\bullet \\ Y_\bullet \arrow[u] \arrow[ur, "id"'] & \end{tikzcd}$$

where the vertical maps are induced from the two inclusions $[0] \rightrightarrows [1]$ in Δ. The natural weak equivalence $|K \circledast X_\bullet| \to K \otimes |X_\bullet|$ then implies that, after the realisation, the compositions $|g||f|$ and $|f||g|$ are weak equivalences. By two-out-of-six we get that $|g|$ and $|f|$ are weak equivalences as well.

It is known [16, Lemma 4.5.1] that $X_\bullet$ admitting an augmentation $\bar{X}_\bullet$ in the sense of Definition 1.23 leads to a retract[6] diagram in $\Delta^{\mathsf{op}}\mathcal{M}$

$$\bar{X}_0 \to X_\bullet \to \bar{X}_0$$

naturally appearing from the extra morphisms in $\bar{X}_\bullet$. In this diagram, both maps are simplicial homotopy equivalences in $\Delta^{\mathsf{op}}\mathcal{M}$; they thus become weak equivalences after applying geometric realisation.

[6] That is, the composition of two maps is the identity morphism.

For the last statement of the proposition, observe that both maps in question are weakly equivalent (that is, weakly equivalent as objects of the category $\mathcal{M}^{[1]}$ of maps in $\mathcal{M}$) to the map

$$\int^{\Delta^{op}} \Delta^\bullet \otimes \delta^*(X_{\bullet\bullet}) \to \int^{\Delta^{op}} \Delta^\bullet \otimes \delta^*(Y_{\bullet\bullet})$$

which finishes the proof. $\square$

2 Fibrations, opfibrations, sections

2.1 Basic notions

Definition 2.1. Let $p : \mathcal{E} \to \mathcal{C}$ be a functor. A morphism $\alpha : x \to y$ in $\mathcal{E}$ is *p-Cartesian*, or simply Cartesian, if, for every morphism $\beta : z \to y$ of $\mathcal{E}$ such that $p(\beta) = p(\alpha)$, there exists a unique morphism $\gamma : z \to x$ such that $\beta = \alpha\gamma$ and $p(\gamma) = id_{p(z)}$.

A morphism $\alpha : x \to y$ in $\mathcal{E}$ is *p-opCartesian* if it is Cartesian for $p^{\mathsf{op}} : \mathcal{E}^{\mathsf{op}} \to \mathcal{C}^{\mathsf{op}}$.

Definition 2.2. A functor $p : \mathcal{E} \to \mathcal{C}$ is called a *Grothendieck fibration* (or simply a fibration) of categories iff the following two conditions are satisfied:

- For every morphism $f : a \to b$ of $\mathcal{C}$ and $y \in \mathcal{E}$ such that $p(y) = b$ there exists a Cartesian morphism $\alpha : x \to y$ in $\mathcal{E}$ covering α, that is, $p(\alpha) = f$.
- The composition of Cartesian morphisms is a Cartesian morphism.

Dually, p is called an *opfibration* of categories iff $p^{\mathsf{op}} : \mathcal{E}^{\mathsf{op}} \to \mathcal{C}^{\mathsf{op}}$ is a fibration of categories.

Example 2.3. From a symmetric monoidal category $\mathcal{M}$, one can obtain an opfibration over the category $\mathbf{Fin}_*$ of finite sets and partially defined maps. The category $\mathcal{M}^{\otimes}$ has objects $(S, \{X_s\}_{s\in S})$ where $S \in \mathbf{Fin}_*$ and each X_s is an object of $\mathcal{M}$; a morphism $(S, \{X_s\}_{s\in S}) \to (T, \{Y_t\}_{t\in T})$ is then a partially defined map $f : S \to T$, and a morphism $\otimes_{s\in f^{-1}(t)} X_s \to Y_t$ for each $t \in T$. The forgetful functor $\mathcal{M}^{\otimes} \to \mathbf{Fin}_*$ is an opfibration. It is possible to characterise exactly the opfibrations arising from symmetric monoidal categories; see [13, 17] for details.

Definition 2.4. Let $p : \mathcal{E} \to \mathcal{C}$ and $q : \mathcal{F} \to \mathcal{C}$ be two (op)fibrations. A *lax morphism* between p and q is a functor $F : \mathcal{E} \to \mathcal{F}$ such that $q \circ F = p$. F is called a *Cartesian morphism* if, in addition, F takes (op)Cartesian morphisms of $\mathcal{E}$ to (op)Cartesian morphisms of $\mathcal{F}$. A *section* of p is a lax morphism s from the (op)fibration $id_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ to p.

We denote by $\mathsf{Sect}(\mathcal{C}, \mathcal{E})$ the category of sections. The morphisms in this category are natural transformations $\alpha : F \to F'$ such that pointwise α_c projects to id_c.

Example 2.5. Any algebra object A in a symmetric monoidal category $\mathcal{M}$ gives a section $A : \mathbf{Fin}_* \to \mathcal{M}^{\otimes}$ by the rule $S \mapsto (A, ..., A) \in \mathcal{M}^{\otimes}(S)$. In fact, consider any section $S : \mathbf{Fin}_* \to \mathcal{M}^{\otimes}$ which is Cartesian along the *inert* [13] maps of $\mathbf{Fin}_*$, that is, those morphisms $S \to T$ which are partially defined identity maps. Then we can naturally extract from S a commutative monoid structure on $S(1)$.

The following is well known (see e.g. [18]):

Construction 2.6. Let $p : \mathcal{E} \to \mathcal{C}$ be an opfibration. For $a \in \mathcal{C}$, denote by $\mathcal{E}(a) = p^{-1}(a)$ the fiber of p over a. Let $f : a \to b$ be a morphism in $\mathcal{C}$ and $x \in \mathcal{E}(a)$. Then we can choose an opCartesian morphism $\alpha : x \to y$ such that $p(\alpha) = f$. This specifies an object $y \in \mathcal{E}(b)$. It can then be shown that the prescription $x \mapsto y$ defines a functor $f_! : \mathcal{E}(a) \to \mathcal{E}(b)$. One can then check that $g_! \circ f_! \cong (g \circ f)_!$ in a coherent way.

There is also an inverse construction, which, given a functor F from $\mathcal{C}$ to **Cat**, produces an opfibration, which we denote[7] $\int F \to \mathcal{C}$ (and call the *Grothendieck construction* of F). An object of $\int F$ is a pair (c, x) of $c \in \mathcal{C}$ and $x \in F(c)$, and a morphism $(c, x) \to (c', x')$ consists of $f : c \to c'$ together with a map $\alpha : F(f)(x) \to x'$ in $F(x')$.

An opfibration $\mathcal{E} \to \mathcal{C}$ is *strictly cleavable*, if, by a choice of opCartesian arrows, the assignment $c \mapsto \mathcal{E}(c)$ can be made into a strict functor $\mathcal{C} \to$ **Cat**. Every opfibration is equivalent to a strictly cleavable one via a Cartesian morphism of (op)fibrations.

Example 2.7. The category Δ/S for a simplicial set S is exactly the domain of the fibration $\int S \to \Delta$, where we view S as a functor $\Delta^{\mathsf{op}} \to \mathbf{Set} \subset \mathbf{Cat}$.

When viewed as a functor $\mathcal{C} \to$ **Cat**, any opfibration can be trivially viewed as a contravariant functor from $\mathcal{C}^{\mathsf{op}}$ to **Cat**. This suggests the following:

Definition 2.8. Given an opfibration $p : \mathcal{E} \to \mathcal{C}$, a *transpose fibration* of p is a fibration $p^\top : \mathcal{E}^\top \to \mathcal{C}^{\mathsf{op}}$, equivalent in $Fib(\mathcal{C}^{\mathsf{op}})$ to the Grothendieck construction of the functor

$$\mathcal{E} : (\mathcal{C}^{\mathsf{op}})^{\mathsf{op}} \to \mathbf{Cat}, \quad c \mapsto \mathcal{E}(c).$$

Construction 2.9. Fix an opfibration $p : \mathcal{E} \to \mathcal{C}$.

Define a category, again denoted as $\mathcal{E}^\top$ as follows:

1. $Ob(\mathcal{E}^\top) = Ob(\mathcal{E})$
2. A morphism from $x \to z$ in $\mathcal{E}^\top$ is an isomorphism class of cospans in $\mathcal{E}$

$$x \longrightarrow y \longleftarrow z$$

   such that the left arrow is fiberwise[8] and the right arrow is opCartesian.

There is an evident functor $p^\top : \mathcal{E}^\top \to \mathcal{C}^{\mathsf{op}}$ which sends maps $x \longrightarrow y \longleftarrow z$ to $p(y \longleftarrow z)$. A morphism of $\mathcal{E}^\top$ is $p^\top$-Cartesian iff it can be represented by a span of the form $y \xrightarrow{id_y} y \longleftarrow z$.

Given a functor $F : \mathcal{D} \to \mathcal{C}$, we can pull back (op)fibrations over $\mathcal{C}$ to $\mathcal{D}$, with the result again being (op)fibrations. The pullback operation $(\mathcal{E} \to \mathcal{C}) \mapsto (F^*\mathcal{E} \to \mathcal{D})$ defines pseudofunctors[9] Fib and $OpFib$ from $\mathbf{Cat}^{\mathsf{op}}$ to **Cat**. Given a section $A : \mathcal{C} \to \mathcal{E}$ of, say, an (op)fibration $\mathcal{E} \to \mathcal{C}$ we obtain from it the section $F^*A : \mathcal{D} \to F^*\mathcal{E}$ of the pullback (op)fibration $F^*\mathcal{E} \to \mathcal{D}$. The operation on sections defines a pullback functor $F^* : \mathsf{Sect}(\mathcal{C}, \mathcal{E}) \to \mathsf{Sect}(\mathcal{D}, F^*\mathcal{E})$.

[7] The symbol $\int$ was already used for coends; in fact, $\int F$ can be reproduced as a certain coend.

[8] That is, $p(x \to y) = id_{p(x)}$.

[9] See [18] for a definition; this only means that the composition of morphisms is preserved up to a (coherent) natural isomorphism.

Proposition 2.10. *Assume given a fibration $\mathcal{E} \to \mathcal{C}$ and a natural transformation $\alpha : F \to G$ of functors $F, G : \mathcal{D} \to \mathcal{C}$. Then*

- *there is a natural Cartesian map of fibrations $R_\alpha : G^*\mathcal{E} \to F^*\mathcal{E}$, which we call the* restriction *map,*
- *given a section $A : \mathcal{C} \to \mathcal{E}$, there is a natural morphism of sections*
$$F^*A \to R_\alpha G^*A.$$

Proof. A direct verification. □

Definition 2.11. Given a fibration $p : \mathcal{F} \to \mathcal{C}$ and an opfibration $q : \mathcal{O} \to \mathcal{C}$, a *power fibration* is a fibration $\mathcal{F}^{\mathcal{O}} \to \mathcal{C}$ satisfying the following universal property: for any $F : \mathcal{D} \to \mathcal{C}$, there is a natural equivalence of categories

$$\mathsf{Sect}(\mathcal{D}, F^*\mathcal{F}^{\mathcal{O}}) \cong \mathsf{Sect}(F^*\mathcal{O}, q^*F^*\mathcal{F}).$$

In functor terms, it corresponds to

$$\mathcal{C}^{\mathsf{op}} \xrightarrow{\mathcal{O}^{\mathsf{op}} \times \mathcal{F}} \mathbf{Cat}^{\mathsf{op}} \times \mathbf{Cat} \xrightarrow{(-)^{\mathsf{op}} \times id} \mathbf{Cat}^{\mathsf{op}} \times \mathbf{Cat} \xrightarrow{Fun(-,-)} \mathbf{Cat}.$$

We can think of $\mathcal{F}^{\mathcal{O}}$ as corresponding to an assignment $c \mapsto Fun(\mathcal{O}(c), \mathcal{E}(c))$, with transition functors being a combination of those for $\mathcal{O}$ and $\mathcal{F}$.

Lemma 2.12. *For a functor $F : \mathcal{D} \to \mathcal{C}$, and $\mathcal{F}, \mathcal{O}$ as above, the natural Cartesian map*

$$(F^*\mathcal{F})^{F^*\mathcal{O}} \to F^*(\mathcal{F}^{\mathcal{O}})$$

is an equivalence over $\mathcal{C}$.

Proof. Clear. □

Definition 2.13. Let $p : \mathcal{E} \to \mathcal{C}$ be an opfibration and $I \in \mathbf{Cat}$ a category.

- A product of I and $p : \mathcal{E} \to \mathcal{C}$ is the functor $I \times p : I \times \mathcal{E} \to \mathcal{C}, \quad (i, x) \mapsto p(x).$
- A powering of p with I is the functor $\mathcal{E}^I \to \mathcal{C}$ defined as the powering (cf. Definition 2.11) of $\mathcal{E} \to \mathcal{C}$ with respect to the trivial *fibration* $I \times \mathcal{C} \to \mathcal{C}$.

