Lie algebra cohomology representing characteristic classes of flags of foliations

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Abstract

We present a complete answer on the Lie algebra cohomology of formal vector fields on the d-dimensional plane with coefficients in the symmetric powers of the coadjoint representation. At the same time we compute the cohomology of the Lie algebra of formal vector fields that preserve a given flag at the origin. The resulting cohomology are known to be responsible for the characteristic classes of the flags of foliations and are well used in the local Riemann-Roch theorem [10].

We use the degeneration theorems of appropriate Hochschild-Serre spectral sequences and provide the method which allows us to avoid one of the most complicated computation in the invariant theory which was used by Gelfand, Feigin and Fuchs in order to cover the case of first symmetric power [17]. The method we provide gives a uniform and beautiful answer for all symmetric powers at the same time.

0 Introduction

0.1 Main results

The main result of this paper concerns the homology computation of the Lie algebra of formal vector fields on the n–dimensional plane (denoted for further by W_n). We compute the cohomology with coefficients in symmetric powers of the coadjoint representation. The answer is simply enough to be repeated here:

Theorem (Theorem 2.3.17). *For all $k \geqslant 1$ the relative cohomology of the Lie algebra W_n (relative to the subalgebra of linear vector fields) with coefficients in k-th symmetric power of coadjoint representation vanishes everywhere except the degree $2n$. The description of relative and absolute cohomology in terms of $\mathbf{gl}_n$-invariants looks as follows:*

$$
\mathrm{H}^i(W_n; S^k W_n^*) = \begin{cases} [S^{n+k}\mathbf{gl}_n]^{\mathbf{gl}_n} \otimes [\Lambda^{i-2n}(\mathbf{gl}_n)]^{\mathbf{gl}_n}, & \text{if } 2n \leqslant i \leqslant n^2 + 2n, \\ 0, & \text{otherwise.} \end{cases}
$$

Recall that the algebra of $\mathbf{gl}_n$–invariants in the symmetric algebra $S^\bullet(\mathbf{gl}_n)$ known to be the free symmetric algebra with n generators of degrees $1, 2, \ldots, n$. Respectively, the algebra of $\mathbf{gl}_n$–invariants in the exterior algebra $\Lambda^\bullet(\mathbf{gl}_n)$ is isomorphic to the free skew-symmetric algebra with n odd generators of degrees $1, 3, 5, \ldots, 2n-1$. See e.g. [30, 12]. In particular, we can identify the $2n$'th cohomology $\mathrm{H}^{2n}(W_n; S^k W_n^*)$ with the subspace of polynomials of degree $n + k$ in the polynomial algebra $\Bbbk[x_1, x_2, \ldots, x_n]$ subject to the following convention on degrees: the generator x_i has degree i. The formulas for generating cocycles are given in Section §A.3.

The conjectural answer in the aforementioned theorem was stated informally in early 70's by B. Feigin, D. Fuchs and I. Gelfand after their cumbersome computation for a particular case of first symmetric power ([17]). Later on, in 1989, the general answer to the problem was stated

(without any proof) by B. Feigin and B. Tsygan in [10] while they were using the desired Lie algebra cohomology in order to prove the local Riemann-Roch theorem. In 2003, V. Dotsenko ([7]) provided a direct homology computation for the linear dual problem in the case $n = 1$. Namely, he computed the homology of the Lie algebra of polynomial vector fields on the line with coefficients in symmetric powers of adjoint representation. We suggest below the uniform method which does not involve many computations with invariants and covers the case of all positive n and all symmetric powers.

At the same time we are going to cover another important homology computation. The problem mentioned below also takes it origin in formal geometry and has important applications in the foliation theory (see e.g. [9].) Namely, we compute the cohomology of the Lie algebra of formal vector fields that preserve a given flag of foliations. More precisely, let us fix the collection of natural numbers $\overline{n} = (n_0, \ldots, n_k)$ and fix a sequence of trivial embedded foliations $\{\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_k\}$ in $\mathbb{R}^{n_0 + \ldots + n_k}$ such that the foliation $\mathcal{F}_{i+1}$ has codimension n_i in $\mathcal{F}_i$ (n_0 is a codimension of $\mathcal{F}_1$). I.e. $\mathcal{F}_i := \mathbb{R}^{n_0 + \ldots + n_{i-1}} \times \mathbb{R}^{n_i + \ldots + n_k}$ and any point $p \in \mathbb{R}^{n_0 + \ldots + n_{i-1}}$ defines you a leaf of $\mathcal{F}_i$. The Lie algebra $W(n_0, \ldots, n_k)$ consists of formal vector fields that infinitesimally preserve all foliations near the origin (see section 1.2 for description of this Lie algebra in terms of coordinates in $\mathbb{R}^n$). Consider the maximal reductive subalgebra in the space of linear vector fields that preserve the aforementioned flag. This subalgebra is isomorphic to the direct sum of matrix algebras $\mathbf{gl}_{n_0} \oplus \ldots \oplus \mathbf{gl}_{n_k}$. We state the answer for the relative cohomology first.

Theorem (Cor.2.2.16 below)**.** *Relative cohomology of the Lie algebra $W(n_0, \ldots, n_k)$ (relatively to sub-algebra $\mathbf{gl}_{n_0} \oplus \ldots \oplus \mathbf{gl}_{n_k}$) are different from zero only in even degrees and coincides with the algebra of $\mathbf{gl}_{n_0} \oplus \ldots \oplus \mathbf{gl}_{n_k}$ invariants in the factor-algebra of the symmetric algebra $S^\bullet(\mathbf{gl}_{n_0} \oplus \ldots \oplus \mathbf{gl}_{n_k})$ by the ideal generated by the set of subspaces $S^{n_0 + \ldots + n_r + 1}(\mathbf{gl}_{n_0} \oplus \ldots \oplus \mathbf{gl}_{n_r})$ for $r = 0, \ldots, k$. The latter ideal is denoted by $I_{n_0, \ldots, n_k}$.*

In particular, we have the following equivalences of vector spaces indexed by homological degree i

$$\mathrm{H}^i(W(n_0, \ldots, n_k), \mathbf{gl}_{n_0} \oplus \ldots \oplus \mathbf{gl}_{n_k}; \Bbbk) = \begin{cases} 0, & \text{if } i = 2k + 1, \\ \left[\frac{S^k(\mathbf{gl}_{n_0} \oplus \ldots \oplus \mathbf{gl}_{n_k})}{I_{n_0, \ldots, n_k}} \right]^{\mathbf{gl}_{n_0} \oplus \ldots \oplus \mathbf{gl}_{n_k}}, & \text{if } i = 2k. \end{cases}$$

The union of these identities gives a graded isomorphism of the corresponding rings.

The theory of $\mathbf{gl}$–invariants provides the following description of the algebra of $\mathbf{gl}_{n_0} \oplus \ldots \oplus \mathbf{gl}_{n_k}$-invariants:

$$\left[S^\bullet(\mathbf{gl}_{n_0} \oplus \ldots \oplus \mathbf{gl}_{n_k}) \right]^{\mathbf{gl}_{n_0} \oplus \ldots \oplus \mathbf{gl}_{n_k}} = \Bbbk \left[\Psi_{01}, \ldots, \Psi_{0n_0}; \ldots; \Psi_{k1}, \ldots, \Psi_{kn_k} \right].$$

Namely, the aforementioned ring is the free commutative algebra generated by the set of generators $\{\Psi_{ij}\}$, index i ranges from 0 to k and j ranges from 1 to n_i. The degree of the generator Ψ_{ij} coincides with the second index j. The intersection of the ideal $I_{n_0, \ldots, n_k}$ with the subalgebra of invariants is generated by monomials $(\Psi_{01}^{\alpha_{01}} \ldots \Psi_{0n_0}^{\alpha_{0n_0}}) \ldots (\Psi_{r1}^{\alpha_{r1}} \ldots \Psi_{rn_r}^{\alpha_{rn_r}})$ such that $\sum_{i=0}^{r} \sum_{j=1}^{n_i} \alpha_{ij} > n_0 + \cdots + n_r$, where the index r ranges from 0 to k as above.

We do not mention here how one should compute the absolute cohomology of the Lie algebra $W(n_0, \ldots, n_k)$ and refer these discussions to Section 2.3 and in particular to Theorem 2.3.18. So far it is enough to state that there exists a straightforward procedure how to get this answer. The particular case $\overline{n} = (1, 1, \ldots, 1)$ where all dimensions are computed in terms of Catalan numbers is considered in Section 4.1.

In order to advertise the description of characteristic classes of flags of foliations via cohomology of the Lie algebra $W(n_1, \ldots, n_k)$ we mention the following particular application.

Let $SL(n + 1, \mathbb{R})$ be a simple Lie Group of square matrices of size $n + 1$ over real numbers with the determinant 1 and Γ its discrete subgroup such that the quotient space $SL(n + 1, \mathbb{R})/\Gamma$ is compact. Let P be a subgroup that fixes a given line in $\mathbb{R}^{n+1}$. Then P acts on $SL(n + 1, \mathbb{R})/\Gamma$ by left translations and the orbits of this action defines a foliation $\mathcal{F}_\mathsf{P}$ of codimension n. The following Corollary is proved in Section 2.4:

Corollary (Statement 2.4.24). *The foliation $\mathcal{F}_\mathsf{P}$ on $SL(n+1,\mathbb{R})/\Gamma$ can not be a subfoliation. I.e., there is a cohomological obstruction to the existence of a flag of foliations $\mathcal{F}_1 \supset \mathcal{F}_2$ on $SL(n+1,\mathbb{R})/\Gamma$ such that the foliation $\mathcal{F}_2$ is concordant to $\mathcal{F}_\mathsf{P}$.*

0.2 Motivations: formal geometry

In this section we remind the connection between our computations and characteristic classes of different structures which appear naturally in differential geometry and algebraic topology. This bridge is known under the name *Formal geometry*. Below we recall the key idea of formal geometry (see e.g. [15]) which is rather simple and was probably first announced in early 1970's by I. M. Gelfand on his famous seminar. We remind only the construction of characteristic classes of a manifold using Lie algebra cohomology of the infinite-dimensional Lie algebra W_n. All other constructions which comes up with characteristic classes of fibrations or foliations on the manifold are speculations around this main construction. For details we refer to the following literature [12]ch.3,[15],[1],[3].

Indeed, with any complex smooth manifold M of dimension n we assign the infinite-dimensional manifold M^{coor} of formal coordinates on M. I.e. the points of M^{coor} are in one-to-one correspondence with the following pairs: a point $p \in M$ together with a system of formal coordinates in the neibourhood p with the origin in this point p. The manifold M^{coor} may be also described as the space of ∞-jets of submersions $M \to \mathbb{C}^n$ with 0 as a target. From one side, the manifold M^{coor} form a bundle over M whose fibers are homotopy equivalent to GL_n. The corresponding principle GL_n-bundle is isomorphic to the principal bundle associated with the tangent bundle on M. On the other hand, the tangent space at each point M^{coor} is isomorphic to the Lie algebra of formal vector fields W_n. The union of these isomorphisms defines a map from the Chevalley complex of the Lie algebra W_n to the De Rham complex of M^{coor}. The relative version of this morphism is a map: $ch : C^\bullet(W_n, \mathbf{gl}_n; \mathbb{C}) \to \Omega^\bullet_{DR}(M^{coor}/GL_n)$ If we pass to the cohomology we get the following characteristic map:

$$ch : \mathrm{H}^\bullet(W_n, \mathbf{gl}_n; \mathbb{C}) \to \mathrm{H}^\bullet(M^{coor}/GL_n) = \mathrm{H}^\bullet(M; \Bbbk).$$

This map is known as the construction of characteristic classes of tangent bundle using the Lie algebra cohomology of infinite-dimensional Lie algebras. The same construction works for real manifolds. The main difference is that one has to replace the group $GL_n(\mathbb{R})$ by it's compact subgroup $\mathsf{O}(n)$ of orthogonal matrices.

One can follow the same procedure on manifolds with additional differential geometric data. For example, if a manifold M admits a foliation $\mathcal{F}$ of codimension d then one can consider the space of formal coordinates on leaves of this foliation $\mathcal{F}$. The tangent space to the corresponding infinite-dimensional manifold will be once again isomorphic to the Lie algebra W_d. This provides the way how one can get the characteristic classes of foliations generalizing the Godbillon-Vey characteristic class. Similar construction may be given if one is interested in the characteristic classes of G-fibration over M with a compact group G (see e.g. [20]). This note was started by considering the following constructions of the same kind:

1. Let us fix the dimension n of a manifold and fix a codimension d of a foliation. With each foliation with prescribed dimensions we assign the infinite-dimensional manifold of charts on the underlying manifold such that the first $n - d$ coordinates should be the coordinates on a leaf.

   In particular, the above situation is interesting even in the case of fibration $f : S \mapsto M$ where the fiber over each point $p \in M$ is a compact manifold of dimension $n - d$. (See e.g.[10] for details and [8, 13, 14] for applications to the local Riemann-Roch theorem).

2. Consider a flag of foliations with a given collection of codimensions. The assigned infinite-dimensional manifold will consists of charts preserving leaves. I.e. the first n_1 coordinates will be the coordi-

nates on a leaf of the foliation with smallest codimension, $n_1 + n_2$ will be coordinates on a leaf of the second foliation and so on. (See [9] for detailed construction.)

The corresponding infinite-dimensional Lie algebra in first example is known to coincide with $W(n - d, d)$. In the second example one has to compute the cohomology of the Lie algebra $W(n_1, \ldots, n_k)$, where $(n_1, \ldots, n_k)$ is the collection of codimensions. All these cohomology problems are covered in this note.

0.3 Outline of the paper

The paper is organized as follows:

Section 1 contains some standard definitions (Subsection 1.1), the description of all infinite-dimensional Lie algebras whose cohomology we compute (Subsection 1.2) and the description of the acyclic Weyl superalgebra (Subsection 1.3).

Section 2 contains statements of all main results of this paper. We start from Subsection 2.1 where we recall the classical result by Gelfand and Fuchs and it's generalization due to the author. In Subsection 2.2 we formulate main Theorems 2.2.10 and 2.2.14 on relative cohomology of Lie algebras $WL(m|n)$ and $W(n_1, \ldots, n_k)$. In Subsection 2.3 we explain the difference between relative and absolute cohomology. A particular cohomological obstruction to have a flag of foliations is explained in Subsection 2.4.

Section 3 contains the proof of the main Theorem 2.2.10. We state several homological results on relative homology of parabolic subalgebra in Subsection 3.1, however, the proofs of these results are postponed to Appendix B. In Subsection 3.2 we use these results in order to describe the Hochschild-Serre spectral sequences assigned with embeddings of parabolic subalgebra into the infinite-dimensional Lie algebras of formal vector fields. In Subsection 3.3 we show how does the degenerations of aforementioned Hochschild-Serre spectral sequences implies main Theorem 2.2.10.

We illustrate the combinatorics of our main result in two particular cases in Section 4. Namely, we show the description of the representing cocycles in the absolute case $W(1, \ldots, 1)$ in Subsection 4.1. We present a simple formulas for the generating cocycles for the Lie algebra cohomology $\mathrm{H}^\bullet(W_n; S^m W_n^*)$ in Subsection 4.2 when $n = 1$. The description of generating cocycles for arbitrary n requires more definitions and, therefore, is presented in Appendix A.3.

Appendix A is concentrated on a careful description of the relative chains and cocycles for infinite-dimensional Lie algebras under consideration.

Apendix B contains general results on cohomology of parabolic Lie subalgebras based on the existence of BGG resolution. The particular case of upper block-triangular matrices leads to Lemmas 3.1.26 and 3.1.28 which was used in the proof of main Theorem 2.2.10.

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1 notations and recollections

1.1 Lie algebra cohomology

The Chevalley–Eilenberg complex $C^\bullet(\mathbf{g}; \Bbbk)$ of a Lie algebra $\mathbf{g}$ is a free differential graded commutative algebra generated by the dual vector space $\mathbf{g}^*$ shifted by 1 with the differential defined on the generators as a map $d : \mathbf{g}^* \to \Lambda^2 \mathbf{g}^*$ linear dual to the commutator in the Lie algebra $\mathbf{g}^*$. The cohomology of this complex counts the cohomology of the Lie algebra $\mathbf{g}$ with trivial coefficients. We refer the reader to the

main monograph [12] on this subject for detailed definitions of cohomologies with arbitrary coefficients. The same monograph also contains the description of most frequently used method of computations and the particular computation for infinite-dimensional Lie algebras. A review of the definition of homology of Lie algebras in terms of derived functors may be found in textbooks on homological algebra (see e.g. [29] ch.7).

In this article we are mostly interested in the Lie algebra cohomology of infinite-dimensional Lie algebras. There are several ways on how to avoid the problems related to the finiteness conditions. One way is to consider a grading on a Lie algebra such that the graded components are finite-dimensional. Then all considered modules should be graded and the duals should be also graded duals. Another possibility is to consider the topology on a Lie algebra. This produces the following changes in the definitions. The duals should be considered as continuous duals. In particular, the Chevalley-Eilenberg complex $C^\bullet(\mathbf{g}; L)$ of a topological Lie algebra $\mathbf{g}$ consists of continuous skew-symmetric maps from the several copies of $\mathbf{g}$ to a topological module L.

