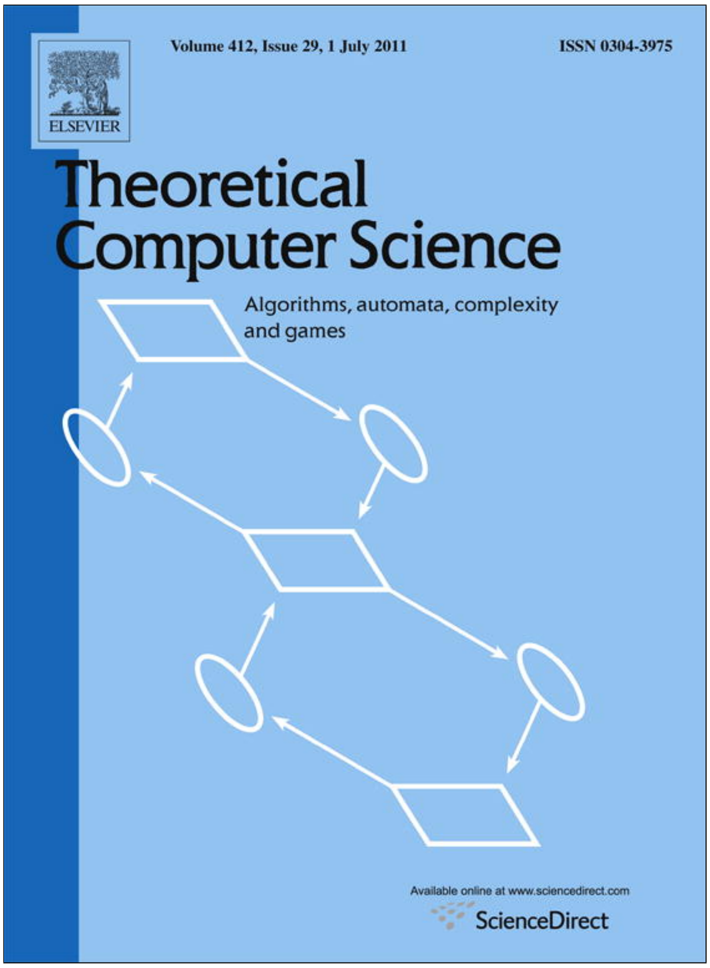
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Boundary properties of graphs for algorithmic graph problems ☆

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ABSTRACT

The notion of a boundary graph property was recently introduced as a relaxation of that of a minimal property and was applied to several problems of both algorithmic and combinatorial nature. In the present paper, we first survey recent results related to this notion and then apply it to two algorithmic graph problems: Hamiltonian cycle and vertex k-colorability. In particular, we discover the first two boundary classes for the Hamiltonian cycle problem and prove that for any $k > 3$ there is a continuum of boundary classes for vertex k-colorability.

1. Introduction

A *graph property* (or *class of graphs*) is a set of graphs closed under isomorphism. A property is *hereditary* if, for any of its graphs G, all the induced subgraphs of G belong to it. In other words, a class is hereditary if it is closed under deletion of vertices from graphs in the class.

Many interesting and important graph properties are hereditary, including, for instance, planar graphs, bipartite graphs, perfect graphs, line graphs, graphs of vertex degree at most k, graphs of tree-width at most k. Many of those properties that are not hereditary have natural hereditary extensions. The minimal hereditary extension of a set of graphs X is called the *hereditary closure* of X. It is unique and consists of all graphs in X and all their induced subgraphs. For instance, for the class of trees, which is not hereditary, the hereditary closure includes all forests, i.e. graphs without cycles, while for the class of cubic graphs the hereditary closure includes all subcubic graphs, i.e. graphs of vertex degree at most 3.

Let us emphasize that in the above list of hereditary classes the example of graphs of degree at most k provides an infinite family of such classes defined for various values of k, and the subcubic graphs are a member of this family for $k = 3$. Similarly, the example of graphs of tree-width at most k provides an infinite family of hereditary classes, and the forests are a member of this family for $k = 1$. Speaking of families of graph classes, let us mention two more important families of hereditary classes: monotone and minor-closed classes.

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A class X of graphs is *monotone* if it is closed under deletion of vertices and edges from graphs in the class. Among examples mentioned before, planar graphs, bipartite graphs, graphs of bounded vertex degree and graphs of bounded tree-width are monotone, while perfect graphs and line graphs are not.

A class X of graphs is *minor-closed* if it is closed under vertex deletion, edge deletion and edge contraction. Directly from the definition it follows that every minor-closed class is monotone, and therefore, is hereditary. Among classes mentioned earlier, only planar graphs and graphs of bounded tree-width are minor-closed; the others are not. There are many other minor-closed graph classes, but the important thing is that any other minor-closed class contains the planar graphs as a subclass. This is due to the following remarkable result proved by Robertson and Seymour [27].

Theorem 1. *Within the family of minor-closed graph classes the planar graphs constitute the unique minimal class of graphs of unbounded tree-width.*

As a consequence of this result, we conclude that within the family of minor-closed graph classes the planar graphs constitute the unique minimal class where many algorithmic problems are NP-hard (provided that $P \neq NP$). This is the case, for instance, for the MAXIMUM INDEPENDENT SET problem, which is NP-hard for planar graphs and polynomial-time solvable for graphs of bounded tree-width.

The situation changes dramatically when we extend the discussion from minor-closed to hereditary classes. The task of identifying minimal classes becomes generally impossible, because the family of hereditary classes is not well-founded with respect to the containment relation, i.e. it contains infinite descending chains of graph classes. For example, it is known that the MAXIMUM INDEPENDENT SET problem is NP-hard in graphs containing no cycles of length at most k for any fixed value of k [24]. With k tending to infinity this creates an infinite descending chain of graph classes where the problem is NP-hard. To overcome this difficulty, Alekseev introduced the notion of a *boundary class* of graphs and identified the first boundary class for the MAXIMUM INDEPENDENT SET problem [3]. Later this notion was applied to some other graph problems of both algorithmic [4,5] and combinatorial [17] nature. In the present paper, we continue this line of research and obtain new results on the boundary classes of graphs for NP-hard graph problems. The paper is organized as follows.