The definitions of products and powers for fibrations are similar.

Proposition 2.14. *Let $\mathcal{E} \to \mathcal{C}$ be a* fibration *with cocomplete fibres, and*

$$\begin{tikzcd} \mathcal{O} \arrow[rr, "P"] \arrow[dr] & & I \times \mathcal{C} \arrow[dl] \\ & \mathcal{C} & \end{tikzcd}$$

be an opCartesian morphism of opfibrations. Then the obvious functor

$$P^* : \mathsf{Sect}(\mathcal{C}, \mathcal{E}^I) \to \mathsf{Sect}(\mathcal{C}, \mathcal{E}^{\mathcal{O}})$$

admits a left adjoint $P_!$.

Proof. The existence of an adjoint is preserved under equivalence and thus we assume that $\mathcal{E} \to \mathcal{C}$ is strictly cleavable. Then, any object of $\mathcal{E}^{\mathcal{O}}$ can be represented as a functor $X : \mathcal{O}(c) \to \mathcal{E}(c)$ for some $c \in \mathcal{C}$. We define $P_! : \mathcal{E}^{\mathcal{O}} \to \mathcal{E}^I$ by setting $P_! X$ to be the left Kan extension [14] of X along the map $P_c : \mathcal{O}(c) \to I$. One can then check it induces the functor which we also denote as $P_!$ on sections left adjoint do P^*. $\square$

2.2 Homotopical Δ-opfibrations

Definition 2.15. A *homotopical structure* on an opfibration $\mathcal{E} \to \mathcal{C}$ consists of a homotopical structure on $\mathcal{E}$, given by a subcategory $\mathcal{W} \subset \mathcal{E}$ of weak equivalences, compatible with the opfibration in the following sense:

1. the image of $\mathcal{W}$ in $\mathcal{C}$ consists of identity morphisms,
2. in a commutative square

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "f"'] & B \arrow[d, "f'"] \\ A' \arrow[r, "\alpha'"] & B' \end{tikzcd}$$

   if we have $f \in \mathcal{W}$ and α, α' are opCartesian, then $f' \in \mathcal{W}$.

Definition 2.16. A Δ-structure on an opfibration $\mathcal{E} \to \mathcal{C}$ consists of a Δ-structure $\otimes : \Delta \times \mathcal{E} \to \mathcal{E}$ such that

1. $\otimes$ is the functor over $\mathcal{C}$,
2. the natural transformation *diag* and unitality isomorphism (see Definition 1.7) are fiberwise.

Definition 2.17. A *homotopical* Δ-opfibration is an opfibration $\mathcal{E} \to \mathcal{C}$ together with a homotopical structure and a Δ-structure, such that for each $c \in \mathcal{C}$, the induced structure on the fiber $\mathcal{E}(c)$ is that of a homotopical Δ-category.

We often call an opfibration with a homotopical Δ-structure as simply a *homotopical* Δ*-opfibration.*

One can see that for an opfibration with homotopical structure, $\mathcal{W} = \coprod_{c \in \mathcal{C}} \mathcal{W}(c)$, with $(\mathcal{E}(c), \mathcal{W}(c))$ being a homotopical category for each $c \in \mathcal{C}$. The transition functors of the opfibration send $\mathcal{W}(c)$ to $\mathcal{W}(c')$.

Definition 2.18. Given an opfibration $\mathcal{E} \to \mathcal{C}$ with a homotopical structure, a morphism $\alpha : x \to y$ of $\mathcal{E}$ is *weakly opCartesian* if it can be factored as an opCartesian morphism $x \to \alpha_! x$ followed by a weak equivalence $\alpha_! x \to y$. Dually, one has the notion of a weakly Cartesian morphism for homotopical fibrations.

Example 2.19. Chain complexes give us homotopical Δ-opfibration $\mathbf{DVect}^{\otimes} \to \mathbf{Fin}_*$. The Δ-structure on the opfibration is essentially explained in Examples 1.10 and 1.16, and the weak equivalences are simply induced from the quasiisomorphisms of $\mathbf{DVect}_k$. In the same way, those simplicial model categories which give us a homotopical Δ-structure (Example 1.21) can as well give us homotopical Δ-opfibrations. If such a category $\mathcal{M}$ in addition possesses a compatible monoidal structure (this is true, for example, both for simplicial presheaves and for simplicial vector spaces), then the associated opfibration $\mathcal{M}^{\otimes} \to \mathbf{Fin}_*$ is a homotopical Δ-opfibration.

Remark 2.20. We can think of a Δ-structure on an opfibration as of a collection of fiberwise Δ-structures $\otimes_c : \Delta \times \mathcal{E}(c) \to \mathcal{E}(c)$ together with 2-squares

$$\begin{tikzcd}
\Delta \times \mathcal{E}(c) \arrow[r, "\otimes_c"] \arrow[d, "f_!"'] \arrow[dr, phantom, "\overset{m_f}{\Rightarrow}"] & \mathcal{E}(c) \arrow[d, "f_!"] \\
\Delta \times \mathcal{E}(c') \arrow[r, "\otimes_{c'}"] & \mathcal{E}(c')
\end{tikzcd}$$

so that each $m_f : - \otimes_{c'} f_! - \to f_!(- \otimes_c -)$ becomes a morphism of Δ-structures. Moreover, $f \mapsto m_f$ is suitably functorial in f.

3 Derived sections

3.1 Simplicial replacements

Definition 3.1 (Cf [3, 16]). Given a category $\mathcal{C}$, its *simplicial replacement*, denoted $\mathbb{C}$, is the *opposite* of $\Delta/N\mathcal{C} = \int N\mathcal{C}$, that is the opposite of the category of simplexes of the simplicial set $N\mathcal{C}$ (cf. Example 2.7).

An object of $\mathbb{C}$ is a sequence $c_0 \to ... \to c_n$ of composable morphisms in $\mathcal{C}$. Any functor $F : \mathcal{D} \to \mathcal{C}$ induces a functor $\mathbb{F} : \mathbb{D} \to \mathbb{C}$: by the rule $\mathbb{F}(d_0 \to ... \to d_n) = Fd_0 \to ... \to Fd_n$. Observe that $\mathbb{F}$ commutes with the projections from $\mathbb{D}$ and $\mathbb{C}$ to Δ^{op}. The following is then evident:

Lemma 3.2. *The assignment $\mathcal{C} \mapsto \mathbb{C}$ defines a functor from* **Cat** *to the full*[10] *subcategory of* $\mathbf{Cat}/(\Delta^{\mathsf{op}})$*, consisting of opfibrations over Δ^{op} with discrete fibers.* □

Notation 3.3. We often denote by $\pi : \mathbb{C} \to \Delta^{\mathsf{op}}$ the natural projection. An object $c_0 \to ... \to c_n$ of $\mathbb{C}$ will be denoted as $\mathbf{c}_{[n]}$ (so that $\pi(\mathbf{c}_{[n]}) = [n]$) or simply as $\mathbf{c}$ when its underlying Δ-object is not important. Given two objects $\mathbf{c}_{[n]}$, $\mathbf{c}'_{[m]}$, and a map $\alpha : c_n \to c'_0$, we denote by $\mathbf{c}_{[n]} *^{\alpha} \mathbf{c}'_{[m]}$ the 'concatenated' object

$$c_0 \to ... \to c_n \overset{\alpha}{\to} c'_0 \to ... \to c'_n.$$

[10]If two opfibrations have discrete fibers, every lax morphism of opfibrations is automatically Cartesian and vice versa.

Lemma 3.4. *There are functors $h_{\mathcal{C}} : \mathbb{C} \to \mathcal{C}$ and $t_{\mathcal{C}} : \mathbb{C} \to \mathcal{C}^{\mathsf{op}}$ given by $\mathbf{c}_{[n]} \mapsto c_0$ or $\mathbf{c}_{[n]} \mapsto c_n$ respectively.* □

Definition 3.5. A map $\zeta : \mathbf{c}_{[n]} \to \mathbf{c}'_{[m]}$ is *anchor* iff its projection in Δ, $\pi(\zeta) : [m] \to [n]$, is an interval inclusion of $[m]$ as first $m+1$ elements of $[n]$, i.e. $\pi(\zeta)(i) = i$ for $0 \le i \le m$. In particular, m should be less or equal than n.

A map $\zeta : \mathbf{c}_{[n]} \to \mathbf{c}'_{[m]}$ is *structural* iff its image under $t_{\mathcal{C}}$ is an identity and the underlying map in Δ^{op} preserves the endpoints: $\pi(\zeta)(m) = n$.

We denote by $A_{\mathbb{C}}$ and $S_{\mathbb{C}}$ the sets of all anchor and structure maps respectively.

Every map $\mathbf{c} \to \mathbf{c}'$ can be uniquely factored as an anchor map $\mathbf{c} \to \mathbf{c}''$ followed by a structural map $\mathbf{c}'' \to \mathbf{c}$.

Proposition 3.6 (Localisation property). *The functor $h_{\mathcal{C}} : \mathbb{C} \to \mathcal{C}$ is a localisation of $\mathbb{C}$ along the set of anchor maps $A_{\mathbb{C}}$, that is, any functor $X : \mathbb{C} \to \mathcal{N}$ which sends $A_{\mathbb{C}}$ to isomorphisms of $\mathcal{N}$ factors essentially uniquely as $X = \tilde{X} \circ h_{\mathcal{C}}$ for $\tilde{X} : \mathcal{C} \to \mathcal{N}$.*

Proof. Is well known (see e.g. [16]). □

Given two functors $\mathcal{D} \xrightarrow{F} \mathcal{C} \xleftarrow{G} \mathcal{B}$, there is a useful categorical notion called the *comma category* F/G [14]. Its objects are triples $(d, b, \alpha : F(d) \to G(b))$ for $d \in \mathcal{D}$ and $b \in \mathcal{B}$. We need the following adaptation of this notion:

Definition 3.7. Given two a diagram $\mathcal{D} \xrightarrow{F} \mathcal{C} \xleftarrow{G} \mathcal{B}$, the associated *simplicial comma object* $\mathbb{F}/\!\!/\mathbb{G}$ is defined as the *opposite* of the category $\int F/\!\!/G$, where $F/\!\!/G : \Delta^{\mathsf{op}} \times \Delta^{\mathsf{op}} \to \mathbf{Set}$ is the bisimplicial set

$$F/\!\!/G([n],[m]) = \{\mathbf{d}_{[n]}, \mathbf{b}_{[m]}, \alpha : F(d_n) \to G(b_0)\}$$

viewed as a contravariant functor to **Cat**.

We often write $\mathbb{D}/\!\!/\mathbb{G}$ or $\mathbb{F}/\!\!/\mathbb{C}$ instead of $\mathbb{F}/\!\!/\mathbb{G}$ if F or G is the identity functor. Given an object $c \in \mathcal{C}$, we also consider $\mathbb{F}/\!\!/c$ where we treat c as a functor $[0] \to \mathcal{C}$ and denote its simplicial replacement by the same letter. The canonical functor $\mathbb{F}/\!\!/\mathbb{G} \to \Delta^{\mathsf{op}} \times \Delta^{\mathsf{op}}$ is an opfibration with discrete fibers $\mathbb{F}/\!\!/\mathbb{G}([n],[m]) = F/\!\!/G([n],[m])$.

There is a *concatenation* functor $con : \Delta \times \Delta \to \Delta$, $([n],[m]) \mapsto [n] * [m] = [n+m+1]$, and we think that $[n]$ is included as first $n+1$ elements of $[n+m+1]$ and $[m]$ as last $[m+1]$ elements. The action of con on morphisms is then evident.

Then we observe the following. There is a diagram in **Cat**

$$\begin{tikzcd} & \mathbb{F}/\!\!/\mathbb{G} \arrow[dl, "pr_{\mathbb{D}}"'] \arrow[d] \arrow[dr, "pr_{\mathbb{B}}"] & \\ \mathbb{D} \arrow[r, "\mathbb{F}"] \arrow[ur, phantom, "\Leftarrow", near end] & \mathbb{C} \arrow[u, phantom, "\Leftarrow\ \Rightarrow"] & \mathbb{B} \arrow[l, "\mathbb{G}"'] \end{tikzcd} \tag{3.1}$$

with the middle map denoted $pr_{\mathbb{F}/\!/\mathbb{G}}$, covering the diagram

$$\begin{tikzcd}
 & \Delta^{\mathsf{op}} \times \Delta^{\mathsf{op}} \arrow[dl, "\pi_1"'] \arrow[d] \arrow[dr, "\pi_2"] \arrow[d, phantom, shift right=4, "\Leftarrow"] \arrow[d, phantom, shift left=4, "\Rightarrow"] & \\
\Delta^{\mathsf{op}} \arrow[r, "id"] & \Delta^{\mathsf{op}} & \Delta^{\mathsf{op}} \arrow[l, "id"']
\end{tikzcd} \tag{3.2}$$

with the middle map acting as $([n],[m]) \mapsto [n] * [m]$. Moreover,

- the left natural transformation $pr_{\mathbb{F}/\!/\mathbb{G}} \to \mathbb{F} \circ pr_{\mathbb{D}}$ is valued in anchor maps $A_{\mathbb{C}}$,
- the right natural transformation $pr_{\mathbb{F}/\!/\mathbb{G}} \to \mathbb{G} \circ pr_{\mathbb{B}}$ is valued in structural maps $S_{\mathbb{C}}$,
- $pr_{\mathbb{B}}$ is an opfibration whose classifying functor $\mathbb{B} \to \mathbf{Cat}$ sends anchor maps to equivalences of categories.