The main computational method we use below is the spectral sequences by Hochschild and Serre. Let us briefly recall the nature of this spectral sequence. Consider an embedding of Lie algebras $i : \mathbf{h} \hookrightarrow \mathbf{g}$ and choose a decomposition $\mathbf{g} \simeq \mathbf{h} \oplus \mathbf{g}/\mathbf{h}$ as $\mathbf{h}$-modules. Then there is a canonical filtration on the Chevalley-Eilenberg complex of the Lie algebra $\mathbf{g}$ associated with this embedding i:

$$C^\bullet(\mathbf{g}; L) = Hom(\Lambda^\bullet \mathbf{g}; L) \supset Hom(\Lambda^\bullet \mathbf{g} \otimes \Lambda^1 \mathbf{g}/\mathbf{h}; L) \supset \ldots \supset Hom(\Lambda^\bullet \mathbf{g} \otimes \Lambda^k \mathbf{g}/\mathbf{h}; M) \supset \ldots \quad (1.1.1)$$

The spectral sequence associated with this filtration is called the Hochschild-Serre spectral sequence and has the following first term $E_1^{p,q} = \mathrm{H}^q(\mathbf{h}; Hom(\Lambda^p(\mathbf{g}/\mathbf{h}); M))$.

We do all computations over the field $\Bbbk$ of zero characteristic. All reasonable applications deals with the case of complex or real numbers. There is no difference between complex and real numbers for the algebraic counterpart of this paper. However, one has to separate these two cases while working with applications via formal geometry for complex or real smooth manifolds respectively.

1.2 Lie algebras of formal vector fields and their subalgebras

The main infinite-dimensional Lie algebra in this article is a Lie algebra of formal vector fields on the n-dimensional plane. We use the standard notation W_n following textbook [12].

Recall the definition. Let us fix coordinates $x_1, \ldots, x_n$ on the n-dimensional plane $\Bbbk^n$. The ring of ∞-jets at the origin $0 \in \Bbbk^n$ coincides with the ring of formal power series $\Bbbk[[x_1, \ldots, x_n]]$ on n variables. We will also use shorter notation $\mathcal{O}_n$ for the same ring. The infinite-dimensional Lie algebra of derivations of this ring is called the Lie algebra of formal vector fields on $\Bbbk^n$ and is denoted by W_n. Any vector field $\nu \in W_n$ admits presentation in coordinates $\nu = \sum_{i=1}^n \nu_i \frac{\partial}{\partial x_i}$, where $\nu_i \in \mathcal{O}_n$. The subspace of vector fields with linear coefficients $\left\{ \sum_{i,j} c_{ij} x_i \frac{\partial}{\partial x_j} \right\}$ form a Lie subalgebra isomorphic to the Lie algebra $\mathbf{gl}_n = \mathbf{gl}_n(\Bbbk)$ of $n \times n$-matrices.

1.2.1 Extension by g-valued functions

With any Lie algebra $\mathbf{g}$ with a commutator $[,]_{\mathbf{g}}$ we can assign a new infinite-dimensional Lie algebra $W_n \ltimes \mathbf{g} \otimes \mathcal{O}_n$ (denoted later by $W(n; \mathbf{g})$). This algebra is a semi-direct product of the Lie algebra of formal vector fields and the Lie algebra of $\mathbf{g}$-valued formal power series on $\Bbbk^n$. In particular, the commutator in this algebra looks as follows:

$$[\eta_1 + g_1 \otimes p_1, \eta_2 + g_2 \otimes p_2] = [\eta_1, \eta_2] + [g_1, g_2]_{\mathbf{g}} \otimes p_1 p_2 + g_2 \otimes \eta_1(p_2) - g_1 \otimes \eta_2(p_1), \quad (1.2.2)$$

where $\eta_i \in W_n$, $g_i \in \mathbf{g}$, $p_i \in \mathcal{O}_n$ with $i \in \{1, 2\}$. This algebra was used in [20] in order to define the characteristic classes of G-bundles (whenever G is a compact group) using the constructions of formal geometry.

1.2.2 Vector fields preserving foliation structures

Let us substitute in the example from the previous Section 1.2.1 instead of $\mathbf{g}$ the infinite-dimensional Lie algebra W_m which does not admit the Lie group at all. However, the geometrical meaning of the Lie algebra $W_n \ltimes W_m \otimes \mathcal{O}_n$ is clear. Indeed, the Lie algebra $W_n \ltimes W_m \otimes \mathcal{O}_n$ is a subalgebra of the Lie algebra W_{n+m} of formal vector fields on $\Bbbk^{n+m}$ consisting of those vector fields that preserve the trivial foliation $\Bbbk^n \times \Bbbk^m$. Where for any $p \in \Bbbk^m$ the product $\Bbbk^n \times \{p\}$ is considered to be a leaf of this foliation. In other words, a vector field $\nu = \sum_{i=1}^{n+m} \nu_i \frac{\partial}{\partial x_i}$ belongs to $W_n \ltimes W_m \otimes \mathcal{O}_n$ whenever for all i from 1 to n the formal power series ν_i does not depend on $x_{n+1}, \ldots, x_{n+m}$.

Let us also recall the generalization of this Lie algebra to the case of a flag of foliations. Namely, let us fix a collection of integers $\overline{n} = (n_0, n_1, \ldots, n_k)$ and consider a collection of trivial embedded foliations $\{\mathcal{F}_1 \supset \ldots \supset \mathcal{F}_k\}$ with prescribed codimensions. Namely, $\mathcal{F}_i := \Bbbk^{n_0 + \ldots + n_{i-1}} \times \Bbbk^{n_i + \ldots + n_k}$ and each leaf of $\mathcal{F}_i$ is equal to $p \times \Bbbk^{n_i + \ldots + n_k}$ for appropriate point $p \in \Bbbk^{n_0 + \ldots + n_{i-1}}$. The Lie algebra $W(n_0, n_1, \ldots, n_k)$ is a Lie subalgebra of $W_{n_0 + n_1 + \ldots + n_k}$ consisting of those vector fields that preserve the aforementioned flag of foliations. In other words, a vector field $\nu = \sum_{i=1}^{n+m} \nu_i \frac{\partial}{\partial x_i}$ belongs to $W(n_0, n_1, \ldots, n_k)$ if and only if for all i the formal power series ν_i depends only on variables $x_1, \ldots, x_{n_0 + \ldots + n_r}$ where r is the integer defined by the inequality $n_0 + \ldots + n_{r-1} < i < n_0 + \ldots + n_r$. $W(n_0, n_1, \ldots, n_k)$ contains a direct sum of matrix subalgebras $\mathbf{gl}_{n_r}$ where r ranges from 0 to k. Each $\mathbf{gl}_{n_r}$ is generated by the fields $x_i \frac{\partial}{\partial x_j}$ where i, j satisfy for some r the inequality $n_0 + \ldots + n_{r-1} < i, j < n_0 + \ldots + n_r$.

1.2.3 Vector fields that are linear in the normal direction to the leaves of a foliation

We also want to give special notations for another class of subalgebras of vector fields. These Lie algebras are very useful for our homological computations:
Define $WL(n|m)$ as a subalgebra of $W(m, n)$ consisting of those fields $\nu = \sum_{i=1}^{n+m} \nu_i \frac{\partial}{\partial x_i}$ such that for all $i = 1, \ldots, n$ the power series ν_i is a polynomial of degree no more than 1. I.e. the intersection of $WL(n|m)$ with W_n is a subspace of W_n spanned by linear and constant vector fields. Similarly, one can define the Lie algebra $WL(n_0, \ldots, n_k | n_{k+1}, \ldots, n_{k+l})$ by the same property. We say that a vector field $\nu = \sum_{i=1}^{n_0 + \ldots + n_k} \nu_i \frac{\partial}{\partial x_i}$ from $W(n_0, \ldots, n_{k+l})$ belongs to $WL(n_0, \ldots, n_k | n_{k+1}, \ldots, n_{k+l})$ if all ν_i are linear or constant for all integer i from the interval $[1, n_0 + \ldots + n_k]$.

Analogously to the case of one group of variables we suggest to consider the extension of the Lie algebra of vector fields preserving a flag by the Lie algebra of $\mathbf{g}$-valued functions. Let $W(n_1, \ldots, n_k; \mathbf{g})$ be the semi-direct product $W(n_1, \ldots, n_k) \ltimes \mathbf{g} \otimes \mathcal{O}_{n_1 + \ldots + n_k}$ with the commutator defined analogously to (1.2.2). Let $WL(n_1, \ldots, n_k | n_{k+1}, \ldots, n_{k+l}; \mathbf{g})$ be it's subalgebra whose intersection with $W_{n_1 + \ldots + n_k}$ consists of linear or constant vector fields.

1.3 Weyl superalgebra

In this section we recall a standard construction from homological algebra. The Weyl Lie superalgebra described below is acyclic, but has a standard filtration such that the corresponding spectral sequence coincides in some cases with the one coming from the universal bundle over the classifying space. One can define characteristic classes of fibrations using a filtered map from a Weyl superalgebra. The details of this construction may be found, for example, in [12, 20].

Let $\mathbf{g}$ be a given Lie algebra. Consider the differential-graded Lie superalgebra $\mathbf{g}[1] \overset{Id}{\to} \mathbf{g}$. Both odd and even parts of this algebra are isomorphic to $\mathbf{g}$. The commutator on the even part $\mathbf{g}$ coincides with the commutator in the initial Lie algebra $\mathbf{g}$, the restriction of the commutator on the odd part is zero and the remaining commutators $\mathbf{g} \otimes \mathbf{g}[1] \to \mathbf{g}[1]$ are prescribed by the adjoint action of $\mathbf{g}$ on itself. This algebra is $\mathbb{Z}$-graded: even part differs from zero in degree 0, odd part differs from zero only in degree -1.

The differential is the identity map $Id : \mathbf{g}[1] \to \mathbf{g}$:

$$\mathbf{g}[1] \overset{Id}{\to} \mathbf{g} := \ldots \longrightarrow 0 \longrightarrow \mathbf{g} \overset{Id}{\longrightarrow} \mathbf{g} \longrightarrow 0 \longrightarrow \ldots$$

The Chevalley-Eilenberg complex $C^{\bullet}(\mathbf{g}[1] \overset{Id}{\to} \mathbf{g})$ of this Lie super-algebra is called *the Weyl superalgebra of* $\mathbf{g}$ and is denoted by $\mathcal{W}^{\bullet}(\mathbf{g})$. The Weyl algebra $\mathcal{W}^{\bullet}(\mathbf{g})$ is acyclic because the initial Lie algebra $\mathbf{g}[1] \overset{Id}{\to} \mathbf{g}$ has no cohomology at all. However, this complex $\mathcal{W}^{\bullet}(\mathbf{g})$ is interested as a filtered complex. Indeed, consider the Hochschild-Serre filtration associated with the embedding of the initial Lie algebra $\mathbf{g}$ into $\mathbf{g}[1] \overset{Id}{\to} \mathbf{g}$. We call this filtration *standard* and denote it by $F^{\bullet}\mathcal{W}^{\bullet}(\mathbf{g})$. In particular, we have the following isomorphisms for the associated graded terms:

$$F^{2k}\mathcal{W}^{\bullet}(\mathbf{g})/F^{2k+1}\mathcal{W}^{\bullet}(\mathbf{g}) = C^{\bullet}(\mathbf{g}; S^k\mathbf{g}^*)[-2k] \quad \text{and} \quad F^{2k-1}\mathcal{W}^{\bullet}(\mathbf{g})/F^{2k}\mathcal{W}^{\bullet}(\mathbf{g}) = 0$$

I.e. for even numbers the associated graded complex coincides with the Chevalley-Eilenberg complex of the initial Lie algebra $\mathbf{g}$ with coefficients in the symmetric power of the coadjoint representation. The associated graded complexes for odd numbers of filtration are empty.

Remark 1.3.3. Let $\mathbf{g}$ be a semi-simple Lie algebra and therefore admits a compact Lie group G. The spectral sequence associated with the standard filtration on the Weyl algebra $\mathcal{W}^{\bullet}(\mathbf{g})$ is an example of transgression and coincides with the Borel-Moore spectral sequence associated with the universal fibration: $EG \overset{G}{\to} BG$. In particular, the cohomology of BG differs from zero only in even degrees and the even cohomology $\mathrm{H}^{2n}(BG)$ coincides with the space of invariants $[S^n\mathbf{g}^*]^{\mathbf{g}}$ of the symmetric power of the coadjoint representation.

Similarly to the Chevalley-Eilenberg complex the Weyl algebra defines a contravariant functor from the category of Lie algebras to the category of filtered dg-algebras. In particular, any morphism $\varphi : \mathbf{g} \to \mathbf{h}$ of Lie algebras defines a morphism of corresponding Weyl superalgebras $\mathcal{W}(\varphi) : \mathcal{W}^{\bullet}(\mathbf{h}) \to \mathcal{W}^{\bullet}(\mathbf{g})$ compatible with standard filtrations. Moreover, the acyclicity of Weyl algebras implies the possibility to define a map in the opposite direction. Namely, let $\mathbf{h}$ be a subalgebra of $\mathbf{g}$. Let φ be a possible inverse map $\mathbf{g} \to \mathbf{h}$, φ not need to be a map of Lie algebras but should be a map of $\mathbf{h}$-modules. Then there is a unique way how to associate the map of filtered complexes from $\mathcal{W}^{\bullet}(\mathbf{h}) \to C^{\bullet}(\mathbf{g}; \Bbbk)$ where the filtration on Weyl superalgebra is standard and the filtration on Chevalley-Eilenberg complex $C^{\bullet}(\mathbf{g}; \Bbbk)$ is the Hochschild-Serre filtration (1.1.1) associated with the embedding $\mathbf{h} \hookrightarrow \mathbf{g}$.

1.3.1 Relative case

Let $\mathbf{h}$ be a Lie subalgebra of $\mathbf{g}$. With any $\mathbf{g}$-module M one can assign the relative Chevalley-Eilenberg complex $C^{\bullet}(\mathbf{g}, \mathbf{h}; M)$ that consists of those chains $c \in \mathrm{Hom}(\Lambda^{\bullet}\mathbf{g}, M)$ such that both c and its differential $d_{CE}(c)$ vanish while restricting at least one of the arguments on $\mathbf{h}$.

We can apply the same procedure for the Weyl super-algebra. Indeed, we define *the relative Weyl superalgebra* $\mathcal{W}^{\bullet}(\mathbf{g}, \mathbf{h})$ to be the relative Chevalley-Eilenberg complex of the Lie superalgebra $\mathbf{g}[1] \overset{Id}{\to} \mathbf{g}$ relatively to the Lie subalgebra $(0 \to \mathbf{h})$. In particular, the components of the associated graded complex with respect to the standard filtration looks as follows:

$$F^{2k}\mathcal{W}^{\bullet}(\mathbf{g}, \mathbf{h})/F^{2k+1}\mathcal{W}^{\bullet}(\mathbf{g}, \mathbf{h}) = C^{\bullet}(\mathbf{g}, \mathbf{h}; S^k\mathbf{g}^*)[-2k] \quad \text{and} \quad F^{2k-1}\mathcal{W}^{\bullet}(\mathbf{g}, \mathbf{h})/F^{2k}\mathcal{W}^{\bullet}(\mathbf{g}, \mathbf{h}) = 0 \quad (1.3.4)$$

Note that relative Weyl dg-algebra is no more acyclic. However, if $\mathbf{h}$ is semi-simple and admits a compact group H then the cohomology of the relative Weyl algebra $\mathcal{W}^{\bullet}(\mathbf{g}, \mathbf{h})$ should be equal to the cohomology of the classifying space $BH \simeq EH/H \simeq \{point\}/H$. That is the total cohomology $\mathrm{H}^{\bullet}(\mathcal{W}^{\bullet}(\mathbf{g}, \mathbf{h}))$ are even and coincides with the ring of invariants $[S^{\bullet}\mathbf{h}]^{\mathbf{h}}$.

2 Relative chains versus truncated Weyl super-algebras

In this section, first, we recall the classical results from [16] concerning the description of the relative cohomology ring of the Lie algebra W_n modulo subalgebra of linear vector fields $\mathbf{gl}_n$ and the similar results obtained by the author in [20] on cohomology ring of the Lie algebra $W_n \ltimes \mathbf{g} \otimes \mathcal{O}_n$. Second, we consider a particular case of the Lie algebras $W(m,n)$ and state similar results for the Lie algebra $WL(m|n)$. Third, we state the main general results which we prove in the next Section 3. All results in this section either follows from careful description of relative chains based on hunting $\mathbf{gl}$-invariants or follows from the theorems from the next section. In both cases we will give direct links either to Appendix A or to Section 3.