In Section 2, we provide a necessary background on this topic, give some motivation to study the notion of boundary classes and consider a number of examples.

In Section 3, we turn to the study of the HAMILTONIAN CYCLE problem. In [5], it was observed that there must exist at least five boundary classes of graphs for this problem, but none of them has been identified so far. In the present paper, we discover the first two boundary classes for the HAMILTONIAN CYCLE problem.

Finally, in Section 4 we study VERTEX k-COLORABILITY. Recently, it was proved in [20] that for $k = 3$ the number of boundary classes is infinite. Moreover, in [21] it was shown that there exists a continuum of boundary classes for this problem. In the present paper, we extend this result to arbitrary values of k.

2. Boundary properties of graphs

All graphs in this paper are finite, undirected, without loops or multiple edges. For a graph G, we denote by $V(G)$ and $E(G)$ the vertex set and the edge set of G, respectively. The neighborhood of a vertex $v \in V(G)$ (i.e., the set of vertices adjacent to v) is denoted $N(v)$. The *degree* of v is the number of its neighbors. If the degree of each vertex of G is exactly 3, we call G a *cubic* graph, and if the degree of G is at most 3, we call G *subcubic*. A vertex of degree 3 will be called a cubic vertex. For a subset of vertices $U \subseteq V(G)$, we denote by $G[U]$ the subgraph of G induced by U, i.e. the subgraph of G with vertex set U and two vertices being adjacent in $G[U]$ if and only if they are adjacent in G. We say that a graph H is an induced subgraph of G (or G contains H as an induced subgraph) if H is isomorphic to $G[U]$ for some $U \subseteq V(G)$.

For a set of graphs M (no matter, finite or infinite), we denote by $Free(M)$ the class of graphs containing no induced subgraphs from the set M and say that M is the set of forbidden induced subgraphs for the class $Free(M)$. Any graph in $Free(M)$ will be called M-free. It is not difficult to see that a class X of graphs is hereditary if and only if $X = Free(M)$ for a set M. Indeed, if $G \in Free(M)$, i.e. if G contains no induced subgraphs from M, then obviously any induced subgraph of G is M-free, which means that $Free(M)$ is a hereditary class for any set M. On the other hand, if X is a hereditary class, then $X = Free(M)$ with M being the set of all graphs that are not in M, which is a trivial observation. A non-trivial task is to find the set of minimal (or equivalently, the minimal set of) forbidden induced subgraphs for X. It is well-known (and not difficult to see) that for any hereditary class X, the minimal set of forbidden induced subgraphs exists and it is unique. If M is a finite set, we will call $Free(M)$ a finitely defined class.

For some classes of graphs, finding the set of minimal forbidden induced subgraphs is a simple task. In particular, it is not difficult to see that the class of graphs of vertex degree at most k is finitely defined for any fixed k. For instance, for the class of subcubic graphs the set of minimal forbidden induced subgraphs consists of 11 graphs on five vertices: in each graph, one vertex is dominating and the remaining vertices induce all possible 4-vertex graphs. However, in general, the problem of finding the set of minimal forbidden induced subgraphs for a hereditary class is far from being trivial, as the example of perfect graphs shows [11].

The importance of the induced subgraph characterization of a hereditary class of graphs can be illustrated by the following example. In 1969, "Journal of Combinatorial Theory" published a paper entitled "An interval graph is a comparability graph" [15]. One year later, the same journal published another paper entitled "An interval graph is not a comparability graph" [12]. With the induced subgraph characterization this situation could not happen, because it is not difficult to see that

$Free(M) \subseteq Free(N)$ if and only if for any graph $G \in N$ there is a graph $H \in M$ such that H is an induced subgraph of G. Therefore, given two hereditary classes of graphs and the induced subgraph characterization for both of them, it is a simple task to decide the inclusion relationship between them. Apparently, in 1969 the induced subgraph characterization was not available for interval or comparability graphs. Nowadays, it is available for both classes.

Now let us shift our discussion from graphs to classes of graphs and ask the following question: is it possible to characterize a family of graph classes in terms of minimal classes that do not belong to the family? More formally, assume we are given a family of graph classes $\mathcal{U}$ (the universe) and a subfamily $\mathcal{A} \subseteq \mathcal{U}$ with the property that if a class X belongs to $\mathcal{A}$ then any subclass of X from the same universe also belongs to $\mathcal{A}$.

(Q) Is it possible to characterize the family $\mathcal{A}$ in terms of minimal classes from $\mathcal{U}$ that do not belong to $\mathcal{A}$?

In the introduction, we have seen already an example in which this question admits an affirmative answer: if $\mathcal{U}$ is the family of minor-closed graph classes and $\mathcal{A}$ is the family of graph classes of bounded tree-width, then the only minimal class in $\mathcal{U}$ which does not belong to $\mathcal{A}$ is the class of planar graphs. Also, provided that $P \neq NP$, the answer to question (Q) remains the same if we replace $\mathcal{A}$ by the family of graph classes for which the MAXIMUM INDEPENDENT SET problem is polynomial-time solvable, retaining the same universe: within the family of minor-closed graph classes, the class of planar graphs is the unique minimal class, where the problem is NP-complete, since in any class of bounded tree-width the problem is polynomial-time solvable.

Let us consider one more example where a family of graph classes admits a characterization in terms of minimal classes that do not belong to the family. This example is of combinatorial nature. It deals with the notion of the *speed of hereditary property*, which is the number of n-vertex graphs in a hereditary class X studied as a function of n. In this example, the universe is the family of all hereditary classes. It is known (see e.g. [1,7]) that this family is partitioned with respect to the speed of hereditary classes into discrete layers. The lowest layer of this hierarchy contains finite classes of graphs, i.e. classes with finitely many graphs. From the Ramsey theorem it follows that there are two minimal classes of graphs that do not belong to this layer: complete graphs and their complements (edgeless graphs). Both these classes are infinite, and any class excluding at least one complete graph and one edgeless graph (i.e. any class of the form $Free(K_n, \overline{K}_m)$) is finite. All classes in all other layers are infinite, and there are infinitely many such layers. The first four lower layers containing infinite classes of graphs are [28]:

- *constant* layer contains classes X with $\log_2 |X_n| = O(1)$,
- *polynomial* layer contains classes X with $\log_2 |X_n| = \Theta(\log_2 n)$,
- *exponential* layer contains classes X with $\log_2 |X_n| = \Theta(n)$,
- *factorial* layer contains classes X with $\log_2 |X_n| = \Theta(n \log_2 n)$.