All this is evident from Definition 3.7: $pr_{\mathbb{D}}$ maps $(\mathbf{d}_{[n]}, \mathbf{b}_{[m]}, \alpha : F(d_n) \to G(b_0))$ to $\mathbf{d}_{[n]}$, $pr_{\mathbb{B}}$ maps it to $\mathbf{b}_{[m]}$, and $pr_{\mathbb{F}/\!/\mathbb{G}}$ maps it to $\mathbb{F}(\mathbf{d}_{[n]}) *^{\alpha} \mathbb{G}(\mathbf{b}_{[m]})$.

Definition 3.8. For an opfibration $\mathcal{E} \to \mathcal{C}$, its *simplicial extension* is a *fibration* $\mathbf{E} \to \mathbb{C}$ which is the pullback of a transpose fibration $\mathcal{E}^{\top} \to \mathcal{C}^{\mathsf{op}}$ along $t_{\mathcal{C}} : \mathbb{C} \to \mathcal{C}^{\mathsf{op}}$.

Remark 3.9. Given two functors $k_1, k_2 : K \to \mathbb{C}$ and a natural transformation $\alpha : k_1 \to k_2$ valued in structural maps $S_{\mathbb{C}}$, we have that the induced Cartesian map of fibrations

$$\alpha^* : k_2^* \mathbf{E} \to k_1^* \mathbf{E}$$

is in fact an equivalence.

We can also pull back $\mathcal{E} \to \mathcal{C}$ to $\mathbb{C}$ by the means of the functor $h_{\mathcal{C}} : \mathbb{C} \to \mathcal{C}$.

Proposition 3.10. *Given an opfibration $p : \mathcal{E} \to \mathcal{C}$, there is a morphism $T : h_{\mathcal{C}}^* \mathcal{E} \to \mathbf{E}$ commuting with functors to $\mathbb{C}$ which sends opCartesian maps of $h_{\mathcal{C}}^* \mathcal{E}$ to Cartesian maps of $\mathbf{E}$ and is universal, i.e. any other functor $G : h_{\mathcal{C}}^* \mathcal{E} \to \mathbf{E}$ over $\mathbb{C}$ with such a property factors through T up to a natural isomorphism.*

Proof. To construct T without passing to cleavable opfibrations, consider the category $\mathcal{X}$ defined as follows.

- An object of $\mathcal{X}$ is a pair $(\mathbf{c}_{[n]}, \alpha)$ where $\mathbf{c}_{[n]} = c_0 \to \ldots \to c_n$ is an object of $\mathbb{C}$ and $\alpha : x \to y$ is an opCartesian map in $\mathcal{E}$ which covers the composition $c_0 \to c_n$ in $\mathcal{C}$ (i.e. $p(\alpha) = c_0 \to c_n$),
- A morphism $(\mathbf{c}_{[n]}, \alpha : x \to y) \to (\mathbf{c}'_{[m]}, \beta : x' \to y')$ is a map $\gamma : x \to x'$ which covers the induced map $c_0 \to c'_0$.

One can check that the natural functor $\mathcal{X} \to \mathbb{C}$ is an opfibration, and it is easy to see that the assignment $(\mathbf{c}, \alpha : x \to y) \mapsto (\mathbf{c}, x)$ defines an equivalence over $\mathbb{C}$ of opfibrations $\mathcal{X} \tilde{\to} h^*_{\mathcal{C}}\mathcal{E}$. The assignment $(\mathbf{c}, \alpha : x \to y) \mapsto (\mathbf{c}, y)$, on the other hand, defines a functor $\mathcal{X} \to \mathbf{E}$ over $\mathbb{C}$ which envoys opCartesian maps of $\mathcal{X}$ to Cartesian maps of $\mathbf{E}$. Inverting the equivalence $\mathcal{X} \tilde{\to} h^*_{\mathcal{C}}\mathcal{E}$, we obtain the desired functor $T : h^*_{\mathcal{C}}\mathcal{E} \to \mathbf{E}$, and one can verify its universal property. $\square$

Proposition 3.11. *Let $\mathcal{E} \to \mathcal{C}$ be a Δ-opfibration. There is a lax* realisation *morphism of fibrations*

$$\begin{tikzcd}
\mathbf{E}^{\Delta^{\mathsf{op}}} \arrow[rr, "|-|"] \arrow[dr] & & \mathbf{E} \arrow[dl] \\
& \mathbb{C} &
\end{tikzcd}$$

so that on each fiber, the functor $\Delta^{\mathsf{op}}\mathbf{E}(\mathbf{c}) \to \mathbf{E}(\mathbf{c})$ is the geometric realisation for the Δ-category $\mathbf{E}(\mathbf{c}) = \mathcal{E}(t_{\mathcal{C}}(\mathbf{c}))$.

Proof. One can assume that $\mathcal{E} \to \mathcal{C}$ is strictly cleavable. Any object of $\mathbf{E}^{\Delta^{\mathsf{op}}}$ is a functor[11] $X : \Delta^{\mathsf{op}} \to \mathbf{E}(\mathbf{c}) = \mathcal{E}(t(\mathbf{c}))$ for some $\mathbf{c} \in \mathbb{C}$. Take its realisation $|X|$ using the induced Δ-structure on $\mathcal{E}(t(\mathbf{c}))$.

Given a morphism $\alpha : X \to Y$ in $\mathbf{E}^{\Delta^{\mathsf{op}}}$ covering $\mathbf{f} : \mathbf{c} \to \mathbf{c}'$ in $\mathbb{C}$, there is an induced map $f : t(\mathbf{c}') \to t(\mathbf{c})$ in $\mathcal{C}$, and then we have a sequence of maps

$$f_!|X| = f_! \int \Delta^\bullet \otimes X \leftarrow \int f_!(\Delta^\bullet \otimes X) \leftarrow \int \Delta^\bullet \otimes f_!X = |f_!X|.$$

Here $f_! : \mathbf{E}(\mathbf{c}') \to \mathbf{E}(\mathbf{c})$ is the functor obtained by the choice of Cartesian arrows for $\mathbf{E}$ (or opCartesian arrows of $\mathcal{E}$), and we also used the maps $f_!(\Delta^\bullet \otimes X) \leftarrow \Delta^\bullet \otimes f_!X$ which appear as explained in Remark 2.20. That defines $|-|$ on morphisms. Its associativity then follows from Proposition 1.19. $\square$

Let I be any category and denote by $\mathbb{I}$ its simplicial replacement.

Definition 3.12. For $X : \mathbb{I} \to \mathcal{M}$, its *realisation* is defined as $|\Pi X|$, where $|-|$ is the geometric realisation for $\mathcal{M}$ and $\Pi : Fun(\mathbb{I}, \mathcal{M}) \to \Delta^{\mathsf{op}}\mathcal{M}$ is the functor left adjoint to the pullback along the canonical projection $\pi : \mathbb{I} \to \Delta^{\mathsf{op}}$.

For any object $i \in I$, there is naturally a map $X(i) \to |\Pi X|$.

Lemma 3.13. *Let I be a category with a terminal object* 1*, and $\mathcal{M}$ be a homotopical Δ-category. Then any $X : \mathbb{I} \to \mathcal{M}$ sending the anchor maps $A_{\mathbb{I}}$ to maps in $\mathcal{W}$, the natural map $X(1) \to |\Pi X|$ is an equivalence.*

Proof. Consider an 'augmented' functor $X^{aug} : \mathbf{i} \mapsto X(\mathbf{i} *^x 1)$ (here x corresponds to the canonical map to the terminal object $t_I(\mathbf{i}) \to 1$). It is then easy to see that there's a canonical equivalence $X^{aug} \to X$ coming from the maps $X(\mathbf{i} *^x 1) \to X(\mathbf{i})$. It then becomes an

[11] We write t for $t_{\mathcal{C}}$ in this proof.

equivalence of realisations. The object ΠX^{aug}, however, can be completed to an augmented simplicial object $\tilde{X}^{aug} : \Delta_+^{\mathsf{op}} \to \mathcal{M}$ defined by the formula

$$\tilde{X}^{aug}_n = \Pi X^{aug}_{n-1}, \;\; n > 0,$$

$$\tilde{X}^{aug}_0 = X(1).$$

in particular, one augmentation map $X(1) \to \tilde{X}^{aug}_1 = \coprod_i X(i \to 1)$ comes from the image $X(1) \to X(1 \to 1)$ of the degeneracy map $1 \to (1 \to 1)$ and the other map

$$\tilde{X}^{aug}_1 = \coprod_i X(i \to 1) \to X(1)$$

is just the coproduct of the natural maps $X(i \to 1) \to X(1)$. By Proposition 1.24 we have the equivalences

$$X(1) \to |\Pi X^{aug}| \to X(1)$$

and we can see that the composite map $X(1) \to |\Pi X^{aug}| \to |\Pi X|$, which is an equivalence, is equal to the map in question. □

Lemma 3.14. *Let I be a category with contractible nerve and $\mathcal{M}$ be a homotopical Δ-category. If a functor $X : \mathbb{I} \to \mathcal{M}$ takes all morphisms of $\mathbb{I}$ to isomorphisms, then the natural map $X(i) \to |\Pi X|$ is an equivalence for any $i \in I$.*

Proof. Fix $i \in I$. Proposition 3.6 implies that the functor X can be factored as $\overline{X} \circ h_I$ with $\overline{X} : I \to \mathcal{M}$. X moreover factors through the fundamental groupoid of I, which is contractible. One can then see that

$$\Pi X_n = \coprod_{\mathbf{i}_{[n]}} X(\mathbf{i}_{[n]}) \cong \coprod_{\mathbf{i}_{[n]}} (\overline{X} \circ h_I)(\mathbf{i}_{[n]}) \cong \coprod_{\mathbf{i}_{[n]}} X(i_0) \cong \coprod_{\mathbf{i}_{[n]}} X(i),$$

and so $|\Pi X| = NI \otimes X(i)$, which is equivalent to $X(i)$, and the map $X(i) \to |\Pi X|$ in question is a homotopy inverse of the projection $NI \otimes X(i) \to X(i)$. □

3.2 Homotopical category of derived sections

Definition 3.15. Given an opfibration $\mathcal{E} \to \mathcal{C}$, its category of *presections* is the category

$$\mathsf{PSect}(\mathcal{C}, \mathcal{E}) := \mathsf{Sect}_{\mathbb{C}}(\mathbb{C}, \mathbf{E}).$$

Recall the functors $h_{\mathcal{C}}$ and T discussed before in Lemma 3.4 and Proposition 3.10.

Proposition 3.16. *The assignment $S \mapsto T \circ (h^*_{\mathcal{C}} S)$ defines a functor $i : \mathsf{Sect}(\mathcal{C}, \mathcal{E}) \to \mathsf{PSect}(\mathcal{C}, \mathcal{E})$. Its essential image consists of the presections sending the anchor maps $A_{\mathbb{C}}$ to Cartesian morphisms in $\mathbf{E}$.*

Proof. Note that for any anchor map $a : \mathbf{c}_{[n]} \to \mathbf{c}_{[k]}$ a map in $h^*_{\mathcal{C}}\mathcal{E}$ is opCartesian over a iff it is an isomorphism $x \xrightarrow{\sim} x$ in $\mathcal{E}(c_0)$. On one hand, the functor T sends such maps to Cartesian maps in $\mathbf{E}$; on the other hand, the pullback section $h^*_{\mathcal{C}} S : \mathbb{C} \to h^*_{\mathcal{C}}\mathcal{E}$ sends anchor maps $A_{\mathbb{C}}$ precisely to identities in $\mathcal{E}$. Further details are then clear. □

Remark 3.17. If $\mathcal{E} \to \mathcal{C}$ is strictly cleavable, then $S \in \mathit{Sect}(\mathcal{C}, \mathcal{E})$ is sent by the functor above to $i(S)$ such that $i(S)(c_0 \xrightarrow{f_1} ... \xrightarrow{f_n} c_n) = (f_n ... f_1)_! S(c_0)$.

Assume now that $\mathcal{E} \to \mathcal{C}$ has a homotopical structure $\mathcal{W}$.

