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On the simplest system with retarding switching and a 2–point critical set

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Abstract

The system considered in this paper consists of two equations $(k = 1,2)$ $\dot{x}(t) = (-1)^{k-1} (0 \le t < \infty)$, $k(0) = 1$, $x(0) = 0$, $x(t) \notin \{0,1\} (-1 \le t < 0)$, that change mutually in every instant t for which $x(t-\tau) \in \{0,1\}$, where $\tau = \text{const} > 0$ is given. In this paper the behavior of the solutions is characterized for every $\tau \in (\frac{4}{3}, \frac{3}{2})$, i. e. in case not covered in [4]; as it was noted there, this behavior turned out to be more complex then when $\tau \in (3/2, \infty)$. Thus the behavior of the solutions of this system with critical set $K = \{0,1\}$ is characterized for every $\tau > 0$.

Keywords. *Functional differential equation, Equation of retarded type, System with retarding switching, Periodic solution*

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In [1] the systems with retarding switching were introduced as a natural generalization of the "raging systems" of T. Vogel [2]. The Vogel's system is defined with two autonomous systems in $\mathbf{R^n}$ ("base systems") which replace each other when the phase point $x(t)$ reaches the given fixed "critical set" K. General systems with retarding switching were defined in [3]. Here the change of the base systems occurs in every instant t when the point $x(t-\tau)$ gets into K, where $\tau = \text{const} > 0$ is given, and the values of the solution are given on some time interval of the length τ as the initial condition, as well as the number of the initial base system.

In [4] the considered system has the form

$$\dot{x}(t) = (-1)^{k-1} \qquad (k = 1, 2; \quad 0 \le t < \infty), \tag{1}$$

whereas the critical set is $K = \{0, 1\}$. We set the continuous initial function $x = \varphi(t)$, for $-1 \le t \le 0$, where $\varphi(0) = 0$, $\varphi(t) \notin \{0, 1\}(-1 \le t < 0)$, and start with the first base equation ($k = 1$).

In [4] the behavior of the solution of the system (1) is fully characterized for $\tau \in [0, \frac{4}{3}] \cup [\frac{3}{2}, \infty)$. When $\tau \in [\frac{4}{3}, \frac{3}{2}]$ there are some calculations for $\tau \in [\frac{4}{3}, \frac{31}{21}]$.

But one critical value - $\tau = \frac{63}{43}$ - in interval $(\frac{16}{11}, \frac{31}{21})$ was lost. That's why coordinates x for turning points mentioned in [4] are correct only for $\tau \in (\frac{16}{11}, \frac{63}{43})$.

If $\tau = \frac{63}{43}$ after switching in points $\frac{63}{43}, \frac{20}{43}, \frac{60}{43}, \frac{14}{43}, \frac{48}{43}, -\frac{10}{43}, 0, -1, -\frac{23}{43}$ the solution goes to $-\infty$;

if $\tau \in (\frac{63}{43}, \frac{31}{21})$ it is periodic with sequential turning points $\tau, \tau-1, 3\tau-3, 5\tau-7, 11\tau-15, 21\tau-31, 43\tau-63, 43\tau-64, \tau-2, -85\tau+124$.

Probably this oversight hindered the author of [4] from finding the general rule of behavior of the solutions of the system (1) when $\tau \in [4/3, 3/2)$.

In this paper the behavior of the solutions of the system (1) is characterized for every $\tau \in (\frac{4}{3}, \frac{3}{2})$, i. e. in case not covered in [4]; as it was noted there, this behavior turned out to be more complex then when $\tau \in (3/2, \infty)$. Thus the behavior of the solution of this system with critical set $K = \{0, 1\}$ is characterized for every $\tau > 0$.

Let us introduce the designations $\tau_k := 3 \cdot 4^k/(2 \cdot 4^k + 1)$ $(k \in \mathbb{N})$; $\theta_k := 3 \cdot (4^{k+1} - 1)/(2 \cdot 4^{k+1} + 1)$ $(k \in \mathbb{N})$ and $\zeta_k := 3 \cdot (2 \cdot 4^k - 1)/(4^{k+1} - 1)$ $(k \in \mathbb{N})$. This are the increasing sequences, with $\tau_1 = \frac{4}{3}, \tau_k \to 3/2$ as $k \to \infty$; $\theta_1 = \frac{15}{11}, \theta_k \to 3/2$ as $k \to \infty$ and $\zeta_1 = \frac{7}{5}, \zeta_k \to 3/2$ as $k \to \infty$. More than $\tau_k < \theta_k < \zeta_k < \tau_{k+1}$ $(\forall k \in \mathbb{N})$.