Each of these layers contains a finite collection of minimal classes. For instance, in the factorial layer there are exactly nine minimal classes [2,7]. Therefore, the family of subfactorial classes can be characterized by nine minimal classes that do not belong to this family. However, the structure of graphs in subfactorial classes is rather simple. The factorial layer is substantially richer. It contains plenty of graph classes of theoretical and practical importance, such as forests, interval, permutation, chordal bipartite, line, threshold graphs, cographs, planar graphs, and even more generally, all minor-closed graph classes (other than the class of all graphs) [25]. Therefore, it would be interesting to characterize the factorial layer in terms of minimal superfactorial classes. However, none of such classes has been identified so far, and possibly, none of such classes exists. To better explain this phenomenon, let us consider the following example.

It is known that the class of bipartite graphs is superfactorial. Moreover, subclasses of bipartite graphs defined by forbidding

(1) either large cycles, such as $(C_{10}, C_{12}, \ldots)$-free bipartite graphs or $(C_8, C_{10}, \ldots)$-free bipartite graphs,
(2) or small cycles, such as C_4-free bipartite graphs or (C_4, C_6)-free bipartite graphs,

are superfactorial. The first sequence can be extended by adding to it the class of $(C_6, C_8, C_{10}, \ldots)$-free bipartite graphs, also known as chordal bipartite graphs, which is still superfactorial [29]. However, by adding to the set of forbidden graphs one more cycle, i.e. C_4, we obtain the class of forests, which is factorial. On the contrary, the second sequence of graph classes can be extended to an infinite chain of superfactorial classes by forbidding more and more cycles. In other words, for any $k \geq 2$, the class of $(C_4, C_6, \ldots, C_{2k})$-free bipartite graphs is superfactorial [19], and only the limit class of this sequence, i.e. the class to which this sequence converges, which is again the class of forests, is factorial. Therefore, in this sequence there is no minimal superfactorial class.

A similar situation arises in the study of algorithmic graph problems that are generally intractable, i.e. NP-hard. Given a problem Π, we call a hereditary class X *Π-tough* if the problem is NP-hard for graphs in X and *Π-easy* otherwise. Assuming that $P \neq NP$ we want to characterize the family of Π-easy classes in terms of minimal Π-tough classes. The first example of a minimal Π-tough class for an algorithmic graph problem can be found in [22]. However, in general, identifying minimal classes is impossible, since in the family of hereditary classes there may exist infinite strictly decreasing sequences of Π-tough classes. For instance, it is known that many algorithmic graph problems, such as MAXIMUM INDEPENDENT SET or MINIMUM DOMINATING SET, are NP-hard in the class $Free(C_3, C_4, \ldots, C_k)$ for any fixed value of k, but solvable in polynomial time in the limit class of this sequence, i.e. in the class of forests. To overcome this difficulty, Alekseev introduced in [3] the notion of a boundary class. Below we define this notion with respect to an arbitrary family $\mathcal{A}$ of hereditary classes closed under taking subclasses.

Definition 1. A class X of graphs is called a limit class for the family $\mathcal{A}$ ($\mathcal{A}$-limit for short) if and only if $X = \bigcap_{i=1}^{\infty} X_i$, where $X_1 \supseteq X_2 \supseteq \cdots$ is a sequence of classes none of which belongs to $\mathcal{A}$.

In the present paper, $\mathcal{A}$ is the family of Π-easy classes for various algorithmic problems Π, in which case we call an $\mathcal{A}$-limit class a Π-limit. We also call a sequence $X_1 \supseteq X_2 \supseteq \cdots$ of Π-tough classes a *decreasing sequence* and say that this sequence converges to the class $X = \bigcap_{i=1}^{\infty} X_i$. Observe that we do not require the classes in the sequence $X_1 \supseteq X_2 \supseteq \cdots$ to be distinct, which means that every class that does not belong to $\mathcal{A}$ is $\mathcal{A}$-limit. On the other hand, this definition allows some classes that belong to $\mathcal{A}$ to be limit for this family. From the previous discussion it follows that the class of forests is a limit class both for the MAXIMUM INDEPENDENT SET and MINIMUM DOMINATING SET problems, although both problems are polynomial-time solvable in this class. However, the class of forests is not a minimal limit class for these problems, since both of them are NP-hard for graphs of vertex degree at most 3 in the class $Free(C_3, C_4, \ldots, C_k)$. Therefore, the class of forests of vertex degree at most 3 is a smaller limit class for both problems. This observation motivates the following key definition.

Definition 2. A minimal $\mathcal{A}$-limit class is called a boundary class for the family $\mathcal{A}$.

The first boundary class for an algorithmic graph problem was found in [3] by Alekseev. He proved that the class of forests every connected component of which has at most 3 leaves is a minimal limit, i.e. boundary, class for the MAXIMUM INDEPENDENT SET problem. Later, this class was shown to be boundary for some other graph problems, not necessarily of algorithmic nature (see e.g. [4,5,17]). However, this class is boundary not for every graph problem. For instance, the HAMILTONIAN CYCLE problem is not of this type. In the next section, we discover the first two boundary classes for this problem. Let us repeat that all the results in this paper are obtained under the assumption that $P \neq NP$.

3. Boundary classes of graphs for the HAMILTONIAN CYCLE problem

In a graph, a Hamiltonian cycle is a cycle containing each vertex of the graph exactly once. Determining whether a graph has a Hamiltonian cycle is an NP-complete problem. Moreover, it is NP-complete for subcubic graphs [14] and for graphs of large girth, i.e. graphs without small cycles [5,6]. Any hereditary class of graphs where the problem is NP-complete will be called HC-tough.

In [5], it was observed that there must exist at least five boundary classes of graphs for the HAMILTONIAN CYCLE problem, but none of them has been identified so far. In what follows, we discover the first two boundary classes for this problem.