Definition 3.18. The *standard homotopical structure* on $\mathsf{PSect}(\mathcal{C}, \mathcal{E})$ is defined by the subcategory of those morphisms $A \to A'$ for which the map $A(\mathbf{c}_{[n]}) \to A'(\mathbf{c}_{[n]})$ is in $\mathcal{W}$ for each $\mathbf{c}_{[n]} \in \mathbb{C}$.

We henceforth assume this homotopical structure whenever dealing with $\mathsf{PSect}(\mathcal{C}, \mathcal{E})$. We denote by $\mathrm{Ho}\,\mathsf{PSect}(\mathcal{C}, \mathcal{E})$ the corresponding localisation.

Definition 3.19. A presection $A : \mathbb{C} \to \mathbf{E}$ is a *derived section* iff A sends anchor maps to weakly Cartesian moprhisms in $\mathbf{E}$.

We denote by $\mathbb{R}\mathsf{Sect}(\mathcal{C}, \mathcal{E})$ the full subcategory of $\mathsf{PSect}(\mathcal{C}, \mathcal{E})$. We restrict the standard homotopical structure from $\mathsf{PSect}(\mathcal{C}, \mathcal{E})$ to $\mathbb{R}\mathsf{Sect}(\mathcal{C}, \mathcal{E})$ and denote by $\mathrm{Ho}\,\mathbb{R}\mathsf{Sect}(\mathcal{C}, \mathcal{E})$ the corresponding localisation.

Lemma 3.20. *Let $A : \mathbb{C} \to \mathbf{E}$ be a derived section and $s : \mathbf{c}_{[n]} \to \mathbf{c}'_{[m]}$ be such a map in $\mathbb{C}$ that its underlying map $s : [m] \to [n]$ in Δ is the surjective left inverse of the inclusion $i : [n] \to [m]$ of $[n]$ as last $n + 1$ objects of $[m]$. Then $A(s)$ is weakly Cartesian in $\mathbf{E}$.*

Proof. Clear. □

We can now formulate the main results of this paper.

Definition 3.21. A functor $F : \mathcal{D} \to \mathcal{C}$ is a *resolution* if it is an opfibration and each fiber $\mathcal{D}(c)$ is contractible (that is, its nerve $N\mathcal{D}(c)$ is contractible).

Remark 3.22. Resolutions $F : \mathcal{D} \to \mathcal{C}$ should properly be called partial resolutions since we do not make the claim about the smoothness (in any sense) of categories defined over $\mathcal{D}$.

Given a functor $F : \mathcal{D} \to \mathcal{C}$, there is an induced morphism

$$\mathbb{F}^* : \mathsf{PSect}(\mathcal{C}, \mathcal{E}) = \mathsf{Sect}(\mathbb{C}, \mathbf{E}) \to \mathsf{Sect}(\mathbb{D}, F^*\mathbf{E}) = \mathsf{PSect}(\mathcal{D}, F^*\mathcal{E})$$

which restricts well to

$$\mathbb{F}^* : \mathbb{R}\mathsf{Sect}(\mathcal{C}, \mathcal{E}) \to \mathbb{R}\mathsf{Sect}(\mathcal{D}, F^*\mathcal{E})$$

and is moreover homotopical.

Denote by $\mathbb{D}(c)$ the simplicial replacement of $\mathcal{D}(c)$.

Definition 3.23. Let $F : \mathcal{D} \to \mathcal{C}$ be a resolution. A presection $A : \mathbb{D} \to \mathbb{F}^*\mathbf{E}$ is *locally constant* iff for any fibre $\mathcal{D}(c)$ over $c \in \mathcal{C}$, the composite functor

$$\mathbb{D}(c) \to \mathbb{D} \xrightarrow{A} \mathbf{E}(c) = \mathcal{E}(c)$$

sends all morphisms of the domain to weak equivalences. A derived presection is locally constant if it is locally constant as a presection.

We denote by $\mathsf{PSect}(\mathcal{D},\mathcal{E})_{lc}$ and $\mathbb{R}\mathsf{Sect}(\mathcal{D},\mathcal{E})_{lc}$ the corresponding full homotopical subcategories of locally constant (pre)sections. It is clear that any (pre)section of the form $\mathbb{F}^*A$ is locally constant.

Theorem 3.24. *Let $\mathcal{E} \to \mathcal{C}$ be a homotopical Δ-opfibration and $F : \mathcal{D} \to \mathcal{C}$ be a resolution (see Definition 3.21). Then after passing to localisations, the pullback functor $\mathbb{F}^* : \mathrm{Ho}\,\mathbb{R}\mathsf{Sect}(\mathcal{C},\mathcal{E}) \to \mathrm{Ho}\,\mathbb{R}\mathsf{Sect}(\mathcal{D},\mathcal{E})$ is full and faithful.*

We can also characterise the homotopical essential image of $\mathbb{F}$. Unfortunately, we only know how to do it for F-special (cf. Definition 5.11) homotopical Δ-opfibrations:

Theorem 3.25. *Let $F : \mathcal{D} \to \mathcal{C}$ be a resolution and $\mathcal{E} \to \mathcal{C}$ be a F-special homotopical Δ-opfibration. Then the functor $\mathbb{F}^* : \mathrm{Ho}\,\mathbb{R}\mathsf{Sect}(\mathcal{C},\mathcal{E}) \to \mathrm{Ho}\,\mathbb{R}\mathsf{Sect}(\mathcal{D},\mathcal{E})_{lc}$ is an equivalence.*

4 The pushforward functor

4.1 Main construction

Fix a functor $F : \mathcal{D} \to \mathcal{C}$ and a homotopical Δ-opfibration $\mathcal{E} \to \mathcal{C}$. Recall the diagram (*3.1*) for $G = id_{\mathcal{C}}$:

$$\begin{tikzcd} & \mathbb{F}/\!\!/\mathbb{C} \arrow[dl, "pr_{\mathbb{D}}"'] \arrow[d] \arrow[dr, "pr_{\mathbb{C}}"] & \\ \mathbb{D} \arrow[r, "\mathbb{F}"] & \mathbb{C} \arrow[l, phantom, shift left=4, "\Leftarrow"] \arrow[r, phantom, shift left=4, "\Rightarrow"] & \mathbb{C}. \arrow[l, "id_{\mathbb{C}}"'] \end{tikzcd} \tag{4.1}$$

The middle map is $pr_{\mathbb{F}/\!\!/\mathbb{C}}$. This diagram gives us in particular the restriction morphism of Proposition 2.10

$$R_{\mathbb{F}} : (\mathbb{F}pr_{\mathbb{D}})^*\mathbf{E} \to pr^*_{\mathbb{F}/\!\!/\mathbb{C}}\mathbf{E}. \tag{4.2}$$

This is a map of fibrations over $\mathbb{F}/\!\!/\mathbb{C}$.

Next, we observe there are equivalences

$$\mathsf{Sect}(\mathbb{F}/\!\!/\mathbb{C}, pr^*_{\mathbb{F}/\!\!/\mathbb{C}}\mathbf{E}) \xleftarrow{\sim} \mathsf{Sect}(\mathbb{F}/\!\!/\mathbb{C}, pr^*_{\mathbb{C}}\mathbf{E}) \xrightarrow{\sim} \mathsf{Sect}(\mathbb{C}, \mathbf{E}^{\mathbb{F}/\!\!/\mathbb{C}}) \tag{4.3}$$

where the right equivalence is just an instance of the universal property of Definition 2.11 (remember that $pr_{\mathbb{C}}$ is an opfibration). The left map comes from the equivalence

$$pr^*_{\mathbb{C}}\mathbf{E} \xrightarrow{\sim} pr^*_{\mathbb{F}/\!\!/\mathbb{C}}\mathbf{E}$$

provided by Remark 3.9. We denote by

$$D_{\mathbb{F}} : \mathsf{Sect}(\mathbb{F}/\!\!/\mathbb{C}, pr^*_{\mathbb{F}/\!\!/\mathbb{C}}\mathbf{E}) \xrightarrow{\sim} \mathsf{Sect}(\mathbb{C}, \mathbf{E}^{\mathbb{F}/\!\!/\mathbb{C}}) \tag{4.4}$$

the resulting equivalence constructed from (*4.3*).

There is a natural 'projection' functor Π over $\mathbb{C}$,

$$\begin{tikzcd} \mathbb{F}/\!\!/\mathbb{C} \arrow[rr, "\Pi"] \arrow[dr, "pr_{\mathbb{C}}"'] & & \Delta^{\mathsf{op}} \times \mathbb{C} \arrow[dl] \\ & \mathbb{C} & \end{tikzcd}$$

which acts as $(\mathbf{d}_{[n]}, \mathbf{c}_{[m]}, \alpha : F(d_n) \to c_0) \mapsto ([n], \mathbf{c}_{[m]})$. Exponentiating and taking sections, we obtain a functor $\Pi^* : \mathsf{Sect}(\mathbb{C}, \mathbf{E}^{\Delta^{\mathsf{op}}}) \to \mathsf{Sect}(\mathbb{C}, \mathbf{E}^{\mathbb{F}/\!\!/\mathbb{C}})$.

Proposition 4.1. *The functor*

$$\Pi^* : \mathsf{Sect}(\mathbb{C}, \mathbf{E}^{\Delta^{\mathsf{op}}}) \to \mathsf{Sect}(\mathbb{C}, \mathbf{E}^{\mathbb{F}/\!\!/\mathbb{C}})$$

admits a homotopical left adjoint

$$\Pi_! : \mathsf{Sect}(\mathbb{C}, \mathbf{E}^{\mathbb{F}/\!\!/\mathbb{C}}) \to \mathsf{Sect}(\mathbb{C}, \mathbf{E}^{\Delta^{\mathsf{op}}}) \tag{4.5}$$

Proof. See Proposition 2.14 for the construction of $\Pi_!$. To observe that it is homotopical, note that for each $\mathbf{c}$, the functor $\mathbb{F}/\!\!/\mathbb{C}(\mathbf{c}) \to \Delta^{\mathsf{op}}$ is a *discrete* opfibration, and the pushforward along it amounts to taking coproducts, which are homotopical. □

Take a $\mathcal{D}$-presection $S : \mathbb{D} \to \mathbb{F}^*\mathbf{E}$. Then apply functors (*4.2*), (*4.4*) and (*4.5*) to obtain

$$B_\bullet(S) := \Pi_! D_{\mathbb{F}}(R_{\mathbb{F}} \circ pr_{\mathbb{D}}^* S) \in Sect(\mathbb{C}, \mathbf{E}^{\Delta}). \tag{4.6}$$

Applying the realisation functor $|-|$ from Proposition 3.11, we get the following:

Definition 4.2. The *derived pushforward* of a presection $A : \mathbb{D} \to \mathbb{F}^*\mathbf{E}$ is defined as

$$\mathbb{F}_!(S) := |B_\bullet(S)| = |\Pi_! D_{\mathbb{F}}(R_{\mathbb{F}} \circ pr_{\mathbb{D}}^* S)|.$$

this defines a homotopical functor $\mathbb{F}_! : \mathsf{PSect}(\mathcal{D}, \mathcal{E}) \to \mathsf{PSect}(\mathcal{C}, \mathcal{E})$.

Remark 4.3. Over an object $\mathbf{c}_{[m]} = c_0 \xrightarrow{f_1} ... \xrightarrow{f_m} c_m$, we have

$$B_n(S)(\mathbf{c}_{[m]}) = \coprod_{\mathbf{d}_{[n]}, \alpha : F(d_n) \to c_0} (f_m ... f_1 \alpha)_! S(\mathbf{d}_{[n]})$$

where $(f_m ... f_1 \alpha)_!$ is the transition functor $\mathcal{E}(F(d_n)) \to \mathcal{E}(c_m)$. This expression is very similar to the bar construction (cf. [3, 16]); the value $\mathbb{F}_! S(\mathbf{c}_{[m]})$ is just the realisation of this simplicial object $B_n(S)(\mathbf{c}_{[m]})$.

4.2 Unit and counit correspondences

Given a $\mathcal{C}$-presection $A : \mathbb{C} \to \mathbf{E}$, use $pr_{\mathbb{F}/\!\!/\mathbb{C}}$ from the diagram (*4.1*) and functors (*4.2*), (*4.4*) and (*4.5*) to obtain

$$B_\bullet^{\mathbb{F}}(A) := \Pi_! D_{\mathbb{F}}(pr_{\mathbb{F}/\!\!/\mathbb{C}}^* A) \in Sect(\mathbb{C}, \mathbf{E}^{\Delta}). \tag{4.7}$$

Denote by $A^{\mathbb{F}}$ the realisation of $B_\bullet^{\mathbb{F}}(A)$.