Theorem 1. *For all $k \in \mathbb{N}$:*

the solution of the problem is periodic and has $4k+2$ switchings on the least period, if $\tau = \tau_k$;

the solution of the problem is periodic and has $2k+4$ switchings on the least period, if $\tau_k < \tau < \theta_k$;

the solution of the problem goes to $-\infty$ after $2k+5$ switchings, if $\tau = \theta_k$;

the solution of the problem is periodic and has $2k+6$ switchings on the least period, if $\theta_k < \tau < \zeta_k$;

the solution of the problem goes to $-\infty$ after $4k+5$ switchings, if $\tau = \zeta_k$;

the solution of the problem is periodic and has $2k+4$ switchings on the least period, if $\zeta_k < \tau < \tau_{k+1}$.

Proof. Let consider, that $\tau \in [4/3, 3/2)$ is given. Let $(a_1, b_1)(= (0,0))$, $(a_2, b_2), \ldots,$ $((\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots)$ be the sequence of the values of (x, t) in the instants of the hit of the solution on the critical set (in instants of the switching, respectively).

If $\tau = \tau_1 = \frac{4}{3}$ the solution is periodic with sequential turning points $\frac{4}{3}, \frac{1}{3}, 1, -\frac{1}{3}$, $\frac{2}{3}, 0$ and it corresponds with the formulation of the Theorem 1.

Let us prove that Theorem 1 is valid for all $\tau \in (4/3, 3/2)$. Denote by J the maximal natural number for which $\alpha_j > 1$ at all odd $j < J$ and $\alpha_j < 1$ at all even $j < J$; $J = \infty$ if α_j satisfy these conditions for all natural j. It is obvious that $J \geq 4$.

Lemma 1. *Let $\tau \in (4/3, 3/2)$, $2k+1 < J$; then $\alpha_{2k} > 0$.*

Proof. Let proof this statement by contradiction. Let assume that there exists $k \in \mathbb{N}$, so that $2k+1 < J$ but $\alpha_{2k} \leq 0$; from all such k choose the smallest. From the definition of J and from our assumption follows, that for all $n \in \mathbb{N}$ $(1 \leq n \leq 2k-2)$ in every segment with ends α_n, α_{n+1} there is only one point from our critical set : $\{1\}$. Taking into account formulas $a_2 = 1$, $\beta_1 < b_3 < \beta_2$, we conclude, that $a_n = 1\,(1 \leq n \leq 2k)$ and $\beta_{n-2} < b_n < \beta_{n-1}\,(3 \leq n \leq 2k+1)$. From $\alpha_{2k-1} > 0$ and $\alpha_{2k} \leq 0$ we obtain, that $a_{2k+1} = 1$, $a_{2k+2} = 0$. According to the definition of solution we have $\beta_n - \beta_n = \tau(\forall n \in \mathbb{N})$; in particular, $\beta_{2k+1} - b_{2k+1} = \tau < 3/2$. But $\beta_{2k+1} - b_{2k+1} = (\beta_{2k+1} - \beta_{2k})_{(}\beta_{2k} - b_{2k+1} = |\alpha_{2k+1} - \alpha_{2k}| + |\alpha_{2k} - a_{2k}| > 2$. This contradiction shows us falseness of our assumption. Lemma 1 is proved. □

Let $j \in [2, J-1]$. It follows from Lemma 1, that $\alpha_j > 0$ for all $j \in [2, J-2]$. Then it follows from the definition of the solution that

$$b_{j+1} - \beta_{j-1} = \beta_{j-1} - b_j. \tag{2}$$

But $\beta_j = b_j + \tau$ for all values of j such that b_j is defined - in particular, for $j = J < \infty$, as $b_J < \beta_{J-1}$. Therefore from (2) we obtain the recurrence relation

$$\beta_{j+1} = -\beta_j + 2\beta_{j-1} + 2\tau, \quad 2 \leq j \leq J-1. \tag{3}$$

