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An application of global Weyl modules of $\mathfrak{sl}_{n+1}[t]$ to invariant theory

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ABSTRACT

We identify the $\mathfrak{sl}_{n+1}$ isotypical components of the global Weyl modules $W(k\omega_1)$ with certain natural subspaces of the polynomial ring in k variables. We then apply the representation theory of current algebras to classical problems in invariant theory.

Introduction

Given a positive integer k, let A_k be the algebra of polynomials with complex coefficients in k variables $t_1, \dots, t_k$ and let S_k be the symmetric group on k letters. The group S_k acts naturally on A_k by permuting the variables. It is a classical result that A_k when regarded as a module for (the algebra of invariants) $A_k^{S_k}$ is free of rank $k!$.

In this paper, we are interested in certain variations of this problem. We regard A_k as being graded by the non-negative integers in the standard way with each generator having degree one and note that $A_k^{S_k}$ is a graded subalgebra. Suppose that we are given a partition $\xi = (\xi_1 \geqslant \xi_2 \geqslant \cdots \geqslant \xi_{n+1} > 0)$ of k and an element $\mathbf{r} \in \mathbf{Z}_+^{\xi_2} \times \cdots \times \mathbf{Z}_+^{\xi_{n+1}}$, where $\mathbf{Z}_+$ is the set of non-negative integers. We define a polynomial $\mathbf{p}(\mathbf{r}) \in A_k$ which can be thought of as being associated to Young symmetrizers of tableaux of

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shape ξ where the filling in the first row is 1. For instance, when $\xi = (k - \ell \geqslant \ell) > 0$, the polynomials are indexed by elements $\mathbf{r} = (r_1, \dots, r_\ell) \in \mathbf{Z}_+^\ell$ and are defined by,

$$\mathbf{p}(\mathbf{r}) = \sum_{\sigma \in S_\ell} \left(\prod_{i=1}^{\ell} \left(t_{2i-1}^{r_{\sigma(i)}} - t_{2i}^{r_{\sigma(i)}} \right) \right). \tag{0.1}$$

Let $M_{k,\xi}$ be the $A_k^{S_k}$-submodule of A_k generated by the elements $\{\mathbf{p}(\mathbf{r})\colon\ \mathbf{r} \in \mathbf{Z}_+^{\xi_2} \times \cdots \times \mathbf{Z}_+^{\xi_{n+1}}\}$. We use the theory of global Weyl modules for the current algebra of $\mathfrak{sl}_{n+1}$ to prove that $M_{k,\xi}$ is free and has a graded basis. The number of elements of a given grade is the coefficient of the corresponding power of q in the Kostka polynomial $\kappa_{1^k,\xi^{tr}}$. The partition ξ defines an irreducible representation of S_k and we show that $M_{k,\xi}$ can be identified with the corresponding multiplicity space in A_k.

The global Weyl modules we use, are indexed by multiples of the first fundamental weight of $\mathfrak{sl}_{n+1}$. These modules are well understood in the literature and many spanning sets and bases are known for them: a Poincare–Birkhoff–Witt type spanning set, a Gelfand–Tsetlin basis defined in [6] and a global canonical basis given in [12]. Our main result gives an explicit way to construct an $A_k^{S_k}$-generating set or basis of $M_{k,\xi}$ from a spanning set or basis of the appropriate global Weyl module. The special case described in (0.1) corresponds to taking a Poincare–Birkhoff–Witt spanning set for the global Weyl modules.

For $\xi = \ell \geqslant \ell$, the freeness of the module $M_{2\ell,\xi}$ was first conjectured by Chari and Greenstein. It was motivated by their study [4,5] of the homological properties of the category of finite-dimensional representations of the current algebra of $\mathfrak{sl}_2$. In [1] the module $M_{k,k-\ell \geqslant \ell}$ is denoted as $\mathbf{M}_{k,\ell}$ and a more specific conjecture is given which identifies a natural basis for it and this conjecture remains open.

Our interest in this result stems from the fact that it plays a crucial role in [2] which establishes a BGG-type duality for locally finite-dimensional representations of the current algebra associated to $\mathfrak{sl}_2$. In Section 1, we give an elementary formulation of the results of this paper without using the notion of global Weyl modules. The definitions and necessary results on global Weyl modules are recalled in Section 2. In Section 3 we prove our main result which gives the isomorphism between the module $M_{k,\xi}$ and the appropriate space of invariants in the global Weyl module.

1. The modules $M_{k,\xi}$

1.1. Throughout this note we let $\mathbf{Z}_+$ (resp. $\mathbf{N}$) denote the set of non-negative (resp. positive) integers and $\mathbf{C}$ the set of complex numbers. For $k \in \mathbf{N}$, we let A_k be the polynomial ring in k-variables equipped with the standard grading. For $r \in \mathbf{Z}_+$ the r-th graded piece is denoted as $A_k[r]$ and is the space of homogeneous polynomials of degree r. We shall use without further mention, the celebrated result of Quillen (see [13] for an exposition) that a projective module for A_k is free.

1.2. We begin with some standard results (see for instance [8]) on the representation theory of the symmetric group S_k on k letters. Let $\mathcal{P}[k]$ be the set of all partitions $\xi = (\xi_1 \geqslant \cdots \geqslant \xi_{n+1} \geqslant 0)$ of k. The irreducible representations of S_k are parametrized by elements of $\mathcal{P}[k]$. By abuse of notation, we shall denote by ξ both a partition of k and an irreducible representation of S_k in the corresponding isomorphism class and set $d_\xi = \dim \xi$. Any representation V of S_k can be written as a sum of finite-dimensional irreducible modules. If V^ξ denotes the isotypical component corresponding to $\xi \in \mathcal{P}[k]$, we have

$$V = \bigoplus_{\xi \in \mathcal{P}[k]} V^\xi,$$

as S_k-modules. We shall also use the notation V^{S_r} to denote the isotypical component corresponding to the trivial representation.

1.3. The group algebra $\mathbf{C}[S_k]$ of S_k is canonically an S_k-module and,

$$\mathbf{C}[S_k] = \bigoplus_{\xi \mathcal{P}[k]} \mathbf{C}[S_k]^{\xi}, \qquad \dim_{\mathbf{C}} \operatorname{Hom}_{S_k}\big(\xi, \mathbf{C}[S_k]^{\xi}\big) = d_{\xi}.$$

The following is well known but we isolate it as a lemma since it is used frequently. Recall that if V and V' are two representations of S_k then one has the diagonal action of S_k on $V \otimes V'$.

