

UNBOUNDED PROBABILITY THEORY AND ITS APPLICATIONS*

V. P. MASLOV[†] AND T. V. MASLOVA[‡]

(Translated by M. A. Shishkova)

Abstract. The order statistics and empirical mathematical expectation (also called the estimate of mathematical expectation in the literature) are considered in the case of infinitely increasing random variables. The Kolmogorov concept, which he used in the theory of complexity, and the relationship with thermodynamics, which was pointed out already by Poincaré, are considered. We compare the mathematical expectation (which is a generalization of the notion of arithmetical mean, and is generally equal to infinity for any increasing sequence of random variables) with the notion of temperature in thermodynamics while deploying a certain analogue of nonstandard analysis. It is shown that there is a relationship with the Van der Waals law of corresponding states. A number of applications of this concept in economics, in internet information networks, and self-teaching systems are also considered.

Key words. theory of complexity, thermodynamics, Bose gas, specific entropy, Wiener quantization, Maxwell rule, nonstandard analysis, law of corresponding states, distribution according to income, Monte Carlo method, queuing theory, World Wide Web

DOI. 10.1137/S0040585X97986084

1. Introduction. Several elements of the unbounded theory of probability were already considered in the papers [1], [2]. The present paper is a continuation of the studies performed in [1], [2].

We here quote, as a more complete version than in previous publications, the writings of the great mathematician H. Poincaré, who was, at the time of his writings, the chairman of the department of probability theory and mathematical statistics at the Sorbonne. In his book *Science et Hypothèse*, in Chapter XI, entitled “Calcul des Probabilités” Poincaré writes:

“Was probability defined? And can it be defined? And if not, how can we dare argue about it? The definition, one may say, is very simple: the probability of an event is the ratio of the number of favorable outcomes to the total number of outcomes.

A simple example shows how incomplete this definition is. Suppose I throw two dice; what is the probability that at least one of them will come up with 6? Each dice can come up with six different values: the total number of outcomes is $6 \times 6 = 36$; the number of favorable ones is 11, thus the probability is 11/36.

Such is the correct solution. But I will be equally correct if I say that the throw of the two dice can result in $(6 \times 7)/2 = 21$ different outcomes; among these 6 are favorable; thus the probability is 6/21.¹

Why is the first method of calculating the probability better than the second one? Whatever the case, our definition doesn't tell us.

*Received by the editors September 19, 2012. This work was supported by Russian Foundation for Basic Research grant 11-01-12058 _ofi_m.

<http://www.siam.org/journals/tvp/57-3/98608.html>

[†]Higher School of Economics, 109028 Moscow, Russia (v.p.maslov@mail.ru).

[‡]Moscow Economical Institute.

¹Note that in this (second) calculation the outcome (1,6) is identified with the outcome (6,1), i.e., it is counted only once; in other words, *we do not distinguish the two dice*. We shall see below that, in thermodynamics, this corresponds to a translation from Boltzman statistics to the statistics of Bose and Einstein.

Thus we must complete the definition by saying "... to the total number of outcomes provided these outcomes are equiprobable." And so we come to a definition of probability in terms of probability itself.

How can we find out whether or not two events are equiprobable? Isn't that the result of some arbitrary convention? If, at the beginning of each problem, we specify what the arbitrary convention is, then all will work out well, we will only need to apply the rules of arithmetics and algebra and perform the corresponding computations, whose result will then leave no room for any doubt. But if we wish to apply this result, we will have to prove that our arbitrary convention was valid, and so we return to the difficulty that we had tried to circumvent.

One might say: a bit of common sense will suffice to decide what specific convention should be chosen. However, Bertrand, out of pure curiosity, studied the following simple problem: "What is the probability that a chord of the circle is longer than the side of an inscribed equilateral triangle?" The great geometer made two conventions, both suggested by common sense, and obtained $1/2$ in one case and $1/3$ in the other.

The conclusion that apparently follows from the above is that probability theory is a useless science, that we should not trust the unclear instinct that we call common sense and to which we appeal in order to specify our conventions." (English translation of [3, pp. 115–116].)

The difficulties related to the dependence of probability theory on common sense, which were already pointed out by Laplace, were overcome by Kolmogorov in his complexity theory. We toss a coin but, instead of talking about the probability of getting heads or tails, we perform a sufficiently large number of trials and then notice that most of the possible combinations lie in the interval $1/2 \pm \sqrt{N}$, where N is the number of trials in the sequences. Here Kolmogorov finds the shortest code that determines an individual sequence of trials, and the length of that code will be close to the value of the entropy.

This means, although Kolmogorov does not say it outright, that his approach is based on entropy.

This remarkable achievement, in the case when the value of the random variable does not increase, completely answers the question asked by Poincaré in his philosophical work quoted above.

Here we carry over Kolmogorov's approach to sequences of random variables in the case when the sequences infinitely increase in length.

First, we consider the case of infinitely growing values of the random variable corresponding to income, i.e., the case of the so-called "Petersburg paradox." A player plays roulette; he can put his stake on black or red and begins by putting l dollars on black. If the roulette stops on red, he plays black again, increasing his stake to el . Having lost, he continues playing black, again multiplying his stake by e , and continues in this way until, after m steps, the roulette stops on black. The stake at that point is e^ml . The player lost

$$\sum_{k=0}^{m-1} e^k l = l \frac{e^m - 1}{e - 1}$$

and won

$$e^ml.$$

His final gain is therefore

$$0.418e^m \cdot l, \quad m > 2.$$

At one time this paradox could not be used by roulette players to obtain guaranteed gain, because they did not have sufficient funds. But today, when money is electronic and high level officials and state banks have practically unlimited credit, the possibility of enrichment in such ways is boundless.² And if society does not have sufficient control over the actions of its officials, this is very dangerous for the economy.

If a gain can be regarded as a random variable, more precisely, as the values of some random variable ξ , then the exponential growth of such a random variable is catastrophic.

Nowadays, we are closer to a dangerous economic catastrophe than to the ecological one everybody is talking about. In 1567, (saint) Pope Pius V wrote the bull “De salutis gregio dominicis” (On the salvation of the Lord’s flock). But neither the inquisition, nor the laws of Shiriat, are capable of saving all of society, which consists of people of different mentalities and different religious beliefs (see Remark 10 below).

We would like to point out that historians do not pay enough attention to such a problem as corruption. Marxist historians regarded ethical wrongdoings and the corruption of high level functionaries as too rough an explanation of social problems.

But already Salustius (86–35 BC) believed that the principal reason for the fall of the first oligarchy was the all engulfing corruption. As an example of the implementation of strict Roman ethical rules he indicated Caton (on the role of ethical rules in society; see [4]).

Remark 1. In dealing with the order statistics of the income of one of the member countries of the European Union, one should consult the statistical data of the country before it becomes a member of the EU. Then one can predict, by using unrestricted probability theory, when this country will “fall out into a condensate” like dew condensing from the mixture of gases constituting air.

2. Dependence on the choice of the origin and isomorphism. Notion of temperature as mean energy.

The great H. Poincaré pointed out the relationship between thermodynamics and temperature. He wrote: “Let me also mention the kinetic theory of gases and the well known hypothesis, according to which it is assumed that each molecule of gas passes over a very complicated trajectory, whereas the phenomena considered in average (the only way that they can be observed) are subject, in view of the law of large numbers, to simple rules, like the laws of Mariotte and Gay-Lussac.

All these theories rest on the law of large numbers, and so the failure of probability theory would lead to that of these laws.” (English translation of [3, p. 117]).³

In our work, we construct the probability theory which is intimately related to the kinetic theory of gases and is not based on the notion of equiprobability of events. It follows Kolmogorov’s main idea, which he formulated in the framework of his complexity theory.

Let us clarify the relationship with Bose gas.

The empirical expectation of a random variable taking the values $\{\lambda_i\}$ has the form

$$(1) \quad \sum_{i=1}^{\infty} \frac{n_i}{n} \lambda_i = \mathcal{E}, \quad \sum_{i=1}^{\infty} \frac{n_i}{n} = 1,$$

²In France, Germany, and other countries court trials constantly take place against high officials accused of this kind of misconduct.

³Poincaré constructed a mathematical model of ideal gas. Kozlov in [5] proved that the Gibbs paradox takes place for this model. Kozlov’s proof stimulated investigations of this important problem by one of the authors (see [6], [7], [8], etc.).

or equivalently

$$(2) \quad \sum_{i=1}^{\infty} n_i \lambda_i = M, \quad M = n\mathcal{E},$$

$$(3) \quad \sum_{i=1}^{\infty} n_i = n.$$

These equations coincide precisely with the equations used to determine the inner energy of an ideal gas if λ_i is the collection of eigenvalues of a certain unbounded self-adjoint operator \widehat{H} , known as Hamiltonian, and N_i is the number of particles at the corresponding level:

$$(4) \quad \sum_{i=1}^{\infty} N_i \lambda_i = E, \quad \sum_{i=1}^{\infty} N_i = N.$$

The mean energy of one particle is

$$E_{\text{average}} = \frac{E}{N}.$$

As is well known, this value is related to the notion of *temperature* T .