3.1. Approaching a limit class

As we mentioned already, the HAMILTONIAN CYCLE problem is NP-complete for subcubic graphs [14] and for graphs of large girth [5,6]. In this section, we strengthen both these results. First, we show that the problem is NP-complete in the class of subcubic graphs, in which every cubic vertex has a non-cubic neighbor. Throughout the paper, we denote this class by Γ.

Lemma 1. *The* HAMILTONIAN CYCLE *problem is NP-complete in the class* Γ.

Proof. Plesńik [26] proved that the HAMILTONIAN CYCLE problem is NP-complete in the class of directed graphs, where every vertex has either indegree 1 and outdegree 2, or indegree 2 and outdegree 1. The lemma is proved by a reduction from the HAMILTONIAN CYCLE problem on such graphs, which we call *Plesńik graphs*. Given a Plesńik graph H, we associate with it an undirected graph from Γ as follows. First, we consider all the *prescribed edges* of H, i.e. directed edges $u \to v$, such that either u has outdegree 1, or v has indegree 1 (or both). We replace every such edge by a *prescribed path* $u \to w \to v$, where w is a new node of indegree and outdegree 1. Then, we erase orientation from all edges, and denote the resulting undirected graph by G.

Clearly, $G \in \Gamma$. Assume H has a directed Hamiltonian cycle. Then the corresponding edges of G form a Hamiltonian cycle in G. Conversely, if G has a Hamiltonian cycle, then it must contain all the prescribed paths, and therefore the corresponding edges of H form a Hamiltonian cycle C in H. Let us show that this cycle respects the orientation of the edges. Let $u \to v$ be a prescribed edge, i.e. an edge on C. If u has outdegree 1, then the edge preceding u on C must be incoming for u and the orientation is respected. If v has indegree 1, then the edge following v on C must be outgoing for v and the orientation is respected again. Since every vertex of H is incident with at least one prescribed edge, the cycle C respects the orientation of all its edges. Together with the obvious fact that the problem belongs to NP this proves the lemma. □

Now we strengthen Lemma 1 as follows. Denote by $Y_{i,j,k}$ the graph represented in Fig. 1 and call any graph of this form a *tribranch*. Also, denote $\mathcal{Y}_p = \{Y_{i,j,k} : i, j, k \leq p\}$ and $\mathcal{C}_p = \{C_k : k \leq p\}$. Finally, let $\mathcal{S}_p$ be the class of $\mathcal{C}_p \cup \mathcal{Y}_p$-free graphs in Γ.

Lemma 2. *For any* $p \geq 1$, *the* HAMILTONIAN CYCLE *problem is NP-complete in the class* $\mathcal{S}_p$.

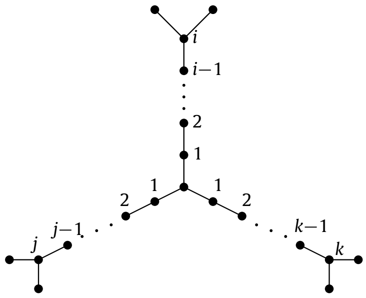

Fig. 1. A tribranch $Y_{i,j,k}$.

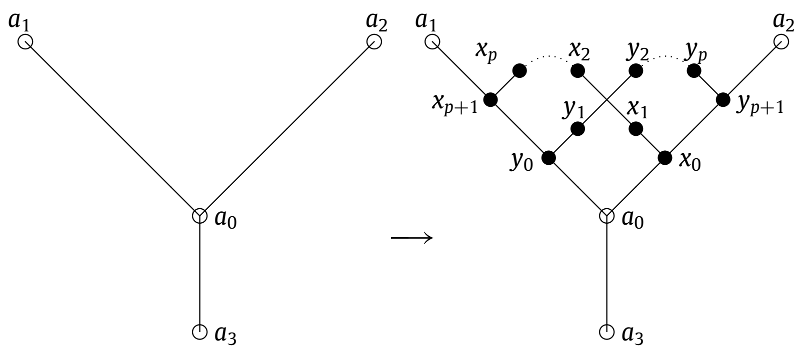

Fig. 2. Transformation F_p.

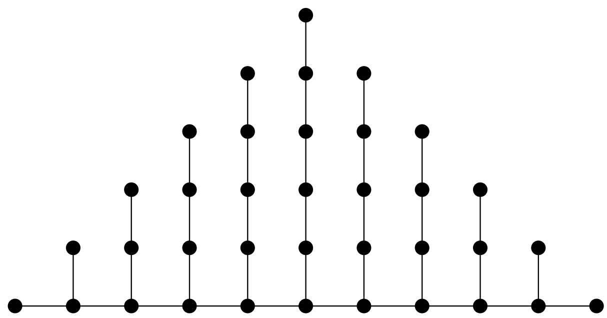

Fig. 3. A caterpillar with hairs of arbitrary length.

Proof. We reduce the problem from the class Γ to $\mathcal{C}_p \cup \mathcal{Y}_p$-free graphs in Γ. Let G be a graph in Γ. Obviously, every edge of G incident to a vertex of degree 2 must belong to any Hamiltonian cycle in G (should G have any). Therefore, by subdividing each of such edges with p new vertices we obtain a graph $G' \in \Gamma$ which has a Hamiltonian cycle if and only if G has. It is not difficult to see that G' is $\mathcal{Y}_p$-free. Moreover, G' has no small cycles (i.e. cycles from $\mathcal{C}_p$) containing at least one vertex of degree 2. If G' has a cycle $C \in \mathcal{C}_p$ each vertex of which has degree 3, we apply to any vertex a_0 of C the transformation F_p represented in Fig. 2, where a_3 denotes a non-cubic neighbor of a_0. It is not difficult to see that F_p transforms G' into a new graph in Γ, which has a Hamiltonian cycle if and only if G has. Moreover, this transformation increases the length of C without producing any new cycle from $\mathcal{C}_p$ or any tribranch from $\mathcal{Y}_p$. Repeated applications of this transformation allow us to get rid of all small cycles. Thus, any graph G in Γ can be transformed in polynomial time into a $\mathcal{C}_p \cup \mathcal{Y}_p$-free graph in Γ, which has a Hamiltonian cycle if and only if G has. Together with the fact that the problem belongs to NP and with its NP-completeness in the class Γ, this proves the lemma. □

3.2. Limit class

The results of the previous section show that $\bigcap_{p \geq 1} \mathcal{S}_p$ is a limit class for the Hamiltonian cycle problem. Throughout the paper we will denote this class by $\mathcal{S}$. In the present section, we describe the structure of graphs in the class $\mathcal{S}$. Let us define a *caterpillar with hairs of arbitrary length* to be a subcubic tree in which all cubic vertices belong to a single path. An example of a caterpillar with hairs of arbitrary length is given in Fig. 3.