Remark 4.4. Again, one can see that explicitly

$$B_n^{\mathbb{F}}(A)(\mathbf{c}_{[m]}) = \coprod_{\mathbf{d}_{[n]}, \alpha : F(d_n) \to c_0} A(\mathbb{F}(\mathbf{d}_{[n]}) *^{\alpha} \mathbf{c}_{[m]}).$$

Lemma 4.5. *There is a natural (in A) correspondence in* $\mathsf{PSect}(\mathcal{C}, \mathcal{E})$

$$\mathbb{F}_!\mathbb{F}^*A \leftarrow A^{\mathbb{F}} \to A$$

coming from the realisation of the correspondence of simplicial presections

$$B_\bullet(\mathbb{F}^*A) \leftarrow B_\bullet^{\mathbb{F}}(A) \to A$$

where the rightmost term is a constant simplicial object. When A is a derived section, the left morphsims in the correspondences above are weak equivalences.

Proof. First, the construction. Given a $\mathcal{C}$-presection $A : \mathbb{C} \to \mathbf{E}$, Proposition 2.10 and the left triangle of the diagram (*4.1*) gives us a map of $pr^*_{\mathbb{F}/\!\!/\mathbb{C}}\mathbb{F}^*\mathbf{E}$-sections over $\mathbb{F}/\!\!/\mathbb{C}$

$$pr^*_{\mathbb{F}/\!\!/\mathbb{C}}A \to R_{\mathbb{F}}(\mathbb{F}pr_{\mathbb{D}})^*A$$

which is an equivalence when A is a derived section. Indeed, over an object $(\mathbf{d}, \mathbf{c}, \alpha)$ of $\mathbb{F}/\!\!/\mathbb{C}$ the map looks like

$$A(\mathbb{F}(\mathbf{d}) *^{\alpha} \mathbf{c}) \to (f_n...f_1\alpha)_!A(\mathbb{F}(\mathbf{d}))$$

with $\mathbf{c} = c_0 \stackrel{f_1}{\to} ... \stackrel{f_n}{\to} c_n$, and this map is an equivalence precisely because of the derived section condition for A. Applying the equivalence $D_{\mathbb{F}}$ of (*4.4*) and then $\Pi_!$ of (*4.5*), we get the map

$$B_\bullet^{\mathbb{F}}(A) \to B_\bullet(\mathbb{F}^*A)$$

between (*4.6*) and (*4.7*) which is again a weak equivalence when A is a derived section.

Proposition 2.10 and the right triangle of the diagram (*4.1*) give us a map

$$pr^*_{\mathbb{F}/\!\!/\mathbb{C}}A \to pr^*_{\mathbb{C}}A$$

and we again apply $\Pi_!D_{\mathbb{F}}$. Observe that $\Pi_!D_{\mathbb{F}}pr^*_{\mathbb{C}}A$ is the following simplicial presection:

$$(\Pi_!D_{\mathbb{F}}pr^*_{\mathbb{C}}A)_n(\mathbf{c}) = \coprod_{\mathbf{d}_{[n]},\alpha:F(d_n)\to c_0} A(\mathbf{c}) \cong N(F/c_0)(n) \otimes A(\mathbf{c}).$$

There is thus a natural map $\Pi_!D_{\mathbb{F}}pr^*_{\mathbb{C}}A \to A$ to the constant simplicial presection A.

The realisation of $\Pi_!D_{\mathbb{F}}pr^*_{\mathbb{C}}A$ is the presection given by the assignment $\mathbf{c} \mapsto N(F/c_0) \otimes A(\mathbf{c})$. On this level as well, we get the map

$$A^{\mathbb{F}} \to N(F/h_{\mathcal{C}}(-)) \otimes A \to A$$

which completes the construction. $\square$

Lemma 4.6. *For $F = id_{\mathcal{C}}$ and a derived section $A : \mathbb{C} \to \mathbf{E}$ both morphisims in the span*

$$id_{\mathbb{C}!}id^*_{\mathbb{C}}A \leftarrow A^{id_{\mathbb{C}}} \to A$$

of Lemma 4.5 are weak equivalences.

Proof. Fix $\mathbf{c} \in \mathbb{C}$. In the case of the identity functor, we see that $B_{\bullet}^{id_{\mathbb{C}}} A(\mathbf{c})$ can calculated as the realisation (cf. Definition 3.12) of the functor $X : \mathbb{C}/\!\!/c_0 \to \mathbf{E}(\mathbf{c})$ defined by the assignment

$$X((\mathbf{c}'_{[k]}, \alpha : c'_k \to c_0)) = A(\mathbf{c}'_{[k]} *^{\alpha} \mathbf{c}).$$

The category $\mathbb{C}/\!\!/c_0$ is the simplicial replacement of the category $\mathcal{C}/c_0$, and the latter has a terminal object. By Lemma 3.13, the natural map $A(c_0 *^{id_{c_0}} \mathbf{c}) = X(c_0) \to |\Pi_! X| = A^{id_{\mathbb{C}}}(\mathbf{c})$ is an equivalence.

There is also an equivalence $A(\mathbf{c}) \to A(c_0 *^{id_{c_0}} \mathbf{c})$ which comes from the degeneracy $\mathbf{c} \to c_0 *^{id_{c_0}} \mathbf{c}$ (cf. Lemma 3.20). One can then see that the composition

$$A(\mathbf{c}) \to A(c_0 *^{id_{c_0}} \mathbf{c}) \to A^{id_{\mathbb{C}}}(\mathbf{c}) \to A(\mathbf{c})$$

is the identity (it is such already on the level of corresponding simplicial objects; also note that the composition $\mathbf{c} \to c_0 *^{id_{c_0}} \mathbf{c} \to \mathbf{c}$ in $\mathbb{C}$ is the identity $id_{\mathbf{c}}$). Thus the $\mathbf{c}$-th component of the map $A^{id_{\mathbb{C}}} \to A$ is an equivalence as a right inverse of an equivalence $A(\mathbf{c}) \to A(c_0 *^{id_{c_0}} \mathbf{c}) \to A^{id_{\mathbb{C}}}(\mathbf{c})$. □

Lemma 4.7. *For a functor $F : \mathcal{D} \to \mathcal{C}$ and a $\mathbb{D}$-presection A, there is a natural (in A) morphism*

$$id_{\mathbb{D}!} id_{\mathbb{D}}^* A \longrightarrow \mathbb{F}^* \mathbb{F}_! A.$$

Proof. By definition, $id_{\mathbb{D}!} id_{\mathbb{D}}^* A$ is the realisation of the simplicially valued presection X which at $d_0 \xrightarrow{g_1} ... \xrightarrow{g_m} d_m$ takes the value

$$[n] \mapsto X_n = \coprod_{\substack{d'_0 \to ... \to d'_n \\ \alpha : d'_n \to d_0}} (F(g_m ... g_1 \alpha))_! A(\mathbf{d}'_{[n]}).$$

In the case when we calculate $\mathbb{F}^* \mathbb{F}_! A$ at $d_0 \xrightarrow{g_1} ... \xrightarrow{g_m} d_m$, we have the following simplicial object Y:

$$[n] \mapsto Y_n = \coprod_{\substack{d'_0 \to ... \to d'_n \\ \beta : F(d'_n) \to F(d_0)}} (F(g_m ... g_1) \beta)_! A(\mathbf{d}'_{[n]}).$$

The assignment of $\alpha : d'_n \to d_0$ to $F\alpha : F(d'_n) \to F(d_0)$ induces the map of sets

$$\{d'_0 \to ... \to d'_n, \alpha : d'_n \to d_0\} \to \{d'_0 \to ... \to d'_n, \beta : F(d'_n) \to F(d_0)\} \tag{4.8}$$

and we obtain a map $X_n \to Y_n$ as X_n and Y_n are the coproducts indexed by the sets in (4.8). Varying $[n] \in \Delta$, we assemble a map $X \to Y$ of simplicial objects, which after realsiations gives the map in question, $id_{\mathbb{D}!} id_{\mathbb{D}}^* A \longrightarrow \mathbb{F}^* \mathbb{F}_! A$. □

We finally prove the main proposition of this section:

Proposition 4.8. *Let $F : \mathcal{D} \to \mathcal{C}$, $A \in \mathbb{R}\mathsf{Sect}(\mathcal{C}, \mathcal{E})$ and $R \in \mathbb{R}\mathsf{Sect}(\mathcal{D}, \mathcal{E})$.*

1. *There is a natural (in A) span*

$$\mathbb{F}_!\mathbb{F}^*A \leftarrow A^{\mathbb{F}} \rightarrow A \tag{4.9}$$

which induces a natural transformation $\epsilon : \mathbb{F}_!\mathbb{F}^* \to id$ *of endofunctors of* $\mathrm{Ho}\,\mathsf{PSect}(\mathcal{C}, \mathcal{E})$.

2. *There is a natural (in R) sequence of morphisms*

$$R \leftarrow R^{id_{\mathbb{D}}} \rightarrow id_{\mathbb{D}!}id_{\mathbb{D}}^*R \rightarrow \mathbb{F}^*\mathbb{F}_!R \tag{4.10}$$

which induces a natural transformation $\eta : id \to \mathbb{F}^*\mathbb{F}_!$ *of endofunctors of* $\mathrm{Ho}\,\mathsf{PSect}(\mathcal{D}, \mathcal{E})$.

3. (Triangle identity) *For each* $A \in \mathrm{Ho}\,\mathbb{R}\mathsf{Sect}(\mathcal{C}, \mathcal{E})$*, the composition in* $\mathrm{Ho}\,\mathsf{PSect}(\mathcal{D}, \mathcal{E})$

$$\mathbb{F}^*A \xrightarrow{\eta\mathbb{F}^*} \mathbb{F}^*\mathbb{F}_!\mathbb{F}^*A \xrightarrow{\mathbb{F}^*\epsilon} \mathbb{F}^*A \tag{4.11}$$

is the identity.

Proof. We proved the first two claims in the preceding lemmas. Only the triangle identity remains. Using the correspondences obtained before, we write a string of morphisms

$$\mathbb{F}^*A \xleftarrow{\sim} (\mathbb{F}^*A)^{id_{\mathbb{D}}} \xrightarrow{\sim} id_{\mathbb{D}!}id_{\mathbb{D}}^*\mathbb{F}^*A \rightarrow \mathbb{F}^*\mathbb{F}_!\mathbb{F}^*A \xleftarrow{\sim} \mathbb{F}^*(A^{\mathbb{F}}) \rightarrow \mathbb{F}^*A$$

with all the weak equivalences drawn as $\xrightarrow{\sim}$ or $\xleftarrow{\sim}$. We can redraw this sequence, obtaining the (potentially non-commutative) diagram

$$\begin{tikzcd}
& (\mathbb{F}^*A)^{id_{\mathbb{D}}} \arrow[r, "\sim"] \arrow[dl, "\sim"'] & id_{\mathbb{D}!}id_{\mathbb{D}}^*\mathbb{F}^*A \arrow[dd] \\
\mathbb{F}^*A & & \\
& \mathbb{F}^*(A^{\mathbb{F}}) \arrow[ul] \arrow[r, "\sim"] & \mathbb{F}^*\mathbb{F}_!\mathbb{F}^*A
\end{tikzcd}$$

The third claim is then equivalent to the commutativity of this diagram. We proceed as follows: writing down in components the simplicial object used to obtain $(\mathbb{F}^*A)^{id_{\mathbb{D}}}$, we see

$$(\mathbb{F}^*A)^{id_{\mathbb{D}}} \longleftrightarrow B_n^{id_{\mathbb{D}}}(\mathbb{F}^*A)(\mathbf{d}_{[m]}) = \coprod_{\mathbf{d}'_{[n]}, \alpha : d'_n \to d_0} A(\mathbb{F}(\mathbf{d}'_{[n]} *^{\alpha} \mathbf{d}_{[m]})).$$

In the same way,

$$\mathbb{F}^*(A^{\mathbb{F}}) \longleftrightarrow (\mathbb{F}^*B_n^{\mathbb{F}}(A))(\mathbf{d}_{[m]}) = \coprod_{\mathbf{d}'_{[n]}, \beta : F(d'_n) \to F(d_0)} A(\mathbb{F}(\mathbf{d}'_{[n]}) *^{\beta} \mathbb{F}(\mathbf{d}_{[m]})).$$

Assigning $\alpha \mapsto F(\alpha)$, we see that there is a natural in A map $(\mathbb{F}^*A)^{id_{\mathbb{D}}} \to \mathbb{F}^*(A^{\mathbb{F}})$. Moreover, a comparison with the construction of Lemma 4.7 reveals that in the resulting diagram

$$\begin{tikzcd}
 & (\mathbb{F}^*A)^{id_{\mathbb{D}}} \arrow[r, "\sim"] \arrow[dl, "\sim"'] \arrow[dd] & id_{\mathbb{D}!}id_{\mathbb{D}}^*\mathbb{F}^*A \arrow[dd] \\
\mathbb{F}^*A & & \\
 & \mathbb{F}^*(A^{\mathbb{F}}) \arrow[r, "\sim"] \arrow[ul] & \mathbb{F}^*\mathbb{F}_!\mathbb{F}^*A
\end{tikzcd}$$

both the left-hand triangle and the right-hand square commute. $\square$

Corollary 4.9. *Assume that for a functor $F : \mathcal{D} \to \mathcal{C}$, the map $\epsilon : \mathbb{F}_!\mathbb{F}^* \to id$ is an equivalence. Then $\mathbb{F}^* : \mathrm{Ho}\,\mathbb{R}\mathsf{Sect}(\mathcal{C},\mathcal{E}) \to \mathrm{Ho}\,\mathbb{R}\mathsf{Sect}(\mathcal{D},\mathcal{E})$ is full and faithful.*

Proof. This result can be proven as a particular case of the following categorical result:

Let $f : \mathcal{M} \rightleftarrows \mathcal{N} : u$ be two functors, and suppose $\mathcal{N}_0 \subset \mathcal{N}$ is a full subcategory such that there are natural transformations $\epsilon : fu|_{\mathcal{N}_0} \xrightarrow{\sim} id_{\mathcal{N}_0}$ and $\eta : id_{\mathcal{M}} \to uf$ defined over $\mathcal{N}_0$ and $\mathcal{M}$ respectively such that the triangle identity is satisfied: $u|_{\mathcal{N}_0} \to ufu|_{\mathcal{N}_0} \to u|_{\mathcal{N}_0}$ is the identity. Then $u|_{\mathcal{N}_0}$ is full and faithful.