Indeed, we all know that in a gas the number of particles is huge: in a vessel of volume $V = 1 \text{ cm}^3$ there are $N \sim 10^{19}$ particles (at room temperature and under pressure of 1 atm). The mean energy of gas per unit volume is kT , where k is the Boltzman constant and T is the temperature of gas in its equilibrium state.

Therefore, in thermodynamics, a mean quantity corresponding to a huge number of tests has been defined in the case when the random variable λ assumes unboundedly growing values. Its growth will be discussed below.

Van der Vaals is famous for his empirical relation (the VdV equation). But all survey articles on thermodynamics claim that Van der Vaals' main discovery is the law of corresponding states. This law consists of the following. Real pure⁴ gases have the same temperature T_c and the same critical pressure P_c .

D. Mendeleev referred to the critical temperature T_c as the temperature of absolute boiling. He meant that the liquid state is impossible at higher temperatures. What is a temperature (in the Kelvin scale)? It is counted off from absolute zero. However, the absolute zero is unaccessible in the sense that no matter how small the temperature (in the Kelvin scale), it still is infinitely far from absolute zero. In this sense it is more natural to consider the logarithm of temperature. Then $\log T|_{T=0} = -\infty$, and it is obviously seen that the distance between the temperature (in the Kelvin scale), no matter how small, and absolute zero is an infinite quantity. Thus, from the mathematical standpoint, we are dealing with "nonstandard analysis" and an infinite temperature, although the latter may equal, say, one hundredth of a degree (in the Kelvin scale).

If we pass to the relative quantities

$$(5) \quad T_{\text{red}} = \frac{T}{T_c}, \quad P_{\text{red}} = \frac{P}{P_c},$$

⁴A pure gas is a gas consisting of identical molecules.

then it turns out that in these reduced coordinates, the isotherms, isochores, and isobars on all the coordinate planes (pressure–volume, temperature–density), and so on behave very similarly for different gases even when the gases have very different critical temperatures T_c . This is true for nitrogen, argon, carbon dioxide, methane, and other gases.

Example 1. Consider the standard Hilbert space H of functions defined on the closed interval $[0, \pi]$, and its basis of orthogonal functions

$$(6) \quad \{e^{inx}\}_{n=-\infty}^{n=\infty},$$

and the two self-adjoint operators

$$(7) \quad \hat{A}_1 = -ih \frac{\partial}{\partial x}, \quad \hat{A}_2 = -2ih \frac{\partial}{\partial x},$$

depending on the parameter $h = 1/k$, where k is an integer.

The initial reference point in the basis $\{e^{inx}\}$ is $n = 1$.

Consider $n_0^{(1)} = k$ and $n_0^{(2)} = \frac{k}{2}$. We see that

$$\hat{A}_1 e^{in_0^{(1)}x} \approx 1, \quad \hat{A}_2 e^{in_0^{(2)}x} \approx 1.$$

Obviously,

$$\{e^{i(n-k)x}\}_{n=-\infty}^{n=\infty} \quad \text{and} \quad \{e^{i(n-k/2)x}\}_{n=-\infty}^{n=\infty}$$

are isomorphic to

$$\{e^{inx}\}_{n=-\infty}^{n=\infty}.$$

Thus, as $h \rightarrow 0$, the reference point of the infinite basis shifts from the point $n = 1$ to the point $n = 1/h$ in the first case \hat{A}_1 and to the point $n = 1/(2h)$ in the second case \hat{A}_2 . On the interval $[0, \pi]$, the operators \hat{A}_1 and \hat{A}_2 will act exactly in the same way on functions x expressed in the basis (6) as the operator \hat{A} of the usual numbering of functions. When we shift the numbering by $n - k$ and $n - k/2$, we come to rapidly oscillating functions. These two spaces are orthogonal to each other and to the initial Hilbert space H (in the sense that $\int \sin \frac{x}{h} \varphi(x) dx \rightarrow 0$ as $h \rightarrow 0$, $\psi(x) \in H$).

Denoting $e^{ikx} = \Psi_c$ and $e^{ik/2x} = \varphi_c$, we see that the space of functions corresponding to the basis of the form Ψ_n/Ψ_c is isomorphic to the basis of functions Ψ_n/φ_c as $h \rightarrow 0$, where $\Psi_n = e^{inx}$ ($h = 0$, $k = \infty$ in the “nonstandard analysis”).

A similar situation occurs for Dirichlet series and for the eigenfunctions of the Schrödinger operator (the Hamilton operator) \hat{H} for the oscillator and so on⁵ (see [10]).

This isomorphism of infinitely large quantities from “nonstandard analysis” explains, in a certain sense, the empirical law of corresponding states due to Van der Waals, who discovered a similar “almost isomorphism.”

It is difficult for a mathematician to explain the ideas of the new thermodynamics to a physicist. The most difficult thing to explain is that such a small temperature as $T = 1/100 K$ is actually an infinitely large quantity. For the expectation (1) of a

⁵Note that the relationship between the solution of the Cauchy problem for the Schrödinger equation of the form $e^{i\hat{H}t}\Psi_0$, where $\Psi_0 \in H$ is the initial condition, and the solution of the heat equation $e^{-\hat{H}t}\Psi_0$, is described in section 2 of the paper [9], which deals with Wiener quantization.

random variable taking unboundedly increasing values (as, for instance, in the case $\lambda_i = i^2$, $i = 1, 2, \dots$), this fact is quite natural in the case when we decompose the natural number M into N summands, as well as in the case of frequency dictionaries [11].

Example 2. Let us recall a famous theorem from number theory, namely the solution of the “partitio numerorum” problem. In this problem we deal with a natural number M that we are to decompose into the sum of N summands (positive integers), for example, $M = 5$, $N = 2$:

$$5 = 1 + 4 = 2 + 3,$$

which gives two variants of the decomposition: $\mathcal{M} = 2$.

If $M = 10^{23}$, $N = 1$, there is only one variant: $\mathcal{M} = 1$. If $M = 10^{23}$ and $N = 10^{23}$, there is also only one variant of the decomposition (the sum of ones): $\mathcal{M} = 1$.

Obviously, for a fixed M there exists a number N_c for which the number of variants \mathcal{M} is maximal (this number is not unique in general). The quantity $\log_2 \mathcal{M}$ is called *Hartley entropy*. The point at which it achieves its maximum is where the maximal entropy occurs.

Suppose we are given a decomposition

$$M = a_1 + \cdots + a_N$$

of the number M into N summands. Denote by N_j the number of summands on the right-hand side of that equation which are exactly equal to j .

Then there will be $\sum_j N_j$ summands, and this sum equals N , since there are N summands total. Further, the sum of all summands that equal j is jN_j , since there are jN_j summands N_j , so that the sum of all the summands is obtained by summing all these expressions over j , i.e., it is $\sum_j jN_j$, which equals M . Namely

$$(8) \quad \sum_{i=1}^{\infty} N_i = N, \quad \sum_{i=1}^{\infty} iN_i = M.$$

The nonuniqueness of the above mentioned maximum and the indeterminacy of their quantity permitted Erdős to obtain his result only with precision $o(\sqrt{M})$.

Thus we have the Erdős theorem for the system of two Diophantine equations

$$(9) \quad \sum_{i=1}^{\infty} N_i = N, \quad \sum_{i=1}^{\infty} iN_i = M,$$

which asserts that the maximum number of solutions N_c of this system is

$$(10) \quad N_c = \beta^{-1} M_c^{1/2} \log M_c + \alpha M_c^{1/2} + o(M_c^{1/2}), \quad \beta = \pi \sqrt{2/3},$$

where the coefficient α is determined by the formula $\beta/2 = e^{-\alpha\beta/2}$.

If we increase the number N in problem (9) while M remains constant, then the number of solutions will decrease. If we count the sums (9) from zero instead of one, i.e.,

$$(11) \quad \sum_{i=0}^{\infty} iN_i = (M - N), \quad \sum_{i=0}^{\infty} N_i = N,$$

then the number of solutions will not decrease and will remain constant.

Let us explain this fact. The Erdős–Lehner problem [12] consists of decomposing the number M_c into $N \leq N_c$ summands.

The decomposition of the number 5 into two summands gives two variants. If we include the number 0, we obtain three variants $5 + 0 = 3 + 2 = 4 + 1$. Thus the inclusion of zero gives us the possibility of saying we are decomposing M into $k \leq n$ summands. Indeed, the decomposition of 5 into three summands includes all the previous variants, $5 + 0 + 0, 3 + 2 + 0, 4 + 1 + 0$, and adds new variants already without zero.

The maximum does not change by much in this case [12], whereas the number of variants does decrease: the zeros will allow the maximum to remain constant, and the entropy will not decrease, it will become constant after the maximum is reached. This remarkable property of the entropy is what allows us to construct the unrestricted probability theory in the general case [13]. In physics, this effect is identical to the so-called Bose condensate effect.

One may ask: How does the “partitio numerorum” in arithmetic differ from the Boltzman–Shannon statistics? If we regard $4 + 1$ and $1 + 4$ as different variants, then we obtain the Boltzman–Shannon statistics. The number of variants \mathcal{M} of the decomposition rapidly increases. Thus the “noncommutativity” of addition gives a huge number of extra variants of the decomposition, and the Hartley entropy, equal to the logarithm of the number of variants, will coincide with the Boltzman–Shannon entropy.