Lemma 3. *A graph G belongs to the class $\mathcal{S}$ if and only if every connected component of G is a caterpillar with hairs of arbitrary length.*

Proof. If every connected component of G is a caterpillar with hairs of arbitrary length, then G is a subcubic graph without induced cycles or tribranches. Therefore, G belongs to $\mathcal{S}$.

Conversely, let G be a connected component of a graph in $\mathcal{S}$. Then, by definition, G is a subcubic tree without tribranches. If G has at most one cubic vertex, then obviously G is a caterpillar with hairs of arbitrary length. If G has at least two cubic vertices, then let P be an induced path of maximum length connecting two cubic vertices, say v and w. Suppose there is a cubic vertex u that does not belong to P. The path connecting u to P meets P at a vertex different from v and w (since otherwise P would not be maximum). But then a tribranch arises. This contradiction shows that every cubic vertex of G belongs to P, i.e., G is a caterpillar with hairs of arbitrary length. □

In the next section, we will prove that $\mathcal{S}$ is a minimal limit class for the Hamiltonian cycle problem. Without loss of generality, we will restrict ourselves to those graphs in $\mathcal{S}$ every connected component of which has the following "canonical" form: T_d ($d \geq 2$) is a caterpillar with a path of length $2d$ (containing all cubic vertices) and $2d - 1$ consecutive hairs of lengths $1, 2, \ldots, d-1, d, d-1, \ldots, 2, 1$. Fig. 3 represents the graph T_5. The following lemma is obvious.

Lemma 4. *Every graph in $\mathcal{S}$ is an induced subgraph of T_d for some $d \geq 2$.*

3.3. Minimality of the limit class

The proof of minimality of the class $\mathcal{S}$ is based on the following lemma.

Lemma 5. *If for every graph G in $\mathcal{S}$, there is a constant $p = p(G)$, such that the* Hamiltonian cycle *problem can be solved in polynomial time for G-free graphs in $\mathcal{S}_p$, then $\mathcal{S}$ is a minimal limit class for the problem.*

Proof. Assume, by contradiction, that for every graph G in $\mathcal{S}$, there is a constant $p = p(G)$, such that the Hamiltonian cycle problem can be solved in polynomial time for G-free graphs in $\mathcal{S}_p$, but $\mathcal{S}$ is not a minimal limit class. Let $\mathcal{X}$ be a limit class which is a subclass of $\mathcal{S}$. Then there must exist a graph G in $\mathcal{S}$ that does not belong to $\mathcal{X}$. Denote by $p = p(G)$ the constant associated with G, and by $\mathcal{Z}$ the class of G-free graphs in $\mathcal{S}_p$. By our assumption, the problem is solvable in polynomial time in $\mathcal{Z}$.

Clearly $\mathcal{X} \subseteq \mathcal{Z}$. Let us show that $\mathcal{Z}$ is also a limit class for the Hamiltonian cycle problem. Since $\mathcal{X}$ is a limit class, we have $\mathcal{X} = \bigcap_n \mathcal{X}_n$ for a sequence $\mathcal{X}_1 \supseteq \mathcal{X}_2 \supseteq \cdots$ of HC-tough graph classes. But then the class $\mathcal{Z}_n := \mathcal{X}_n \cup \mathcal{Z}$ is HC-tough for each n, $\mathcal{Z}_k \supseteq \mathcal{Z}_{k+1}$ for each k, and $\mathcal{Z} = \bigcap_n \mathcal{Z}_n$.

We observe that the class $\mathcal{Z}$ is defined by finitely many forbidden induced subgraphs. Indeed, the set of forbidden subgraphs for this class consists of G, finitely many cycles and tribranches, and the set of forbidden graphs for the class Γ. To characterize the class Γ in terms of forbidden induced subgraphs, we need to exclude 11 graphs containing a dominating vertex of degree 4 (which is equivalent to bounding vertex degree by 3), and finitely many subcubic graphs containing a cubic vertex with three cubic neighbors.

Since the set of forbidden induced subgraphs for the class $\mathcal{Z}$ is finite, there must exist an n, such that $\mathcal{Z}_n$ contains none of the forbidden graphs for $\mathcal{Z}$. But then $\mathcal{Z}_n = \mathcal{Z}$, which contradicts the assumption that the problem is polynomial-time solvable in the class $\mathcal{Z}$, while $\mathcal{Z}_n$ is an HC-tough class of graphs. This contradiction proves the lemma. □

We now apply Lemma 5 to prove the key result of this section.

Lemma 6. *For each graph $T \in \mathcal{S}$, there is a constant p such that the* Hamiltonian cycle *problem can be solved in polynomial time for T-free graphs in $\mathcal{S}_p$.*

Proof. By Lemma 4, T is an induced subgraph of T_d for some d. We define $p = 3 \times 2^d$, and will prove the lemma for T_d-free graphs in $\mathcal{S}_p$. Obviously, this class contains all T-free graphs in $\mathcal{S}_p$.

Let G be a T_d-free graph in $\mathcal{S}_p$. Without loss of generality, we will assume that G has no vertices of degree 1, since otherwise there is no Hamiltonian cycle in G. Let us call an edge of G *black*, if it belongs to any Hamiltonian cycle in G (should such a cycle exist). Similarly, we will call an edge of H *white*, if it does not belong to any Hamiltonian cycle in G. We will show that every vertex of G is incident to at least 2 black edges. This is obviously true for vertices of degree 2. Therefore, let v be a cubic vertex of G.