Lemma. *Let $\xi, \xi' \in \mathcal{P}[k]$. Then*

$$\dim\big(\xi \otimes \xi'\big)^{S_k} = \begin{cases} 0, & \xi \neq \xi', \\ 1, & \xi = \xi'. \end{cases}$$

In particular, given a basis $\mathbf{e}_1, \ldots, \mathbf{e}_{d_\xi}$ of ξ there exists a dual basis $\{\mathbf{e}^1, \ldots, \mathbf{e}^{d_\xi}\}$ such that $(\xi \otimes \xi)^{S_k}$ is spanned by the element $\sum_{s=1}^{d_\xi} \mathbf{e}_s \otimes \mathbf{e}^s$. $\square$

1.4. Consider the natural right action of S_k on the ring A_k given by permuting the variables which clearly preserves the graded pieces of A_k. The subalgebra $A_k^{S_k}$ of invariants is a graded subalgebra of A_k and is a polynomial algebra in k homogeneous elements where the s-th element has degree s. We also regard A_k as a right module over the ring $A_k^{S_k}$ and note that it commutes with the action of S_k. Hence for any partition ξ of k, the S_k-isotypical component A_k^{ξ} is also a free module for $A_k^{S_k}$ of rank d_ξ^2. We have an isomorphism of $A_k^{S_k}$ and S_k-modules,

$$A_k \cong \bigoplus_{\xi \in \mathcal{P}[k]} A_k^{\xi}.$$

If I_k is the augmentation ideal in $A_k^{S_k}$, then there exists an isomorphism of S_k-modules,

$$A_k / I_k A_k^{S_k} \cong \mathbf{C}[S_k].$$

The augmentation ideal I_k is graded and hence $\mathbf{C}[S_k]$ acquires a $\mathbf{Z}_+$-grading. It is known that the graded multiplicity of the irreducible representation ξ is given by the Kostka polynomial $\kappa_{1^k, \xi^{tr}}$.

1.5. Fix a partition $\xi = (\xi_1 \geqslant \xi_2 \geqslant \cdots \geqslant \xi_{n+1} > 0)$ of k with $(n+1)$-parts and set $k_i = \xi_i - \xi_{i+1}$, $1 \leqslant i \leqslant n+1$ where we understand that $k_{n+1} = \xi_{n+1}$. It will be convenient to think of A_k as a tensor product of algebras

$$A_k = A_1^{\otimes k_1} \otimes A_2^{\otimes k_2} \otimes \cdots \otimes A_{n+1}^{\otimes k_{n+1}},$$

where we understand that $A_\ell^{\otimes k_\ell} = \mathbf{C}$ if $k_\ell = 0$. Let

$$S_\xi = S_{\xi_1} \times \cdots \times S_{\xi_{n+1}},$$

be the corresponding Young subgroup of S_k. An element $\mathbf{r} \in \mathbf{Z}_+^{\xi_1} \times \cdots \times \mathbf{Z}_+^{\xi_{n+1}}$ defines a filling of the Young diagram of ξ in an obvious way. Namely, writing $\mathbf{r} = (\mathbf{r}_1, \ldots, \mathbf{r}_{n+1})$ where $\mathbf{r}_j = (r_{j,1}, \ldots, r_{j,\xi_j})$ for $1 \leqslant j \leqslant n+1$, we fill the j-th row of the Young diagram of ξ with the integers $r_{j,1}, \ldots, r_{j,\xi_j}$.

For each $1 \leqslant \ell \leqslant n+1$ with $k_\ell \neq 0$ define elements $\mathbf{a}_\ell(\mathbf{r}) \in A_\ell^{\otimes k_\ell}$ by

$$\mathbf{a}_\ell(\mathbf{r}) = \sum_{\tau \in S_\ell} \operatorname{sgn}(\tau) t_{\tau(1)}^{r_{1,\xi_{\ell+1}+1}} \cdots t_{\tau(\ell)}^{r_{\ell,\xi_{\ell+1}+1}} \otimes \cdots \otimes \sum_{\tau \in S_\ell} \operatorname{sgn}(\tau) t_{\tau(1)}^{r_{1,\xi_\ell}} \cdots t_{\tau(\ell)}^{r_{\ell,\xi_\ell}},$$

and if $k_\ell = 0$ set $\mathbf{a}_\ell(\mathbf{r}) = 1$. Finally, set

$$\mathbf{p}(\mathbf{r}) = \sum_{\sigma \in S_\xi} \mathbf{a}_1(\sigma \mathbf{r}) \otimes \cdots \otimes \mathbf{a}_{n+1}(\sigma \mathbf{r}).$$

Definition. Let $M_{k,\xi}$ be the $A_k^{S_k}$-submodule of $A_k^{S_\xi}$ generated by the elements

$$\left\{\mathbf{p}(\mathbf{r}):\ \mathbf{r} = (\mathbf{r}_1, \dots, \mathbf{r}_{n+1}) \in \left\{0^{\xi_1}\right\} \times \mathbf{Z}_+^{\xi_2} \times \cdots \times \mathbf{Z}_+^{\xi_{n+1}}\right\}.$$

1.6. We compute an example to illustrate the preceding definitions and to relate it to the modules in the introduction. Suppose that $n = 1$ and $\xi = (k - m \geqslant m > 0)$ so that $k_1 = k - 2m$, $k_2 = m$. The polynomials we are interested in are indexed by elements $\mathbf{r} = (0^{k-m}, (r_1, \dots, r_m)) \in \{0^{k-m}\} \times \mathbf{Z}_+^m$ and live in $A_1^{\otimes k-2m} \otimes A_2^{\otimes m}$. Then $\mathbf{a}_1(\mathbf{r}) = 1$ while

$$\mathbf{a}_2(r) = \left(t_2^{r_1} - t_1^{r_1}\right) \otimes \cdots \otimes \left(t_2^{r_m} - t_1^{r_m}\right) \in A_2^{\otimes m}.$$