The entropy arising in arithmetic is the Bose–Einstein entropy, which is due to commutativity and symmetry. Let us recall the formula for entropy of a nonequilibrium Bose gas, derived in the textbook [14] (formula 54.6):

$$(12) \quad S = \sum_j G_j [(1 + \bar{n}_j) \log(1 + \bar{n}_j) - \bar{n}_j \log \bar{n}_j],$$

where \bar{n}_j are the average numbers of fillings of the quantum states $n_j = N_j/G_j$ (cf. [15]). Here G_j is the infinitely increasing value of random variables; see [1] and (37).

We have

$$(13) \quad \begin{aligned} \sum_j N_j &= \sum_j G_j \bar{n}_j = N, \\ \sum_j \varepsilon_j N_j &= \sum_j \varepsilon_j G_j \bar{n}_j = E. \end{aligned}$$

From this, applying the method of Lagrange multipliers to the maximum of the entropy (12) under conditions (13), we obtain⁶

$$(14) \quad \bar{n}_j = \frac{1}{e^{a+b\varepsilon_j} - 1}, \quad \varepsilon_j = E_0 j.$$

Here $a = -\mu/T$, $b = 1/T$, T is the temperature, and $\mu \leq 0$ is the chemical potential. Thus the Lagrange multipliers are expressed in terms of the temperature and the chemical potential of the gas.

⁶We repeat the derivation word for word [14]. A rigorous proof of fractional dimensions was first obtained by Vershik [16]. Also see [17].

Since, according to the Bose–Einstein formulas, sums of the form

$$\sum_{i=0}^{\infty} \lambda_i n_i = M, \quad \sum_{i=0}^{\infty} n_i = n$$

are replaced by integrals, it is impossible to distinguish the point $i = 0$, and the condensation effect must occur in a neighborhood of this point, not at the point $i = 0$ itself. But for $D \leq 2$, the Euler–Maclaurin formula cannot be applied (and so we cannot pass from sums to integrals), and therefore the point $i = 0$ can indeed be distinguished.

However, will the extra “particles” condense precisely near the point $i = 0$ for $N > N_c$? Actually, this question consists of the following: Will the number of zeros in the problem under consideration as $M \rightarrow \infty$ be much larger than the number of ones; i.e., would

$$N_1 = o(N_0)$$

hold even for $N > N_c$?

A computer calculation gives a negative answer. Experts in number theory also answer negatively. Moreover when $D > 2$ and the Euler–Maclaurin formula can be applied, for $i \gg 1$ we can pass to the integral and it is impossible to speak of one point $i = 0$, as is done in the textbook [14].

Actually, along with the convergence to the integral (for instance, in the three-dimensional case) as $N \rightarrow \infty$ and $\mu \rightarrow 0$, the bell-shaped function $\sqrt{N}e^{-Nx^2}$ also increases, where

$$(15) \quad \frac{1}{\log N} > x \geq 0$$

(cf. [18]).

In this general argument it is important to understand the meaning of G_j in the general case.

In the mathematical literature, there are two definitions of G_j : the statistical one, corresponding to quantum statistics, and the one from mathematical physics, corresponding to the D -dimensional Laplace equation in the D -dimensional volume. In exactly the same way, the Landau–Lifshits textbook has two definitions of the Bose gas. One appears in the part called “Statistical Physics” and the other (which uses the symmetric eigenfunctions of the Schrödinger operator) appears in the part entitled “Quantum Mechanics.” In the mathematical literature, the first is named after Weyl and the second after Courant. Let us look at both definitions in more detail.

The $2D$ -dimensional phase space is discretized into a lattice and G_j is determined by the formula

$$(16) \quad G_i = \frac{\Delta p_j \Delta q_j}{(2\pi\hbar)^D}.$$

Here \hbar is the Planck constant.

By assigning to G_i the multiplicities of the spectrum of the Schrödinger operator, we obtain a correspondence between the symmetric (with respect to permutations) eigenfunctions of the N -particle Schrödinger equation and the combinatorial calculations of the Bose statistics in the book [14].

A single-particle ψ -function satisfies the free Schrödinger equation with Dirichlet conditions on the walls of the vessel. According to the classical Courant formula, the number of eigenvalues less than a given $\lambda \gg 1$ (the spectral density $\rho(\lambda)$) has the following asymptotics:

$$(17) \quad \rho(\lambda) = \frac{Vm^{D/2}\lambda^{D/2}}{\Gamma(D/2+1)(2\pi)^{D/2}\hbar^D}(1+o(1)) \quad \text{as } \lambda \rightarrow \infty$$

(V is the D -dimensional volume).

Specifically, by means of this correspondence we establish the relationship between the Bose–Einstein combinatorics [14], the definition of the N -particle Schrödinger equation, and the multiplicity of the single-particle Schrödinger equation.

A spectrum of the single-particle Schrödinger equation without interaction potential coincides up to the factor with the spectrum of the Laplace operator.

In the case of the Laplace operator and the Schrödinger equation, it makes no sense to consider fractional dimensions. In the statistical case, fractional dimensions do make sense, but their meaning is different. They determine the average number of degrees of freedom, which can be different for molecules of different velocities, so that the average value of the degrees of freedom for the whole gas can be fractional. It corresponds to the mean energy, which is the temperature. For the unrestricted probability theory that we consider here, it is precisely the fractional “dimension” (understood as an average of the number of degrees of freedom over the entire “general” population) that makes sense.

One of the authors proved that both the degeneration energy for the Bose gas and the number of particles in the degeneration state coincide up to normalization with the critical values N_c and E_c (i.e., for $\mu = 0$) of the classical noninteracting gas. In the D -dimensional case, the degeneration energy has the form

$$(18) \quad E_d = \int_0^\infty \frac{\frac{|p|^2}{2m} d\varepsilon}{e^{\frac{|p|^2}{2m}/T_c} - 1},$$

where

$$(19) \quad d\varepsilon = C \frac{|p|^2}{2m} dp_1 \cdots dp_D,$$

where C is a certain constant, which allows us to determine the coefficients C and Λ^D in the formula

$$(20) \quad E_c = C_1 \Lambda^D T_d^{2+\gamma} \zeta(1 + D/2) \Gamma(1 + D/2), \quad \gamma = D/2 + 1.$$

It is more convenient to pass to another normalization

$$(21) \quad d\varepsilon' = \frac{1}{\Gamma(D/2+1)} d\varepsilon$$

for the energy and, respectively,

$$(22) \quad d\varepsilon'' = \frac{1}{\Gamma(D/2)} d\varepsilon$$

for the number of particles N .

Passing to the Van der Waals coordinate (5), we obtain the coefficient $C\Lambda^D$ of the form $\tilde{\Lambda}^{\gamma-\gamma_c}$, where γ_c is determined from the experimental values of the quantity $Z|_{\mu=0} = \zeta(\gamma_c+2)/\zeta(\gamma_c+1)$, and $\zeta(x)$ is the Riemann zeta function. In what follows, we omit the tilde on the coefficient Λ and the index r (reduced) on the parameters T and P .

After this, it is more convenient to use the polylogarithm $\text{Li}_{2+\gamma}(e^{-\mu/T})$ for the energy and the polylogarithm $\text{Li}_{1+\gamma}(e^{-\mu/T})$ for the number of particles:

$$(23) \quad E = T^{2+\gamma} \text{Li}_{2+\gamma}(a),$$

$$(24) \quad N = T^{1+\gamma} \text{Li}_{1+\gamma}(a),$$

where a is the activity ($a = e^{-\mu/T}$).

Then the compressibility factor is equal to

$$(25) \quad Z = \frac{E}{NT} = \frac{\text{Li}_{2+\gamma}(a)}{\text{Li}_{1+\gamma}(a)} = \frac{M}{Tn}.$$

The coefficient Λ is here eliminated.

3. Basic series and phase transition. To each degree of freedom we assign its basis sequence, whose terms correspond in thermodynamics to pure gases with identical molecules. Namely, we consider the following basis family of random variables $\{i^{D/2}\}$, corresponding to the number D of degrees of freedom.

The variational series

$$(26) \quad G_i = i^\gamma$$

is called the *basis series*⁷ of dimension D .

Let n_i be the value of the outcome corresponding to $p_i = n_i/n$.

A sample value (estimate) of the mathematical expectation is

$$\frac{\sum \varepsilon_i n_i G_i}{n} = \mathcal{E}, \quad \varepsilon_i = E_0 i.$$

We assume that $\mathcal{E} \rightarrow \infty$ as $n \rightarrow \infty$.

It is useful to introduce parameter γ as follows:

$$D = 2(\gamma + 1), \quad \gamma = \frac{D}{2} - 1.$$

We shall consider the one-parameter family of entropies S_γ as the logarithm of the number of solutions of the system

$$(27) \quad \sum_{i=1}^{\infty} G_i n_i = n, \quad E_0 \sum_{i=1}^{\infty} i G_i n_i \leq n \mathcal{E}.$$

The asymptotic relation $n(\mathcal{E})$ between n and \mathcal{E} as $n \rightarrow \infty$ for which the maximal number of solutions of (27) (up to $o(N)$) will be called the *point of maximum deaccumulation* of the estimate (sample value) of the expectation, while the families of solutions $\{n_i\}/n$ of (27) are called *typical*. The value n for which the maximal deaccumulation is achieved is denoted by N_c .