In turn, the categorical result is proven as follows. The functoriality of u supplies us with maps $u(x,y) : \mathcal{N}_0(x,y) \to \mathcal{M}(ux,uy)$. Given a map $\alpha : ux \to uy$, we define $v(x,y)\alpha$ to be the map fitting in the commutative square

$$\begin{tikzcd}
fux \arrow[r, "f\alpha"] \arrow[d, "\epsilon_x"', "\sim"] & fuy \arrow[d, "\epsilon_y", "\sim"'] \\
x \arrow[r, "v(x,y)\alpha"'] & y.
\end{tikzcd}$$

This defines the map $v(x,y) : \mathcal{M}(ux,uy) \to \mathcal{N}_0(x,y)$ which is inverse to $u(x,y)$. $\square$

Note in particular that in the situation like above, for $A \in \mathbb{R}\mathsf{Sect}(\mathcal{C},\mathcal{E})$, $\mathbb{F}_!\mathbb{F}^*A$ is again a derived section.

5 Case of a resolution

In this section, we prove our main results, Theorems 3.24 and 3.25. Recall the definition of a resolution, Definition 3.21.

Lemma 5.1. *For $F : \mathcal{D} \to \mathcal{C}$ a resolution, the functor $\mathbb{F}^*$ reflects the condition of being a derived section. That is, if $\mathbb{F}^*A$ is a derived section, then A is one as well.*

Proof. If $\mathbb{F}^*A$ is a derived section for $A \in \mathsf{PSect}(\mathcal{C},\mathcal{E})$, then take any anchor map $\mathbf{c}' \to \mathbf{c}$ and find an anchor map $\mathbf{d}' \to \mathbf{d}$ such that $\mathbb{F}(\mathbf{d}' \to \mathbf{d}) = \mathbf{c}' \to \mathbf{c}$ (this is possible due to F being an opfibration with contractible, and hence nonempty, fibers). Then since $\mathbb{F}^*A(\mathbf{d}' \to \mathbf{d}) = A(\mathbf{c}' \to \mathbf{c})$, we get that A is a derived section. $\square$

5.1 Fullness and faithfullness

The main result of this section which we use to prove Theorem 3.24 is the following proposition:

Proposition 5.2. *Let $F : \mathcal{D} \to \mathcal{C}$ be a resolution. Then for any homotopical Δ-opfibration $\mathcal{E} \to \mathcal{C}$, the counit transformation*

$$\epsilon : \mathbb{F}_!\mathbb{F}^* A \to id_{\mathrm{Ho}\,\mathsf{PSect}(\mathcal{C},\mathcal{E})} A$$

is an isomorphism in $\mathrm{Ho}\,\mathsf{PSect}(\mathcal{C}, \mathcal{E})$ *for any derived section A.*

The proof will be carried out in several steps. Note that for an opfibration $F : \mathcal{D} \to \mathcal{C}$ and an object $c \in \mathcal{C}$, we can take two categories F/c, the comma category of F and c (viewed as a functor $[0] \to \mathcal{C}$), and $\mathcal{D}(c)$, the fiber of F at c. There is a functor which sends $d \in \mathcal{D}(c)$ to $(d, id_c : F(d) \xrightarrow{=} c) \in F/c$ and it has a left adjoint given by choosing, for each object $(d, f : F(d) \to c) \in F/c$, an opCartesian morphism $d \to f_! d$ covering f. A similar pattern occurs a few times in this section, and this motivates us to introduce the following technical notion:

Definition 5.3. For a category $\mathcal{D}$, a functor $F : \mathcal{D} \to \mathcal{C}$ and an object $c \in \mathcal{C}$, a (F, c)-*transition structure* consists of

1. two categories I, J and functors $\mathcal{I} : I \to \mathcal{D}$, $\mathcal{J} : J \to \mathcal{D}$,
2. a functor $\mathrm{R} : J \to I$ in $\mathbf{Cat}/\mathcal{D}$.

These data are subject to the following conditions:

- R admits a left adjoint L in $\mathbf{Cat}$,
- $\mathcal{J}$ maps J to the fiber $\mathcal{D}(c)$, so that $F\mathcal{J}$ factors through c.

In the notation of this definition, we sometimes write $(\mathcal{I}, \mathcal{J}, \mathrm{R})$ to denote a given (F, c)-transition structure.

Example 5.4. The transition structures of importance for us are the following:

1. For an opfibration $F : \mathcal{D} \to \mathcal{C}$ and an object $c \in \mathcal{C}$, there is a (F, c)-transition structure given by $I = F/c$ and $J = \mathcal{D}(c)$ outlined just before Definition 5.3.
2. If $F : \mathcal{D} \to \mathcal{C}$ is an opfibration and $d \in \mathcal{D}$, one can have the following $(F, F(d))$-transition structure: $I = \mathcal{D}/d$ and $J = \mathcal{D}(F(d))/d$. The right adjoint R is the evident inclusion; the left adjoint L is given by factoring any morphism $d' \to d$ as 'opCartesian followed by fiberwise' pair of morphisms.
3. Any (F, c) structure $(\mathcal{I}, \mathcal{J}, \mathrm{R})$ induces a $(F \circ \mathcal{I}, c)$ structure $(id_I, \mathrm{R}, \mathrm{R})$ with the same right adjoint R. Thus the first example gives us a (F_c, c)-structure where $F_c : \mathcal{D}/c \to \mathcal{C}$ is the functor $(d, f : Fd \to c) \mapsto Fd$. For this structure, $I = \mathcal{D}/c$, $J = \mathcal{D}(c)$ and R acts in the same way as before.

Remark 5.5. Consider the unit map $\eta(i) : i \to \mathrm{RL}i$ for any $i \in I$. Apply $F \circ \mathcal{I}$ to this map and obtain $\bar{\eta}(i) : F\mathcal{I}(i) \to c$. For any opfibration $\mathcal{E} \to \mathcal{C}$ we then have a well-defined 'restriction' functor

$$R_c : I^*\mathbb{F}^*\mathbf{E} \to \mathbf{E}(c) = \mathcal{E}(c)$$

where we denote by the same letter $\mathcal{I}$ the induced functor $\mathbb{I} \to \mathbb{D}$. Concretely, this functor sends $(\mathbf{i}_{[n]}, x \in \mathcal{E}(F\mathcal{I}(i_n)))$ to $\bar{\eta}(i)_!x$, using a (chosen) opCartesian lift $x \to \bar{\eta}(i)_!x$ in $\mathcal{E}$ covering $\bar{\eta}(i) : F\mathcal{I}(i_n) \to c$.

Construction 5.6. Assume given $c_0 \to \ldots \to c_n = \mathbf{c} \in \mathbb{C}$. Denote by $\mathbf{c}_!$ the natural transition functor

$$\mathbf{c}_! : \mathbf{E}(c_0) \cong \mathcal{E}(c_0) \to \mathcal{E}(c_n) \cong \mathbf{E}(c_n).$$

Consider also the simplicial comma object (Definition 3.7) $\mathbb{I}/\!\!/\mathbb{R}$, where $\mathbb{R} : \mathbb{J} \to \mathbb{I}$ is the simplicial replacement of R. Using the diagram (*3.1*) and postcomposing with functors to $\mathbb{D}$ we obtain a new diagram

$$\begin{tikzcd} & \mathbb{I}/\!\!/\mathbb{R} \arrow[dl, "pr_{\mathbb{I}}"'] \arrow[d] \arrow[dr, "pr_{\mathbb{J}}"] & \\ \mathbb{I} \arrow[r, "\mathcal{I}"] & \mathbb{D} \arrow[ul, phantom, "\Leftarrow"] \arrow[ur, phantom, "\Rightarrow"] & \mathbb{J} \arrow[l, "\mathcal{J}"'] \end{tikzcd} \tag{5.1}$$

and we henceforth denote the middle map again by $pr_{\mathbb{I}/\!\!/\mathbb{R}}$.

For $B \in \mathsf{PSect}(\mathcal{D}, \mathcal{E})$ and a given (F, c_0)-structure, we get the following diagram

$$\begin{tikzcd} & \mathbb{I}/\!\!/\mathbb{R} \arrow[dl, "pr_{\mathbb{I}}"'] \arrow[d] \arrow[dr, "pr_{\mathbb{J}}"] & \\ \mathbb{I} \arrow[r, "\mathbf{c}_!R_{c_0}I^*B"'] & \mathbf{E}(\mathbf{c}) \arrow[ul, phantom, "\Leftarrow"] \arrow[ur, phantom, "\Rightarrow"] & \mathbb{J} \arrow[l, "\mathbf{c}_!\mathcal{J}^*B"] \end{tikzcd}$$

with the middle map $\mathbf{c}_!pr^*_{\mathbb{I}/\!\!/\mathbb{R}}B$. Thus we have the span

$$pr^*_{\mathbb{I}}\mathbf{c}_!R_{c_0}I^*B \longleftarrow \mathbf{c}_!pr^*_{\mathbb{I}/\!\!/\mathbb{R}}B \longrightarrow pr^*_{\mathbb{J}}\mathbf{c}_!\mathcal{J}^*B.$$

Pushing this forward to the span given by projections,

$$\Delta^{\mathsf{op}} \xleftarrow{\pi_1} \Delta^{\mathsf{op}} \times \Delta^{\mathsf{op}} \xrightarrow{\pi_2} \Delta^{\mathsf{op}},$$

we obtain a span of bisimplicial objects in $\mathbf{E}(c)$:

$$\pi_1^*\Pi(\mathbf{c}_!R_{c_0}I^*B) \longleftarrow \Pi\mathbf{c}_!pr^*_{\mathbb{I}/\!\!/\mathbb{R}}B \longrightarrow \pi_2^*\Pi(\mathbf{c}_!\mathcal{J}^*B). \tag{5.2}$$

Here Π is the pushforward functor, simplicial (along $\mathbb{I} \to \Delta^{\mathsf{op}}$ and same for $\mathbb{J}$) or bisimplicial (along $\mathbb{I}/\!\!/\mathbb{R} \to \Delta^{\mathsf{op}} \times \Delta^{\mathsf{op}}$). We implicitly used the Beck-Chevalley morphisms, such as $\Pi pr^*_{\mathbb{I}} \to \pi_1^*\Pi$, for pullbacks and pushforwards; they arise from commutative squares like

$$\begin{tikzcd} \mathbb{I}/\!\!/\mathbb{R} \arrow[r, "pr_{\mathbb{I}}"] \arrow[d] & \mathbb{I} \arrow[d] \\ \Delta^{\mathsf{op}} \times \Delta^{\mathsf{op}} \arrow[r, "\pi_1"] & \Delta^{\mathsf{op}} \end{tikzcd}$$

by taking associated pullback functors on functor categories and then replacing some of them by left adjoints.