2.1 known results

Recall one of the first cohomological computation in formal geometry due to Gelfand and Fuchs:

Theorem 2.1.5. *([16]) The space of relative cochains of formal vector fields W_n with constant coefficients is isomorphic to the truncated cohomology of the classifying space BSU_n, in particular there are no relative cochains of odd degree:*

$$\frac{\mathrm{H}^{\bullet}(BSU_n)}{\mathrm{H}^{>2n}(BSU_n)} = \frac{[S^{\bullet}(\mathbf{gl}_n)]^{\mathbf{gl}_n}}{[S^{>n}(\mathbf{gl}_n)]^{\mathbf{gl}_n}} = \frac{\mathcal{W}^{\bullet}(\mathbf{gl}_n, \mathbf{gl}_n)}{F^{2n+1}\mathcal{W}^{\bullet}(\mathbf{gl}_n, \mathbf{gl}_n)} \xrightarrow{\simeq} C^{\bullet}(W_n, \mathbf{gl}_n; \Bbbk)$$

This computation is based on the description of relative chains as $\mathbf{gl}_n$-invariants. We refer to the original paper [16] and to the Appendix A.2 for details. The similar computation with $\mathbf{gl}_n$-invariants was done by the author in [20] and leads to the following result:

Theorem 2.1.6. *([20]) The ring of relative chains of the Lie algebra of formal vector fields extended by $\mathbf{g}$-valued functions coincides with the quotient of the relative Weyl superalgebra $\mathcal{W}^{\bullet}(\mathbf{gl}_n \oplus \mathbf{g}, \mathbf{gl}_n)$ modulo the $(2n+1)$-st part of standard filtration:*

$$\mathcal{W}^{\bullet}(\mathbf{gl}_n \oplus \mathbf{g}, \mathbf{gl}_n)/F^{2n+1}\mathcal{W}^{\bullet}(\mathbf{gl}_n \oplus \mathbf{g}, \mathbf{gl}_n) \xrightarrow{\simeq} C^{\bullet}(W_n \ltimes \mathbf{g} \otimes \mathcal{O}_n, \mathbf{gl}_n; \Bbbk).$$

We will go further and state the analogous result for the Lie subalgebra of constant and linear vector fields extended by $\mathbf{g}$-valued functions. In Section 1.2.3 these algebras are denoted by $WL(m|0; \mathbf{g})$ or $WL(m|; \mathbf{g})$ for simplicity.

Theorem 2.1.7. *The relative chain complex of the Lie algebra $WL(m|0; \mathbf{g})$ is isomorphic to the truncated Weyl superalgebra of the Lie algebra $\mathbf{g}$:*

$$\mathcal{W}^{\bullet}(\mathbf{g})/F^{2m+1}\mathcal{W}^{\bullet}(\mathbf{g}) \xrightarrow{\simeq} C^{\bullet}(WL(m|0; \mathbf{g}), \mathbf{gl}_m; \Bbbk). \tag{2.1.8}$$

In particular, we can substitute the Lie algebra of vector fields on $\Bbbk^n$ as a possible $\mathbf{g}$ and get the following:

$$\mathcal{W}^{\bullet}(W_n, \mathbf{gl}_n)/F^{2n+1}\mathcal{W}^{\bullet}(W_n, \mathbf{gl}_n) \xrightarrow{\simeq} C^{\bullet}(WL(m|n), \mathbf{gl}_m \oplus \mathbf{gl}_n; \Bbbk). \tag{2.1.9}$$

This observation becomes crucial for further implications. Namely, the computation of the cohomology of the Lie algebra $WL(m|n)$ for different m will be enough in order to get the description of the spectral sequence associated with the standard filtration on the relative Weyl algebra $\mathcal{W}^{\bullet}(W_n, \mathbf{gl}_n)$.

2.2 Main theorems

Denote by π the natural surjection $WL(m|n) \twoheadrightarrow W_n \twoheadrightarrow \mathbf{gl}_n$ of $\mathbf{gl}_n$-modules.

Theorem 2.2.10. *The morphism from Weyl superalgebras associated with the projection π leads to the following quasi-isomorphism of dg-algebras:*

$$\mathrm{H}^\bullet(BSU_n)/\mathrm{H}^{\geqslant 2(n+m)+1}(BSU_n) = \mathcal{W}^\bullet(\mathbf{gl}_n, \mathbf{gl}_n)/F^{2(n+m)+1}\mathcal{W}^\bullet(\mathbf{gl}_n, \mathbf{gl}_n) \xrightarrow{quis}$$
$$\xrightarrow{quis} C^\bullet(WL(m|n), \mathbf{gl}_m \oplus \mathbf{gl}_n; \Bbbk). \quad (2.2.11)$$

In other words, the relative cohomology of the Lie algebra $WL(m|n)$ is isomorphic to the truncated ring of polynomials $\Bbbk^{\bullet \leqslant 2(m+n)}[\Psi_2, \ldots, \Psi_{2n}]$ of degrees less or equal than $2m + 2n$. Where the polynomial ring is generated by n even variables $\Psi_2, \ldots, \Psi_{2n}$ with $\deg \Psi_{2i} = 2i$.

Proof. The proof of this theorem is postponed to Section 3 (page 18). In this section we will only explain why this result implies all others. $\qquad\square$

Together with Equation (2.1.9) we get the main corollary of this paper:

Corollary 2.2.12. *For all $m \geqslant 1$ the relative cohomology of the Lie algebra W_n with coefficients in the m-th symmetric power of the coadjoint representation are different from zero only in degree $2n$ and is equal to the space of invariants $[S^{m+n}(\mathbf{gl}_n)]^{\mathbf{gl}_n}$:*

$$\mathrm{H}^{2n}(W_n, \mathbf{gl}_n; S^m(W_n^*)) = [S^{m+n}(\mathbf{gl}_n)]^{\mathbf{gl}_n}, \quad and \quad \mathrm{H}^{i \neq 2n}(W_n, \mathbf{gl}_n; S^m(W_n^*)) = 0$$

Proof. The isomorphism (2.1.9) between the Chevalley-Eilenberg complex of the Lie algebra $WL(m|n)$ and the truncated Weyl superalgebra for W_n implies the following short exact sequence of complexes that relates two subsequent values of m:

$$0 \to \frac{F^{2m-1}\mathcal{W}^\bullet(W_n, \mathbf{gl}_n)}{F^{2m+1}\mathcal{W}^\bullet(W_n, \mathbf{gl}_n)} \to C^\bullet(WL(m|n), \mathbf{gl}_m \oplus \mathbf{gl}_n; \Bbbk) \xrightarrow{p_m} C^\bullet(WL(m-1|n), \mathbf{gl}_{m-1} \oplus \mathbf{gl}_n; \Bbbk) \to 0$$

We denote the surjective map by p_m. As was mentioned in (1.3.4) the standard filtration on the Weyl algebra has the same odd and even parts: $F^{2m} = F^{2m-1}$. and the associated graded F^{2m}/F^{2m+1} is isomorphic to the Chevalley-Eilenberg complex with coefficients in the m-th symmetric power of the coadjoint representation. Theorem 2.2.10 implies that relative cohomology of the Lie algebra $WL(m|n)$ is the truncation of the polynomial ring $\mathrm{H}^\bullet(BSU_n) = \Bbbk[\Psi_2, \ldots, \Psi_{2n}]$ by polynomials of degree $2(m + n)$. Therefore, the map p_m is surjective on cohomology rings and is also the truncation by polynomials of degree $2(m - 1 + n)$. Hence, the map p_m should be surjective on cohomology rings and we get a short exact sequence of cohomologies:

$$\mathrm{H}^\bullet(W_n, \mathbf{gl}_n; S^m W_n^*)[2m] \hookrightarrow \mathrm{H}^\bullet(WL(m|n), \mathbf{gl}_m \oplus \mathbf{gl}_n; \Bbbk) \xrightarrow{p_m} \mathrm{H}^\bullet(WL(m-1|n), \mathbf{gl}_{m-1} \oplus \mathbf{gl}_n; \Bbbk)$$
$$\downarrow \simeq \qquad\qquad\qquad \downarrow \simeq$$
$$\Bbbk^{\bullet \leqslant 2(m+n)}[\Psi_2, \ldots, \Psi_{2n}] \twoheadrightarrow \Bbbk^{\bullet \leqslant 2(m-1+n)}[\Psi_2, \ldots, \Psi_{2n}]$$

Thus, we conclude that the shifted cohomology of the cochain complex $C^\bullet(W_n, \mathbf{gl}_n; S^m W_n^*)[2m]$ are different from zero only in degree $2(m + n)$ and coincides with the polynomials of degree $2(m + n)$. Making the opposite homological shift we get the following answer:

$$\mathrm{H}^i(W_n, \mathbf{gl}_n; S^m W_n^*) = \begin{cases} \mathrm{H}^{2(m+n)}(BSU_n) = [S^{m+n}(\mathbf{gl}_n)]^{\mathbf{gl}_n} = \\ \{\text{polynomials of degree } 2(n+m) \text{ in } \Bbbk[\Psi_2, \ldots, \Psi_{2n}]\}, & \text{if } i = 2n, \\ 0, \text{ otherwise.} \end{cases} \quad (2.2.13)$$

$\qquad\square$

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We also state here the generalization of Theorem 2.2.10 to the Lie algebra related to a flag of foliations with given codimensions.

Theorem 2.2.14. *For any given collection of codimensions $(m; n_1, \ldots, n_k$, arbitrary Lie algebra $\mathbf{g}$ and a $(\mathbf{gl}_m \oplus \mathbf{gl}_{n_1} \oplus \ldots \oplus \mathbf{g})$-equivariant projection $\pi : WL(m|n_1, \ldots, n_k; \mathbf{g}) \twoheadrightarrow \mathbf{gl}_{n_1} \oplus \ldots \mathbf{gl}_{n_k} \oplus \mathbf{g}$ we get the following quasi-isomorphism (2.2.15) between the quotient of the Weyl algebra and the Chevalley-Eilenberg complex of the Lie algebra of vector fields:*

$$
\frac{\mathcal{W}^\bullet(\mathbf{gl}_{n_1} \oplus \ldots \oplus \mathbf{gl}_{n_k} \oplus \mathbf{g}, \mathbf{gl}_{n_1} \oplus \ldots \oplus \mathbf{gl}_{n_k})}{I_{n_1,\ldots,n_k}(\mathbf{g})} \xrightarrow{quis} \mathcal{W}^\bullet(W(n_1, \ldots, n_k; \mathbf{g}), \mathbf{gl}_{n_1} \oplus \ldots \oplus \mathbf{gl}_{n_k})/F^{2m+1} \qquad (2.2.15)
$$

$$
\Big\| \qquad\qquad \searrow^{quis} \qquad \downarrow^{quis}
$$

$$
\frac{\mathrm{H}^\bullet(BSU_{n_1} \times \ldots \times BSU_{n_k}) \otimes \mathcal{W}^\bullet(\mathbf{g})}{I_{n_1,\ldots,n_k}} \qquad\qquad C^\bullet(WL(m|n_1, \ldots, n_k; \mathbf{g}), \mathbf{gl}_m \oplus \ldots \mathbf{gl}_{n_k}; \Bbbk).
$$

Where the ideal $I_{n_1,\ldots,n_k}(g)$ is generated by symmetric powers $S^{(m+1+\sum_{i=1}^{j} n_i)}(\mathbf{gl}_{n_1} \oplus \ldots \oplus \mathbf{gl}_{n_j})$ with j ranges from 1 to k and the symmetric power $S^{(m+1+\sum_{i=1}^{k} n_i)}(\mathbf{gl}_{n_1} \oplus \ldots \oplus \mathbf{gl}_{n_k} \oplus \mathbf{g})$. Moreover, quasi-iso (2.2.15) is compatible with filtrations: Standard filtration on Weyl algebra and the Hochschild-Serre filtration on Chevalley-Eilenberg complex. What means that the same result remains valid in the non-relative case (see Theorem 2.3.18).

Proof. The proof repeats the one of Theorem 2.2.10. However, one may prove this theorem by induction on the number k of foliations in the flag. The key ingredient will be Computation (2.2.13) and the consecutive degeneration of Hochschild-Serre spectral sequences based on the following embeddings of Lie algebras:

$$
WL(m|n_1, \ldots, n_k; \mathbf{g}) \supset WL(m|n_1, \ldots, n_k) \supset \ldots \supset WL(m|n_1).
$$

We omit straightforward but rather technical details. $\qquad\square$

In particular, Theorem 2.2.14 describes the cohomology ring of the Lie algebra of vector fields preserving a given flag of foliations.

Corollary 2.2.16. *There exists an isomorphism between the truncated ring of characteristic classes and the relative cohomology ring of the Lie algebra of vector fields preserving a given flag of foliations:*

$$
\frac{\mathrm{H}^\bullet(BSU_{n_1} \times \ldots \times BSU_{n_k})}{I_{n_1,\ldots,n_k}} \xrightarrow{quis} \mathrm{H}^\bullet(W(n_1, \ldots, n_k), \mathbf{gl}_{n_1} \oplus \ldots \oplus \mathbf{gl}_{n_k}; \Bbbk),
$$

where the ideal $I_{n_1,\ldots,n_k}$ is generated by the union of subspaces $\mathrm{H}^{>2(n_1+\ldots+n_r)}(BSU_{n_1} \oplus \ldots \oplus BSU_{n_r})$ for $r = 0, \ldots, k$.

In particular, we have the following equivalences of vector spaces indexed by the homological degree i

$$
\mathrm{H}^i(W(n_1, \ldots, n_k), \mathbf{gl}_{n_1} \oplus \ldots \oplus \mathbf{gl}_{n_k}; \Bbbk) =
\begin{cases}
0, \text{ if } i \text{ is odd,} \\
\left[\dfrac{S^l(\mathbf{gl}_{n_1} \oplus \ldots \oplus \mathbf{gl}_{n_k})}{\left(\prod_{j=1}^{k} S^{n_1+\ldots+n_j+1}(\mathbf{gl}_{n_1} \oplus \ldots \oplus \mathbf{gl}_{n_j}) \right)} \right]^{\mathbf{gl}_{n_1} \oplus \ldots \oplus \mathbf{gl}_{n_k}}, \text{ if } i = 2l.
\end{cases}
$$

The union of these identities gives a graded isomorphism of corresponding rings.

Proof. Substitution $m = 0$ and $\mathbf{g} = 0$ in Theorem 2.2.14. $\qquad\square$

2.3 Absolute case

So far we have been discussing only relative cohomologies because we think that this is the core of the construction. Below we explain what should be done in order to compute the absolute cohomology. The general prescription looks as follows: in all statements from the previous Section 2.2 one has to replace the relative Weyl algebra for matrix Lie algebras by the absolute one's. For example, all maps in Diagram (2.2.15) remains being quasi-iso if one replaces the relative complexes by absolute one's.

Let us state certain main corollaries which we find important for applications in the geometry of foliations:

Theorem 2.3.17. *The absolute cohomology of the Lie algebra of formal vector fields W_n with coefficients in the m'th symmetric power ($m \geqslant 1$) of the coadjoint representation is the tensor product of the relative cohomology and the cohomology $\mathrm{H}^{\bullet}(\mathbf{gl}_n; \Bbbk)$ of the matrix Lie algebra:*

$$\mathrm{H}^i(W_n; S^m W_n^*) = \begin{cases} [S^{n+m}\mathbf{gl}_n]^{\mathbf{gl}_n} \otimes [\Lambda^{i-2n}(\mathbf{gl}_n)]^{\mathbf{gl}_n}, & \text{if } 2n \leqslant i \leqslant n^2 + 2n, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Consider the Hochschild-Serre spectral sequence associated with the canonical embedding $\mathbf{gl}_n \hookrightarrow W_n$. The first term of this sequence is $E_1^{pq} = \mathrm{H}^q(\mathbf{gl}_n; \Bbbk) \otimes \mathrm{H}^p(W_n, \mathbf{gl}_n; S^k W_n^*)$. Computation 2.2.13 implies that for $p \neq 2n$ the corresponding space E_1^{pq} is zero and, therefore, this spectral sequence degenerates in the first term. $\qquad\square$

Theorem 2.3.18. *The absolute cohomology of the Lie algebra of vector fields preserving a given flag of foliations may be computed via the cohomology of the truncated Weyl algebra. We have a quasi-isomorphism:*

$$\frac{\mathcal{W}^{\bullet}(\mathbf{gl}_{n_1} \oplus \ldots \oplus \mathbf{gl}_{n_k})}{I_{n_1,\ldots,n_k}} \xrightarrow{quis} C^{\bullet}(W(n_1,\ldots,n_k); \Bbbk)$$

where the ideal $I_{n_1,\ldots,n_k}$ is generated by symmetric powers $S^{(1+\sum_{i=1}^{j} n_i)}(\mathbf{gl}_{n_1} \oplus \ldots \oplus \mathbf{gl}_{n_j})$ with j ranges from 1 to k.

Proof. The implication from the relative case (Corollary 2.2.16) is standard. The method used in [16, 20] for the same implications related with the cohomology of Lie algebras W_n and $W(n; \mathbf{g})$ respectively is also applied in our situation. $\qquad\square$

In Section 4.1 we will show a particular computation of the generating series of cohomology of a truncated Weyl algebra for the case $n_i = 1$ for all i. From the geometrical point of view this cohomology corresponds to characteristic classes of a full flag of foliations.

2.4 Characteristic classes of flags of foliations

In this subsection we first recall the standard construction of characteristic classes of (flags) of foliations and then show how does our cohomological computation predicts a cohomological obstruction to have a flag of foliations.

As we have mentioned in the introduction the concept of formal geometry assigns to a foliation $\mathcal{F}$ of codimension n on a smooth manifold X a characteristic map

$$ch : \mathrm{H}^{\bullet}(W_n, \mathsf{o}(n); \mathbb{R}) \longrightarrow \mathrm{H}^{\bullet}_{DR}(X)$$

where $\mathsf{o}(n)$ is the Lie algebra of the subgroup of orthogonal matrices of size $n \times n$. If the foliation $\mathcal{F}$ is framed, i.e. the trivialization of the normal bundle is fixed, then the corresponding characteristic classes comes from the characteristic map

$$ch : \mathrm{H}^{\bullet}(W_n; \mathbb{R}) \longrightarrow \mathrm{H}^{\bullet}_{DR}(X)$$

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with the source space being isomorphic to the absolute cohomology of the Lie algebra W_n. For a foliation of codimension 1 relative and absolute characteristic maps coincide and lead to a definition of the Godbillon-Vey class ([19],[3]). Therefore, we call the absolute cohomology by generalized Godbillon-Vey classes.