⁷Similarly to thermodynamics, the pure gases correspond to the periodic table, and each pure gas is associated with a certain value Z_c , for example, up to 0.01%.

The binary logarithm of the number of solutions of (27) is the Hartley entropy S . Thus a, b are the Lagrange multipliers (see (12)–(14)) for finding the maximum of S_γ under conditions (27):

$$(28) \quad dS = a dn + b d(\mathcal{E}n), \quad \mathcal{E}n = M.$$

We have locally

$$(29) \quad d(\mathcal{E}n) = \frac{1}{b} dS - \frac{a}{b} dn.$$

Denote the quantity $1/b$ by T , and the quantity a/b by μ ($\mu \leq 0$). In thermodynamics, there is an additional term $P dV$, where P is the pressure and V is the volume.

Remark 2. In the economic calculation of income (see below), the variable T_c is called the *minimal wage*, while $\bar{\mu} = -\mu$ is the nominal percentage or the deaccumulation coefficient. We also introduce the notation $b = 1/T$. To the value $n = N_c$ corresponds $T = 1$. For a given E , the maximum of deaccumulation is attained at the value of $n(E)$, which we denote by N_c . Volume of goods Q corresponds to the volume V , and the price of goods to the pressure P (Irving Fisher's correspondence law).

The Hartley entropy S_γ for $\gamma > 0$ has the form

$$(30) \quad S = n \left[Z(\gamma)(2 + \gamma) + \frac{\mu}{T} \right].$$

For $\mu = 0$ and $n > N_c$, the entropy S can be extended continuously by a constant (a phenomenon that, in the particular case $D = 3$, is equivalent to the physical notion of “Bose–Einstein condensate”).

The relation $n > n_c$ is equivalent to $T_r < 1$. Here for $n \gg k \gg 1$ (see (15)), the number of values greater than the critical one is $\sum_{i=k}^{\infty} n_i = n_c$. The remaining $n - n_c$ values accumulate within the interval $i \leq k \ll n$, which becomes relatively smaller as $n \rightarrow \infty$ and $\mu \rightarrow 0$. Here $k \gg 1$ is the *support of the bell-shaped function*.

The passage to the integral is accompanied by the appearance of a bell-shaped function that progressively approaches the δ -function (Bose condensate). Therefore in formula (30) for $n > n_c$ or $T_r < 1$ the values of dn (and dV in thermodynamics) can be set equal to zero. As a result, the entropy of the “liquid” phase assumes the form

$$(31) \quad d(\mathcal{E}n) = T_r dS.$$

On the plane $\{M, Z = \frac{M}{nT}\}$, the “gas” critical isotherm

$$(32) \quad M = \text{Li}_{\gamma_c+2}(e^{-\mu/T_c}), \quad Z = \frac{\text{Li}_{\gamma_c+2}(e^{-\mu/T_c})}{\text{Li}_{\gamma_c+1}(e^{-\mu/T_c})}$$

is practically the same for all pure gases, each of which is associated with $Z_c = \zeta(\gamma_c + 2)/\zeta(\gamma_c + 1)$. For argon $Z_c = 0.29$, for methane $Z_c = 0.29$, and for oxygen and other gases, the value of Z_c is determined experimentally. The “base” spinodal (i.e., the set of points at $\mu = 0$) on the plane $\{M, Z\}$ is the interval $Z = Z_c$, $M = \{0, 1\}$. The isotherm $T \leq 1$ demonstrates a good overlapping on the interval $-\infty < \mu < \mu_l$ up to the phase transition “gas–liquid” for $\mu = \mu_l$.

On the “liquid” isotherm, the value $n = T^{\gamma_c+1} \zeta(\gamma_c + 1)$ is constant. Therefore, it is a straight line passing through the point $M = T^{\gamma_c+2}$, $Z = Z_c$ (the origin is $M = 0$, $Z = 0$).

Remark 3. The Bose condensation in income statistics means that when we increase the number of trials $n > n_c$, we necessarily come to the low income group if the general population is the population of the country, while the order statistics is determined by the value of the random variable (expressing the income of each citizen; see section 7.1).

4. Negative values of the parameter γ . Under the assumption that the transition to the liquid phase does not occur at $T_r = 1$, we equate the chemical potentials μ and $\tilde{\mu}$ for the “liquid” and “gas” phases on the isotherm $T_r = 1$.

Then we determine a point μ , i.e., the point of transition to the “liquid” phase for $T_r < 1$, by equating the chemical potentials of the “liquid” and “gas” phases.

Remark 4. Consider the case $\gamma < 0$ and $\tilde{\mu} = 0$.

Let us introduce the notation

$$F(\xi) = \left(\frac{1}{\xi} - \frac{1}{e^\xi - 1} \right).$$

For $\gamma < 0$ and $\tilde{\mu} = 0$ we have (according to (14))

$$(33) \quad N_c = \Lambda^{\gamma-\gamma_c} \sum_{j=1}^{\infty} \frac{j^\gamma}{e^{bj} - 1} = \Lambda^{\gamma-\gamma_c} \left\{ \sum_{j=1}^{\infty} j^\gamma \frac{1}{bj} - \sum_{j=1}^{\infty} j^\gamma F(bj) \right\}, \quad b = \frac{E_0}{T}.$$

Since the function $f(x) = x^\gamma F(bx)$ monotonically decreases, we have

$$(34) \quad \begin{aligned} \sum_{j=1}^{\infty} j^\gamma F(bj) &= \sum_{j=1}^{\infty} f(j) \leq \int_0^{\infty} f(x) dx \\ &= \int_0^{\infty} x^\gamma F(bx) dx = b^{-\gamma-1} \int_0^{\infty} x^\gamma F(x) dx \end{aligned}$$

(the Nazaikinsky inequality).

Thus, we have

$$(35) \quad N|_{\tilde{\mu}=0} = \Lambda^{\gamma-\gamma_c} b^{-1} \zeta(1 - \gamma) + O(b^{-1-\gamma}), \quad \gamma < 0,$$

where ζ is the Riemann zeta function.

In this section we find that the liquid isotherm-isobar point as the quantity $\varkappa = -\mu/T$ slowly tends to zero.

First, we recall that although N_c is large, it is still finite, and hence we must use the parastatistical correction. We recall the relation

$$(36) \quad \Omega = -\Lambda^{\gamma-\gamma_c} T \sum_k \log k^{\gamma+1} \left(\frac{1 - \exp \frac{\mu - \varepsilon_k}{T_r} N}{1 - \exp \frac{\mu - \varepsilon_k}{T}} \right), \quad \varepsilon_k = E_0 k,$$

for the Ω -potential.

We regularize $\sum G_i$ by the photon statistical weight $\frac{1}{e^{bj} - 1}$ and we obtain

$$\sum \frac{G_i}{e^{bj} - 1} = N \Lambda^{\gamma_c - \gamma}.$$

If we set

$$E_0 = \frac{T_c}{N} \zeta(1 - \gamma) \Lambda^{\gamma - \gamma_c}, \quad -1 < \gamma < 0$$

$(E_0 \rightarrow 0 \text{ as } N \rightarrow \infty)$, we obtain

$$\frac{\Lambda^{\gamma - \gamma_c} \sum(G_i)/(e^{bj} - 1)}{\sum N_i} = \rho = 1.$$

We obtain the following value of *relative entropy in the case of Bose statistics* [1]:

$$(37) \quad S_B = \sum(p_i + q_i) \log(p_i + q_i) - p_i \log p_i - q_i \log q_i,$$

where $p_i = N_i/N$, $q_i = \Lambda^{\gamma - \gamma_c} G_i (e^{bj} - 1)^{-1}/N$ in the case of $\gamma < 0$ (holes).