1.7. We shall prove,

Theorem 1. *Let $k \in \mathbf{N}$ and let $\xi = (\xi_1 \geqslant \xi_2 \geqslant \cdots \geqslant \xi_{n+1} > 0)$ be a partition of k into $(n+1)$-parts. The $A_k^{S_k}$ module $M_{k,\xi}$ is graded free and the number of elements in the basis of degree s is the coefficient of q^s in the Kostka polynomial $\kappa_{1^k, \xi^{tr}}$. Moreover, we have an isomorphism of graded S_k and $A_k^{S_k}$-modules*

$$A_k^\xi \cong M_{k,\xi} \otimes \xi,$$

where the action is given by,

$$a(m \otimes v) = am \otimes v, \qquad \sigma(m \otimes v) = m \otimes \sigma v, \quad a \in A_k^{S_k},\ \sigma \in S_k,\ m \in M_{k,\xi},\ v \in \xi.$$

In particular, if I_k is the augmentation ideal in $A_k^{S_k}$, we have a $\mathbf{Z}_+$-graded isomorphism of vector spaces

$$M_{k,\xi}/I_k M_{k,\xi} \cong \operatorname{Hom}_{S_k}\left(\xi, \mathbf{C}[S_k]\right).$$

We deduce this theorem from the results of the next two sections.

2. Representations of $\mathfrak{sl}_{n+1}[t]$

In this section we first recall several results on the representation theory of $\mathfrak{sl}_{n+1}$ which can be found in any standard book (see [8,11] for instance). We then define a family of representations (the global Weyl modules) of the infinite-dimensional Lie algebra $\mathfrak{sl}_{n+1} \otimes \mathbf{C}[t]$ and recall some of their properties which were first established in [7] and [9].

2.1. Let $\mathfrak{sl}_{n+1}$ be the Lie algebra of $(n+1) \times (n+1)$-matrices of trace zero and $\mathfrak{h}$ (resp. $\mathfrak{n}^+$, $\mathfrak{n}^-$) be the subalgebra of diagonal (resp. strictly upper, lower triangular) matrices. Let $e_{i,j}$ be the $(n+1) \times (n+1)$-matrix with one in the (i,j)-th position and zero elsewhere. The elements

$$x_i^+ = e_{i,i+1}, \qquad h_i = e_{i,i} - e_{i+1,i+1}, \qquad x_i^- = e_{i+1,i}, \quad 1 \leqslant i \leqslant n,$$

generate $\mathfrak{sl}_{n+1}$ as a Lie algebra. Let $\Omega : \mathfrak{sl}_{n+1} \to \mathfrak{sl}_{n+1}$ be the anti-automorphism satisfying

$$\Omega\left(x_i^\pm\right) = x_i^\mp, \qquad \Omega(h_i) = h_i. \tag{2.1}$$

2.2. Let $\mathcal{P}^n$ be the set of all partitions with at most $(n+1)$-parts and for $1 \leqslant i \leqslant n$, let $\omega_i \in \mathcal{P}^n$ be the partition 1^i. An element $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_{n+1} \geqslant 0)$ of $\mathcal{P}^n$ defines a finite-dimensional irreducible representation $V(\lambda)$ of $\mathfrak{sl}_{n+1}$ and two partitions $\lambda, \lambda' \in \mathcal{P}^n$ define isomorphic representations iff $\lambda_i - \lambda_{i+1} = \lambda'_i - \lambda'_{i+1}$ for all $1 \leqslant i \leqslant n$. The trivial representation of $\mathfrak{sl}_{n+1}$ corresponds to taking the empty partition.

It is well known that $V(\lambda)$ is generated as an $\mathfrak{sl}_{n+1}$-module by an element v_λ with defining relations:

$$\mathfrak{n}^+ v_\lambda = 0, \qquad h_i v_\lambda = (\lambda_i - \lambda_{i+1}) v_\lambda, \qquad \left(x_i^-\right)^{\lambda_i - \lambda_{i+1}+1} v_\lambda = 0, \quad 1 \leqslant i \leqslant n.$$

Further, any irreducible finite-dimensional representation of $\mathfrak{sl}_{n+1}$ is isomorphic to $V(\lambda)$ for some $\lambda \in \mathcal{P}^n$.

2.3. We say that a representation V of $\mathfrak{sl}_{n+1}$ is locally finite-dimensional if it is a direct sum of finite-dimensional representations. By Weyl's theorem, this is equivalent to requiring that V be isomorphic to a direct sum of modules $V(\lambda)$, $\lambda \in \mathcal{P}^n$. The Lie algebra $\mathfrak{h}$ acts diagonalizably on a locally finite-dimensional module V, and we have

$$V = \bigoplus_{\mathbf{r} \in \mathbf{Z}^n} V_{\mathbf{r}}, \qquad V_{\mathbf{r}} = \{v \in V\colon h_i v = r_i v\}, \quad \mathbf{r} \in \mathbf{Z}^n.$$

In the case when ξ is a partition, we also use the notation,

$$V_\xi = \left\{v \in V\colon h_i v = (\xi_i - \xi_{i+1}) v,\ 1 \leqslant i \leqslant n\right\}, \qquad V_\xi^{\mathfrak{n}^+} = \left\{v \in V_\xi\colon \mathfrak{n}^+ v = 0\right\}.$$

Note that $V(\lambda)^{\mathfrak{n}^+} = \mathbf{C} v_\lambda$ for all $\lambda \in \mathcal{P}^n$ and that a locally finite-dimensional representation is generated as an $\mathfrak{sl}_{n+1}$-module by the spaces $V_\xi^{\mathfrak{n}^+}$, $\xi \in \mathcal{P}^n$. The following is elementary.