The same construction assigns to a flag of foliations of codimensions $(n_1, \ldots, n_k)$ on a smooth manifold X a characteristic map

$$ch : \mathrm{H}^\bullet(W(n_1, \ldots, n_k), \mathsf{o}(n_1) \oplus \ldots \mathsf{o}(n_k); \mathbb{R}) \longrightarrow \mathrm{H}^\bullet(X)$$

and whenever all foliations are framed we get the characteristic map from the absolute Lie algebra cohomology:

$$ch : \mathrm{H}^\bullet(W(n_1, \ldots, n_k); \mathbb{R}) \longrightarrow \mathrm{H}^\bullet(X)$$

The construction of characteristic maps becomes very explicit whenever a foliation is determined by a system of *determining forms*:

Definition 2.4.19. The system of smooth 1-forms $\omega_1, \ldots, \omega_n$ on a manifold X determines a framed foliation of codimension n iff the following conditions are satisfied:

($\imath$) The set $\omega_1, \ldots, \omega_n$ is linearly independent at each point. Equivalently, the n-form $\omega_1 \wedge \ldots \wedge \omega_n$ has no zero's on X.

($\imath\imath$) For all i there exists a collection of 1-forms η_{ij} such that $d\omega_i = \sum_{j=1}^n \omega_j \wedge \eta_{ij}$, where d is the De Rham differential. Equivalently, the product $\omega_1 \wedge \ldots \wedge \omega_n \wedge d\omega_i$ is zero for all $i = 1, \ldots, n$.

If the system $\omega_1, \ldots, \omega_n$ is a system of determining forms of a foliation $\mathcal{F}$ on a manifold X, then the restriction of ω_i on any component of an intersection of a leaf of $\mathcal{F}$ and any open contractible set on X is identically zero for all i. Note, that the system of determining forms $\omega_1, \ldots, \omega_n$ defines a flag of foliations $\{\mathcal{F}_1 \supset \ldots \supset \mathcal{F}_k\}$ of codimensions $n_1, n_1 + n_2, \ldots, \sum_{i=1}^k n_i = n$ iff for all $r = 1, \ldots, k$ the collection $\omega_1, \ldots, \omega_{n_1 + \ldots + n_r}$ is a system of determining forms of the foliation $\mathcal{F}_r$. This is equivalent to the following condition:

$$\forall i \leqslant \sum_{j=1}^r n_r \text{ there exists } \eta_{ij} \in \Omega_{DR}^1(X) \text{ such that } d\omega_i = \sum_{j=1}^{n_1 + \ldots + n_r} \omega_j \wedge \eta_{ij}.$$

The matrix units e_{ij} with $1 \leqslant i, j \leqslant n$ form a basis of the Lie algebra $\mathbf{gl}_n$. We define a linear map

$$ch_\omega : \mathbf{gl}_{n_1} \oplus \ldots \oplus \mathbf{gl}_{n_k} \to \Omega_{DR}^1(X)$$

by mapping a matrix unit e_{ij} from the r'th factor $\mathbf{gl}_{n_r}$ to the 1-form $\eta_{n_1 + \ldots + n_{r-1} + i, n_1 + \ldots + n_{r-1} + j}$. Recall that the Weyl superalgebra $\mathcal{W}^\bullet(\mathbf{gl}_{n_1} \oplus \ldots \oplus \mathbf{gl}_{n_k})$ is a free acyclic skew-commutative algebra, therefore, any linear map of generators e_{ij} is extended in a unique way to the map of dg-algebras:

$$ch_\omega : \mathcal{W}^\bullet(\mathbf{gl}_{n_1} \oplus \ldots \oplus \mathbf{gl}_{n_k}) \longrightarrow \Omega_{DR}^\bullet(X)$$

Statement 2.4.20. *The map ch_ω assigned to a system of determining forms of a flag of foliations on a manifold factors through the truncated Weyl superalgebra and determines characteristic classes of this flag of foliations:*

$$\mathcal{W}^\bullet(\mathbf{gl}_{n_1} \oplus \ldots \oplus \mathbf{gl}_{n_k}) \longrightarrow \frac{\mathcal{W}^\bullet(\mathbf{gl}_{n_1} \oplus \ldots \oplus \mathbf{gl}_{n_k})}{I_{n_1, \ldots, n_k}}$$

$$\downarrow{quis} \qquad \searrow^{ch_\omega}$$

$$C^\bullet(W(n_1, \ldots, n_k); \mathbb{R}) \xrightarrow{ch} \Omega_{DR}^\bullet(X)$$

where the ideal $I_{n_1, \ldots, n_k}$ and the vertical quasi-iso were defined in Theorem 2.3.18.

Remark 2.4.21. The construction of characteristic classes via system of determining forms does not require any knowledge of infinite-dimensional Lie algebras and their cohomology. (See e.g. [6] ch.6.) However, the proofs and the exposition becomes clear while working with Gelfand-Fuchs cohomology. See e.g. [19] for the comparison of these two languages.

Let us fix two integers m and n. The commutative diagram of embeddings of Lie algebras:

$$\begin{array}{ccccc}
\mathbf{gl}_m & \hookrightarrow & \mathbf{gl}_n \oplus \mathbf{gl}_m & \hookrightarrow & \mathbf{gl}_{m+n} \\
\downarrow & & \downarrow & & \downarrow \\
W_m & \hookrightarrow & W(m,n) & \hookrightarrow & W_{m+n}
\end{array}$$

defines the following commutative diagram of dg-algebras:

$$\begin{array}{ccccc}
\mathcal{W}^\bullet(\mathbf{gl}_m)/I_m & \longleftarrow & \frac{\mathcal{W}^\bullet(\mathbf{gl}_n \oplus \mathbf{gl}_m)}{I_{m,n}} & \longleftarrow & \mathcal{W}^\bullet(\mathbf{gl}_{m+n})/I_{m+n} \\
\downarrow {\scriptstyle quis} & & \downarrow {\scriptstyle quis} & & \downarrow {\scriptstyle quis} \\
C^\bullet(W_m;\Bbbk) & \longleftarrow & C^\bullet(W(m,n);\Bbbk) & \longleftarrow & C^\bullet(W(m+n);\Bbbk)
\end{array}$$

This construction is universal and should be compatible with the characteristic maps of foliations. Indeed, for a flag of two foliations $\mathcal{F}' \supset \mathcal{F}$ on a manifold X of codimensions m and $m+n$ respectively we have the following commutative diagram:

$$\begin{array}{ccccc}
\mathrm{H}^\bullet(W_m;\mathbb{R}) & \longleftarrow & \mathrm{H}^\bullet(W(m,n);\mathbb{R}) & \longleftarrow & \mathrm{H}^\bullet(W(m+n);\mathbb{R}) \\
& \searrow {\scriptstyle ch(\mathcal{F}')} \quad \downarrow {\scriptstyle ch(\mathcal{F}' \supset \mathcal{F})} \quad \swarrow {\scriptstyle ch(\mathcal{F})} & \\
& \mathrm{H}^\bullet(X) &
\end{array}$$
(2.4.22)

Corollary 2.4.23. *A foliation $\mathcal{F}$ on a manifold X of codimension n may not be included into a flag of foliations $\mathcal{F}' \supset \mathcal{F}$ if it has a nonzero characteristic class of degree greater than $n^2 + 2$.*

Proof. Let us count the top possible degrees of elements in the truncated Weyl superalgebras. Indeed, elements of maximal degree in $\frac{\mathcal{W}^\bullet(\mathbf{gl}_n)}{I_n}$ belongs to the subspace $\Lambda^{top}(\mathbf{gl}_n^*) \otimes S^n(\mathbf{gl}_n)$. Therefore, $\mathrm{H}^i(W_n;\Bbbk) = 0$ for $i > n^2 + 2n$. Also by degree reasons we know that $\mathrm{H}^i(W(n-d,d);\Bbbk) = 0$ for $i > (n-d)^2 + d^2 + 2n$. The maximum of expressions $(n-d)^2 + d^2 + 2n$ for $d = 1,\ldots,n-1$ is reached when $d = 1$ or $d = n-1$ and is equal to $n^2 + 2$. Consequently,

$$\forall i > n^2 + 2 \text{ and } \forall d < n \quad \mathrm{H}^i(W(n-d,d);\Bbbk) = 0.$$

Therefore, the nonzero element in the image of $ch(\mathcal{F})$ of homological degree greater than n^2+2 contradicts with the existence of commutative diagram (2.4.22) □

Let us show that the example of a foliation mentioned in the introduction satisfies the conditions of Corollary 2.4.23. Denote by $SL(n+1,\mathbb{R})$ the Lie group of matrices with real coefficients of determinant 1. Let Γ be a discrete subgroup of $SL(n+1,\mathbb{R})$ such that the quotient space $SL(n+1,\mathbb{R})/\Gamma$ is compact. Let P be a subgroup that fixes a given line in $\mathbb{R}^{n+1}$. Any orbit of the left action of P on the quotient space $SL(n+1,\mathbb{R})/\Gamma$ defines a leaf of a foliation of codimension n on $SL(n+1,\mathbb{R})/\Gamma$. Denote the latter foliation by $\mathcal{F}_H$.

Statement 2.4.24. *The foliation $\mathcal{F}_P$ on the compact space $SL(n+1,\mathbb{R})/\Gamma$ has a nontrivial characteristic class of the top degree $n^2 + 2n$ and, therefore, for $n \geqslant 2$ may not be a subfoliation.*

Proof. The main feature of examples of foliations coming from the action of a Lie subgroup H on a compact quotient of a Lie group G by a discrete subgroup Γ is that most of computations inherits the structure of computations for the corresponding Lie algebras $\mathbf{h} = Lie(H) \subset \mathbf{g} = Lie(G)$. Standard arguments shows that their exists a collection of 1-forms $\omega_{i,j} \in \Omega^1(SL(n+1,\mathbb{R})/\Gamma)$ where i,j range from 1 to $n+1$, $\sum_{i=1}^{n+1} \omega_{i,i} = 0$ and $d\omega_{i,j} = \sum_{k=1}^{n+1} \omega_{ik}\omega_{kj}$. The foliation $\mathcal{F}_\mathsf{P}$ is defined by the system of determining forms $\omega_{1,n+1},\ldots,\omega_{n,n+1}$. The direct check shows that there exists a class in $\mathrm{H}^{n^2+2n}(W_n; \mathbb{R})$ whose image under characteristic map is equal to the product $\wedge_{(i,j) \neq (n+1,n+1)} \omega_{i,j} \in \Omega^{(n+1)^2-1}(SL(n+1,\mathbb{R})/\Gamma)$. The latter product is a volume form on the space $SL(n+1,\mathbb{R})/\Gamma$ and, therefore, represents a nontrivial cohomological class. $\qquad\square$

3 The core of the proof

Let us state several technical results and show afterwords how do they imply Theorem 2.2.10. The idea is to compute the relative cohomology of the Lie algebra W_{m+n} of all vector fields and then show that the inclusion $WL(m|n) \hookrightarrow W_{m+n}$ produces the surjective map of relative cohomology rings in opposite direction. The surjectivity follows from the degenerations of the Hochschild-Serre spectral sequences associated with embeddings of a parabolic subalgebra. Therefore, we start this section from general results on cohomologies of parabolic subalgebras.

3.1 Parabolic subalgebra

Consider the Lie algebra of matrices $\mathbf{gl}_{m+n}$. Let $\mathbf{p}_{mn}$ be it's parabolic subalgebra with maximal reductive subalgebra isomorphic to the Lie algebras of matrices with two blocks $\mathbf{gl}_m \oplus \mathbf{gl}_n$. Note that $\mathbf{p}_{mn}$ is embedded into the Lie algebra $WL(m|n)$ and, moreover, $\mathbf{p}_{mn}$ is isomorphic to the intersection $WL(m|n) \cap \mathbf{gl}_{m+n}$ inside W_{m+n}. In this subsection we state general homological results on relative homology of parabolic subalgebra $\mathbf{p}_{mn} \hookrightarrow \mathbf{gl}_{m+n}$. All proofs are postponed to Appendix B.

Denote the vector space $\Bbbk^m$ by V and the vector space $\Bbbk^n$ by U, such that $\mathbf{gl}_m = \mathbf{gl}(V)$, $\mathbf{gl}_n = \mathbf{gl}(U)$, $\mathbf{gl}_{m+n} = \mathbf{gl}(V \oplus U)$ and the parabolic subalgebra $\mathbf{p}_{mn}$ is isomorphic to $\mathbf{gl}(V) \oplus \mathbf{gl}(U) \oplus V^* \otimes U$. Recall, that the set of irreducible $\mathbf{gl}_n = \mathbf{gl}(U)$-modules are numbered by dominant highest weights $\lambda = \{\lambda_1 \geqslant \ldots \geqslant \lambda_n\}$ with all $\lambda_i \in \mathbb{Z}$. Moreover, if λ_n and, consequently, all λ_i are nonnegative integers then the collection λ is called a Young diagram and the corresponding irreducible $\mathbf{gl}_n$ module is called polynomial (or holomorphic). This means that the matrix coefficients of this module are polynomial. Moreover, this module may be defined using Schur-Weyl duality: Let $|\lambda| = \sum \lambda_i$ be the number of boxes in the Young diagram λ. Let Σ_λ be an irreducible representation of the symmetric group $S_{|\lambda|}$ assigned to the same diagram λ. Then the irreducible polynomial $\mathbf{gl}(U)$-module with the highest weight λ is isomorphic to the image of the Schur functor

$$S^\lambda(U) := U^{\otimes|\lambda|} \otimes_{S_{|\lambda|}} \Sigma_\lambda \tag{3.1.25}$$

(We refer to [11] for details on Schur-Weyl duality and to [27] ch1.8 for the detailed definition of a Schur functor). Schur functors are defined for Young diagrams of arbitrary length. However, if the length $l(\lambda) > \dim(U)$ then $S^\lambda(U)$ is zero, otherwise it is a nontrivial irreducible $\mathbf{gl}(U)$-module. The symmetric power $S^i U$ is also an example of a Schur functor assigned to the Young diagram $\{i\}$ of length 1. The trivial module $\Bbbk$ is isomorphic to $S^0 U$.

We also consider $S^\lambda U$ to be an irreducible $\mathbf{p}_{mn}$-module where the action of $\mathbf{gl}_m$ is trivial.

Lemma 3.1.26. *The higher relative cohomology of the Lie algebra $\mathbf{p}_{mn}$ with coefficients in a $\mathbf{p}_{mn}$-module $Hom(S^\lambda U; L)$ vanishes for all Young diagrams λ of length less than or equal to $n = \dim(U)$ and for all finite-dimensional $\mathbf{gl}_{m+n}$-modules L. The zero cohomology is isomorphic to the space of $\mathbf{gl}_{m+n}$-invariant*

maps between $S^\lambda(U \oplus V)$ and L:

$$\mathrm{H}^{>0}(\mathbf{p}_{mn}, \mathbf{gl}_m \oplus \mathbf{gl}_n; Hom(S^\lambda(U); L)) = 0,$$
$$\mathrm{H}^0(\mathbf{p}_{mn}, \mathbf{gl}_m \oplus \mathbf{gl}_n; Hom(S^\lambda(U); L)) \simeq Hom_{\mathbf{gl}_{m+n}}(S^\lambda(V \oplus U); L).$$

In particular, if $\lambda = 0$ we have the following degeneration property for all $\mathbf{gl}_{m+n}$-modules L:

$$\mathrm{H}^{>0}(\mathbf{p}_{mn}, \mathbf{gl}_m \oplus \mathbf{gl}_n; L) = 0 \quad \text{and} \quad \mathrm{H}^0(\mathbf{p}_{mn}, \mathbf{gl}_m \oplus \mathbf{gl}_n; L) = [L]^{\mathbf{gl}_{m+n}} \tag{3.1.27}$$

Proof. The proof of this Lemma is postponed to Appendix B where items 1 and 3 of Theorem B.2.47 covers the generalization of this statement for an arbitrary parabolic subalgebra. Corollary B.2.51 explains what one get in the case we are interested in. $\qquad\square$

Lemma 3.1.28. *The relative Hochschild-Serre spectral sequence associated with an embedding $\mathbf{p}_{mn} \hookrightarrow \mathbf{gl}_{m+n}$ with coefficients in any finite-dimensional $\mathbf{gl}_{m+n}$-module L degenerates in the first term. More precisely, we have*

$$\mathrm{H}^p(\mathbf{p}_{mn}^-, \mathbf{gl}_m \oplus \mathbf{gl}_n; \Lambda^q(\mathbf{n}_{mn}^+)^* \otimes L) = \begin{cases} 0, & \text{if } p \neq q, \\ \mathrm{H}^{2p}(\mathbb{C}Gr(m, m+n)) \otimes [L]^{\mathbf{gl}_{m+n}} = \\ \quad = \mathrm{H}^{2p}(\mathbf{gl}_{m+n}, \mathbf{gl}_m \oplus \mathbf{gl}_n; L), & \text{if } p = q. \end{cases}$$

Where $\mathbb{C}Gr(m, m+n) = \frac{SU_{m+n}}{SU_m \times SU_n}$ is the Grassmanian on m-dimensional complex subspaces in $m+n$-dimensional one.

Proof. This lemma is also proven in Appendix B using the parabolic BGG resolution. Part 2 of Theorem B.2.47 is a generalization of this lemma to the case of an arbitrary parabolic subalgebra of a reductive Lie algebra.