We use the Euler–Maclaurin formula with parameter γ being taken into account and obtain [29]

$$\sum_{j=1}^n \left(\frac{j^\gamma}{e^{bj+\varkappa} - 1} - \frac{k j^\gamma}{e^{bkj+\varkappa}} \right) = \frac{1}{\alpha} \int_0^\infty \left(\frac{1}{e^{bx+\varkappa} - 1} - \frac{k}{e^{bx+\varkappa} - 1} \right) dx^\alpha + R,$$

where $\alpha = \gamma + 1$, $k = N$, and $b = E_0/T$. Here the remainder R satisfies the estimate

$$|R| \leq \frac{1}{\alpha} \int_0^\infty |f'(x)| dx^\alpha, \quad \text{where } f(x) = \frac{1}{e^{bx+\varkappa} - 1} - \frac{k}{e^{k(bx+\varkappa)} - 1}.$$

We calculate the derivative and obtain

$$(38) \quad \begin{aligned} f'(x) &= \frac{bk^2 e^{k(bx+\varkappa)}}{(e^{k(bx+\varkappa)} - 1)^2} - \frac{be^{bx+\varkappa}}{(e^{bx+\varkappa} - 1)^2}, \\ |R| &\leq \frac{1}{\alpha b^\alpha} \int_0^\infty \left| \frac{k^2 e^{k(y+\varkappa)}}{(e^{k(y+\varkappa)} - 1)^2} - \frac{e^{y+\varkappa}}{(e^{y+\varkappa} - 1)^2} \right| dy^\alpha. \end{aligned}$$

We also have

$$\frac{e^y}{(e^y - 1)^2} = \frac{1}{y^2} + \psi(y), \quad \text{where } \psi(y) \text{ is smooth and } |\psi(y)| \leq C(1 + |y|)^{-2}.$$

Substituting this formula into (38), we obtain

$$\begin{aligned} |R| &\leq \frac{1}{\alpha b^\alpha} \int_0^\infty |\psi(k(y+\varkappa)) - \psi(y+\varkappa)| dy^\alpha \\ &\leq \frac{k^{-\alpha}}{b^\alpha} \int_{k\varkappa}^\infty |\psi(y)| dy^\alpha + \frac{1}{b^\alpha} \int_\varkappa^\infty |\psi(y)| dy \leq C b^{-\alpha} \end{aligned}$$

with a certain constant C . For example, if $\varkappa \sim (\log k)^{-1/4}$, then $|R|$ preserves the estimate $|R| \sim O(b^{-\alpha})$. Therefore, we have $k = N_c$ and $T = T_c$ and obtain the following formula for the integral at $\mu = 0$:

$$M = \frac{\Lambda^{\gamma - \gamma_c}}{\alpha \Gamma(\gamma + 2)} \int \frac{\xi d\xi^\alpha}{e^{b\xi} - 1} = \frac{\Lambda^{\gamma - \gamma_c}}{b^{1+\alpha}} \int_0^\infty \frac{\eta d\eta^\alpha}{e^\eta - 1},$$

where $\alpha = \gamma + 1$. Thus,

$$b = \frac{1}{M^{1/(1+\alpha)}} \left(\frac{\Lambda^{\gamma-\gamma_c}}{\alpha\Gamma(\gamma+2)} \int_0^\infty \frac{\xi d\xi^\alpha}{e^\xi - 1} \right)^{1/(1+\alpha)}.$$

We obtain (see [20])

$$\begin{aligned} \int_0^\infty \left\{ \frac{1}{e^{b\xi} - 1} - \frac{k}{e^{kb\xi} - 1} \right\} d\xi^\alpha &= \frac{1}{b^\alpha} \int_0^\infty \left(\frac{1}{e^\xi - 1} - \frac{1}{\xi} \right) d\xi^\alpha \\ &+ \frac{1}{b^\alpha} \int_0^\infty \left(\frac{1}{\xi} - \frac{1}{\xi(1 + (k/2)\xi)} \right) d\xi^\alpha - \frac{k^{1-\alpha}}{b^\alpha} \int_0^\infty \left\{ \frac{k^\alpha}{e^{k\xi} - 1} - \frac{k^\alpha}{k\xi(1 + (k/2)\xi)} \right\} d\xi^\alpha \\ &= \frac{c(\gamma)}{b^\alpha} (k^{1-\alpha} - 1). \end{aligned}$$

Since $k \gg 1$, we finally have by setting $k = N|_{\bar{\mu}/T=o(1)}$,

$$(39) \quad N|_{\bar{\mu}/T=o(1)} = (\Lambda^{\gamma-\gamma_c} c(\gamma))^{1/(1+\gamma)} T, \quad \text{where } c(\gamma) = \int_0^\infty \left(\frac{1}{\xi} - \frac{1}{e^\xi - 1} \right) \xi^\gamma d\xi.$$

$N = T_c^{\gamma_c+1} \zeta(\gamma_c + 1)$ agrees with the linear relation $N = A(\gamma)T_c$, where $A(\gamma) = (\Lambda^{\gamma-\gamma_c} c(\gamma))^{1/(1+\gamma)}$ for $P < 0$.

We can normalize the activity a at the point T_c and can find a_0 by adjusting the liquid and gas branches at T_c for the pressure (i.e., for M) so that the phase transition on the critical isotherm could not occur at $T_r = 1$.

Further, we normalize the activity for $T_r < 1$ by the value of a_0 calculated below. Then the chemical potentials (the Gibbs thermodynamical potential for the liquid and gas branches in thermodynamics) coincide, and hence no “gas–liquid” phase transition occurs at $T_r = 1$ (cf. section 5).

Now for the isochor–isotherm of “incompressible liquid” to take place, we must also construct it for a density less than $N_c = \zeta(\gamma_c + 1)$, namely, for

$$N(T) = T_r^{\gamma_c+1} \zeta(\gamma_c + 1).$$

Here the values $\gamma_c = \gamma_c(T_r)$ are obtained from the experimental spinodal on the isotherm $T = T_r$ (see [30]). We obtain the value $\gamma(T_r)$ from the implicit equation

$$A(\gamma) = T_r^{\gamma_c} \zeta(\gamma_c + 1), \quad \gamma_c = \gamma_c(T_r).$$

We thus obtain the spinodal curve (i.e., the points where $\tilde{\mu} \sim T(\log N(T_r))^{-1/4}$) for each $T_r < 1$ in the region of negative γ :

$$(40) \quad \Lambda^{-(\gamma-\gamma_c)/(1+\gamma)} c(\gamma)^{1/(1+\gamma)} = T_r^{\gamma_c} \zeta(\gamma_c + 1), \quad \gamma_c = \gamma_c(T_r).$$

From the two values of γ of the solution (40) we take the least (the largest in the absolute value) and denote it by $\gamma(T_r)$. In particular, for $T_r = 1$ we denote $\gamma_0 = \gamma(1)$.

The second solution takes place for some metals (mercury, cesium, etc.) and corresponds to the metastable state. The actual interaction between particles gives a more significant deviation of the second solution from γ_0 (see [19]).

Let $a_g = e^{-\mu/T}$ be the gas activity, and let $a_l = e^{-\tilde{\mu}/T}$ be the liquid activity. We give the condition that M and the activities coincide at the point of phase transition (see section 5):

$$(41) \quad T_r^{\gamma_c} \text{Li}_{2+\gamma_c}(a_g) = \Lambda^{-|\gamma(T_r)|-\gamma_c} T_r^{-|\gamma(T_r)|} \text{Li}_{2-|\gamma(T_r)|} \left(\frac{a_l}{a_0} \right),$$

$$(42) \quad \frac{\Lambda^{\gamma_0 - \gamma_c}}{\zeta(2 + \gamma_0)} \text{Li}_{2+\gamma_0}(a_0) = 1 \quad \text{since } T_r = 1,$$

$$(43) \quad a_g = \frac{a_l}{a_0}.$$

These two equalities determine the value of the chemical potential $\mu = \tilde{\mu} = T \log a_g$ at which the “gas–liquid” phase transition occurs.

Let $T_0 = \min_{-1 < \gamma < 0} A(\gamma)$. Thus, for each $T_0 < T < T_c$ we obtain the value of the liquid reduced activity $a_r = a_l/a_0$ (where a_l is the liquid activity), which corresponds to the Van der Waals normalization (see (5)).

In thermodynamics, the critical values T_c , P_c , and ρ_c were experimentally calculated for almost all gases, and hence the critical number of degrees of freedom can be given in advance. The parameter Λ ($1.6 < \Lambda < 3$, $T > 1/3$) is obtained from the coincidence of binodals at the triple point determined experimentally.

Remark 5. In probability theory and economics, these quantities can be determined only by indirect methods (for example, by examining the income growth rate (see below) or by analysis of a sufficiently large set of debt-duration pairs (duration is the period of debt repayment); see [21]). But analogy with the Van der Waals law of corresponding states we hope that the scenarios of “phase transitions” are sufficiently similar for different states (see [22]).

5. Infinitely small dissipation taken into account and the Maxwell rule.

A. S. Mishchenko and one of the authors [23] considered the case of transition to the two-dimensional Lagrangian manifold in the four-dimensional phase space, where the pressure P and the temperature T (intensive variables) play the role of coordinates, and the extensive variables, the volume V , and the entropy S play the role of momenta of the Lagrangian manifold, where the entropy is an action generating the Lagrangian structure.

It turns out that this complicated transformation, which leads to the Gibbs thermodynamical potential, can be performed only by using an extension to the region of negative values of γ . Then one can justify the Maxwell transition by introducing a small dissipation (viscosity). The introduction of infinitely small dissipation allows one to simultaneously solve the problems of critical exponents without using the scaling hypothesis, which underlies the renormalization group method. Let us show this.

There is no viscosity in thermodynamics. However, in general, thermodynamical equilibrium cannot occur without infinitely small dissipation. Therefore, it is natural to introduce a small viscosity and then to let it tend to zero.

The geometric quantization of the Lagrangian manifold (see [24, section 11.4]) is usually associated with the introduction of the constant \hbar . The case of pure imaginary number \hbar was called the Wiener (or tunnel) quantization by one of the authors [25].