Lemma. *Let V be a locally finite-dimensional $\mathfrak{sl}_{n+1}$-module and let $\xi \in \mathcal{P}^n$. Then*

$$V_\xi = V_\xi^{\mathfrak{n}^+} \oplus \left(\mathfrak{n}^- V \cap V_\xi\right). \qquad \square$$

2.4. Recall that if U is any vector space there is a natural action of S_k on the k-fold tensor product $U^{\otimes k}$ which just permutes the factors. We will need the following result, the first two parts of which are elementary and the third part is the famous Schur–Weyl duality between representations of $\mathfrak{sl}_{n+1}$ and those of S_k

Proposition. *Let $k, n \in \mathbf{N}$.*

(i) *The module $V(\omega_1)$ is isomorphic to the natural representation of $\mathfrak{sl}_{n+1}$ with standard basis $e_1, \ldots, e_{n+1}$ and action given by $e_{i,j} e_\ell = \delta_{j,\ell} e_i$. The assignment $(e_i, e_j) = \delta_{i,j}$ defines a symmetric bilinear from $(\ ,\ ) : V(\omega_1) \otimes V(\omega_1) \to \mathbf{C}$ satisfying,*

$$\left(xv, v'\right) = \left(v, \Omega(x) v'\right), \quad v, v' \in V(\omega_1),\ x \in \mathfrak{sl}_{n+1}.$$

(ii) *For $1 \leqslant i \leqslant n+1$, we have*

$$\left(V(\omega_1)^{\otimes i}\right)_{(\omega_i)}^{\mathfrak{n}^+} = \mathbf{C}(e_1 \wedge e_2 \wedge \cdots \wedge e_i),$$

and an isomorphism of $\mathfrak{sl}_{n+1}$-modules $\bigwedge^i V(\omega_1) \cong V(\omega_i)$ which maps $e_1 \wedge \cdots \wedge e_i$ to v_{ω_i}.

(iii) *Let $k \in \mathbf{N}$. The natural right action of S_k on $V(\omega_1)^{\otimes k}$ commutes with the left action of $\mathfrak{sl}_{n+1}$ and as an $(\mathfrak{sl}_{n+1}, S_k)$-bimodule, we have*

$$V(\omega_1)^{\otimes k} \cong \bigoplus_{\xi \in \mathcal{P}^n \cap \mathcal{P}[k]} V(\xi) \otimes \xi.$$

Equivalently, the space $(V(\omega_1)^{\otimes k})_\xi^{\mathfrak{n}^+}$ is an irreducible S_k-submodule of $V(\omega_1)^{\otimes k}$ and we have an isomorphism of S_k-modules

$$\left(V(\omega_1)^{\otimes k}\right)_\xi^{\mathfrak{n}^+} \cong \xi. \qquad \square$$

2.5. Denote by $(\ ,\ )_k : V(\omega_1)^{\otimes k} \times V(\omega_1)^{\otimes k} \to \mathbf{C}$ the symmetric non-degenerate bilinear form given by extending linearly the assignment,

$$(e_{i_1} \otimes \cdots \otimes e_{i_k}, e_{j_1} \otimes \cdots \otimes e_{j_k})_k = (e_{i_1}, e_{j_1}) \cdots (e_{i_k}, e_{j_k}),$$

where $i_s, j_s \in \{1, \ldots, n+1\}$ for $1 \leqslant s \leqslant k$. The form is clearly S_k-invariant and satisfies

$$\left(x\mathbf{v}, \mathbf{v}'\right)_k = \left(\mathbf{v}, \Omega(x)\mathbf{v}'\right)_k, \quad \mathbf{v}, \mathbf{v}' \in V(\omega_1)^{\otimes k},\ x \in \mathfrak{g}.$$

In particular since $\Omega(h_i) = h_i$ for all $1 \leqslant i \leqslant n$, we have

$$\left(V(\omega_1)_\xi^{\otimes k}, V(\omega_1)_{\xi'}^{\otimes k}\right)_k = 0, \quad \xi, \xi' \in \mathcal{P}^n,\ \xi \neq \xi',$$

and so the restriction of the form to $V(\omega_1)_\xi^{\otimes k}$ is non-degenerate.

Lemma. *For all $\xi \in \mathcal{P}^n \cap \mathcal{P}[k]$ the restriction of the form $(\ ,\ )_k$ to $(V(\omega_1)^{\otimes k})_\xi^{\mathfrak{n}^+} \times (V(\omega_1)^{\otimes k})_\xi^{\mathfrak{n}^+}$ is non-degenerate.*

Proof. Let $\mathbf{v}, \mathbf{v}' \in V(\omega_1)_\xi^{\otimes k}$ be such that $(\mathbf{v}, \mathbf{v}')_k \neq 0$ and assume that $\mathfrak{n}^+\mathbf{v} = 0$. By Lemma 2.3, we may write $\mathbf{v}' = \mathbf{v}_1 + \mathbf{v}_2$, where $\mathbf{v}_1 \in V_\xi^{\mathfrak{n}^+}$ and $\mathbf{v}_2 = \sum_{i=1}^n e_{i+1,i}\mathbf{u}_i$ for some $\mathbf{u}_i \in V(\omega_1)^{\otimes k}$. Hence we get

$$(\mathbf{v}, \mathbf{v}_2)_k = \left(\mathbf{v}, \sum_{i=1}^n e_{i+1,i}\mathbf{u}_i\right)_k = \sum_{i=1}^n (e_{i,i+1}\mathbf{v}, \mathbf{u}_i)_k = 0,$$

i.e., $(\mathbf{v}, \mathbf{v}_1)_k = (\mathbf{v}, \mathbf{v}')_k \neq 0$ and the lemma is proved. $\square$

2.6. Given $k \in \mathbf{N}$ and a partition $\xi = (\xi_1 \geqslant \cdots \geqslant \xi_{n+1} > 0)$ of k set $k_i = \xi_i - \xi_{i+1}$, and let

$$\mathbf{e}(\xi) = e_1^{\otimes k_1} \otimes (e_1 \wedge e_2)^{\otimes k_2} \otimes \cdots \otimes (e_1 \wedge e_2 \wedge \cdots \wedge e_{n+1})^{\otimes k_{n+1}} \in \left(V(\omega_1)^{\otimes k}\right)_\xi^{\mathfrak{n}^+}.$$

We note the following consequence of Proposition 2.4.