The cohomology of the complex Grassmanian $\mathbb{C}Gr(m, m+n)$ is known to coincide with the cohomology ring $\mathrm{H}^q(\mathbf{gl}_{m+n}, \mathbf{gl}_m \oplus \mathbf{gl}_n; \Bbbk)$ (see e.g. [12]). These cohomology are known to be even and numbered by m-n-shuffle permutations (see Section B.2.1 for details). $\qquad\square$

3.2 Degeneration of Hochschild-Serre spectral sequences

In this section we prove the degenerations of Hochschild-Serre spectral sequences corresponding to the embedding $\mathbf{p}_{mn} \hookrightarrow W_{m+n}$ and $\mathbf{p}_{mn} \hookrightarrow WL(m|n)$ in the first term and prove the surjectivity map between these spectral sequences by applying Lemmas from previous Section 3.1. We use the same notation and description of the decomposition of Lie algebras as in Appendix A and refer for the detailed description of chains. Only sketched version is given in this section.

As above we denote $V = \Bbbk^m$, $U = \Bbbk^n$ and $V \oplus U = \Bbbk^{m+n}$. We have the following decompositions as $\mathbf{gl}_n \oplus \mathbf{gl}_m = \mathbf{gl}(V) \oplus \mathbf{gl}(U)$-modules:

$$\begin{aligned}
\mathbf{p}_{mn} &\simeq \mathbf{n}_{mn}^- \oplus \mathbf{gl}_m \oplus \mathbf{gl}_n \simeq V^* \otimes U \oplus V \otimes V^* \oplus U \otimes U^*, \\
\mathcal{O}_m &\simeq \widehat{\bigoplus_{i \geqslant 0}} S^i V^*, \\
W_{m+n} &\simeq \widehat{\bigoplus_{i \geqslant 0}} S^i(V \oplus U)^* \otimes (V \oplus U), \\
WL(m|n) &\simeq (V \oplus V^* \otimes V) \oplus \widehat{\bigoplus_{i \geqslant 0}} S^i(V \oplus U)^* \otimes U,
\end{aligned} \tag{3.2.29}$$

where $\mathbf{p}_{mn} := WL(m|n) \cap \mathbf{gl}_{m+n} \hookrightarrow W_{m+n}$ is a parabolic subalgebra of matrices with two diagonal blocks and lower diagonal matrices; $\mathbf{n}_{mn}^-$ (resp. $\mathbf{n}_{mn}^+$) are lower (resp. upper) block-triangular nilpotent matrices.

The embedding $\mathbf{p}_{mn} \hookrightarrow WL(m|n)$ produces the Hochschild-Serre filtration on the relative chain complex $C^\bullet(WL(m|n), \mathbf{gl}_m \oplus \mathbf{gl}_n; \Bbbk)$ and we denote by $E_r^{p,q}(WL(m|n))$ the corresponding spectral sequence.

Similarly, we denote by $E_r^{p,q}(W_{m+n})$ the relative Hochschild-Serre spectral sequence associated with the embedding $\mathbf{p}_{mn} \hookrightarrow W_{m+n}$. Since the Lie algebra $WL(m|n)$ is embedded into W_{m+n} we have the map of aforementioned spectral sequences in the opposite direction:

$$\pi_{m,n}^r : E_r^{p,q}(W_{m+n}) \longrightarrow E_r^{p,q}(WL(m|n))$$

Below we prove that these spectral sequences degenerates in the first terms and the map $\pi_{m,n}^1$ is surjective.

Lemma 3.2.30. *The relative Hochschild-Serre spectral sequence $E_r^{p,q}(W_{m+n})$ with the embedding $\mathbf{p}_{mn} \hookrightarrow W_{m+n}$ degenerates in the first term and*

$$E_1^{p,q}(W_{m+n}) = \mathrm{H}^{2q}(\mathbb{C}Gr(m, m+n)) \otimes \mathrm{H}^{p-q}(W_{m+n}, \mathbf{gl}_{m+n}; \Bbbk) \tag{3.2.31}$$

Proof. Lemma 3.1.28 implies the following description of the first term:

$$E_1^{pq} = \mathrm{H}^q(\mathbf{p}_{mn}, \mathbf{gl}_m \oplus \mathbf{gl}_n; \Lambda^p\left(\frac{W_{m+n}}{\mathbf{p}_{mn}}\right)^*) = \mathrm{H}^q\left(\mathbf{p}_{mn}, \mathbf{gl}_m \oplus \mathbf{gl}_n; \Lambda^p\left(\mathbf{n}_{mn}^+ \oplus \frac{W_{m+n}}{\mathbf{gl}_{m+n}}\right)^*\right) =$$

$$= \mathrm{H}^q\left(\mathbf{p}_{mn}, \mathbf{gl}_m \oplus \mathbf{gl}_n; \Lambda^q(\mathbf{n}_{mn}^+)^*\right) \otimes \left[\Lambda^{p-q}\left(\frac{W_{m+n}}{\mathbf{gl}_{m+n}}\right)^*\right]^{\mathbf{gl}_{m+n}} =$$

$$= \mathrm{H}^{2q}\left(\mathbf{gl}_{m+n}, \mathbf{gl}_m \oplus \mathbf{gl}_n; \Bbbk\right) \otimes C^{p-q}(W_{m+n}, \mathbf{gl}_{m+n}; \Bbbk).$$

Theorem 2.1.5 implies that the space of relative cochains $C^{p-q}(W_{m+n}, \mathbf{gl}_{m+n}; \Bbbk)$ vanishes for odd $p - q$. Therefore, E_1^{pq} vanishes when $p + q$ is odd and the spectral sequence degenerates in the first term. $\qquad\square$

Lemma 3.2.32. *The relative Hochschild-Serre spectral sequences associated with the embedding $\mathbf{p}_{mn} \hookrightarrow WL(m|n)$ degenerates in the first term:*

$$E_1^{p,q>0}(WL(m|n)) = 0 \ \text{ and } \ E_1^{p,0}(WL(m|n)) = \mathrm{H}^p(WL(m|n), \mathbf{gl}_m \oplus \mathbf{gl}_n; \Bbbk).$$

Moreover, the morphism $\pi_{mn}^r : E_r^{p,q}(W_{m+n}, \mathbf{p}_{mn}) \to E_r^{p,q}(WL(m|n), \mathbf{p}_{mn})$ of the relative Hochschild-Serre spectral sequences coming from the inclusion of Lie algebras $\pi_{mn} : WL(m|n) \hookrightarrow W_{m+n}$ is surjective. In particular,

$$\mathrm{H}^\bullet(W_{m+n}, \mathbf{gl}_{m+n}; \Bbbk) = E_1^{\bullet,0}(W_{m+n}) \twoheadrightarrow E_1^{\bullet,0}(WL(m|n)) = \mathrm{H}^\bullet(WL(m|n), \mathbf{gl}_m \oplus \mathbf{gl}_n; \Bbbk).$$

Proof. Decomposition (3.2.29) into the direct sum of finite-dimensional $\mathbf{gl}(U) \times \mathbf{gl}(V)$-modules predicts that $\mathbf{p}_{mn}$-module $\frac{WL(m|n)}{\mathbf{p}_{mn}}$ is a direct sum of modules $L_i \otimes U$ with L_i being an irreducible finite-dimensional $\mathbf{gl}(U \oplus V)$-module and U an irreducible tautological $\mathbf{gl}(U)$-module. Therefore, it's dual satisfy the condition from Lemma 3.1.26. Let us show that the exterior algebra $\Lambda^\bullet\left(\frac{WL(m|n)}{\mathbf{p}_{mn}}\right)^*$ also decomposes into the direct sum of modules $Hom(S^\lambda U, L_i)$ such that we can apply Lemma 3.1.26. This follows from the general argument in the spirit of Howe duality [22]:

Indeed, let A and B be a pair of vector spaces. Then the exterior algebra $\Lambda^\bullet(A \otimes B)$ has a multiplicity free decomposition $\oplus_\lambda S^{\lambda^t}(A) \otimes S^\lambda(B)$ as $\mathbf{gl}(A) \times \mathbf{gl}(B)$-module, where sum is taken over all possible Young diagrams; $S^\lambda()$ denotes the Schur functor associated with the Young diagram λ (after (3.1.25)). The Young diagram transposed to λ is denoted by λ^t.

Consequently, we have the following decomposition:

$$\Lambda^{\bullet}\left(\frac{WL(m|n)}{\mathbf{p}_{mn}}\right)^{*} = \Lambda^{\bullet}\left((V\oplus U)^{*}\underset{i\geqslant 2}{\oplus} S^{i}(V\oplus U)\otimes U^{*}\right) =$$

$$= \Lambda^{\bullet}\left((V\oplus U)^{*}\right)\bigotimes\Lambda^{\bullet}\left(\underset{i\geqslant 2}{\oplus} S^{i}(V\oplus U)\otimes U^{*}\right) =$$

$$= \Lambda^{\bullet}\left((V\oplus U)^{*}\right)\bigotimes\left(\underset{\lambda}{\oplus} S^{\lambda^{t}}\left(\underset{i\geqslant 2}{\oplus} S^{i}(V\oplus U)\right)\otimes S^{\lambda}U^{*}\right) =$$

$$\bigoplus_{\lambda}\underbrace{\Lambda^{\bullet}\left((V\oplus U)^{*}\right)\bigotimes S^{\lambda^{t}}\left(\underset{i\geqslant 2}{\oplus} S^{i}(V\oplus U)\right)}_{\mathbf{gl}(U\oplus V)-\text{module}}\bigotimes\underbrace{S^{\lambda}U^{*}}_{\mathbf{gl}(U)-\text{module}}$$

Now we are able to apply Lemma 3.1.26 in order to compute the first term of the Hochschild-Serre spectral sequence. The vanishing of the higher cohomology in Lemma 3.1.26 implies that for $q > 0$ we have

$$E_{1}^{pq}(WL(m|n)) = \mathrm{H}^{q}\left(\mathbf{p}_{mn},\mathbf{gl}_{m}\oplus\mathbf{gl}_{n};\Lambda^{p}\left(\frac{WL(m|n)}{\mathbf{p}_{mn}}\right)^{*}\right) = 0.$$

For $q = 0$ we have the following collection of identities and inclusion:

$$E_{1}^{\bullet,0}(WL(m|n) := \mathrm{H}^{0}\left(\mathbf{p}_{mn},\mathbf{gl}_{m}\oplus\mathbf{gl}_{n};\Lambda^{\bullet}\left(\frac{WL(m|n)}{\mathbf{p}_{mn}}\right)^{*}\right) =$$

$$= \mathrm{H}^{0}\left(\mathbf{p}_{mn},\mathbf{gl}_{m}\oplus\mathbf{gl}_{n};\left(\bigoplus_{\lambda}\left(\Lambda^{\bullet}\left((V\oplus U)^{*}\right)\bigotimes S^{\lambda^{t}}\left(\underset{i\geqslant 2}{\oplus} S^{i}(V\oplus U)\right)\right)\bigotimes S^{\lambda}U^{*}\right)\right) \subset$$

$$\subset \left[\Lambda^{\bullet}\left((V\oplus U)^{*}\right)\bigotimes\left(\underset{\lambda}{\oplus} S^{\lambda^{t}}\left(\underset{i\geqslant 2}{\oplus} S^{i}(V\oplus U)\right)\otimes S^{\lambda}(V\oplus U)^{*}\right)\right]^{\mathbf{gl}_{m+n}} =$$

$$= \left[\Lambda^{\bullet}\left((V\oplus U)^{*}\right)\bigotimes\Lambda^{\bullet}\left(\underset{i\geqslant 2}{\oplus} S^{i}(V\oplus U)\otimes(V\oplus U)^{*}\right)\right]^{\mathbf{gl}_{m+n}} =$$

$$= \left[\Lambda^{\bullet}\left(\frac{W_{m+n}}{\mathbf{gl}_{m+n}}\right)^{*}\right]^{\mathbf{gl}_{m+n}} \simeq \mathrm{H}^{\bullet}(W_{n+m},\mathbf{gl}_{m+n};\Bbbk). \quad (3.2.33)$$

The inclusion also follows from Lemma 3.1.26. Note, that this is indeed an inclusion and not an isomorphism because if the length of the diagram λ is greater than n then $S^{\lambda}U = 0$, however, $S^{\lambda}(U\oplus V)$ may be different from zero. The middle identity in (3.2.33) follows from the Howe decomposition for the exterior algebra of $\frac{W_{m+n}}{\mathbf{gl}_{m+n}}$. Once again, we identify relative cochains on W_{m+n} and cohomology using Theorem 2.1.5.

Finally, we get that the $E_{1}^{p,q}(WL(m|n)$ vanishes for $q > 0$ and there is a surjection from the zero line of the first term of the spectral sequence $E_{1}^{\bullet,0}(W_{m+n}) = \mathrm{H}^{\bullet}(W_{m+n},\mathbf{gl}_{m+n};\Bbbk)$ to the the zero line of the spectral sequence $E_{1}^{\bullet,0}(WL(m|n)) = \mathrm{H}^{\bullet}(WL(m|n),\mathbf{gl}_{m}\oplus\mathbf{gl}_{n};\Bbbk)$. $\qquad\square$

3.3 Final conclusions

The degeneration of Hochschild-Serre spectral sequences associated with embeddings $\mathbf{p}_{mn}\hookrightarrow WL(m|n)$ and $\mathbf{p}_{mn}\hookrightarrow W_{m+n}$ implies the following corollary:

Corollary 3.3.34. *There exists a commutative diagram of surjections:*

$$
\begin{array}{ccc}
\mathrm{H}^{\bullet}(W_{m+n}, \mathbf{gl}_m \oplus \mathbf{gl}_n; \Bbbk) & \xrightarrow{\ \kappa_{mn}\ } & \begin{array}{l}\mathrm{H}^{\bullet}(W_{m+n}, \mathbf{gl}_{m+n}; \Bbbk) = \\ = \mathrm{H}^{\bullet \leqslant 2(m+n)}(BSU_{m+n})\end{array} \\
& \searrow^{\pi^*_{mn}} \qquad \swarrow_{\tilde{\pi}^*_{mn}} & \\
& \mathrm{H}^{\bullet}(WL(m|n), \mathbf{gl}_m \oplus \mathbf{gl}_n; \Bbbk) &
\end{array}
$$

Morphism π^*_{mn} is the one associated with the natural embedding $\pi_{mn} : WL(m|n) \hookrightarrow W_{m+n}$. Morphism κ_{mn} is the augmentation of the cohomology of Grassmanian $\mathrm{H}^{\bullet}(\mathbf{gl}_{m+n}, \mathbf{gl}_m \oplus \mathbf{gl}_n; \Bbbk)$.

Moreover the surjections $\tilde{\pi}^*_{mn}$ are compatible for different m and n. That is, for any pair of tuples $m \leqslant m'$ and $n \leqslant n'$ there exists a commutative diagram of surjections:

$$
\begin{array}{ccccc}
\mathrm{H}^{\bullet}(W_{m'+n'}, \mathbf{gl}_{m'} \oplus \mathbf{gl}_{n'}; \Bbbk) & \xrightarrow{\kappa_{m'n'}} & \mathrm{H}^{\bullet \leqslant 2(m'+n')}(BSU_{m'+n'}) & \xrightarrow{\tilde{\pi}^*_{m'n'}} & \mathrm{H}^{\bullet}(WL(m'|n'), \mathbf{gl}_{m'} \oplus \mathbf{gl}_{n'}; \Bbbk) \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle p_{m' \to m}} \\
\mathrm{H}^{\bullet}(W_{m+n}, \mathbf{gl}_m \oplus \mathbf{gl}_n; \Bbbk) & \xrightarrow{\kappa_{mn}} & \mathrm{H}^{\bullet \leqslant 2(m+n)}(BSU_{m+n}) & \xrightarrow{\tilde{\pi}^*_{mn}} & \mathrm{H}^{\bullet}(WL(m|n), \mathbf{gl}_m \oplus \mathbf{gl}_n; \Bbbk)
\end{array}
\quad (3.3.35)
$$

Proof. Morphism π^*_{mn} is surjective because the morphism of the first terms of Hochschild-Serre spectral sequences $E_1^{p,q}(W_{m+n}) \xrightarrow{\pi^1_{mn}} E_1^{p,q}(WL(m|n))$ is surjective. The projection κ_{mn} is the projection on the zero line $E_1^{\bullet,0}(W_{m+n})$ and the map π^*_{mn} is the map of zero lines of the first terms of spectral sequences.