Denote by H the subgraph of G induced by the set of vertices of distance at most d from v. Since the degree of each vertex of H is at most 3, the number of vertices in H is less than p. Since H belongs to $\mathcal{S}_p$, it cannot contain small cycles and small tribranches (i.e. graphs from the set $\mathcal{C}_p \cup \mathcal{Y}_p$). Moreover, H cannot contain large cycles and large tribranches, because the size of H is too small (less than p). Therefore, H belongs to $\mathcal{S}$, and obviously H is connected. Thus, H is a caterpillar with hairs of arbitrary length. Observe that each leaf u in H is at distance exactly d from v, since otherwise u has degree 1 in G.

Let P be a path in H connecting two leaves and containing all vertices of degree 3. If every vertex of P (except the endpoints) has degree 3, then $H = T_d$, which is impossible because G is T_d-free. Therefore, P must contain a vertex of degree 2. Let v_i be a vertex of degree 2 on P closest to v, and let $(v = v_0, v_1, \ldots, v_i)$ be the path connecting v_i to $v = v_0$ (along P). Then the edge $v_i v_{i-1}$ is black, as it is incident to a vertex of degree 2. By the choice of v_i, the vertex v_{i-1} has degree 3, and hence it has a neighbor u that does not belong to P. Since u does not belong to P, it has degree 2, and therefore, the edge uv_{i-1} is also black, which implies that the edge $v_{i-1}v_{i-2}$ is white. In its turn, this implies that $v_{i-2}v_{i-3}$ is black, and therefore, as before, $v_{i-3}v_{i-4}$ is white. By induction, we conclude that the colors of the edges of the path $(v = v_0, v_1, \ldots, v_i)$

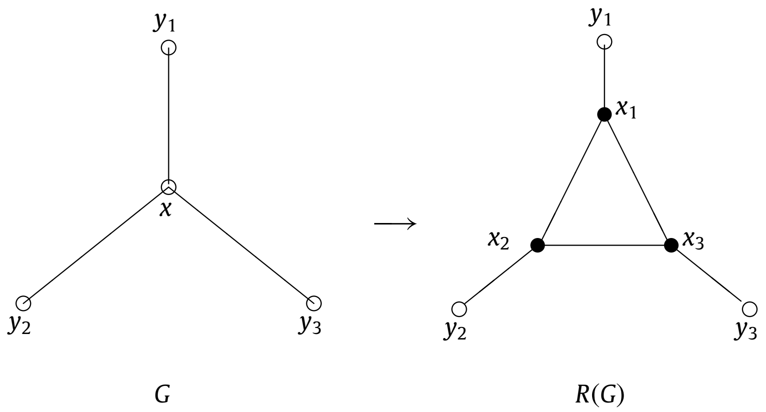

Fig. 4. Transformation *R*.

alternate. If the edge v_0v_1 is white, then the other two edges incident to $v = v_0$ are black. If the edge v_0v_1 is black, then the edge connecting v to the vertex outside P is also black.

Thus, we proved that every vertex of G is incident to at least 2 black edges. Clearly, G has a Hamiltonian cycle if and only if no vertex is incident to 3 black edges, and the subgraph of G formed by the black edges is connected. □

From Lemmas 5 and 6 we conclude with the following theorem.

Theorem 2. *$\mathcal{S}$ is a boundary class for the* Hamiltonian cycle *problem.*

3.4. One more boundary class

To obtain one more boundary class, we use the transformation R represented in Fig. 4. It is not difficult to see that a graph G has a Hamiltonian cycle if and only if $R(G)$ has. Let us denote by $R(\mathcal{S})$ the class of graphs obtained from graphs in $\mathcal{S}$ by application of transformation R to each cubic vertex.

Theorem 3. *$R(\mathcal{S})$ is a boundary class for the* Hamiltonian cycle *problem.*

3.5. Open problems

In the previous sections we revealed the first two boundary classes of graphs for the Hamiltonian cycle problem. The existence of one more boundary class for this problem arises from the fact that Hamiltonian cycle is NP-complete in the class of chordal bipartite graphs (i.e. in the class $Free(C_3, C_5, C_6, C_7, \ldots)$) [23]. This fact implies that there must exist a boundary subclass of chordal bipartite graphs, i.e. a class X together with a sequence $X_1 \subseteq X_2 \subseteq X_3 \ldots$ of subclasses of chordal bipartite graphs such that $X = \cap X_i$ and the problem is NP-complete in each class in the sequence $X_1 \subseteq X_2 \subseteq X_3 \ldots$. In fact, $\mathcal{S}$ is a subclass of chordal bipartite graphs. But we claim that $\mathcal{S}$ is not equal to X. Indeed, each class X_i in the sequence must contain a C_4, since otherwise X_i is a subclass of forests where the problem is polynomial-time solvable. But if each class contains a C_4, then X also must contain a C_4, which is not the case for the class $\mathcal{S}$. Some hints regarding the structure of graphs in a boundary class of chordal bipartite graphs are given in the following two observations.

Observation 1. *Let $X_1 \subseteq X_2 \subseteq X_3 \ldots$ be a sequence of subclasses of chordal bipartite graphs such that the* Hamiltonian cycle *problem is NP-complete in each class in the sequence. Then the class $X = \cap X_i$ must contain a fork F_p (the graph obtained from a star $K_{1,p}$ by subdividing one edge exactly once) for all values of p and a domino (the graph obtained from a chordless cycle C_6 by adding an edge connecting two vertices of distance 3).*

Proof. Every connected domino-free chordal bipartite graph is distance-hereditary [8], and the clique-width of distance-hereditary graphs is at most 3 [13]. Also, the clique-width is bounded by a constant in the class of F_p-free chordal bipartite graphs for any value of p [18]. It is known [9] that the Hamiltonian cycle problem can be solved for graphs of bounded clique-width in polynomial time. Therefore, each class in the sequence $X_1 \subseteq X_2 \subseteq X_3 \cdots$ must contain a domino and all forks F_p. Consequently, the class $X = \cap X_i$ must contain a domino and all forks F_p. □

Finally, we observe that for each boundary class of bipartite graphs, there must exist a respective class of split graphs, i.e. graphs partitionable into an independent set and a clique. Indeed, a bipartite graph $G = (V_1, V_2, E)$ has a Hamiltonian cycle only if $|V_1| = |V_2|$. If in such a graph we replace V_1 (or V_2) by a clique, then the split graph obtained in this way has a Hamiltonian cycle if and only if G has. Therefore, any result on the Hamiltonian cycle problem in bipartite graphs can be transformed into a respective result in split graphs.