Lemma. *The set $\{\mathbf{e}(\xi)\sigma\colon \sigma \in S_k\}$ spans $(V(\omega_1)^{\otimes k})_\xi^{\mathfrak{n}^+}$.* $\square$

From now on, we fix elements $\sigma_\ell \in S_k$, $1 \leqslant \ell \leqslant d_\xi - 1$ with $\sigma_1 = \mathrm{id}$ such that the set $\{\mathbf{e}(\xi)\sigma_\ell \colon 1 \leqslant \ell \leqslant d_\xi\}$ is a basis of $(V(\omega_1)^{\otimes k})_\xi^{\mathfrak{n}^+}$. We also use Lemma 2.5 and fix a dual basis $\{\mathbf{e}_\ell \colon 1 \leqslant \ell \leqslant d_\xi\}$ of $(V(\omega_1)^{\otimes k})_\xi^{\mathfrak{n}^+}$ satisfying

$$\big(\mathbf{e}_j, \mathbf{e}(\xi)\sigma_m\big)_k = 0, \quad 1 \leqslant j \neq m \leqslant d_\xi, \qquad \big(\mathbf{e}_m, \mathbf{e}(\xi)\sigma_m\big)_k = 1. \tag{2.2}$$

2.7. We now introduce the main tools to prove Theorem 1 and we begin with some general definitions. Given a complex Lie algebra $\mathfrak{a}$ and an indeterminate t, let $\mathfrak{a}[t] = \mathfrak{a} \otimes \mathbf{C}[t]$ be the Lie algebra with commutator given by,

$$[a \otimes f, b \otimes g] = [a, b] \otimes fg, \quad a, b \in \mathfrak{a}, \ f, g, \in \mathbf{C}[t].$$

We identify without further comment the Lie algebra $\mathfrak{a}$ with the subalgebra $\mathfrak{a} \otimes 1$ of $\mathfrak{a}[t]$. Clearly, $\mathfrak{a}[t]$ has a natural $\mathbf{Z}_+$-grading given by the powers of t and this also induces a $\mathbf{Z}_+$-grading on $\mathbf{U}(\mathfrak{a}[t])$. A $\mathbf{Z}_+$-graded bimodule for the pair $(\mathfrak{a}[t], A_k)$ is a complex vector space $V = \bigoplus_{s \in \mathbf{Z}_+} V_s$ which admits a left action of $\mathfrak{a}[t]$ and a commuting right action of A_k such that both actions are compatible with the grading, i.e.,

$$\big(\mathfrak{a} \otimes t^r\big) V[s] \subset V[s+r], \qquad V[s]A_k[r] \subset V[s+r], \quad s, r \in \mathbf{Z}_+.$$

An elementary way to construct such modules is as follows: let U_s, $1 \leqslant s \leqslant k$ be $\mathfrak{a}$-modules and define on $(U_1 \otimes \cdots \otimes U_k \otimes A_k)$ the structure of a bimodule for the pair $(\mathfrak{a}[t], A_k)$ by:

$$(u_1 \otimes \cdots \otimes u_k \otimes a)b = u_1 \otimes \cdots \otimes u_k \otimes ab, \tag{2.3}$$

$$\big(x \otimes t^r\big)(u_1 \otimes \cdots \otimes u_k \otimes a) = \sum_{s=1}^{k} \left(\bigotimes_{j=1}^{s-1} u_j \right) \otimes x u_s \left(\bigotimes_{j=s+1}^{k} u_j \right) \otimes t_s^r a, \tag{2.4}$$

where $a, b \in A_k$, $u_s \in U_s$, $1 \leqslant s \leqslant k$, $r \in \mathbf{Z}_+$ and $x \in \mathfrak{a}$. The grading on $(U_1 \otimes \cdots \otimes U_k \otimes A_k)$ is induced by the grading on A_k, i.e., for an integer r, the r-th-graded piece is $(U_1 \otimes \cdots \otimes U_k) \otimes A_k[r]$. These modules are clearly free as right A_k (and so also as right $A_k^{S_k}$) modules.

2.8. Consider the special case when all the U_i's are the same, say $U_i = U$ for $1 \leqslant i \leqslant k$. The induced diagonal (right) action of S_k on $U^{\otimes k} \otimes A_k$ commutes with the right action of A_k and hence for each $\xi \in \mathcal{P}[k]$, the isotypical component $(U^{\otimes k} \otimes A_k)^\xi$ is a $A_k^{S_k}$-module and we have an isomorphism of $A_k^{S_k}$-modules

$$U^{\otimes k} \otimes A_k \cong \bigoplus_{\xi \in \mathcal{P}[k]} \big(U^{\otimes k} \otimes A_k\big)^\xi.$$

If $\dim U < \infty$ then $(U^{\otimes k} \otimes A_k)^\xi$ is a free $A_k^{S_k}$-module of finite rank. The following is easily checked by using the explicit definition of the action of $\mathfrak{a}[t]$ given in (2.4).

Lemma. *The space $(U^{\otimes k} \otimes A_k)^{S_k}$ is a $\mathbf{Z}_+$-graded $(\mathfrak{a}[t], A_k^{S_k})$-submodule of $U^{\otimes k} \otimes A_k$.* $\square$

2.9. Apply the preceding construction to the case when $\mathfrak{a} = \mathfrak{sl}_{n+1}$ and $U = V(\omega_1)$. Define an A_k-valued bilinear form

$$\langle\ ,\ \rangle : \left(V(\omega_1)^{\otimes k} \otimes A_k\right) \times \left(V(\omega_1)^{\otimes k} \otimes A_k\right) \to A_k,$$

by extending linearly the assignment,

$$\left\langle \mathbf{v} \otimes f, \mathbf{v}' \otimes g\right\rangle = \left(\mathbf{v}, \mathbf{v}'\right)_k fg,$$

where $\mathbf{v}, \mathbf{v}' \in V(\omega_1)^{\otimes k}$, and $f, g \in A_k$. Notice that the form $\langle\ ,\ \rangle$ is not S_k-invariant. The following is proved by a direct calculation using the formulae in (2.3) and (2.4).

Lemma. *For all $x \in \mathfrak{sl}_{n+1}$, $f \in \mathbf{C}[t]$, we have*

$$\left\langle (x \otimes f)(\mathbf{v} \otimes g), \mathbf{v}' \otimes g'\right\rangle = \left\langle \mathbf{v} \otimes g, \left(\Omega(x) \otimes f\right)\left(\mathbf{v}' \otimes g'\right)\right\rangle,$$

where $\mathbf{v}, \mathbf{v}' \in V(\omega_1)^{\otimes k}$ and $g, g' \in A_k$. Further, for all $\mathbf{v} \in V(\omega_1)^{\otimes k}$, the assignment

$$\langle\ , \mathbf{v}\rangle : V(\omega_1)^{\otimes k} \otimes A_k \to A_k$$

is a map of A_k-modules. □

2.10. The following lemma gives an interpretation of the polynomials $\mathbf{p}(\mathbf{r})$ in terms of the representation theory of current algebras. The proof is straightforward but tedious. It is included since the lemma is crucial for the proof of the main theorem in Section 3. We remark that $[e_{j,1}, e_{k,1}] = 0 = [e_{1,j}, e_{1,k}]$ for $1 \leqslant j, k \leqslant n+1$ which implies that the products in the statement and proof of the lemma are well defined. We fix a partition $\xi = (\xi_1 \geqslant \cdots \geqslant \xi_{n+1} > 0)$ of k and use freely the notation of Section 2.6.