All horizontal arrows in Diagram (3.3.35) are surjective by aforementioned results. The middle vertical arrow is surjective because the map $\mathrm{H}^{\bullet}(BSU_{N'}) \to \mathrm{H}^{\bullet}(BSU_N)$ is surjective whenever $N' > N$. The commutativity of the Diagram implies that the map $p_{m' \to m}$ is also surjective. $\qquad \square$

Finally, we can show how does Corollary 3.3.34 imply the main result on the cohomology of the Lie algebra $WL(m|n)$:

Proof of Theorem 2.2.10. Theorem 2.1.7 explains that the chain complex $C^{\bullet}(WL(m|n), \mathbf{gl}_m \oplus \mathbf{gl}_n; \Bbbk)$ is quasi-iso to the truncated Weyl superalgebra $\mathcal{W}^{\bullet}(W_n, \mathbf{gl}_n)/F^{2m+1}\mathcal{W}^{\bullet}(W_n, \mathbf{gl}_n)$. Let us fix the integer parameter n and vary the parameter m. Corollary 3.3.34 implies that for different $m' \geqslant m$ the morphism $p_{m' \to m}$ is surjective and, therefore, the homology $\mathrm{H}^{\bullet}(WL(m|n), \mathbf{gl}_m \oplus \mathbf{gl}_n; \Bbbk)$ are bounded from above by the homology of the inverse limit of truncated Weyl algebras. The inverse limit of truncated Weyl algebras when m goes to infinity is the nontruncated Weyl algebra $\mathcal{W}^{\bullet}(W_n, \mathbf{gl}_n)$ whose homology coincides with the ring of characteristic classes $\mathrm{H}^{\bullet}(BSU_n)$. Hence, the cohomology ring $\mathrm{H}^{\bullet}(WL(m|n), \mathbf{gl}_m \oplus \mathbf{gl}_n; \Bbbk)$ is a quotient of the polynomial ring $\Bbbk[\Psi_2, \ldots, \Psi_{2n}]$. The same compatibility conditions for all possible $m' \geqslant m$ implies that the map $\tilde{\pi}^*_{mn} : \mathrm{H}^{\bullet \leqslant 2(m+n)}(BSU_{m+n}) \to \mathrm{H}^{\bullet}(WL(m|n), \mathbf{gl}_m \oplus \mathbf{gl}_n; \Bbbk)$ factors through the map $\mathrm{H}^{\bullet \leqslant 2(m+n)}(BSU_{m+n}) \to \mathrm{H}^{\bullet \leqslant 2(m+n)}(BSU_n)$ which sends additional generators $\Psi_{2(n+1)}, \ldots, \Psi_{2(m+n)}$ to zero and survives $\Psi_2, \ldots, \Psi_{2n}$. Finally, we ends up with the isomorphism $\mathrm{H}^{\bullet}(WL(m|n), \mathbf{gl}_m \oplus \mathbf{gl}_n; \Bbbk)$ and the truncated polynomial ring $\Bbbk^{\bullet \leqslant 2(n+m)}[\Psi_2, \ldots, \Psi_{2n}]$. $\qquad \square$

The proof of Theorem 2.2.14 follows from the same degeneration properties of the Hochschild-Serre spectral sequences corresponding to the parabolic subalgebra $\mathbf{p} := WL(m|n_1, \ldots, n_k; \mathbf{g}) \cap \mathbf{gl}_{m+n_1+\ldots+n_k}$.

4 Particular computations

4.1 dimension series for absolute cohomology of $W(1, \ldots, 1)$.

Relative cohomological classes of the Lie algebra $W(n_1, \ldots, n_k)$ (relative to the product of orthogonal groups $O(n_1) \times \ldots \times O(n_k)$) are in one-to-one correspondance with the continuous characteristic classes

of flags of foliations. The absolute cohomology of the Lie algebra $W(n_1, \ldots, n_k)$ corresponds to the case of framed foliations. We omit the detailed construction based on the concept of formal geometry and refer the reader to [9]. In this section we will show an example of application of Theorem 2.3.18 in order to describe these cohomological classes. Corollary 4.1.40 below describes the absolute cohomology classes of the Lie algebra $W(1, \ldots, 1)$ with trivial coefficients.

Remark 4.1.36. The orthogonal group $O(1)$ is trivial and therefore the normal bundle of the foliation of codimension 1 is trivial. Similarly, the relative normal bundles of $\mathcal{F}_{i+1} \subset \mathcal{F}_i$ for a flag of foliations $\{\mathcal{F}_1 \supset \mathcal{F}_2 \supset \cdots \supset \mathcal{F}_k\}$ should be trivial whenever the corresponding codimension is equal to 1. Hence, for the case of full flags (i.e. all codimensions are equal to 1) we have the trivial orthogonal group and the space of characteristic classes are the same for framed and nonframed foliations.

Let ζ_i (respectively ξ_i) be the i'th odd (respectively even) generator of the Weyl superalgebra $\mathcal{W}^\bullet(\underbrace{\mathbf{gl}_1 \oplus \ldots \oplus \mathbf{gl}_1}_{k \text{ copies}})$. I.e. ζ_i (resp. ξ_i) is a basis of $\Lambda^1 \mathbf{gl}_1^*$ (resp. $S^1 \mathbf{gl}_1^*$) associated with the i'th copy of $\mathbf{gl}_1$. Let I be the ideal in the aforementioned Weyl superalgebra generated by subspaces $S^{j+1}\left(\underbrace{\mathbf{gl}_1 \oplus \ldots \oplus \mathbf{gl}_1}_{j \text{ copies}}\right)$ with j ranges from 1 to k. I equals to the ideal $I_{1,\ldots,1}$ which was defined in Corollary 2.2.16 or Theorem 2.3.18 for arbitrary collection of codimensions.

Theorem 4.1.37. *The set of monomials*

$$\left\{ \zeta_{\alpha_1} \ldots \zeta_{\alpha_s} \xi_1^{i_1} \xi_2^{i_2} \ldots \xi_{\alpha_s}^{i_{\alpha_s}} \; : \; 1 \leqslant \alpha_1 < \ldots < \alpha_s \leqslant N \text{ and } \begin{array}{l} \forall k \leqslant \alpha_s \quad i_1 + \ldots + i_k \leqslant k, \\ i_1 + \ldots + i_{\alpha_s} = \alpha_s \end{array} \right\} \qquad (4.1.38)$$

form a basis of cohomological classes of the truncated Weyl superalgebra $\dfrac{\mathcal{W}^\bullet(\overbrace{\mathbf{gl}_1 \oplus \ldots \oplus \mathbf{gl}_1}^{N \text{ copies}})}{I_{1,\ldots,1}}$. *The Poincare generating series of the homology of the truncated Weyl superalgebra is as follows:*

$$\sum_{q \geqslant 0} q^k \dim \mathrm{H}^k(\mathcal{W}^\bullet(\mathbf{gl}_1 \oplus \ldots \oplus \mathbf{gl}_1)/I_{1,\ldots,1}) = 1 + \sum_{n=1}^{N} q^{2n+1}(1+q)^{n-1} C(n), \qquad (4.1.39)$$

where $C(n) = \frac{1}{(n+1)}\binom{2n}{n}$ *is the n'th Catalan number.*

Proof. Consider the second term of the spectral sequence associated with the standard filtration on the truncated Weyl superalgebra:

$$E_2 = \mathbb{k}[\zeta_1, \ldots, \zeta_N; \xi_1, \ldots \xi_N]/(I_{1,\ldots,1}), \quad d_2 = \sum_{i=1}^{N} \xi_i \frac{\partial}{\partial \zeta_i}.$$

All higher differentials in this spectral sequence vanish. Recall that the ideal I is generated by the following collection of monomials:

$$\xi_1^{i_1} \cdot \ldots \cdot \xi_k^{i_k} \text{ with } i_1 + \ldots + i_k > k$$

Let us prove that monomials yielding the restriction (4.1.38) in the statement of Theorem form a basis of the cohomology of the complex given by the second term (E_2, d_2). The differential d_2 is a direct sum of N commuting differentials $\xi_i \frac{\partial}{\partial \zeta_i}$. Let us compute the cohomology of E_2 with respect to the differential $\xi_N \frac{\partial}{\partial \zeta_N}$. In other words, we consider the spectral sequence associated with the bicomplex where differential d_2 is the sum of two commuting differentials $\xi_N \frac{\partial}{\partial \zeta_N}$ and $\left(\sum_{i=1}^{N-1} \xi_i \frac{\partial}{\partial \zeta_i}\right)$. The differential $\xi_N \frac{\partial}{\partial \zeta_N}$ maps

monomials to monomials. Therefore, representing cocycles may be also chosen to be monomials. Consider a monomial $f = \zeta_1^{\epsilon_1} \ldots \zeta_N^{\epsilon_N} \xi_1^{i_1} \ldots \xi_N^{i_N}$. In order to be nonzero in the fraction of Weyl algebra by monomial ideal $I_{1,\ldots,1}$ we have the following restrictions:

$$\forall i = 1 \ldots N \ \epsilon_i \in \{0,1\}, \quad \text{and} \ \forall k = 1 \ldots N \text{ we have } i_1 + \ldots + i_k \leqslant k.$$

There are two possibility for the monomial f to represent a nonzero cohomological class with respect to the differential $\xi_N \frac{\partial}{\partial \zeta_N}$. Either $\epsilon_N = i_N = 0$ or $\epsilon_N = 1$, $i_N \geqslant 1$ and $i_1 + \ldots + i_N = N$. The first case implies that we can forget about variables ξ_N and ζ_N and deal with the problem of chasing cocycles for the Weyl algebra with the less number of generating $\mathbf{gl}_1$. This case is covered by induction arguments. Let us show that in the second case all monomials are survived with respect to the differential $\delta := \sum_{i=1}^{N-1} \xi_i \frac{\partial}{\partial \zeta_i}$ and by higher differentials in this spectral sequence. Indeed, the condition $i_N \geqslant 1$ and $i_1 + \ldots + i_N = N$ means that for all i monomial $\xi_i f$ belongs to the ideal I and $\delta f = 0$. As one knows from the homotopy transfer theorem (see e.g. [18],[25]) or just from the description of the spectral sequence (e.g. [29]) all higher differentials δ_k in the spectral sequence should be of the form $\delta d^{-1} \delta \ldots d^{-1} \delta$. Hence, if δf is zero on the level of chains then $\delta_k(f) = 0$ for all k and, consequently, monomials (4.1.38) represents the generating set of cocycles.

In order to count the Poincare series it remains to count the generating series of the set (4.1.38) of monomials representing the linear independent homological classes. Recall that the number of monomials $\{\xi_1^{i_1} \ldots, \xi_n^{i_n}\}$ of degree n subject to the condition $i_1 + \ldots + i_k \leqslant k$ for all $k \leqslant n$ is equal to the n'th Catalan number $C(n) = \frac{1}{(n+1)} \binom{2n}{n}$. This description of Catalan numbers may be found in [28]. Therefore, the generating series of monomials of the form $\zeta_1^{\epsilon_1} \ldots \zeta_n^{\epsilon_n} \xi_1^{i_1} \ldots \xi_n^{i_n}$, that belong to the set (4.1.38), is equal to $(1+q)^{n-1} q^1 q^{2n} C(n)$ where the factor $(1+q)^{n-1}$ means that ϵ_i for $i < n$ may be either 0 or 1, factor q^1 comes from the degree of ζ_n and the factor q^{2n} represents the degree of the monomial $\xi_1^{i_1} \ldots \xi_n^{i_n}$ which is $2(i_1 + \ldots + i_n) = 2n$. The final set of monomials is the union of the aforementioned sets with n ranging from 1 to N. $\qquad\square$

Consider a flag of foliations $\mathcal{F}_1 \supset \mathcal{F}_2 \supset \cdots \supset \mathcal{F}_N$ whose codimensions $\mathcal{F}_{i+1}$ in $\mathcal{F}_i$ are equal to 1 for all i. Let $\omega_1, \ldots, \omega_N$ be the system of determining forms that defines this flag of foliations. I.e. we have $\forall k = 1, \ldots, N \ d\omega_k = \sum_{i=1}^{k} \omega_i \wedge \nu_{ik}$ and the tangent space to the leaves of the foliation $\mathcal{F}_k$ is annihilated by forms $\omega_1, \ldots, \omega_k$. The space of characteristic classes of a flag of foliations are described by Theorem 4.1.37:

Corollary 4.1.40. *The Poincare series of the cohomology of the Lie algebra $W(1, \ldots, 1)$ with trivial coefficients is given by Identity 4.1.39. The corresponding characteristic classes of flags of foliations generalizes the Godbillon-Vey class. What means, that each monomial in $\zeta_i = \nu_{ii}$ and $\xi_i = d\nu_{ii}$ from Theorem 4.1.37 produces a characteristic class of a flag of foliations which is defined by a collection of 1-forms ω_k with $d\omega_k = \sum_{i=1}^{k} \omega_i \wedge \nu_{ik}$. In particular, the corresponding cohomological class does not depend on the choice of ω_i's.*

Proof. We use Theorem 2.3.18 in order to identify the cohomology ring of the Lie algebra $W(1, \ldots, 1)$ and the truncated Weyl superalgebra. The generalization of Godbillon-Vey class is also straightforward. We refer to [12] for detailed description of the Godbillon-Vey class using the Lie algebra homology of formal vector fields on the line. $\qquad\square$

4.2 Formulas for cocycles in the case of a line ($n = 1$)

For the case of vector fields on a line there are some simplifications of direct description of cochains representing cocycles for the Lie algebra cohomology $\mathrm{H}^\bullet(W_n; S^m W_n^*)$. These simplifications are based on a direct description of the space of chains which is simpler for the case of line. Indeed, let us identify the space of q-linear functionals on the Lie algebra W_1 with the ring of polynomials $\Bbbk[y_1, \ldots, y_q]$ using

the following identification of basis in these two spaces. With a monomial $f := y_1^{r_1} \ldots y_q^{r_q}$ we associate a q-linear functional $D_f : W_1^{\otimes q} \to \Bbbk$ in the following way:

$$D_f : \left(\sum_{r \geqslant 0} a_{1r} x^r \frac{\partial}{\partial x}, \ldots, \sum_{r \geqslant 0} a_{qr} x^r \frac{\partial}{\partial x} \right) \mapsto r_1! \ldots r_q! a_{1r_1} \ldots a_{qr_q}. \qquad (4.2.41)$$

Then the space of chains $C^p(W_1; S^m W_1^*) = Hom(\Lambda^p W_1 \otimes S^m W_1; \Bbbk)$ are identified with the subspace of polynomials in $p + m$ variables $\Bbbk[y_1, \ldots, y_p; z_1, \ldots, z_m]$, such that they are skew-symmetric with respect to the permutations of the first p variables and symmetric with respect to the permutations of the last m variables. In this notations the Chevalley-Eilenberg differential $d : C^p(W_1; S^m W_1^*) \to C^{p+1}(W_1; S^m W_1^*)$ is described by the following formula (see e.g. the description given in [12][§2.3] for $m = 0$):

$$dP(y_1, \ldots, y_{p+1}; z_1, \ldots, z_m) =$$
$$\sum_{1 \leqslant s < t \leqslant p+1} (-1)^{s+t-1} (y_s - y_t) P(y_s + y_t, y_1, \ldots, \hat{y}_s, \ldots, \hat{y}_t, \ldots, y_{p+1}; z_1, \ldots, z_m) +$$
$$+ \sum_{\substack{1 \leqslant s \leqslant p+1 \\ 1 \leqslant t \leqslant m}} (-1)^s (y_s - z_t) P(y_1, \ldots, \hat{y}_s, \ldots, y_{p+1}; y_s + z_t, z_1, \ldots, \hat{z}_t, \ldots, z_m).$$

Theorem 4.2.42. *The cochains*

$$a_{2m} := (y_1^2 - y_2^2) z_1 \ldots z_m \in C^2(W_1; S^m W_1^*),$$
$$a_{3m} := (y_1 - y_2)(y_2 - y_3)(y_3 - y_1) z_1 \ldots z_m \in C^3(W_1; S^m W_1^*)$$

represents the basis in the cohomology $\mathrm{H}^\bullet(W_1; S^m W_1^*)$.

Proof. It is a direct check that $d(a_{2m}) = d(a_{3m}) = 0$ what means that they are cocycles. Moreover, a_{2m} belongs to the relative chain complex $C^2(W_1, \mathbf{gl}_1; S^m W_1^*)$ and is equal to the cocycle $\xi_{1^{m+1}, m}$ which is defined in Proposition A.3.45 on page 24.

Alternatively, one can easily verify that the natural pairing between cocycle a_{2m} and the cycle in $\mathrm{H}_\bullet(W_1, \mathbf{gl}_1; S^m W_1)$ given in [7] is different from zero for all m. This will imply that the corresponding cohomological class are different from zero. Theorem 2.3.17 implies that there is only one nontrivial cohomological class in this dimension. $\qquad \square$

A Chains on the Lie algebra W_n and gl-invariant's

This section contains the detailed explanations of the known results and their generalizations mentioned in Section 2.1. We will describe the action of matrix Lie subalgebras on the Lie algebras W_n, $W(m; \mathbf{g})$, $WL(m|; \mathbf{g})$, $WL(m|n)$ and use this description in order to identify the space of **gl**-invariant chains with the truncated Weyl algebras. This description of the relative chain complexes has been written in details in [16], [12] for the Lie algebra W_n and [20] deals with the Lie algebra $W(m; \mathbf{g})$.