Remark 5.7. Let us write the terms of the span (*5.2*) explicitly. For $\mathbf{c} = c_0 \xrightarrow{f_1} \ldots \xrightarrow{f_n} c_n$, we find that

$$\Pi(\mathbf{c}_! R_{c_0} I^* B)_m = \coprod_{\mathbf{i}_{[m]}} (f_n \ldots f_1 \bar{\eta}(i_m))_! B(I \mathbf{i}_{[m]}),$$

where $\bar{\eta}(i_m)$ is induced from the unit of $\mathrm{L} \dashv \mathrm{R}$ (Remark 5.5). Next,

$$\Pi(\mathbf{c}_! \mathcal{J}^* B)_l = \coprod_{\mathbf{j}_{[l]}} (f_n \ldots f_1)_! B(\mathcal{J} \mathbf{j}_{[l]}),$$

and, finally,

$$(\Pi \mathbf{c}_! pr^*_{\mathbb{I}/\!\!/\mathbb{R}} B)_{ml} = \coprod_{\mathbf{i}_{[m]},\, \mathbf{j}_{[l]},\, \alpha: i_m \to \mathrm{R} j_0} (f_n \ldots f_1)_! B(I(\mathbf{i}_{[m]}) *^{I\alpha} \mathcal{J}(\mathbf{j}_{[l]})).$$

Proposition 5.8. *For* $c_0 \xrightarrow{f_1} \ldots \xrightarrow{f_n} c_n = \mathbf{c} \in \mathbb{C}$*, a* $(F : \mathcal{D} \to \mathcal{C}, c_0)$*-transition structure* $(I, \mathcal{J}, R)$*, and any* $B \in \mathbb{R}\mathsf{Sect}(\mathcal{D}, F^*\mathcal{E})$ *there is a natural (in* B*) span of* weak equivalences *in* $\mathbf{E}(\mathbf{c})$

$$|\Pi(\mathbf{c}_! R_{c_0} I^* B)| \longleftarrow ||\Pi \mathbf{c}_! pr^*_{\mathbb{I}/\!\!/\mathbb{R}} B|| \longrightarrow |\Pi(\mathbf{c}_! \mathcal{J}^* B)| \tag{5.3}$$

which comes from a natural (in B*) span (5.2) of bisimplicial objects in* $\mathbf{E}(\mathbf{c})$*.*

Proof. We need to prove that after realisations, both arrows become equivalences. Consider the bisimplicial object

$$\Pi \pi_1^* (\mathbf{c}_! R_{c_0} I^* B)_{ml} = \coprod_{\mathbf{i}_{[m]},\, \mathbf{j}_{[l]},\, \alpha: i_m \to \mathrm{R} j_0} (f_n \ldots f_1 \bar{\eta}(i_m))_! B(I(\mathbf{i}_{[m]}))$$

Our left hand side map in (*5.2*) passes through this object, as it is equal to the composition

$$\Pi \mathbf{c}_! pr^*_{\mathbb{I}/\!\!/\mathbb{R}} B \to \Pi \pi_1^* (\mathbf{c}_! R_{c_0} I^* B) \to \pi_1^* \Pi(\mathbf{c}_! R_{c_0} I^* B). \tag{5.4}$$

Writing down the simplicial objects explicitly, we see that the first map in (*5.4*) arises from the action of B on anchor maps and is a termwise weak equivalence of bisimplicial objects because B is a derived section. Realising the second map $\Pi \pi_1^* (\mathbf{c}_! R_{c_0} I^* B) \to \pi_1^* \Pi(\mathbf{c}_! R_{c_0} I^* B)$ in (*5.4*) along the second simplicial argument, we obtain a map in $\Delta^{\mathsf{op}} \mathbf{E}(\mathbf{c})$, whose m-th component is

$$|\Pi \pi_1^* (\mathbf{c}_! R_{c_0} I^* B)|_m \cong N(i_m \backslash \mathrm{R}) \otimes \Pi(\mathbf{c}_! R_{c_0} I^* B)_m \to \Pi(\mathbf{c}_! R_{c_0} I^* B)_m. \tag{5.5}$$

Observe that because of the adjunction $\mathrm{L} \dashv \mathrm{R}$, the category $i_m \backslash \mathrm{R} = (\mathrm{R}^{\mathsf{op}} / i_m)^{\mathsf{op}}$ has an initial object (the unit at i_m) and is thus contractible, so the map (*5.5*) and thus (*5.4*) and the left-hand side map in (*5.2*) are all weak equivalences.

We now have to prove that the right-hand side map $\Pi \mathbf{c}_! pr^*_{\mathbb{I}/\!\!/\mathbb{R}} B \longrightarrow \pi_2^* \Pi(\mathbf{c}_! \mathcal{J}^* B)$ in (*5.2*) becomes an equivalence after realisations. For each fixed $\mathbf{j}_{[l]}$, we have a map of simplicial objects, written in components as

$$\coprod_{\mathbf{i}_{[m]},\, \alpha: i_m \to \mathrm{R} j_0} (f_n \ldots f_1)_! B(I(\mathbf{i}_{[m]}) *^{I\alpha} \mathcal{J}(\mathbf{j}_{[l]})) \longrightarrow (f_n \ldots f_1)_! B(\mathcal{J}(\mathbf{j}_{[l]})); \tag{5.6}$$

because L/j_0 has a terminal object, Lemma 3.13 and Lemma 3.20 imply that the map (*5.6*) is a weak equivalence after being realised. We conclude that the morphism

$$|\Pi \mathbf{c}_! pr^*_{\mathbb{I}/\!\!/\mathbb{R}} B| \longrightarrow |\pi_2^* \Pi(\mathbf{c}_! \mathcal{J}^* B)| \cong \Pi(\mathbf{c}_! \mathcal{J}^* B)$$

is an equivalence of simplicial objects in $\mathbf{E}(\mathbf{c})$, where we took the realisation of bisimplicial objects along the first argument. Proposition 1.24 then implies that the double realisation

$$||\Pi \mathbf{c}_! pr^*_{\mathbb{I}/\!\!/\mathbb{R}} B|| \longrightarrow ||\pi_2^* \Pi(\mathbf{c}_! \mathcal{J}^* B)| \cong |\Pi(\mathbf{c}_! \mathcal{J}^* B)|,$$

taken in any order, is a weak equivalence. □

We are now ready to prove Proposition 5.2. Fix $\mathbf{c}_{[n]} \in \mathbb{C}$. For $A \in \mathbb{R}\mathsf{Sect}(\mathcal{C}, \mathcal{E})$ there are functors $A^{aug}_{\mathbf{c}}$ and $A_{\mathbf{c}}$ (cf. the proof of Lemma 4.5):

$$A^{aug}_{\mathbf{c}} : \mathbb{F}/\!\!/c_0 \to \mathbf{E}(\mathbf{c}_{[n]}), \quad (\mathbf{d}_{[m]}, \alpha : Fd_m \to c_0) \mapsto A(\mathbb{F}(\mathbf{d}_{[m]}) *^{\alpha} \mathbf{c}_{[n]}),$$
$$A_{\mathbf{c}} : \mathbb{F}/\!\!/c_0 \to \mathbf{E}(\mathbf{c}_{[n]}), \quad (\mathbf{d}_{[m]}, \alpha : Fd_m \to c_0) \mapsto A(\mathbf{c}_{[n]}).$$

There is an obvious natural transformation $A^{aug}_{\mathbf{c}} \to A_{\mathbf{c}}$. Pushing it forward to Δ^{op} and realising gives us a map $A^{\mathbb{F}}(\mathbf{c}) \to N(F/c_0) \otimes A(\mathbf{c})$ so that the obvious composition

$$A^{\mathbb{F}}(\mathbf{c}) \to N(F/c_0) \otimes A(\mathbf{c}) \to A(\mathbf{c})$$

is the $\mathbf{c}$-th component of the right-hand map of the counit correspondence (*4.9*).

Lemma 5.9. *The morphism $N(F/c_0) \otimes A(\mathbf{c}) \to A(\mathbf{c})$ is a weak equivalence.*

Proof. There is an adjunction $F/c_0 \rightleftharpoons \mathcal{D}(c_0)$ and $\mathcal{D}(c_0)$ is contractible, thus F/c_0 is contractible as well because adjunctions of categories are known to induce homotopy equivalences between the associated nerves [15]. □

Now recall Example 5.4(3) where we work over F/c_0, with $I = F/c_0$, $J = \mathcal{D}(c_0)$ and $\mathrm{R} : \mathcal{D}(c_0) \to F/c_0$ being the evident functor. Also take the trivial opfibration $\mathcal{E}(c_n) \times \mathcal{C} \to \mathcal{C}$. Both $A_{\mathbf{c}}$ and $A^{aug}_{\mathbf{c}}$ are then sections over $\mathbb{F}/\!\!/c_0$ of the trivial fibration $\mathbf{E}(\mathbf{c}_{[n]}) \times \mathbb{F}/\!\!/c_0 \to \mathbb{F}/\!\!/c_0$.

Lemma 5.10. *The map $A^{\mathbb{F}}(\mathbf{c}) \to N(F/c_0) \otimes A(\mathbf{c})$ is a weak equivalence.*

Proof. The obvious natural transformation $A^{aug}_{\mathbf{c}} \to A_{\mathbf{c}}$, when plugged in the left hand side of the span (*5.3*) for the transition structure of the Example 5.4(3), gives us the map in question, $A^{\mathbb{F}}(\mathbf{c}) \to N(F/c_0) \otimes A(\mathbf{c})$. The right-hand side of span (*5.3*) gives the map

$$|\Pi(\mathrm{R}^* A^{aug}_{\mathbf{c}})| \to N(\mathcal{D}(c_0)) \otimes A(\mathbf{c}) \tag{5.7}$$

so by Proposition 5.8 we are done if the map (*5.7*) is a weak equivalence. Observe however that

$$\Pi(\mathrm{R}^* A^{aug}_{\mathbf{c}})_m = \coprod_{\mathbf{d}_{[m]} \in \mathbb{D}(c_0)} A(\mathbb{F}(\mathbf{d}_{[m]}) * \mathbf{c}) = \coprod_{\mathbf{d} \in \mathbb{D}(c_0)} A(id^m_{c_0} * \mathbf{c})$$

with $id^m_{c_0}$ being the degenerate m-simplex $c_0 \overset{id_{c_0}}{\to} \ldots \overset{id_{c_0}}{\to} c_0$. Because A is a derived section, Lemma 3.20 implies that the obvious map $A(id^m_{c_0} * \mathbf{c}) \to A(\mathbf{c})$ is a weak equivalence, so that

$$\Pi(\mathrm{R}^* A^{aug}_{\mathbf{c}})_m \to N(\mathcal{D}(c_0))_m \otimes A(\mathbf{c}) = \Pi(\mathrm{R}^* A_{\mathbf{c}})_m$$

is a weak equivalence as well. □

Varying $\mathbf{c}$, we obtain the proof of Proposition 5.2. With Corollary 4.9, we get that $\mathbb{F}^*$ is fully faithful on homotopy level, which is exactly the contents of Theorem 3.24.

5.2 Essential surjectivity

Our second main result, Theorem 3.25, needs a technical condition of speciality. To state it, we need to define a few auxiliary things. First, take any opfibration $F : \mathcal{D} \to \mathcal{C}$. When F is viewed as a functor $\mathcal{C} \to \mathbf{Cat}$, we can compose it with the endodfunctor $\mathbf{Cat} \to \mathbf{Cat}$ which is the simplicial replacement functor. On the level of opfibrations, define the category[12] $\mathbb{O}_{\mathcal{C}}(\mathcal{D})$ as follows. An object of $\mathbb{O}_{\mathcal{C}}(\mathcal{D})$ is an object $c \in \mathcal{C}$ and $\mathbf{d} \in \mathbb{D}(c)$. A morphism $(c, \mathbf{d}_{[n]}) \to (c', \mathbf{d}'_{[m]})$ consists of a map $f : c \to c'$ and an equivalence class of pairs (β, γ) where

- $\beta : \mathbf{d}_{[n]} \Rightarrow \mathbf{d}^0_{[n]}$ is some natural transformation in $Fun([n], \mathcal{D})$ with domain $\mathbf{d}_{[n]}$ and so that each $\beta_i : d_i \to d^0_i$ is an opCartesian morphism in $\mathcal{D}$ lying over $f : c \to c'$,
- $\gamma : \mathbf{d}^0_{[n]} \to \mathbf{d}'_{[m]}$ is a morphism in $\mathbb{D}(c')$,
- and the equivalence relation is as follows. Two pairs $(\beta^0 : \mathbf{d}_{[n]} \Rightarrow \mathbf{d}^0_{[n]}, \gamma^0 : \mathbf{d}^0_{[n]} \to \mathbf{d}'_{[m]})$ and $(\beta^1 : \mathbf{d}_{[n]} \Rightarrow \mathbf{d}^1_{[n]}, \gamma^1 : \mathbf{d}^1_{[n]} \to \mathbf{d}'_{[m]})$ are equivalent if, after applying the functor $\pi : \mathbb{D}(c') \to \Delta^{\mathsf{op}}$, we have that $\pi\gamma^0 = \pi\gamma^1$.

In all, we obtain an opfibration $\mathbb{O}_{\mathcal{C}}(\mathcal{D}) \to \mathcal{C}$ whose fibers are $\mathbb{D}(c)$ and whose transition functors are given by the simplicial replacements of $f_!$, the transition functors of $F : \mathcal{D} \to \mathcal{C}$ associated to $f : c \to c'$.