Lemma. *Let $\mathbf{r} = (\{0^{\xi_1}\}, \mathbf{r}_2, \ldots, \mathbf{r}_{n+1})$, where $\mathbf{r}_j = (r_{j,1}, \ldots, r_{j,\xi_j}) \in \mathbf{Z}_+^{\xi_j}$, $2 \leqslant j \leqslant n+1$. We have*

$$\left\langle \mathbf{y}(\mathbf{r})\left(e_1^{\otimes k} \otimes 1\right), \mathbf{e}(\xi)\right\rangle = \mathbf{p}(\mathbf{r}),$$

where $\mathbf{y}(\mathbf{r}) \in \mathbf{U}(\mathfrak{n}^-[t])$ is the element

$$\mathbf{y}(\mathbf{r}) = \prod_{j=2}^{n+1} \left(e_{j,1} \otimes t^{r_{j,1}}\right) \cdots \left(e_{j,1} \otimes t^{r_{j,\xi_j}}\right).$$

Proof. Using Lemma 2.9 we have

$$\left\langle \mathbf{y}(\mathbf{r})\left(v_{\omega_1}^{\otimes k} \otimes a\right), \mathbf{e}(\xi)\right\rangle = \left\langle v_{\omega_1}^{\otimes k} \otimes a, \mathbf{x}(\mathbf{r})\mathbf{e}(\xi)\right\rangle,$$

where

$$\mathbf{x}(\mathbf{r}) = \prod_{j=2}^{n+1} \left(e_{1,j} \otimes t^{r_{j,1}}\right) \cdots \left(e_{1,j} \otimes t^{r_{j,\xi_j}}\right) = \prod_{\ell=2}^{n+1} \prod_{s=\xi_{\ell+1}+1}^{\xi_\ell} \mathbf{x}_{\ell,s}(\mathbf{r}),$$

$$\mathbf{x}_{\ell,s}(\mathbf{r}) = \left(e_{1,2} \otimes t^{r_{2,s}}\right) \cdots \left(e_{1,\ell} \otimes t^{r_{\ell,s}}\right), \quad 2 \leqslant \ell \leqslant n+1,\ \xi_{\ell+1} + 1 \leqslant s \leqslant \xi_\ell.$$

Given $\sigma \in S_\xi$ it is clear that $\mathbf{x}(\mathbf{r}) = \mathbf{x}(\sigma\mathbf{r})$ although it is not true that $\mathbf{x}_{\ell,s}(\mathbf{r}) = \mathbf{x}_{\ell,s}(\sigma\mathbf{r})$. To compute $\mathbf{x}(\mathbf{r})\mathbf{e}(\xi)$, we note first that

$$\mathbf{x}_{\ell,s}(\mathbf{r})(e_1 \wedge \cdots \wedge e_\ell) = \left(\sum_{\tau \in S_\ell} \operatorname{sgn}(\tau) t_{\tau(2)}^{r_{2,s}} \cdots t_{\tau(\ell)}^{r_{\ell,s}}\right) e_1^{\otimes \ell},$$

and hence

$$\mathbf{x}_{\ell,\xi_{\ell+1}+1}(\mathbf{r})(e_1 \wedge \cdots \wedge e_\ell) \otimes \cdots \otimes \mathbf{x}_{\ell,\xi_\ell}(\mathbf{r})(e_1 \wedge \cdots \wedge e_\ell) = \mathbf{a}(\mathbf{r}) e_1^{\otimes \ell}.$$

Since $(e_{1,k} \otimes t^r)(e_{1,k} \otimes t^s)(e_1 \wedge \cdots \wedge e_m) = 0$ for all $2 \leqslant m \leqslant n+1$, we get

$$\begin{aligned}&\big(e_{1,k} \otimes t^{r_{k,1}}\big) \cdots \big(e_{1,k} \otimes t^{r_{k,\xi_k}}\big)(e_1 \wedge \cdots \wedge e_m)^{\otimes \xi_k} \\ &\quad = \sum_{\sigma \in S_{\xi_k}} \big(e_{1,k} \otimes t^{r_{k,\sigma(1)}}\big)(e_1 \wedge \cdots \wedge e_m) \otimes \cdots \otimes \big(e_{1,k} \otimes t^{r_{n+1,\xi_{\sigma(k)}}}\big)(e_1 \wedge \cdots \wedge e_m).\end{aligned}$$

This now implies that

$$\begin{aligned}\mathbf{x}(\mathbf{r})\mathbf{e}(\xi) = \sum_{\sigma \in S_\xi} &e_1^{\otimes k_1} \otimes \mathbf{x}_{2,\xi_3+1}\big(\sigma(\mathbf{r})\big)(e_1 \wedge e_2) \otimes \cdots \otimes \mathbf{x}_{2,\xi_2}\big(\sigma(\mathbf{r})\big)(e_1 \wedge e_2) \otimes \cdots \\ &\otimes \mathbf{x}_{n+1,1}\big(\sigma(\mathbf{r})\big)(e_1 \wedge \cdots \wedge e_{n+1}) \otimes \cdots \otimes \mathbf{x}_{n+1,\xi_{n+1}}\big(\sigma(\mathbf{r})\big)(e_1 \wedge \cdots \wedge e_{n+1}), \\ = \mathbf{p}&(\mathbf{r}) e_1^{\otimes k},\end{aligned}$$

and the proof of the lemma is complete. □

2.11. For $k \in \mathbf{Z}_+$, let

$$W(k) = \big(V(\omega_1)^{\otimes k} \otimes A_k\big)^{S_k}. \tag{2.5}$$

Then $W(k)$ is a $\mathbf{Z}_+$-graded $(\mathfrak{sl}_{n+1}, A_k^{S_k})$-bimodule. Since $V(\omega_1)^{\otimes k} \otimes A_k$ and hence also, $W(k)$ is a locally finite-dimensional $\mathfrak{sl}_{n+1}$-module, we can write

$$W(k) = \bigoplus_{\xi \in \mathcal{P}^n \cap \mathcal{P}[k]} W(k)_\xi, \qquad W(k)_\xi = W(k)_\xi^{\mathfrak{n}^+} \oplus \big(\mathfrak{n}^- W(k) \cap W(k)_\xi\big). \tag{2.6}$$