A.1 gl-decompositions

Let V be a vector space of dimension m. Let $x_1, \ldots, x_m$ be the basis of linear coordinates on this space. Then as a $\mathbf{gl}_m = \mathbf{gl}(V)$ module the linear span of constant derivatives $\frac{\partial}{\partial x_i}$ is isomorphic to V and the space of polynomial functions of degree k is spanned by monomials $x_1^{i_1} \ldots x_m^{i_m}$ with $i_1 + \ldots + i_m$ and is isomorphic to the k-th symmetric power of V^*. Therefore, the space of functions $\mathcal{O}_m = \mathcal{O}(V)$ on V

is isomorphic as a $\mathbf{gl}(V)$-module to the completed direct sum $\widehat{\oplus_{k\geqslant 0}}S^kV^*$. Consequently, we have the following decompositions as $\mathbf{gl}_m$-modules:

$$W_m \simeq \widehat{\underset{k\geqslant 0}{\oplus}} S^kV^* \otimes V, \qquad \mathbf{gl}_m \simeq V^* \otimes V$$

$$W(m;\mathbf{g}) = W_m \ltimes \mathbf{g} \otimes \mathcal{O}_m \simeq_{\mathbf{gl}_m} \left(\widehat{\underset{k\geqslant 0}{\oplus}} S^kV^* \otimes V\right) \oplus \widehat{\underset{k\geqslant 0}{\oplus}} S^kV^* \otimes \mathbf{g},$$

$$WL(m|;\mathbf{g}) \simeq_{\mathbf{gl}_m} \left(\widehat{\underset{k=0,1}{\oplus}} S^kV^* \otimes V\right) \oplus \widehat{\underset{k\geqslant 0}{\oplus}} S^kV^* \otimes \mathbf{g} \simeq_{\mathbf{gl}_m} (V^* \oplus \mathbf{gl}(V)) \oplus \widehat{\underset{k\geqslant 0}{\oplus}} S^kV^* \otimes \mathbf{g}.$$

Let us now describe the space of relative chains in the aforementioned Lie algebras. First, we remind, that these infinite-dimensional Lie algebras are topological. The underlying topology coming from the topology on formal power series. The linear continuous functions on the formal power series on a vector space is the polynomial ring on the dual space. In particular, we have the following identity:

$$\left(\widehat{\underset{k\geqslant 0}{\oplus}} S^kV\right)^* \simeq \left(\underset{k\geqslant 0}{\oplus} S^kV^*\right)$$

We use the usual dual sign $*$ having in mind that one should take the linear continuous dual.

A.2 Graphs representing chains on W_m

Consequently, we have the following description of the relative chain complexes:

$$C^\bullet(W_m,\mathbf{gl}_m;\Bbbk) = \left[\Lambda^\bullet\left(\frac{W_m}{\mathbf{gl}_m}\right)^*\right]^{\mathbf{gl}_m} = \left[\left(\widehat{\underset{k=0,2,3,\dots}{\oplus}} S^kV^* \otimes V\right)\right]^{\mathbf{gl}(V)} \simeq$$

$$\simeq \left[\Lambda^\bullet\left(\underset{k=0,2,3,\dots}{\oplus} S^kV \otimes V^*\right)\right]^{\mathbf{gl}(V)} \simeq \underset{\{p_0,p_2,p_3,\dots\}}{\oplus} \left[\underset{k=0,2,3,\dots}{\otimes} \Lambda^{p_k}(S^kV \otimes V^*)\right]^{\mathbf{gl}(V)}. \quad \text{(A.2.43)}$$

The index $k = 1$ is omitted in the right-hand side of Isomorphism A.2.43 because we factorize W_n by the subspace $\mathbf{gl}(V) \simeq V^* \otimes V$. The main observation due to Gelfand and Fuchs in [16] is the description of the aforementioned ring of invariants. They observed that all nonzero invariants come from the subring

$$\left[\underset{p_0}{\oplus} \Lambda^{p_0}V^* \otimes \Lambda^{p_0}(S^2V \otimes V^*)\right]^{\mathbf{gl}(V)}.$$

Let us explain their result using the language of graphs as one always do while working with $\mathbf{gl}$-invariants. (see e.g. [21] for the definition of the graph-complex in the similar problem of describing the cohomology of the Lie algebra of Hamiltonian vector fields). Consider an oriented graph Γ yielding the following conditions:

($\imath$) Each vertex has exactly one outgoing edge;

($\imath\imath$) The number of incoming edges in each vertex differs from 1. I.e. we allow $0, 2, 3, \dots$ incoming edges.

Assign with Γ the relative cochain c_Γ on the Lie algebra W_n using the right-hand side of Isomorphism (A.2.43). Suppose that Γ has p_0 vertices with no incoming edges, p_2 vertices with 2 incoming edges and so on. Then Γ defines a $\mathbf{gl}(V)$-invariant in the space $\underset{k=0,2,3,\dots}{\otimes} \Lambda^{p_k}(S^kV \otimes V^*)$ where each edge in Γ defines a $\mathbf{gl}(V)$-invariant pairing between appropriate factors V^* and V. To be strict one has to do the following: first, fix an order of vertices: $v_1,\dots,v_{\sum p_i}$; second, attach to each vertex with k incoming edges a factor $S^kV \otimes V^*$; third, each edge $v_i \to v_j$ defines a $\mathbf{gl}(V)$-invariant pairing between V^* and V coming from contravariant argument of $S^{in(v_i)}V \otimes V^*$ and one of covariant arguments of $S^{in(v_j)}V \otimes V^*$; forth, make an alternation with respect to the vertices of the same valency. (See e.g. [12] for the detailed

description.) The latter alternation procedure produces two following necessary conditions on Γ in order to have a nonzero cochain c_Γ:

(1) The outgoing arrows from different vertices with no incoming edges have different outcoming vertices.

(2) The number of vertices with no inputs is less or equal to the dimension of V.

The condition (1) comes from the following fact: The permutation of two vertices of Γ with no inputs should send c_Γ to $-c_\Gamma$, however the permutation of two incoming edges in any vertex should not change the cochain c_Γ. Therefore, whenever the first condition is not satisfied we get $c_\Gamma = -c_\Gamma$ and thus $c_\Gamma = 0$. The condition (2) corresponds to the fact that $\Lambda^{>dim(V)}(V^*) = 0$.

The conditions (1) and (2) are already enough to describe all possible graphs representing nontrivial cochains. Let Γ^c be a connected component of the graph Γ representing a nonzero cocycle c_Γ. Then Γ^c has only vertices with 0 or 2 inputs. Moreover, the number of vertices with no inputs and the number of vertices with 2 inputs should be the same. The graph Γ^c looks as a wheel, see Picture 1 below. We denote by Γ_r the corresponding connected graph with $2r$ vertices. Finally, we get, that $c_\Gamma \neq 0$ only if $\Gamma = \Gamma_{r_1} \sqcup \ldots \sqcup \Gamma_{r_k}$ with $r_1 + \ldots + r_k \leqslant dim(V)$. The direct check shows that all these Γ are linearly independent. The map $\Psi_{2r} \to c_{\Gamma_r}$ defines an isomorphism from the truncated ring $\mathrm{H}^{\bullet \leqslant 2n}(BSU_n) = \Bbbk^{\bullet \leqslant 2n}[\Psi_2, \ldots, \Psi_{2n}]$ to the relative cochain complex $C^\bullet(W_n, \mathbf{gl}_n; \Bbbk)$ and we get the conclusion of Theorem 2.1.5.
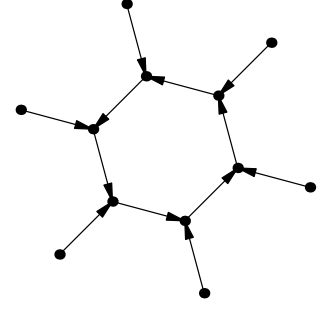

Figure 1: Graph-wheel Γ_6

A.2.1 Chains on $WL(m|; \mathbf{g})$.

The relative chain complex of the Lie algebra $WL(m|; \mathbf{g})$ has the following description in terms of $\mathbf{gl}$-invariants:

$$C^\bullet(WL(m|; \mathbf{g}), \mathbf{gl}_m; \Bbbk) \simeq \left[\Lambda^\bullet \left((V^* \oplus \mathbf{gl}(V)) \oplus \widehat{\bigoplus_{k \geqslant 0}} S^k V^* \otimes \mathbf{g} \right)^* \right] =$$

$$= \bigoplus_{\{q_0; p_0, p_1, \ldots\}} \left[\Lambda^{q_0} V^* \otimes \bigotimes_{k \geqslant 0} \Lambda^{p_k} \left(S^k V \otimes \mathbf{g}^* \right) \right]^{\mathbf{gl}(V)}$$

Same arguments shows that there are nontrivial invariants only in the case $p_k = 0$ for $k > 1$ and $p_1 = q_0 \leqslant dim(V)$. So we ends up with an isomorphism with the truncated Weyl algebra

$$C^\bullet(WL(m|; \mathbf{g}), \mathbf{gl}_m; \Bbbk) \simeq \bigoplus_{p \geqslant 0, 1 \leqslant q \leqslant m} [\Lambda^q V \otimes \Lambda^p(\mathbf{g}^*) \otimes \Lambda^q(V \otimes \mathbf{g}^*)]^{\mathbf{gl}(V)} \simeq$$

$$\simeq \Lambda^\bullet(\mathbf{g}^*) \otimes S^{\bullet \leqslant dim(V)}(\mathbf{g}^*) \simeq \mathcal{W}^\bullet(\mathbf{g})/F^{2m+1}$$

The description of the relative chains of the Lie algebra $W(m; g)$ is somehow the union of the descriptions of the chains on W_m and chains on $WL(m|; \mathbf{g})$. We omit the details and refer to the previous paper [20].

A.3   formulas for cocycles

All previous sections of Appendix A was presented here in order to be able to give a description of cocycles representing the cohomology classes of $\mathrm{H}^\bullet(W_n, \mathbf{gl}_n; S^m W_n^*)$. The main idea is to describe the images of cocycles coming from the cohomology of $\mathrm{H}^\bullet(W_{n+m}, \mathbf{gl}_{n+m}; \Bbbk)$.

In order to distinguish with the previous case we denote the canonical n-dimensional vector space $\Bbbk^n$ by U and the m-dimensional vector space is denoted by V. Consider the $\mathbf{gl}_{m+n}$-invariant

$$\xi_r \in \Lambda^r(V^* \oplus U^*) \otimes \Lambda^r\left(S^2(V \oplus U) \otimes (V^* \oplus U^*)\right)$$

represented by the wheel graph Γ_r. Let $\varphi(\xi_r)$ be the restriction of this invariant onto the subspace

$$\Lambda^r(V^* \oplus U^*) \otimes \Lambda^r(S^2(V \oplus U) \otimes U^*)$$

The latter subspace is decomposed into the following direct sum:

$$\Lambda^r(V^* \oplus U^*) \otimes \Lambda^r(S^2(V \oplus U) \otimes U^*) \simeq$$
$$\simeq \bigoplus_{\substack{s_1+s_2=r \\ t_1+t_2+t_3=r}} \Lambda^{s_1}V^* \otimes \Lambda^{s_2}U^* \otimes \Lambda^{t_1}(S^2V \otimes U^*) \otimes \Lambda^{t_2}(V \otimes U \otimes U^*) \otimes \Lambda^{t_3}(S^2U \otimes U^*) \quad \text{(A.3.44)}$$

and the symmetry conditions implies that the nontrivial $\mathbf{gl}(V)$ invariants exists for $t_1 = 0$, $s_1 = t_1$ and $s_2 = t_2$ respectively. Moreover, we have the following description of the subspace of $\mathbf{gl}(V)$-invariants:

$$\left[\Lambda^s V^* \otimes \Lambda^{r-s}U^* \otimes \Lambda^s(V \otimes U \otimes U^*) \otimes \Lambda^{r-s}(S^2U \otimes U^*)\right]^{\mathbf{gl}(V)} \simeq$$
$$\simeq \Lambda^{r-s}U^* \otimes \Lambda^{r-s}(S^2U \otimes U^*) \otimes [\Lambda^s V^* \otimes \Lambda^s(V \otimes U \otimes U^*)]^{\mathbf{gl}(V)} \simeq$$
$$\simeq \Lambda^{r-s}U^* \otimes \Lambda^{r-s}(S^2U \otimes U^*) \otimes S^s(U \otimes U^*).$$

We denote by $\xi_{r,s}$ the image of the projection tr_s of $\varphi(\xi_r)$ onto the aforementioned subspace of $\mathbf{gl}(V)$ invariants:

$$tr_s : \left[\Lambda^r(V^* \oplus U^*) \otimes \Lambda^r(S^2(V \oplus U) \otimes U^*)\right]^{\mathbf{gl}(V)} \twoheadrightarrow \Lambda^r U^* \otimes \Lambda^r(S^2U \otimes U^*) \otimes S^s(U \otimes U^*)$$

where parameter s may vary from 1 to r. The underlying graphs associated with the invariant $\xi_{r,s}$ also look like a union of wheels. The main difference with the graphs Γ_r is that now some vertices on the ring of a wheel will have no incoming edges from the outside of the ring. (See Picture 2).

Proposition A.3.45. *For any partition* $\lambda = \{\lambda_1 \geqslant \ldots \geqslant \lambda_k\}$ *of the integer* $(m + n)$ *the cochain*

$$\xi_{\lambda,n} := \sum_{s_1+\ldots+s_k=m} \xi_{\lambda_1,s_1}\cdot\ldots\cdot\xi_{\lambda_k,s_k} \in \left[\Lambda^n U^* \otimes \Lambda^n(S^2U \otimes U^*) \otimes S^m(U \otimes U^*)\right]^{\mathbf{gl}(U)}$$

Figure 2: Graph representing $\xi_{6,1}$

defines a cocycle in $C^{2n}(W_n, \mathbf{gl}_n; S^m W_n^*)$. *Moreover, the cocycles* $\xi_{\lambda,n}$ *with* $l(\lambda) \leqslant n$ *are linearly independent and form a set of representatives of all different cohomological classes.*

Proof. The proof is a straightforward description of the images of cocycles under surjections:

$$C^{\bullet}(W_{m+n}, \mathbf{gl}_{m+n}; \Bbbk) = \mathrm{H}^{\bullet}(W_{m+n}, \mathbf{gl}_{m+n}; \Bbbk) \twoheadrightarrow \mathrm{H}^{\bullet}(WL(m|n), \mathbf{gl}_m \oplus \mathbf{gl}_n; \Bbbk) \twoheadrightarrow \mathrm{H}^{\bullet}(W_n, \mathbf{gl}_n; S^k W_n^*).$$

The linear independence of $\xi_{\lambda,n}$ may be checked by showing that the pairing with the appropriate collection of chains is nondegenerate. $\square$
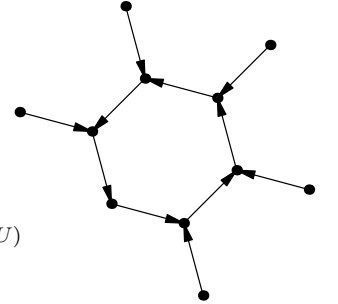
B Relative cohomology of parabolic Lie subalgebra

In this section we will explain some vanishing cohomological results for the cohomology of a parabolic subalgebra. All these results are simple applications of the BGG resolution. We formulate the key corollaries in the full generality. That is, we work with a semisimple or reductive Lie algebra $\mathbf{g}$ and it's negative parabolic subalgebra $\mathbf{p}_I$. However, the applications we need in order to compute the Lie algebra cohomology of certain class of vector fields is a very specific case: $\mathbf{g} = \mathbf{gl}_{m+n}$ and Levi subalgebra of parabolic subalgebra $\mathbf{p}_I$ is the subset of matrices with two blocks $\mathbf{h}_I = \mathbf{gl}_m \oplus \mathbf{gl}_n$.

B.1 Notation

Let $\mathbf{g}$ be a semisimple (reductive) Lie algebra with a chosen Cartan decomposition $\mathbf{g} = \mathbf{n}^- \oplus \mathbf{h} \oplus \mathbf{n}^+$. Let R be the associated root system, S the subset of simple roots and W the corresponding Weyl group. The subalgebra $\mathbf{p}$ is called parabolic if it contains the negative Borel subalgebra $\mathbf{b}^- = \mathbf{h} \oplus \mathbf{n}^-$. Let us fix a subset I of simple roots and assign to it a standard parabolic subalgebra $\mathbf{p}_I$. Let R_I be the subset of roots generated by I. R_I^+ and R_I^- denotes the subsets of positive and negative roots of R_I.

Now we are able to specify the notations for subalgebras related to a chosen parabolic subalgebra $\mathbf{p}_I$:

$$
\begin{aligned}
\mathbf{p}_I &:= \quad \mathbf{n}^- \oplus \mathbf{h} \oplus \bigoplus_{\alpha \in R_I^+} \mathbf{g}_\alpha \text{ (Standard parabolic subalgebra)}, \\
\mathbf{h}_I &:= \quad \mathbf{h} \oplus \bigoplus_{\alpha \in R_I} \mathbf{g}_\alpha \text{ (Maximal reductive subalgebra of } \mathbf{p}_I), \\
\mathbf{n}_I^+ &:= \quad \bigoplus_{\alpha \in R^+ \setminus R_I^+} \mathbf{g}_\alpha \\
\mathbf{n}_I^- &:= \quad \bigoplus_{\alpha \in R^- \setminus R_I^-} \mathbf{g}_\alpha \text{ (Nilradical of } \mathbf{p}_I).
\end{aligned}
$$

We have a decomposition $\mathbf{g} = \mathbf{n}_I^- \oplus \mathbf{h}_I \oplus \mathbf{n}_I^+ = \mathbf{p}_I \oplus \mathbf{n}_I^+$ which we also call the generalized Cartan decomposition.

In general, for any parabolic subalgebra $\mathbf{p}$ it is possible to choose a Borel subalgebra and a set of generators I such that $\mathbf{p} = \mathbf{p}_I$. (See e.g. [11, 23] for details on parabolic Lie subalgebras.) Moreover, it is known that the corresponding description of BGG category $\mathcal{O}$ is absolutely parallel to the standard one (see e.g. [24] for details on BGG). In particular, the parabolic BGG resolution was introduced in [26] and may be shortly summarized in the following way:

Let $P^+ \subset P$ be the set of integral dominant weights for the root system R and Cartan subalgebra $\mathbf{h}$. Let $P_I^+ \subset P_I = P$ be the set of integral dominant weights for the root system R_I and same Cartan subalgebra $\mathbf{h}$. I.e. the elements of P^+ are in one-to-one correspondence with irreducible finite-dimensional $\mathbf{g}$-modules and the elements of P_I^+ correspond to irreducible finite-dimensional $\mathbf{h}_I$-modules. By $W_I \subset W$ we denote the Weyl group associated with R_I and by W^I we denote the right coset $W_I \backslash W$ represented by elements of the minimal length in W.