Let us apply the Wiener quantization to thermodynamics. The thermodynamical potential $G = \mu N$ is the action $\mathbf{S} = \int p dq$ on the two-dimensional Lagrangian manifold Λ^2 in the four-dimensional phase space q_1, q_2, p_1, p_2 , where q_1 and q_2 are the pressure P and the temperature T , respectively, p_1 is equal to the volume V , and p_2 is equal to the entropy S with the opposite sign. All other potentials, i.e., the internal energy E , the free energy F , and the enthalpy W , are the results of projection of the

Lagrangian manifolds on the coordinate planes p_1, p_2

$$(44) \quad \begin{aligned} E &= - \int \vec{q} d\vec{p}, \quad \vec{q} = \{q_1, q_2\}, \quad \vec{p} = \{p_1, p_2\}, \\ W &= - \int (q_2 dp_2 + q_1 dp_1), \quad F = \int (q_1 dp_1 - q_2 dp_2). \end{aligned}$$

In the Wiener quantization, we have

$$N = \varepsilon \frac{\partial}{\partial \mu}, \quad V = \varepsilon \frac{\partial}{\partial p}, \quad S = -\varepsilon \frac{\partial}{\partial T}.$$

Hence the role of time t is played by $\log(-\mu/T)$

$$G = \mu N \sim \varepsilon \frac{\mu}{T} \frac{\partial}{\partial(\mu/T)} = \varepsilon \frac{\partial}{\partial \log(\mu/T)}.$$

Note that the tunnel quantization of the Van der Waals equation (VdW) gives the Maxwell rule as $\varepsilon \rightarrow 0$ (see below).

As seen below, the critical point and spinodal points are focal point and hence do not “pass” into classics, i.e., into the VdW model, as $\varepsilon \rightarrow 0$. The spinodal points, similarly to the turning points in quantum mechanics, are approximated by the Airy function, and the critical point, which is the point of origination of two turning points (two Airy functions), is approximated by the Weber function (see [26]). The point of shock wave origination in the Bürgers equation is expressed precisely in terms of the Weber function as $\varepsilon \rightarrow 0$. If the transition as $\varepsilon \rightarrow 0$ is performed outside these points, then the VdW–Maxwell model is obtained. But at these points themselves the limit transition is violated. Therefore, the so-called “classical” critical Landau exponents [14] turn out to seriously disagree with experiments. The Weber function gives singularities of the form $\varepsilon^{-1/4}$, and the Airy function those of the form $\varepsilon^{-1/6}$.

We now consider the Bürgers equations in more detail.

Consider the heat equation

$$(45) \quad \frac{\partial u}{\partial t} = \frac{\varepsilon}{2} \frac{\partial^2 u}{\partial x^2}, \quad x \in \mathbf{R}, \quad t \geq 0,$$

where $\varepsilon > 0$ is a small parameter. It is known that all linear combinations

$$(46) \quad u = \lambda_1 u_1 + \lambda_2 u_2$$

of the solutions u_1 and u_2 to (45) are solutions of this equation.

We perform the change

$$(47) \quad u = \exp \left(-\frac{w(x, t)}{\varepsilon} \right)$$

and obtain the nonlinear equation

$$(48) \quad \frac{\partial w}{\partial t} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 - \frac{\varepsilon}{2} \frac{\partial^2 w}{\partial x^2} = 0,$$

which is called the integrated Bürgers equation.⁸ Obviously, any solution u_i of (45) is associated with the solution $w_i = -\varepsilon \log u_i$ of (48), $i = 1, 2$. The solution (46) of (45)

⁸The regular Bürgers equation can be derived from (48) by differentiating with respect to x and performing the change $v = \partial w / \partial x$.

is associated with the solution of (48)

$$w = -\varepsilon \log \left(e^{-\frac{w_1+\mu_1}{\varepsilon}} + e^{-\frac{w_2+\mu_2}{\varepsilon}} \right),$$

where $\mu_i = -\varepsilon \log \lambda_i$ ($i = 1, 2$). Since

$$\lim_{\varepsilon \rightarrow 0} w = \min(w_1, w_2),$$

we obtain the $(\min, +)$ algebra of tropical mathematics [27].

To determine the solutions for $t > t_{\text{cr}}$, Hopf proposed considering the Burgers equation

$$(49) \quad \frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} - \frac{\varepsilon}{2} \frac{\partial^2 v}{\partial x^2} = 0, \quad v|_{t=0} = p_0(x),$$

The (generalized) solution of the equation

$$(50) \quad \frac{\partial p}{\partial t} + p \frac{\partial p}{\partial x} = 0, \quad p|_{t=0} = p_0(x)$$

is the function $p_{\text{gen}} = \lim_{\varepsilon \rightarrow 0} v$ (the Riemann waves).

The solution v of the Burgers equation is expressed in terms of the logarithmic derivative

$$(51) \quad v = -\varepsilon \frac{\partial}{\partial x} \log u$$

of the solution u of the heat equation

$$(52) \quad \frac{\partial u}{\partial t} = \frac{\varepsilon}{2} \frac{\partial^2 u}{\partial x^2}, \quad u|_{t=0} = \exp \left\{ -\frac{1}{\varepsilon} \int_{-\infty}^x p_0(x) dx \right\}.$$

Thus, the initial problem is reduced to studying the logarithmic limit of the solution of the heat equation. It is known that the solution of problem (52) has the form

$$(53) \quad u = (2\pi\varepsilon t)^{-1/2} \int_{-\infty}^{\infty} \exp \left\{ -\left((x-\xi)^2 + 2t \int_{-\infty}^{\xi} p_0(\xi) d\xi \right) / 2th \right\} d\xi.$$

The asymptotic behavior of the integral (53) is calculated by the Laplace method. For $t < t_{\text{cr}}$, we have

$$(54) \quad u = (|J|^{-1/2} (\xi(x, t), t) + O(\varepsilon)) \exp \left\{ -\frac{1}{\varepsilon} S(x, t) \right\}.$$

Here $S(x, t) = \int_{-\infty}^{r(t)} p dx$, and the integral is calculated along the Lagrangian curve Λ^t , and the point $r(x)$ is a point on Λ^t . For $t > t_{\text{cr}}$, there are three points $r_1(x)$, $r_2(x)$, $r_3(x)$ on Λ^t whose projections on the axis x coincide or, in other words, the equation $Q(t, \xi) = x$ for $x \in (x_1, x_2)$ has three solutions $\xi_1(x, t)$, $\xi_2(x, t)$, $\xi_3(x, t)$.

We introduce the notation $S(x, t) = \int_{-\infty}^{r(x)} p dx$ for $x < x_1$, $x > x_2$, and $S(x, t) = \min(S_1, S_2, S_3)$, $S_j = \int_{-\infty}^{r_j(x)} p dx$, where $j = 1, 2, 3$ for $x \in [x_1, x_2]$.

The above argument allows us to obtain a generalized discontinuous solution of problem (50) for times $t > t_{\text{cr}}$. It is given by the function $p = p(x, t)$ determining

essential regions [25] on the curve Λ^t . We note that, in particular, this implies the rule of equal areas applied in hydrodynamics for defining the front of a shock wave whose evolution is described by (50). We note that this exactly corresponds to the Maxwell rule for the VdW equation.

The solution $v = v(x, \varepsilon)$ of the Burgers equation at the critical point $x = p^3$ is calculated by the formula

$$(55) \quad v(x, \varepsilon) = \varepsilon \frac{\partial \log u(x)}{\partial x} = \frac{\int_0^\infty \exp\left\{\frac{-x\xi - \xi^4/4}{\varepsilon}\right\} \xi d\xi}{\int_0^\infty \exp\left\{\frac{-x\xi - \xi^4/4}{\varepsilon}\right\} d\xi}.$$

As $x \rightarrow 0$, after the change $\frac{\xi}{\sqrt[4]{\varepsilon}} = \eta$, we obtain

$$(56) \quad v(\varepsilon, x) \xrightarrow{x \rightarrow 0} \sqrt[4]{\varepsilon} \cdot \text{const.}$$

What does this mean from the standpoint of classics and classical measurements, i.e., in the case where the so-called condition for being “semiclassical” in the book [28] is satisfied (i.e., in the case where we are outside of the focal point)? For the Laplace transformation this means that we are in the region where the asymptotic Laplace method can be applied, i.e., where

$$(57) \quad u(x) = \frac{1}{\sqrt{\varepsilon}} \int_0^\infty e^{-\frac{px - \tilde{S}(p)}{\varepsilon}} dp.$$

Moreover, the solution of the relation

$$(58) \quad x = \frac{\partial \tilde{S}}{\partial p}$$

is not degenerate, i.e., $\frac{\partial^2 \tilde{S}}{\partial p^2} \neq 0$ at the point $\frac{\partial \tilde{S}}{\partial p} = x$, and the reduced integral (57) is bounded as $\varepsilon \rightarrow 0$. For this integral to have a zero of the order of $\varepsilon^{1/4}$, we must integrate it over x with the fractional derivative $D^{-1/4}$. The value of $D^{-1/4}$ applied to unity, $D^{-1/4}1$, is approximately equal to $x^{1/4}$.