Since the action of $\mathfrak{s}l_{n+1}$ action and S_k on $W(k)$ commute, we have

$$W(k)_\xi = \big(V(\omega_1)_\xi^{\otimes k} \otimes A_k\big)^{S_k}, \qquad W(k)_\xi^{\mathfrak{n}^+} = \big(\big(V(\omega_1)^{\otimes k}\big)_\xi^{\mathfrak{n}^+} \otimes A_k^\xi\big)^{S_k}. \tag{2.7}$$

Since the action of $\mathfrak{sl}_{n+1}$ on $W(k)$ also commutes with the action of $A_k^{S_k}$, we see that the subspaces $W(k)_\xi$, $W(k)_\xi^{\mathfrak{n}^+}$ and $(\mathfrak{n}^- W(k) \cap W(k)_\xi)$ are all $\mathbf{Z}_+$-graded $A_k^{S_k}$-submodules of $W(k)$. Further, the direct sums in (2.6) are of $A_k^{S_k}$-submodules. Since $W(k)$ is obviously a free $A_k^{S_k}$-module we see that $W(k)_\xi^{\mathfrak{n}^+}$ is also a free $A_k^{S_k}$-module and hence graded free.

2.12. We shall need the following result. As usual we denote by $\mathbf{U}(\mathfrak{a})$ the universal enveloping algebra of the Lie algebra $\mathfrak{a}$.

Theorem 2. *Let $k \in \mathbf{N}$.*

(i) *The element $v_{\omega_1}^{\otimes k}$ generates $W(k)$ as a module for $\mathfrak{sl}_{n+1}[t]$, and*

$$W(k) = \mathbf{U}\big(\mathfrak{n}^-[t]\big)\big(v_{\omega_1}^{\otimes k} \otimes A_k^{S_k}\big).$$

(ii) *For $\xi \in \mathcal{P}^n$, the $A_k^{S_k}$-module $W(k)_\xi^{\mathfrak{n}^+}$ is graded free and the number of elements in a basis with grade s is the coefficient of q^s in the Kostka polynomial $\kappa_{1^k,\xi^{tr}}$.* $\square$

Part (i) of the theorem was proved in [7] for $\mathfrak{sl}_2$ and in [9] for $\mathfrak{sl}_{n+1}$. Recall that I_k is the augmentation ideal in $A_k^{S_k}$ and consider the graded $\mathfrak{sl}_{n+1}[t]$-module, $W(k)/I_kW(k)$. It was shown in [6] that the subspace $(W(k)/I_kW(k))_\xi^{\mathfrak{n}^+}$ has a graded basis and also that the number of elements in the basis of grade s is the coefficient of q^s in the Kostka polynomial $\kappa_{1^k,\xi^{tr}}$. Part (ii) follows since $W(k)_\xi^{\mathfrak{n}+}$ is graded free as an $A_k^{S_k}$-module and we have an isomorphism of graded spaces,

$$W(k)_\xi^{\mathfrak{n}+}/I_kW(k)_\xi^{\mathfrak{n}^+} \cong \big(W(k)/I_kW(k)\big)_\xi^{\mathfrak{n}^+}.$$

2.13. The modules $W(k)$ are special examples of a family of modules called the global Weyl modules which were defined and studied in [7] for arbitrary simple Lie algebras and further studied in [6,10,14]. For $\mathfrak{sl}_{n+1}$ it is proved in [3] that the global Weyl modules can be realized as sitting inside a suitable space of invariants. However, except in the special case considered in this paper, the global Weyl modules are strictly smaller than the space of invariants.

3. The main result and Proof of Theorem 1

3.1. Fix $k \in \mathbf{N}$, $\xi = (\xi_1 \geqslant \xi_2 \geqslant \cdots \geqslant \xi_{n+1} > 0)$ a partition of k and set $k_i = \xi_i - \xi_{i+1}$, $1 \leqslant i \leqslant n+1$. We use freely the notation of the earlier sections, in particular we let $\sigma_s \in S_k$, $1 \leqslant s \leqslant d_\xi$ be as defined in Section 2.6. Theorem 1 is clearly a consequence Theorem 2 and the following result.

Theorem 3. *The restriction of the map $\langle\ ,\mathbf{e}(\xi)\rangle : V(\omega_1)^{\otimes k} \otimes A_k \to A_k$ gives an isomorphism of $A_k^{S_k}$-modules $W(k)_\xi^{\mathfrak{n}^+} \to M_{k,\xi}$. For $\sigma \in S_k$ we have*

$$M_{k,\xi}\sigma = \big\langle W(k)_\xi^{\mathfrak{n}^+}, \mathbf{e}(\xi)\sigma\big\rangle, \tag{3.1}$$

and further,

$$\big\langle W(k)_\xi^{\mathfrak{n}^+}, \big(V(\omega_1)^{\otimes k}\big)_\xi^{\mathfrak{n}^+}\big\rangle = A_k^\xi = \bigoplus_{s=1}^{d_\xi} M_{k,\xi}\sigma_s, \tag{3.2}$$

as S_k and $A_k^{S_k}$-modules.

Proof. By Lemma 2.9, we have that $\langle\ ,\mathbf{e}(\xi)\rangle : V(\omega_1)^k \otimes A_k \to A_k$ is a map of A_k-modules. Hence, using (2.5) and the remarks in Section 2.11, we see that the restriction to $W(k)$ and hence to $W(k)_\xi^{\mathfrak{n}^+}$ is a map of $A_k^{S_k}$-modules. Let $\{\mathbf{e}_s\colon 1 \leqslant s \leqslant d_\xi\}$ be the basis of $(V(\omega_1)^{\otimes k})_\xi^{\mathfrak{n}^+}$ defined in Section 2.6. Using Eq. (2.7) and Lemma 1.2, we may write any non-zero element $\mathbf{v} \in W(k)_\xi^{\mathfrak{n}^+}$ as

$$\mathbf{v} = \sum_{s=1}^{d_\xi} \mathbf{e}_s \otimes g_s, \quad g_s \in A_k^\xi,\ g_1 \neq 0. \tag{3.3}$$

Using Eq. (2.2), we get

$$\langle \mathbf{v}, \mathbf{e}(\xi) \rangle = g_1 \neq 0,$$

which proves that the map $\langle\ , \mathbf{e}(\xi)\rangle : W(k)_\xi^{\mathfrak{n}^+} \to A_k$ is injective and that its image is contained in A_k^ξ. In particular it follows from Theorem 2 that $\langle W(k)_\xi^{\mathfrak{n}^+}, \mathbf{e}(\xi)\rangle$ is a free $A_k^{S_k}$-module of rank d_ξ.