Any weight $\lambda \in P_I^+$ defines a finite-dimensional irreducible $\mathbf{h}_I$-module $L_I(\lambda)$ with the highest weight λ and the parabolic Verma module $M_I(\lambda) := U(\mathbf{g}) \otimes_{U(\mathbf{p}_I)} L_I(\lambda)$.

Theorem ([26]). *Let $\lambda \in P^+$ be a regular dominant weight, $L(\lambda)$ be the corresponding irreducible $\mathbf{g}$-module with highest weight λ. Then there is an exact sequence*

$$
\ldots \to \bigoplus_{\omega \in W^I, l(\omega) = k} M_I(\omega \cdot \lambda) \to \ldots \to M_I(\lambda) \twoheadrightarrow L(\lambda) \to 0 \tag{B.1.46}
$$

For a nonregular integral dominant weight $\lambda \in P^+$ a similar resolution exists. The k-th terms is isomorphic to the direct sum of $M_I(\omega \cdot \lambda)$ such that ω is an element of the minimal length which sends λ to $\omega \cdot \lambda$.

B.2 Applications of BGG resolution

We are interested in the applications of BGG for the cohomological computations over parabolic subalgebras. Indeed, we are going to prove the following:

Theorem B.2.47. *Let $\mathbf{p}_I \subset \mathbf{g}$ be a parabolic subalgebra in $\mathbf{g}$, $\mathbf{h}_I$ it's Levi subalgebra as above. Then for any finite-dimensional $\mathbf{g}$-module L we have the following*

1. *the higher relative cohomology of the parabolic subalgebra $\mathbf{p}_I$ with coefficients in L vanishes and the zero'th cohomology are $\mathbf{g}$-invariants:*

$$\mathrm{H}^{>0}(\mathbf{p}_I, \mathbf{h}_I; L) = 0 \ and \ \mathrm{H}^0(\mathbf{p}_I, \mathbf{h}_I; L) = [L]^{\mathbf{g}}. \tag{B.2.48}$$

2. *For any pair $\lambda \in P^+$ and $\mu \in P_I^+$ of integral weights the relative extension groups between irreducible $\mathbf{g}$-module $L(\lambda)$ and irreducible $\mathbf{p}_I$-module $L_I(\lambda)$ vanishes if λ and μ are in a different linkage class. Moreover, if there exists $\omega \in W^I$ such that $\omega \cdot \lambda = \mu$ then the cohomology $\mathrm{H}^{\bullet}(\mathbf{p}_I, \mathbf{h}_I; Hom(L(\lambda), L_I(\mu)))$ is one dimensional and has homological degree $l(\omega)$:*

$$\mathrm{H}^{\bullet}(\mathbf{p}_I, \mathbf{h}_I; Hom(L(\lambda), L_I(\mu))) = \begin{cases} \Bbbk[-l(\omega)], \ if \ \omega \cdot \lambda = \mu \ for \ \omega \in W^I, \\ 0, \ otherwise. \end{cases}$$

3. *The relative Hochschild-Serre spectral sequence associated with the embedding $\mathbf{p}_I \hookrightarrow \mathbf{g}$ with coefficients in L degenerates in the first term. In other words, we have the following collection of isomorphisms:*

$$\mathrm{H}^{\bullet}(\mathbf{g}, \mathbf{h}_I; L) \simeq \mathrm{H}^{\bullet}(\mathbf{p}_I, \mathbf{h}_I; L \otimes \Lambda^{\bullet}(\mathbf{n}_I^-)) \simeq \mathrm{H}^{\bullet}(\mathbf{g}, \mathbf{h}_I; \Bbbk) \otimes [L]^{\mathbf{g}} \simeq \left(\underset{w \in W^I}{\oplus} \Bbbk[-2l(w)] \right) \otimes [L]^{\mathbf{g}} \ \ (\text{B.2.49})$$

where W^I is the right coset $W_I \backslash W$ represented by elements of the minimal length in the Weyl group W and by $\Bbbk[-2l(w)]$ we mean the one-dimensional vector space shifted in homological degree $2l(w)$.

Proof. Consider the category $(\mathbf{p}_I, \mathbf{h}_I)$-*mod* of $\mathbf{p}_I$-modules that are semi-simple as $\mathbf{h}_I$-modules. I.e. each module is a sum (probably infinite) of finite-dimensional $\mathbf{h}_I$-modules. Then the Lie algebra cohomology which we are looking for may be considered as the derived Hom-functor in this category. In particular, we can use a BGG resolution $B_\lambda^{\bullet}$ of the irreducible $\mathbf{g}$-module $L(\lambda)$ as defined in (B.1.46) in order to compute the derived hom between $L(\lambda)$ and arbitrary module $N \in mod\text{-}(\mathbf{p}_I, \mathbf{h}_I)$:

$$\mathrm{H}^{\bullet}(\mathbf{p}_I, \mathbf{h}_I; Hom(L(\lambda), N)) = \mathrm{H}^{\bullet}(\mathbf{p}_I, \mathbf{h}_I; Hom(\underset{\omega \in W^I}{\oplus} M_I(\omega \cdot \lambda)[-l(\omega)], N)) =$$

$$\mathrm{H}^{\bullet}\left(\underset{\omega \in W^I}{\oplus} Hom_{\mathbf{h}_I}(L_I(\omega \cdot \lambda), N)[-l(\omega)] \right) \ \ (\text{B.2.50})$$

In particular, if N is a trivial $\mathbf{g}$-module then $Hom_{\mathbf{h}_I}(L_I(\omega \cdot \lambda), \Bbbk)$ differs from zero only for $\omega = Id$ and $\lambda = 0$. What means that we have checked Identity (B.2.48) for irreducible $\mathbf{g}$-module N. The semisimplicity of the category of finite-dimensional $\mathbf{g}$-modules implies that Identity B.2.48 is true for all finite-dimensional $\mathbf{g}$-modules and item 1 is proven.

If N is isomorphic to an irreducible $\mathbf{h}_I$-module $L_I(\mu)$ with the highest weight μ then there exists at most one $\omega \in W^I$ such that $\omega \cdot \lambda = \mu$. Consequently, the group $Ext^{\bullet}_{(\mathbf{p}_I, \mathbf{h}_I)\text{-}mod}(L(\lambda), L_I(\mu))$ is at most one-dimensional and item 2 is proven.

Let us consider Identity (B.2.50) for the case $N = \Lambda^{\bullet}((n_I^+)^*) \simeq \Lambda^{\bullet}(n_I^-)$ in order to prove item 3. Let us describe the possible values of ω and λ when there are nontrivial $\mathbf{h}_I$-homomorphisms between $L_I(\omega \cdot \lambda)$ and the exterior algebra $\Lambda^{\bullet}(n_I^-)$.

Let us first consider in details the most degenerate case: $\mathbf{p}_I$ coincides with Borel subalgebra $\mathbf{b}_-$. Recall that $\mathbf{n}^- \simeq \oplus_{\alpha \in R^-} \mathbf{g}_\alpha$ is the semisimple decomposition of $\mathbf{n}^-$ as $\mathbf{h}$-modules. Each $\mathbf{g}_\alpha$-is one-dimensional and we get the $\mathbf{h}$-decomposition of the exterior algebra:

$$\Lambda^\bullet \mathbf{n}^- \simeq \underset{S \subset R^-}{\oplus} \Lambda_{\alpha \in S} \mathbf{g}_\alpha$$

In particular, $\Lambda^{top} \mathbf{n}^-$ has the weight -2ρ (the sum of all negative roots). Therefore, the set X of $\mathbf{h}$-eigen values in $\Lambda^\bullet n^-$ is the intersection of the convex hull of the orbit $W \cdot 0$ for the dot-action and the root lattice. In particular, for any integral dominant weight $\lambda \in P^+$ the intersection of the orbit $W \cdot \lambda$ of dot-action and X is nontrivial only if $\lambda = 0$. In order to capture the possible values of ω we have to recall another definition of the length. (See e.g. [5].) The length $l(\omega)$ of an element ω of the Weyl group may be defined as the cardinality of the set $\omega(R^+) \cap R^-$. I.e. ω sends exactly $l(\omega)$ positive roots to negative roots. Denote by $n_\omega^- := \oplus_{\alpha \in (\omega(R^+) \cap R^-)} \mathbf{g}_\alpha$ and $n_\omega^+ := \oplus_{\alpha \in (R^+ \cap \omega(R^-))} \mathbf{g}_\alpha$ the corresponding dual subspaces in n^- and n^+ respectively. For any given element $\omega \in W$ we get the isomorphisms:

$$Hom_\mathbf{h}(\Bbbk_{\omega \cdot 0}, \Lambda^\bullet(\mathbf{n}^-)) = Hom_\mathbf{h}(\Bbbk_{\omega \cdot 0}, \Lambda^{l(\omega)}(\mathbf{n}_\omega^-)) \simeq \Bbbk,$$

where $\Bbbk_{\omega \cdot 0}$ is the one-dimensional $\mathbf{h}$-module of weight $\omega \cdot 0$. The semisimplicity of the category of finite-dimensional $\mathbf{g}$-modules implies the case of general module L:

$$\mathrm{H}^\bullet(\mathbf{b}_-, \mathbf{h}; Hom(L, \Lambda^\bullet \mathbf{n}^-)) = \mathrm{H}^\bullet(\mathbf{b}_-, \mathbf{h}; Hom(\underset{\omega \in W}{\oplus} M(\omega \cdot 0)[-l(\omega)], \Lambda^\bullet \mathbf{n}^-)) \otimes [L]^\mathbf{g} =$$

$$= \mathrm{H}^\bullet \left(\underset{\omega \in W}{\oplus} Hom_\mathbf{h}(\Bbbk_{\omega \cdot 0}, \Lambda^\bullet \mathbf{n}^-)[-l(\omega)] \right) \otimes [L]^\mathbf{g} =$$

$$= \mathrm{H}^\bullet \left(\underset{\omega \in W}{\oplus} Hom_\mathbf{h}(\Bbbk_{\omega \cdot 0}, \Lambda^{l(\omega)} \mathbf{n}_\omega^-[-l(\omega)])[-l(\omega)] \right) \otimes [L]^\mathbf{g} = \left(\underset{\omega \in W}{\oplus} \Bbbk[-2l(\omega)] \right) \otimes [L]^\mathbf{g}.$$

Thus part 3 is proven for $\mathbf{p}_I$ being a Borel subalgebra.

The same arguments works in the case of arbitrary parabolic subalgebra $\mathbf{p}_I$. It is enough to mention that possible $\mathbf{h}$-weights in the exterior algebra $\Lambda^\bullet \mathbf{n}_I^-$ are bounded from above by the weights of $\Lambda^\bullet \mathbf{n}^-$ and, therefore, the derived hom from any nontrivial $\mathbf{g}$-representation $L(\lambda)$ to $\Lambda^\bullet \mathbf{n}_I^-$ is zero. Moreover, the representatives in the right coset $W_I \backslash W$ were chosen to have a minimal length. In particular, this follows that $\mathbf{n}_\omega^-$ belongs to $\mathbf{n}_I^-$ iff $\omega \in W^I$ and, therefore, there is exactly one nontrivial $\mathbf{h}_I$-homomorphism from $L_I(\omega \cdot 0)$ to $\Lambda^\bullet n_I^-$ which factors through the subspace $\Lambda^{l(\omega)} n_\omega^-$. □

B.2.1   Particular case of matrices with two blocks

Let us explain what we get in the case $\mathbf{g} = \mathbf{gl}_{m+n}$ and $\mathbf{p}_I = \mathbf{p}_{mn}$ which is of our main interest. In this case $\mathbf{h}$ consists of diagonal matrices and has dimension $m + n$; $\mathbf{h}_I$ is isomorphic to $\mathbf{gl}_m \oplus \mathbf{gl}_n$. Integral dominant weights are numbered by non-increasing sequences of integer numbers:

$$P^+ := \{\lambda = (\lambda_1 \geqslant \ldots \geqslant \lambda_{m+n}) : \lambda_i \in \mathbb{Z}\} \text{ and } P_I^+ := \{\lambda = (\lambda_1 \geqslant \ldots \geqslant \lambda_m; \lambda_{m+1} \geqslant \ldots \geqslant \lambda_{m+n}) : \lambda_i \in \mathbb{Z}\}$$

The entire Weyl group W is the symmetric group S_{m+n} and the subgroup W_I is a product $S_m \times S_n$ of two symmetric groups. Consequently, the right coset $W^I := W_I \backslash W = S_m \times S_n \backslash S_{m+n}$ consists of m-n–shuffle permutations. I.e. $\omega \in S_{m+n}$ belongs to W^I iff $\omega(1) < \ldots < \omega(m)$ and $\omega(m+1) < \ldots < \omega(m+n)$

The dot-action $\omega \cdot \lambda$ is the standard action shifted by $-\rho$. In our case of $\mathbf{g} = \mathbf{gl}_{m+n}$ the half-sum of positive roots ρ is equal to $\frac{1}{2}(m + n - 1, m + n - 3, \ldots, 3 - m - n, 1 - m - n)$. Consequently, for $\omega \in S_{m+n}$ the dot-action looks as follows:

$$\omega \cdot \lambda = \omega(\lambda + \rho) - \rho = (\lambda_{\omega(1)} + 1 - \omega(1), \ldots, \lambda_{\omega(n)} + n - \omega(n))$$

Corollary B.2.51. *Let $L_I(\mu)$ be an irreducible polynomial $\mathbf{gl}_n$-module with the highest weight $\mu = (\mu_1 \geqslant \ldots \geqslant \mu_n \geqslant 0)$. Consider $L_I(\mu)^* = L_I(-\mu)$ as an irreducible $\mathbf{p}_{mn}$-module. I.e. the action of $\mathbf{gl}_m$ and of the nilpotent part $\mathbf{n}_I^-$ is trivial. Then for any irreducible $\mathbf{gl}_{m+n}$-module $L(\lambda)$ with the highest weight $\lambda = (\lambda_1 \geqslant \ldots \geqslant \lambda_{m+n})$ the relative cohomology $\mathrm{H}^\bullet(\mathbf{p}_{mn}, \mathbf{gl}_m \oplus \mathbf{gl}_n; Hom(L(\lambda), L_I(-\mu)))$ vanishes except the case $\lambda = -\mu$ where they are one dimensional in homological degree 0:*

$$\mathrm{H}^{>0}(\mathbf{p}_{mn}, \mathbf{gl}_m \oplus \mathbf{gl}_n; Hom(L(\lambda), L_I(-\mu))) = 0 \ \ for \ all \ \ \lambda, \mu$$

$$and \ \ \mathrm{H}^0(\mathbf{p}_{mn}, \mathbf{gl}_m \oplus \mathbf{gl}_n; Hom(L(\lambda), L_I(-\mu))) = \Bbbk \ \ iff \ \ \lambda_1 = \ldots = \lambda_m = 0 \ \& \ \lambda_{m+i} = -\mu_{n-i}.$$

Proof. We apply item 2 of Theorem B.2.47 for the case $\mathbf{g} = \mathbf{gl}_{m+n}$, $\mathbf{p}_I = \mathbf{p}_{mn}$ and $\mathbf{h}_I = \mathbf{gl}_m \oplus \mathbf{gl}_n$.

The weight $(0, \ldots, 0 \geqslant -\mu_n \geqslant \ldots \geqslant -\mu_1)$ is dominant for the entire Lie algebra $\mathbf{g}$. Hence, the dot-action $\omega \cdot (-\mu)$ for a nontrivial shuffle permutation $\omega \neq Id$ will be no more a dominant weight because in each dot-orbit of the Weyl group there is no more than one integral dominant weight. Therefore, $\omega \cdot \lambda = -\mu$ is equivalent to $\omega = Id$ and $\lambda = -\mu$. $\qquad\square$

Let us reformulate Corollary B.2.51 in terms of Schur functors as we need it in Lemma 3.1.26. Indeed, if μ is a highest weight of a polynomial $\mathbf{gl}_n$-module, that is μ is a Young diagram, then $L_I(\mu) = S^\mu(U)$. Let $L(\mu) := S^\mu(V \oplus U)$ be the corresponding irreducible polynomial $\mathbf{gl}_{m+n}$-module with the same highest weight. Thus Corollary B.2.51 implies coincidence of dimensions:

$$\dim \mathrm{H}^0(\mathbf{p}_{mn}, \mathbf{gl}_m \oplus \mathbf{gl}_n; Hom(S^\mu U; L(\lambda))) = \delta_{\lambda,\mu} = \dim Hom_{\mathbf{gl}_{m+n}}(S^\mu(U \oplus V); L(\lambda)).$$

Since the category of finite-dimensional $\mathbf{gl}_{m+n}$-modules is semi-simple we get an isomorphism for arbitrary $\mathbf{gl}_{m+n}$-module L:

$$\mathrm{H}^0(\mathbf{p}_{mn}, \mathbf{gl}_m \oplus \mathbf{gl}_n; Hom(S^\mu U; L)) \simeq Hom_{\mathbf{gl}_{m+n}}(S^\mu(U \oplus V); L). \qquad (\text{B.2.52})$$

This is the reformulation of Corollary B.2.51 given in Lemma 3.1.26.

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