4. Vertex graph colorability

Vertex coloring of a graph G is an assignment of colors to its vertices in such a way that adjacent vertices receive different colors. The minimum number of colors needed to color G is called the *chromatic number* of G and is denoted $\chi(G)$. The VERTEX COLORABILITY problem consists in deciding, for a graph G and an integer k, whether G admits a coloring with at most k colors. If k is a fixed number, we refer to the problem as VERTEX k-COLORABILITY. It is well-known that VERTEX COLORABILITY and VERTEX k-COLORABILITY (for each $k \geq 3$) are NP-complete. A hereditary class of graphs where VERTEX COLORABILITY is NP-complete will be called vc-tough and a class where VERTEX k-COLORABILITY is NP-complete k-vc-tough .

Boundary classes for the VERTEX 3-COLORABILITY problem have been studied in [20] and [21]. In particular, [20] proves that the number of boundary classes for this problem is infinite, while [21] strengthens this result by showing that the number of such classes is continuum. In the present paper, we extend this result to arbitrary values of k, i.e. we prove that for any $k > 3$, the boundary classes for the VERTEX k-COLORABILITY problem form a continuum set. It is interesting to observe that all boundary classes known for the VERTEX 3-COLORABILITY problem are also boundary for VERTEX COLORABILITY. In contrast, for values of k strictly greater than 3, we construct a continuum set of boundary classes of graphs none of which is boundary for VERTEX COLORABILITY.

To prove our result we use the following notations. Given two graphs G_1 and G_2, we denote by $G_1 \circ G_2$ the graph with the vertex set $V(G_1) \cup V(G_2)$ and the edge set $E(G_1) \cup E(G_2) \cup \{(x, y) : x \in V(G_1), y \in V(G_2)\}$. In particular, $G \circ K_0 = G$. Given a class of graphs X, we denote by $\mathcal{X}^p$ the class of graphs containing all induced subgraphs of graphs in $\{G \circ K_p : G \in \mathcal{X}\}$. Note that for any integer $p \geq 0$, the class $\mathcal{X}^p$ contains $\mathcal{X}$.

We start by proving two auxiliary results, the first of which is valid for an arbitrary NP-hard graph problem Π.

Lemma 7. *Let $\mathcal{B}$ be a Π-limit class and $\mathcal{X}$ a finitely defined class containing $\mathcal{B}$. Then for any nonnegative integer p there is a decreasing sequence of Π-tough subclasses of $\mathcal{X}^p$ converging to $\mathcal{B}^p$.*

Proof. First, we prove the lemma for $p = 0$. Let $Forb(\mathcal{X})$ be the set of minimal forbidden induced subgraphs for $\mathcal{X}$ and $\{\mathcal{B}_i\}$ a decreasing sequence of Π-tough classes converging to $\mathcal{B}$. Since $\mathcal{B}$ is contained in $\mathcal{X}$, for any graph $G \in Forb(\mathcal{X})$ there exists a natural number i_G such that for any $i > i_G$ the inclusion $\mathcal{B}_i \subseteq Free(G)$ holds. Let i^* be the maximum in $\{i_G : G \in Forb(\mathcal{X})\}$. Then $\mathcal{B}_{i^*} \supseteq \mathcal{B}_{i^*+1} \supseteq \dots$ is a decreasing sequence of Π-tough subclasses of $\mathcal{X}$ converging to $\mathcal{B}$.

To prove the lemma for $p > 0$, let us observe that all members of the sequence $\mathcal{B}^p_{i^*}, \mathcal{B}^p_{i^*+1}, \dots$ are Π-tough subclasses of $\mathcal{X}^p$. Obviously, this sequence converges to $\mathcal{B}^p$. □

Lemma 8. *Let $\mathcal{B}$ be a boundary class for the* VERTEX *3*-COLORABILITY *problem. If there is a finitely defined class $\mathcal{X}$ of graphs of chromatic number at most 4 containing $\mathcal{B}$, then $\mathcal{B}$ also is a boundary class for* VERTEX COLORABILITY.

Proof. Obviously, the VERTEX k-COLORABILITY problem polynomially reduces to VERTEX COLORABILITY on the same graph. Therefore, $\mathcal{B}$ is a limit class for VERTEX COLORABILITY. Now let us show that $\mathcal{B}$ is a boundary class for this problem.

Assume to the contrary that there is a boundary class $\mathcal{B}'$ which is properly contained in $\mathcal{B}$. Let us consider an arbitrary decreasing sequence of vc-tough classes converging to $\mathcal{B}'$. Similarly as in Lemma 7 we can show that some member of this sequence is contained in $\mathcal{X}$ as a subclass. Since the chromatic number of every graph in $\mathcal{X}$ is at most 4, we conclude that for any $k \geq 4$ VERTEX k-COLORABILITY is trivial for graphs in $\mathcal{X}$. Therefore, VERTEX 3-COLORABILITY and VERTEX COLORABILITY are polynomially equivalent for graphs in $\mathcal{X}$. Therefore, $\mathcal{B}'$ is a limit class for VERTEX 3-COLORABILITY, which contradicts the assumption of the lemma. This contradiction shows that $\mathcal{B}$ is a minimal limit (i.e. boundary) class for VERTEX COLORABILITY. □

Now we turn to the main results of the section.

Theorem 4. *Let $k \geq 3$ be an integer and $\mathcal{B}$ a boundary class for the* VERTEX *k*-COLORABILITY *problem. If there is a finitely defined class $\mathcal{X}$ of graphs of chromatic number at most $k + 1$ containing $\mathcal{B}$, then for any natural p the class $\mathcal{B}^p$ is boundary for* VERTEX *$(k + p)$*-COLORABILITY.

Proof. Since $\mathcal{B}$ is a boundary class for VERTEX k-COLORABILITY, there must exist a decreasing sequence $\{\mathcal{B}_i\}$ of k-vc-tough classes converging to $\mathcal{B}$. Obviously, for an arbitrary graph G the formula $\chi(G \circ K_p) = \chi(G) + p$ holds. Therefore, $\chi(G \circ K_p) \leq p + k$ if and only if $\chi(G) \leq k$. This implies that the class $\mathcal{B}^p_i$ is $(k + p)$-vc-tough for any i. As a result, $\mathcal{B}^p$ is a limit class for $(k + p)$-COLORABILITY. Let us show that $\mathcal{B}^p$ is a minimal limit class for this problem.