According to the indeterminacy principle [9], the correspondence between the differentiation operator and a small parameter of the form $D \rightarrow 1/\varepsilon$ is preserved for the ratio $-\varepsilon \frac{\partial u / \partial x}{u}$, although the leading term of the asymptotic expansion in the difference of $\frac{\partial^2 u / \partial x^2}{u}$ and $\frac{(\partial u / \partial x)^2}{u^2}$ is eliminated.

In the case of thermodynamics, the role of x is played by the pressure P and the role of the momentum p is played by the volume V . Therefore, $V \sim P^{1/4}$, i.e.,

$$(59) \quad P_c \sim (V - V_c)^4.$$

This is precisely the jump of the critical exponent. The other critical exponents can be obtained similarly [9]. A comparison with experimental data is discussed in the same paper.

6. Independent events, conditional probability, and logarithmic scaling.

DEFINITION 1. *We will call two events independent if the Hartley entropies corresponding to the events add together.*

Let us consider the conditional probability for independent events.

It is well known that the *conditional probability* of an event under a condition is the probability of the simultaneous occurrence of the event and the condition normed by the probability of the condition:

$$\mathbf{P}(A | B) = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(B)},$$

where A is the event and B is the condition.

The formula

$$(60) \quad \mathbf{P}(B) = \sum_{i=1}^k \mathbf{P}(B \cap A_i) = \sum_{i=1}^k \mathbf{P}(B | A_i) \mathbf{P}(A_i), \quad \prod A_i = \Omega,$$

for nonintersecting events $\{A_i\}$ is called the *total probability* formula.

A similar formula holds in the new concept of probability. It suffices to prove it for $k = 2$. This, according to Kolmogorov's complexity theory, must be the equality of entropies with conditional probability $\{\alpha, \beta = 1 - \alpha\}$ taken into account. Namely, for $\tilde{\mu} = 0$ we have

$$(61) \quad Z_c(\gamma_c + 2) = \alpha(\gamma_1^c + 2)Z_1^c + \beta(\gamma_2^c + 2)Z_2^c,$$

where $Z_c = \zeta(\gamma + 2)/\zeta(\gamma + 1)$, $\alpha + \beta = 1$.

From these formulas, we obtain the value of γ_c for "mixtures" such as those given in (60) by using the basis variational series.

The above construction can easily be expanded to the general case of formula (60).

Logarithmic scaling determines, in the order statistics (variational series), a bijection between the numbers $A \geq 0$ and $\log A = a$, $B \geq 0 \rightarrow b = \log B$. Product AB corresponds, under logarithmic scaling, to the sum $\log A + \log B$.

Addition under logarithmic scaling. The summing algorithm is given by

$$(62) \quad A \oplus B = \frac{1}{\log n} \log(n_c^{(1)} + n_c^{(2)})$$

with the restriction (61) on the growth of n taken into account.

By means of this bijection, we can encode any numbers A and B , as well as the semifield AB and the correspondence $A + B \rightarrow A \oplus B$.

The addition (62) as $n \rightarrow \infty$ undergoes a "phase jump" from the "gas" state to the "liquid" state, i.e., from an "open state" to a "closed" one.

Remark 6. Note that the volume V , one of the main notions in thermodynamics, is absent in our considerations, and the role of density is played by the number of particles N . During the gas-liquid phase transition, the density undergoes a jump and therefore, in our case, so does the quantity N . In thermodynamics, it is natural to assume that the number of particles is constant, and a jump in density is due to the jump in the volume V . Then the chemical potential μ , multiplied by the number of particles, is equal to the Gibbs thermodynamical potential, which remains continuous at the point of phase transition (the Maxwell rule). Therefore, the chemical potential of the liquid and the gas phases must coincide.

Remark 7. This entropy, according to Kolmogorov's complexity theory, is intimately linked with encodings of minimal length. Thus the phase transition of the "gas" entropy (30) into the "liquid" entropy (31) is a transition to the new encoding. We cannot decrease the number of trials: *the number of trials is irreversible*. In our

case an increase in the number of trials in the state of “liquid phase” gives us nothing, because the extra trials fall into the “Bose condensate” of the “Bose liquid.”

Remark 8. In thermodynamics, the volume V is also introduced, but the geometry of the volume V is neglected in principle, i.e., the problem is invariant under the variations of its shape. There is no external field in this problem, and hence the one-particle potential in the Gibbs distribution must have the form $\varphi(r^3/V)$ if the molecule radius is “infinitely small.” And since the volume is large, the leading term as $V \rightarrow \infty$ has the form

$$(63) \quad -\frac{\alpha r^3}{V},$$

where α is a parameter which can be expressed in terms of a new parameter q . The parameter q is called the heat of transition from the gas phase into the liquid phase [14] as $T \rightarrow 0$. Since an additional parameter must be independent from the shape of the vessel, it suffices to use the volume of a ball of radius R for determining this parameter.

Taking into account this potential changes the pressure and the compressibility factor Z , which allows one to take into account the reflection from the vessel walls in the general form. We have

$$(64) \quad \int_0^R e^{-r^3 \xi k/R} dr^3 = \frac{1}{k(\beta - \beta_c)} \int_0^{k(\beta - \beta_c)} e^{-y} dy = \frac{1}{k(\beta - \beta_c)} (e^{(\beta_c - \beta)k} - 1),$$

where $\xi = q(\beta - \beta_c)$, $\beta = 1/T$, $\beta_c = 1/T_c$.

Hence the k th term in the definition of polylogarithm

$$\text{Li}_s(a) = \sum_{k=1}^{\infty} \frac{a^k}{k^s}$$

becomes

$$\frac{1}{\xi} \cdot \frac{(e^{\xi k} - 1)a^k}{k^{3+\gamma_c}},$$

whence

$$(65) \quad P = -\frac{T^{2+\gamma_c}}{\xi} \{ \text{Li}_{3+\gamma_c}(ae^{-\xi}) - \text{Li}_{3+\gamma_c}(a) \},$$

$$(66) \quad Z = \frac{\text{Li}_{3+\gamma_c}(ae^{-\xi}) - \text{Li}_{3+\gamma_c}(a)}{\text{Li}_{2+\gamma_c}(ae^{-\xi}) - \text{Li}_{2+\gamma_c}(a)}.$$

Now the condition for the spinodal in the domain $\gamma < 0$ becomes

$$(67) \quad A(\gamma) = T^{\gamma_c} \frac{|\text{Li}_{2+\gamma_c}(e^{-\xi}) - \text{Li}_{2+\gamma_c}(1)|}{\xi}.$$

7. Applications.

7.1. Distribution according to incomes. One of the most important problems of contemporary mathematical statistics is that the “general population” of measurable random variables is very large, and the samples are also large, but much smaller than the general population.

In the USSR the sample used in social questions was usually 3.5 thousand people.

As an example of the general population, one could consider the set of all citizens of the Russian Federation. We would like to calculate their mean income.

There is a quantity fixed by the state: the minimal wage and the nominal percentage $\tilde{\mu}$.

On the other hand, the income of many citizens is less than the minimum wage, e.g., the family members of the worker (dependents), as well as the unemployed. If we construct the order series, the “order statistics” according to increasing incomes⁹ and divide it up into clusters of 1%–3%, then, denoting the values of the random variable of ξ -income by G_i , we obtain the variational series

$$\left\{ \frac{G_i}{T} \right\}_{i=0}^{\infty}.$$

Suppose that the number of trials (size of the sample) is n , while n_i is the number of outcomes for which the number G_i/T is obtained; then, according to the usual definition, the estimate of the expectation of the sample, multiplied by n trials, is equal to

$$(68) \quad \sum i n_i \frac{G_i}{T} = M, \quad \sum n_i = n.$$

Thus the estimate of the expectation of the sample is

$$(69) \quad E = \frac{M}{n}.$$

Obviously, if the G_i take unboundedly large values, then a randomly chosen sample cannot serve even as an approximate “estimate” of the correct expectation.

Thus, in determining the average per capita income, we must divide the general population into parts, distinguishing the region that characterizes (on average) the studied stratum of society, for which we accept the hypothesis of homogeneous distribution. Sometimes one considers separate households in their development (longitudinal studies).

Sometimes one also uses the following methodology for measuring the distribution of the population according to income. Here the initial data are regular reports on 50 000 people, of which 2000 are muscovites, not supported by any documents. The choice of these respondents is purposely left unchanged (new ones are added only when some of the previous ones disappear). There are no very rich people among them, because participation is purely voluntary. There may be some people among them who live below the poverty line, because the reports are modestly paid for.

The obtained data is extrapolated by comparing the socio-economic characteristics of the respondents (region and type of city, town, or village, gender, age, size of the family, profession, etc.) to the data given by census.

No estimates of precision of such studies are made, either in Russia or abroad. There is only a comparison with “theoretically expected” distributions (lognormal with a Pareto tail). Pareto was the first to propose the power distribution as the description of the distribution of incomes in accordance with empirical data. There have also been theoretical discussions about the possibility of interpreting the observed discrepancies by the presence of numerous maxima.

⁹This can actually be done only in the countries where the incomes can be calculated from the paid taxes, which means that the incomes are not hidden as a rule.

However, it is impossible to partition an infinite set into a finite family of finite sets.

Therefore, in our case it is natural to call the estimate of an expectation “an empirical expectation.”