For any opfibration $F : \mathcal{D} \to \mathcal{C}$, denote by $F^*F : F^*\mathcal{D} \to \mathcal{D}$ the pullback opfibration of F along F. Then from F^*F we obtain the opfibration $\mathbb{O}_{\mathcal{D}}(F^*\mathcal{D}) \to \mathcal{D}$ constructed as above, and denote its pullback along the first element map $h_{\mathcal{D}} : \mathbb{D} \to \mathcal{D}$ (see Lemma 3.4) by $\mathbb{O}(F^*\mathcal{D}) \to \mathbb{D}$. Finally, take the power fibration

$$(\mathbb{F}^*\mathbf{E})^{\mathbb{O}(F^*\mathcal{D})} \to \mathbb{D}.$$

The Δ-structure, as usual, gives us the lax realisation morphism

$$\begin{tikzcd}
(\mathbb{F}^*\mathbf{E})^{\mathbb{O}(F^*\mathcal{D})} \arrow[rr, "|-|"] \arrow[dr] & & \mathbb{F}^*\mathbf{E} \arrow[dl] \\
& \mathbb{D} &
\end{tikzcd}$$

defined by taking $X \in (\mathbb{F}^*\mathbf{E})^{\mathbb{O}(F^*\mathcal{D})}$, which is a functor $\mathbb{D}(F(d_0)) \to \mathbf{E}(\mathbf{d}_{[n]})$ for some $\mathbf{d}_{[n]} \in \mathbb{D}$, and realising it (cf. Definition 3.12). There is also, however, the 'evaluation' map

$$\begin{tikzcd}
(\mathbb{F}^*\mathbf{E})^{\mathbb{O}(F^*\mathcal{D})} \arrow[rr, "ev"] \arrow[dr] & & \mathbb{F}^*\mathbf{E} \arrow[dl] \\
& \mathbb{D} &
\end{tikzcd}$$

given by sending the same X to $X(d_0)$, since $d_0 \in \mathbb{D}(F(d_0))$. The inclusion $X(d_0) \to \coprod_{d \in \mathbb{D}(F(d_0))} X(d)$ defines a natural transformation $i : ev \Rightarrow |-|$.

[12]The dependence of the definition of $\mathbb{O}_{\mathcal{C}}(\mathcal{D})$ on F is implicit in the notation.

Definition 5.11. Given a resolution $F : \mathcal{D} \to \mathcal{C}$, a homotopical Δ-opfibration $\mathcal{E} \to \mathcal{C}$ is *F-special* iff for each $X \in (\mathbb{F}^*\mathbf{E})^{\mathbb{O}(F^*\mathcal{D})}$, which, when viewed as a functor $\mathbb{D}(F(d_0)) \to \mathbf{E}(\mathbf{d}_{[n]})$, sends all maps of $\mathbb{D}(F(d_0))$ to weak equivalences in $\mathbf{E}(\mathbf{d})$, the X-th component of the natural transformation i,

$$i_X : ev(X) \to |X|.$$

is a weak equivalence

The condition of speciality is satisfied when, for example, each fiber of the fibration $\mathbf{E} \to \mathbb{C}$ is a model category, and taking a realisation of any simplicial object $X : \Delta^{\mathsf{op}} \to \mathcal{E}(c)$ amounts to calculating its homotopy colimit (see [4] for the discussion of locally constant functors in this setting).

The result of this section is the following. Let $F : \mathcal{D} \to \mathcal{C}$ be a resolution.

Proposition 5.12. *For a F-special (Definition 5.11) homotopical Δ-opfibration $\mathcal{E} \to \mathcal{C}$ and a locally constant $B \in \mathbb{R}\mathsf{Sect}(\mathcal{D}, \mathcal{E})$, the map*

$$id_{\mathbb{D}!}id_{\mathbb{D}}^*B \to \mathbb{F}^*\mathbb{F}_!B$$

is a weak equivalence.

We will prove that for each $\mathbf{d}$, the map $id_{\mathbb{D}!}id_{\mathbb{D}}^*B(\mathbf{d}) \to \mathbb{F}^*\mathbb{F}_!B(\mathbf{d})$ is an equivalence.

Definition 5.13. Given a $(F : \mathcal{D} \to \mathcal{C}, c)$-structure $(\mathcal{I}, \mathcal{J}, \mathrm{R})$ and a $(F' : \mathcal{D} \to \mathcal{C}', c')$-structure $(\mathcal{I}', \mathcal{J}', \mathrm{R}')$, a *morphism* from the first to the second one consists of

- a functor $G : \mathcal{C} \to \mathcal{C}'$ in $\mathcal{D}\backslash\mathbf{Cat}$ with $G(c) = c'$.
- a commutative square in $\mathbf{Cat}/\mathcal{D}$

$$\begin{tikzcd} I \arrow[d, "\lambda"'] & J \arrow[l, "\mathrm{R}"'] \arrow[d, "\mu"] \\ I' & J'. \arrow[l, "\mathrm{R}'"'] \end{tikzcd}$$

Example 5.14. In Example 5.4, there is a morphism from the second to the first example as soon as $c = F(d)$. In detail: we have a $(F, F(d))$ transition structure $\mathrm{L} : \mathcal{D}/d \rightleftharpoons \mathcal{D}(F(d))/d : \mathrm{R}$ and $(F, c = F(d))$ transition structure $\mathrm{L}' : F/c \rightleftharpoons \mathcal{D}(c) : \mathrm{R}'$. In the notation of the definition, G is simply given by $id_{\mathcal{C}}$ (this works because $F(d) = c$), λ is given by mapping $\alpha : d' \to d$ to $(d', F(\alpha) : F(d') \to c)$ and μ is the evident functor $\mathcal{D}(c)/d \to \mathcal{D}(c)$. In this case, even more is true: the square with left adjoints

$$\begin{tikzcd} \mathcal{D}/d \arrow[r, "\mathrm{L}"] \arrow[d, "\lambda"'] & \mathcal{D}(c)/d \arrow[d, "\mu"] \\ F/c \arrow[r, "\mathrm{L}'"] & \mathcal{D}(c). \end{tikzcd}$$

commutes up to isomorphism.

Remark 5.15. Given functors $p : A \to B$, $q : A' \to B$ and $r : A' \to A$ such that $pr = q$, for any other functor $X : A \to \mathcal{M}$ to a cocomplete category, there is a natural map $q_! r^* X \to p_! X$ where as usual, r^* denotes pullback and $p_!, q_!$ denote pushforward functors (left adjoint to pullbacks p^* and q^*).

Lemma 5.16. *Fix $\mathbf{c} \in \mathbb{C}$ and $\mathbf{c}' \in \mathbb{C}'$. Let $(\mathcal{I}, \mathcal{J}, R)$ be a $(F : \mathcal{D} \to \mathcal{C}, c_0)$-structure and $(\mathcal{I}', \mathcal{J}', R')$ be a $(F' : \mathcal{D} \to \mathcal{C}', c_0')$-structure. For any morphism $(G : \mathcal{C}' \to \mathcal{C}, \lambda, \mu)$ of these transition structures such that $\mathbb{G}(\mathbf{c}') = \mathbf{c}$, a homotopical Δ-opfibration $\mathcal{E} \to \mathcal{C}$ and a presection $B : \mathbb{D} \to \mathbb{F}^*\mathbf{E}$, there is an induced morphism of spans*

$$\begin{tikzcd} \pi_1^*\Pi(\mathbf{c}'_! R_{c_0'} I'^* B) \arrow[d] & \Pi \mathbf{c}'_! pr^*_{\mathbb{I}'/\!\!/\mathbb{R}'} B \arrow[l] \arrow[r] \arrow[d] & \pi_2^*\Pi(\mathbf{c}'_! \mathcal{J}'^* B) \arrow[d] \\ \pi_1^*\Pi(\mathbf{c}_! R_{c_0} I^* B) & \Pi \mathbf{c}_! pr^*_{\mathbb{I}/\!\!/\mathbb{R}} B \arrow[l] \arrow[r] & \pi_2^*\Pi(\mathbf{c}_! \mathcal{J}^* B). \end{tikzcd} \tag{5.8}$$

Proof. The maps exist due to Remark 5.15. To get the rightmost map of (*5.8*), apply π_2^* to

$$\Pi(\mathbf{c}'_! \mathcal{J}'^* B) \to \Pi(\mathbf{c}_! \mathcal{J}^* B)$$

which we get due to the fact that $\mu^* \mathbf{c}_! \mathcal{J}^* B = \mathbf{c}'_! \mathcal{J}'^* B$. The middle map of (*5.8*) is obtained in this way as well, and so is the leftmost map (observe that due to the conditions imposed, both restriction functors R_{c_0} and $R_{c_0'}$ agree).

One can then check the commutativity of the squares obtained through a direct computation. For example, observe that the middle map, in components

$$\coprod_{\substack{\mathbf{i}'_{[m]},\, \mathbf{j}'_{[l]},\\ \alpha' : i'_m \to \mathrm{R} j'_0}} \mathbf{c}'_! B(I'(\mathbf{i}'_{[m]}) *^{I'\alpha'} \mathcal{J}'(\mathbf{j}'_{[l]})) \to \coprod_{\substack{\mathbf{i}_{[m]},\, \mathbf{j}_{[l]},\\ \alpha : i_m \to \mathrm{R} j_0}} \mathbf{c}_! B(I(\mathbf{i}_{[m]}) *^{I\alpha} \mathcal{J}(\mathbf{j}_{[l]}))$$

is induced by the maps of sets indexing the coproducts, given by $(\mathbf{i}', \mathbf{j}', \alpha') \mapsto (\lambda(\mathbf{i}'), \mu(\mathbf{j}'), \lambda\alpha')$ (cf. the proof of Lemma 4.7). $\square$

Corollary 5.17. *Given a map of two transition structures and a* derived section *B, the following are equivalent*

1. $|\Pi(\mathbf{c}'_! R_{c_0'} I'^* B)| \to |\Pi(\mathbf{c}_! R_{c_0} I^* B)|$ *is a weak equivalence,*
2. $|\Pi(\mathbf{c}'_! \mathcal{J}'^* B)| \to |\Pi(\mathbf{c}_! \mathcal{J}^* B)|$ *is a weak equivalence.*

Proof. Evident. $\square$

We now apply that to Example 5.14. Observe that for $\mathbf{d} \in \mathbb{D}$ with $\mathbf{c} = \mathbb{F}(\mathbf{d})$, the map

$$id_{\mathbb{D}!} id^*_{\mathbb{D}} B(\mathbf{d}) \to \mathbb{F}^* \mathbb{F}_! B(\mathbf{d})$$

exactly corresponds to the first morphism in Corollary 5.17. Writing $\mathbf{d}$ instead of $\mathbf{c}'$, observe that the objects in the second map of Corollary 5.17 are

$$\Pi(\mathbf{d}_! \mathcal{J}'^* B)_m = \coprod_{\mathbf{d}'_m \in \mathbb{D}(F(d_0)),\ d'_m \to d_0} \mathbb{F}(\mathbf{d})_! B(\mathbf{d}'_{[m]}),$$

$$\Pi(\mathbf{c}_! \mathcal{J}^* B)_m = \coprod_{\mathbf{d}'_m \in \mathbb{D}(F(d_0))} \mathbb{F}(\mathbf{d})_! B(\mathbf{d}'_{[m]}).$$

The realisation of the first object is equivalent to $\mathbb{F}(\mathbf{d})_! B(d_0)$. It is easy to check that for a F-special homotopical Δ-opfibration the functor

$$\mathbb{D}(F(d_0)) \to \mathbf{E}(\mathbb{F}(\mathbf{d})), \quad \mathbf{d}' \mapsto \mathbb{F}(\mathbf{d})_! B(\mathbf{d}')$$

which sends all morphisms to weak equivalences has its realisation equivalent to $\mathbb{F}(\mathbf{d})_! B(d_0)$ and this implies that the map $|\Pi(\mathbf{d}_! \mathcal{J}'^* B)| \to |\Pi(\mathbf{c}_! \mathcal{J}^* B)|$ is an equivalence.

Corollary 5.18 (proof of Theorem 3.25). $\mathbb{F}_!$ *sends locally constant sections to derived sections, and*

$$\mathbb{F}_! : \mathrm{Ho}\, \mathbb{R}\mathsf{Sect}_{lc}(\mathcal{D}, \mathcal{E}) \rightleftarrows \mathrm{Ho}\, \mathbb{R}\mathsf{Sect}(\mathcal{C}, \mathcal{E}) : \mathbb{F}^*$$

is an equivalence of categories for a special homotopical Δ*-fibration* $\mathcal{E} \to \mathcal{C}$.

Proof. We proved that the unit correspondence gives an isomorphism $id \to \mathbb{F}^*\mathbb{F}_!$ of functors on $\mathrm{Ho}\, \mathsf{PSect}_{lc}(\mathcal{D}, \mathcal{E})$. Using Lemma 5.1, we see that then $\mathbb{F}_!$ preserves derived section condition for locally constant sections. This allows us to restrict the unit $id \to \mathbb{F}^*\mathbb{F}_!$ to the derived sections. $\square$

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