Assume the opposite. Then, by Lemma 7, there must exist a decreasing sequence $\{\mathcal{C}_i\}$ of $(k + p)$-vc-tough subclasses of $\mathcal{X}^p$ converging to a proper subclass of $\mathcal{B}^p$. For each i, all graphs in $\mathcal{C}_i$ are of the form $G \circ K_i$ with $G \in \mathcal{X}$ and $0 \leq i \leq p$. Let us split $\mathcal{C}_i$ into two parts: $\mathcal{C}_i{}'$ containing graphs of the form $G \circ K_i$ with $i < p$ and $\mathcal{C}_i{}''$ containing graphs of the form $G \circ K_p$. Since the chromatic number of graphs in $\mathcal{X}$ is at most $k + 1$, all graphs in $\mathcal{C}_i{}'$ are $k + p$ colorable, i.e. VERTEX $(k + p)$-COLORABILITY is trivial for graphs in $\mathcal{C}_i{}'$. On the other hand, this problem is NP-complete in the class $\mathcal{C}_i$. Therefore, it is also NP-complete for graphs in $\mathcal{C}_i{}''$.

Each graph in $\mathcal{C}_i{}''$ has a unique (up to isomorphism) representation in the form $G \circ K_p$ that can be found in polynomial time. By deleting the vertices of K_p from each graph in $\mathcal{C}_i{}''$ we obtain a set of graphs, say $\mathcal{D}_i$. Since VERTEX $(k+p)$-COLORABILITY is NP-complete for graphs in $\mathcal{C}_i{}''$, we conclude that VERTEX k-COLORABILITY is NP-complete for graphs in $\mathcal{D}_i$. Obviously, it is also NP-complete in the hereditary closure $[\mathcal{D}_i]$ of $\mathcal{D}_i$. Since $\{\mathcal{C}_i\}$ converges to a proper subclass of $\mathcal{B}^p$, we conclude that

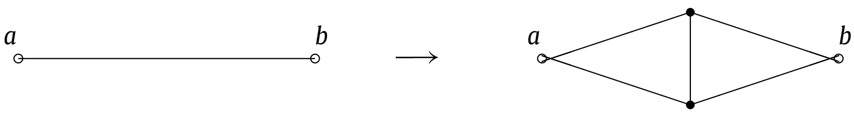

Fig. 5. Replacement of an edge by $K_4 - e$.

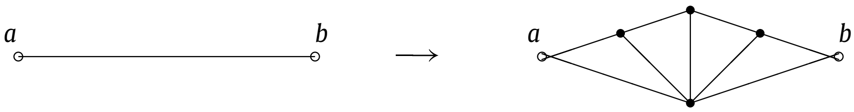

Fig. 6. Replacement of an edge by $P_5 \circ K_1$.

the sequence $[\mathcal{D}_i]$ converges to a proper subclass of $\mathcal{B}$ contradicting the assumption that $\mathcal{B}$ is a boundary class for VERTEX k-COLORABILITY. This contradiction completes the proof of the theorem. □

In order to reveal a continuum set of boundary classes for VERTEX k-COLORABILITY with $k > 3$, we will use a construction from [21] which describes a continuum set of boundary classes for VERTEX 3-COLORABILITY. This construction is based on the following two operations: replacement of an edge by $K_4 - e$ (Fig. 5) and replacement of an edge by $P_5 \circ K_1$ (Fig. 6).

Let $\pi = \{\pi_1, \pi_2, \dots, \pi_k\}$ be an arbitrary binary sequence. A π-garland is the graph obtained from P_{2k+1} by replacing, for each $i \in \{1, 2, \dots, k\}$, its ith and $(2k + 1 - i)$th edges by $K_4 - e$ if $\pi_i = 0$ and by $P_5 \circ K_1$ if $\pi_i = 1$.

Denote by D_π the graph obtained from a triangle and three copies of the π-garland by identifying, for $i = 1, 2, 3$, the ith vertex of the triangle with one of the end vertices (i.e. vertices of degree 2) of the ith copy of the π-garland.

Now let $\pi = \{\pi_1, \pi_2, \dots\}$ be an infinite binary sequence and $\pi^{(k)} = \{\pi_1, \dots, \pi_k\}$. The class $\mathcal{D}_\pi$ contains all graphs each connected component of which belongs to the hereditary closure of the set $\bigcup_{k=1}^{\infty}\{D_{\pi^{(k)}}\}$. The following theorem was proved in [21].

Theorem 5. *For an arbitrary infinite binary sequence π the class $\mathcal{D}_\pi$ is boundary for the* VERTEX 3-COLORABILITY *problem.*

Now we extend this result to VERTEX k-COLORABILITY with $k > 3$, which is the main result of this section.

Theorem 6. *For an arbitrary infinite binary sequence π and for any $p > 3$ the class $\mathcal{D}_\pi^p$ is boundary for the* VERTEX $(3 + p)$-COLORABILITY *problem.*

Proof. Note that for any π the class $\mathcal{D}_\pi$ is a subclass of K_5-free graphs of vertex degree at most 4. This class is finitely defined and from the well-known Brooks' Theorem [10] we know that all graphs in this class are 4 colorable. Therefore, by Theorem 4, for any $p > 0$ the class $\mathcal{D}_\pi^p$ is boundary for VERTEX $(3 + p)$-COLORABILITY.

Notice that by Lemma 8 and Theorem 5 $\mathcal{D}_\pi$ is a boundary class for VERTEX COLORABILITY. However, for $p > 0$ the class $\mathcal{D}_\pi^p$ is not boundary for VERTEX COLORABILITY since it properly contains the boundary class $\mathcal{D}_\pi$. □

From the proof of Theorem 6 we know that there exist boundary classes for the VERTEX k-COLORABILITY problem with $k > 3$ which are not boundary for VERTEX COLORABILITY. No such classes are known for $k = 3$ and we conjecture that no such classes exist. Proving or disproving this conjecture is a challenging research problem.

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