We called the quantity

$$(70) \quad \lim_{n \rightarrow \infty} \frac{\log M}{\log n} = D$$

the *dimension* in the situation corresponding to quantum mechanics and the *number of degrees of freedom* in the statistical case.

It turns out that for any M (under our assumption (70) there exists a critical number of measurements $N_c \ll M$ such that, when the number of measurements increases above this critical value $N > N_c$ with our asymptotic precision (e.g., $O(N^\alpha)$, where $\alpha < 1$), we will reach the level of the unemployed or very low income groups, which, within the limits of our accuracy, does not influence the value of N_c .

Remark 9. Let us explain what we mean by a closed monetary system. This is not the analogue of the situation of the ruble in the USSR, although the ruble was not convertible. The USSR did not have a market economy; in other words, it did not have an economic system based on entropy.

Another type of closed system is the closed system of currency in postwar Great Britain. At that time the exchange of the pound for other currencies was forbidden. Similarly, in Israel, the exchange of the shekel was only allowed within the limit of \$500 U.S. dollars.

A rough analogy with liquid and gas may be carried out here. If we open the cork of a bottle of gas, the gas will leave the bottle, unlike the contents of a bottle of water. The surface tension, which is the boundary of the liquid, is similar to the boundary of a country with a closed currency system.

7.2. Problems of queuing theory on the Web. Here for the general population we take into consideration all the communicators: computers, i-pads, i-phones, smart phones, etc.

When there is only one communication channel, then a message sent before the others will arrive first. Here the order is preserved, and the Shannon theory is applicable; in particular we have the Shannon entropy, which is equal to the Boltzman entropy. But if the messages are gathered in packets, then this principle can no longer be applied, and we arrive at a specific situation. In problems of queuing theory, the value of the random variable defined as the number of dispatched packets, corresponding to the given general population, grows without a bound. We have in mind the order statistics and the corresponding variational series (see [13]).

A supercomputer connected to the World Wide Web is similar to a member state in a worldwide union of states. Sufficient conditions for the computer’s dropping out of the Web follow from the unrestricted probability theory described above.

Remark 10. Under the total reign of the World Wide Web, when even high level functionaries must confess to it under control of its lie detector, ethical rules prescribed by various religions as “rules of the game” and ensuring the existence of human society, can be monitored via the Web.

7.3. Relationship with the Monte-Carlo method. As an example of the relationship to the Monte-Carlo method, let us describe a specific problem for a self-teaching audio-guide, a device that describes paintings in art museums. This device

is oriented to three degrees of freedom: motion in the direction “north–south,” motion in the direction “east–west,” and rotation about its axis. However, the device must also take into account the directions along which there are no paintings (e.g., the directions of certain corridors).

On the one hand, three degrees of freedom give values of the random variable that grow like $\{i^{3/2}\}$. On the other hand, monitoring of corridors without paintings is naturally ensured by a separate program in its self-teaching software.

The application of the Monte-Carlo method for choosing a pseudorandom variable in the computer language Java preserves the number of degrees of freedom. For a sufficiently large number of trials (i.e., visitors to the museum), the device’s software, supposedly adapted to the picture gallery, can become, as the result of self-teaching, a software adapted to the architecture of that specific museum. And this is the passage from an open system to a closed one.

If the usual rounding procedure is used, i.e., if pseudorandom variables of some finite length are considered, then testing of an initial program is not sufficiently perfect, and hence the device cannot meet all requirements of the consumer. In the example of an art museum, this fact is not catastrophic. Only the reputation of the firm producing this device will suffer. But if such a device is used as a guide for a blind man, then such a rounding imperfection can lead to a catastrophe.

Acknowledgments. The authors express their deep gratitude to Academician A. A. Guseinov, and to professors G. I. Arkhipov, V. S. Vorobiev, V. N. Chubarikov, and M. S. Kosolapov for fruitful discussions. We also especially express our thanks to the editor T. Tolozova for valuable remarks about the composition of our humanities-mathematical paper.

REFERENCES

- [1] V. P. MASLOV, *Integral equations and phase transitions in stochastic games. An analogy with statistical physics*, Theory Probab. Appl., 48 (2004), pp. 359–367.
- [2] V. P. MASLOV, *Nonlinear averaging axioms in financial mathematics and stock price dynamics*, Theory Probab. Appl., 48 (2004), pp. 723–733.
- [3] H. POINCARÉ, *Science et Hypothèse, Valeur de la Science, Science et Méthode, Dernières Pensées*.
- [4] A. A. GUSEINOV, *Great Prophets and Thinkers, Ethical Studies from Moses to the Present Day*, Veche Editions, Moscow, 2009 (in Russian).
- [5] V. V. KOZLOV, *Kinetics of collisionless gas: Equalization of temperature, growth of the coarse-grained entropy and the Gibbs paradox*, Regul. Chaotic Dyn., 14 (2009), pp. 535–540.
- [6] V. P. MASLOV, *Solution of the Gibbs paradox in the framework of classical mechanics (statistical physics) and crystallization of the gas C₆₀*, Math. Notes, 83 (2008), pp. 716–722.
- [7] V. P. MASLOV, *Solution of the Gibbs paradox using the notion of entropy as a function of fractal dimension*, Russian J. Math. Phys., 17 (2010), pp. 251–261.
- [8] V. P. MASLOV, *Gibbs paradox, liquid phase as an alternative to the Bose condensate, and homogeneous mixtures of new ideal gases*, Math. Notes, 89, (2011), pp. 366–373.
- [9] V. P. MASLOV, *Critical exponents as the Wiener quantization of thermodynamics*, Theoret. Math. Phys., 170 (2012), pp. 384–393.
- [10] V. P. MASLOV, *Degeneration in the transition from the discrete spectrum to the continuous one and the transition from quantum mechanics to classical mechanics*, Dokl. Akad. Nauk SSSR, 114 (1957), pp. 957–960 (in Russian).
- [11] T. V. MASLOVA, *A refinement of the Zipf law for frequency dictionaries*, Scientific and Technical Information Processing, 11 (2006), pp. 37–44 (in Russian).
- [12] P. ERDŐS AND J. LEHNER, *The distribution of the number of summands in the partitions of a positive integer*, Duke Math. J., 8 (1941), pp. 335–345.
- [13] V. P. MASLOV, *Unbounded probability theory compatible with the probability theory of numbers*, Math. Notes, 91 (2012), pp. 603–609.
- [14] L. D. LANDAU AND E. M. LIFSHITS, *Statistical Physics*, Nauka, Moscow, 1964 (in Russian).

- [15] V. P. MASLOV, *Mathematical aspects of weakly nonideal Bose and Fermi gases on a crystal base*, Functional Anal. Appl., 37 (2003), pp. 94–102.
- [16] A. M. VERSHIK, *Statistical mechanics of combinatorial partitions, and their limit shapes*, Functional Anal. Appl., 30 (1996), pp. 90–105.
- [17] V. P. MASLOV AND V. E. NAZAIKINSKII, *On the distribution of integer random variables related by a certain linear inequality*: I, Math. Notes, 83 (2008), pp. 211–237.
- [18] V. P. MASLOV, *Bose condensate in the D-dimensional case, in particular, for D = 2*, Russian J. Math. Phys., 19 (2012), pp. 1–10.
- [19] K. I. SHMULOVICH AND L. MERCURY, *Geochemical processes at negative pressures*, Electronic Scientific Information Journal “Herald of the Departure of Earth Sciences RAS,” 1 (2006), pp. 1–3.
- [20] V. P. MASLOV, *Threshold Levels in Economics*, preprint, arXiv:0903.4783v2 [q-fin.ST], 2009.
- [21] V. P. MASLOV, *Theorems on the debt crisis and the occurrence of inflation*, Math. Notes, 85 (2009), pp. 146–150.
- [22] V. P. MASLOV, *Quantum Economics*, Nauka, Moscow, 2006 (in Russian).
- [23] V. P. MASLOV AND A. S. MISCHENKO, *Geometry of a Lagrangian manifold in thermodynamics*, Russian J. Math. Phys., 10 (2003), pp. 161–172.
- [24] N. HURT, *Geometric Quantization in Action*, Reidel, Dordrecht, 1983.
- [25] V. P. MASLOV, *Asymptotic Methods and Perturbation Theory*, Nauka, Moscow, 1988 (in Russian).
- [26] V. P. MASLOV, *Nonstandard characteristics in asymptotic problems*, Russian Math. Surveys, 38 (1983), pp. 1–42.
- [27] G. L. LITVINOV, *The Maslov dequantization, idempotent and tropical mathematics: A very brief introduction*, in Idempotent Mathematics and Mathematical Physics, Contemp. Math. 377, Amer. Math. Soc., Providence, RI, 2005, pp. 1–17.
- [28] L. D. LANDAU AND E. M. LIFSHITS, *Quantum Mechanics*, Nauka, Moscow, 1976 (in Russian).
- [29] V. P. MASLOV, *Unbounded probability theory and multistep relaxation processes*, Math. Notes, 93 (2013), pp. 451–459.
- [30] V. P. MASLOV, *Bose–Enstein type distribution for nonideal gas. Two-liquid model of the supercritical state and its applications*, Math. Notes, 94 (2013) (to appear).