We now prove the first equality in (3.2). For this, we begin by showing that

$$\langle W(k)_\xi^{\mathfrak{n}^+}, \mathbf{e}(\xi) \rangle \sigma = \langle W(k)_\xi^{\mathfrak{n}^+}, \mathbf{e}(\xi)\sigma \rangle. \tag{3.4}$$

Choose $\mathbf{v}$ as in Eq. (3.3) and let $\sigma \in S_k$. We have

$$\langle \mathbf{v}, \mathbf{e}(\xi)\sigma \rangle = \langle \mathbf{v}\sigma, \mathbf{e}(\xi)\sigma \rangle = \left\langle \sum_{s=1}^{\ell} \mathbf{e}_s\sigma \otimes g_s\sigma, \mathbf{e}(\xi)\sigma \right\rangle = \left(\sum_{s=1}^{\ell} \big(\mathbf{e}_s, \mathbf{e}(\xi)\big)_k g_s \right)\sigma = \langle \mathbf{v}, \mathbf{e}(\xi) \rangle \sigma$$

where the first equality follows from Eq. (2.5) and the penultimate equality is a consequence of the S_k invariance of $(\ ,\ )_k$. Hence (3.4) is established.

Recalling from Lemma 2.6 that the elements $\{\mathbf{e}(\xi)\sigma\colon\ \sigma \in S_k\}$ span $(V(\omega_1)^{\otimes k})_\xi^{\mathfrak{n}^+}$, we observe that

$$\sum_{\sigma \in S_k} \langle W(k)_\xi^{\mathfrak{n}^+}, \mathbf{e}(\xi)\sigma \rangle = \langle W(k)_\xi^{\mathfrak{n}^+}, \big(V(\omega_1)^{\otimes k}\big)_\xi^{\mathfrak{n}^+} \rangle \subset A_k^\xi.$$

To prove the reverse inclusion, suppose that $a \in A_k^\xi$ and assume that it generates an irreducible S_k-submodule N and let $a_1, \dots, a_{d_\xi}$ be a basis of N where $a_1 = a$. Then there exists a non-zero element

$$\mathbf{v} \in \Big(\big(V(\omega_1)^{\otimes k}\big)_\xi^{\mathfrak{n}^+} \otimes N \Big)^{S_k} \subset W(k)_\xi^{\mathfrak{n}^+},$$

and by Lemma 1.2 we can write $\mathbf{v} = \sum_{s=1}^{d_\xi} \mathbf{v}_s \otimes a_s$ where the elements $\{\mathbf{v}_s\colon\ 1 \leqslant s \leqslant d_\xi\}$ are a basis for $(V(\omega_1)^{\otimes k})_\xi^{\mathfrak{n}^+}$. By Lemma 2.5 we may choose a dual basis $\{\mathbf{v}'_s\colon\ 1 \leqslant s \leqslant d_\xi\}$ of $(V(\omega_1)^{\otimes k})_\xi^{\mathfrak{n}^+}$ and we find now that

$$\langle \mathbf{v}, \mathbf{v}'_1 \rangle = a_1 = a.$$

Since A^ξ is completely reducible, this proves that

$$A^\xi = \sum_{s=1}^{d_\xi} \langle W(k)_\xi^{\mathfrak{n}^+}, \mathbf{e}(\xi)\sigma_s \rangle.$$

Recall from Section 1.4 that A^ξ is a free $A_k^{S_k}$-module of rank d_ξ^2. Since we have shown earlier in this proof that $\langle W(k)_\xi^{\mathfrak{n}^+}, \mathbf{e}(\xi)\sigma_s\rangle$ is free $A_k^{S_k}$-module of rank d_ξ, it follows that in fact

$$A_k^\xi = \bigoplus_{s=1}^{d_\xi} \langle W(k)_\xi^{\mathfrak{n}^+}, \mathbf{e}(\xi)\sigma_s \rangle.$$

To complete the proof of the theorem, we must prove that

$$\left\langle W(k)_\xi^{\mathfrak{n}^+}, \mathbf{e}(\xi)\right\rangle = M_{k,\xi}.$$

Since $\langle \mathfrak{n}^- W(k)_\xi, \mathbf{e}(\xi)\rangle = 0$, it suffices by Lemma 2.3 to prove that $\langle W(k)_\xi, \mathbf{e}(\xi)\rangle = M_{k,\xi}$. Using $e_{i,j} v_{\omega_1} = 0$, $j \neq 1$ and the fact that

$$W(k) = \mathbf{U}\big(\mathfrak{n}^-[t]\big)\big(v_{\omega_1}^{\otimes k} \otimes A_k^{S_k}\big),$$

we see that $W(k)_\xi$ is spanned by elements of the form $\mathbf{y}(\mathbf{r})(v_{\omega_1}^{\otimes k} \otimes a)$ where $a \in A_k^{S_k}$, $\mathbf{r} = (\{0^{\xi_1}\}, \mathbf{r}_2, \ldots, \mathbf{r}_{n+1})$, where $\mathbf{r}_j = (r_{j,1}, \ldots, r_{j,\xi_j}) \in \mathbf{Z}_+^{\xi_j}$, $2 \leqslant j \leqslant n+1$, and,

$$\mathbf{y}(\mathbf{r}) = \prod_{j=2}^{n+1} \big(e_{j,1} \otimes t^{r_{j,1}}\big) \cdots \big(e_{j,1} \otimes t^{r_{j,\xi_j}}\big).$$

Lemma 2.10 shows that

$$\left\langle \mathbf{y}\big(v_{\omega_1}^{\otimes k} \otimes a\big), \mathbf{e}(\xi)\right\rangle = \mathbf{p}(\mathbf{r})a,$$

which proves

$$M_{k,\xi} = \big\langle W(k)_\xi, \mathbf{e}(\xi)\big\rangle,$$

as required. □

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