Moreover the definition of the solution directly implies the expression for α in terms of β:

$$\alpha_j = 1 + (-1)^j[\tau - 2(\beta_j - \beta_{j-1})], \quad 2 \le j \le J. \tag{4}$$

The formula

$$\beta_j = \frac{6j + 1 - (-2)^j}{9}\tau - \frac{(-2)^{j-1} - 1}{3}, \quad 1 \le j \le J,$$

follows from the relation (3) and the initial data $\beta_1 = \tau$, $\beta_2 = \tau + 1$. This formula together with (4) and the equality $\alpha_1 = \tau$ yields that

$$\alpha_j = \frac{2^j - (-1)^j}{3}\tau - 2^{j-1} + 1, \quad 1 \le j \le J. \tag{5}$$

One can see that both sequences $\{\alpha_j\}(j \in [1, J]$ even) and ($j \in [1, J]$ odd) are decreasing. Hence J is the smallest even number for which the value α_j calculated according to the formula (5), becomes ≤ 1. The values $\tau_k, k \in \mathbb{N}$ for which $\alpha_{2k+1} = 1$, i.e.

$$\frac{2^{2k+1} + 1}{3}\tau_k - 2^{2k} + 1 = 1.$$

are the critical ones. We find from here that $\tau_k = 3 \cdot 4^k/(2 \cdot 4^k + 1)$. Now let's consider possible cases.

If $\tau = \tau_k$, then $\alpha_{2k+1} = 1$, that's why because in $[\alpha_{2k}, \alpha_{2k+1}]$ there are no other points from critical set, the solution continues with the same time intervals but symmetrically relatively $\{x = 1/2\}$, i. e. the solution is periodical and has $2k + 1 + 2k + 1 = 4k + 2$ switchings on the least period (Fig. 1)[1].

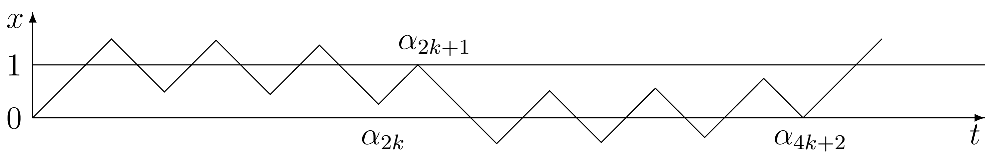

Fig. 1.

Let then $\tau_k < \tau < \tau_{k+1}$. Therefore $\alpha_{2k+1} > 1$ and $\alpha_{2k+3} < 1$. The further behavior of the solution depends on that, if α_{2k+3} is greater than zero or no. So we have another set of critical points, for which

$$\frac{2^{2k+3} + 1}{3}\theta_k - 2^{2k+2} + 1 = 0,$$

[1]All figures were rendered for $k = 3$

wherefrom we receive $\theta_k = 3 \cdot (4^{k+1} - 1)/(2 \cdot 4^{k+1} + 1)$.

If $\tau_k < \tau < \theta_k$, i. e. $\alpha_{2k+3} < 0$, then after switchings in points α_{2k+2} and α_{2k+3} there is one another switching in point $\alpha_{2k+4} < 0$ and the solution gets to the beginning of it's period after $2k + 4$ switchings (Fig. 2).

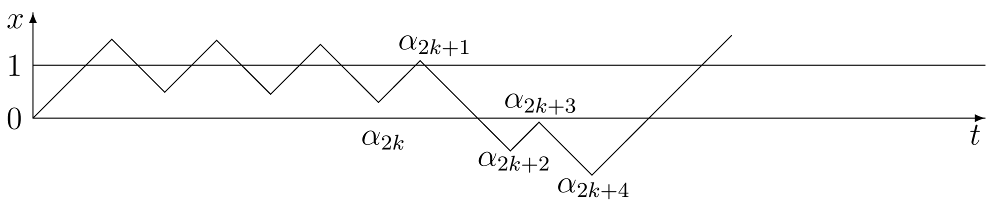

Fig. 2.

If $\tau = \theta_k$, i. e. $\alpha_{2k+3} = 0$, then we can verify by direct calculating, that $\alpha_{2k+4} = -1$. Therefore $\alpha_{2k+5} < 0$ after which nothing does not prevent the solution to go to $-\infty$. It has performed $2k + 5$ switchings (Fig. 3).

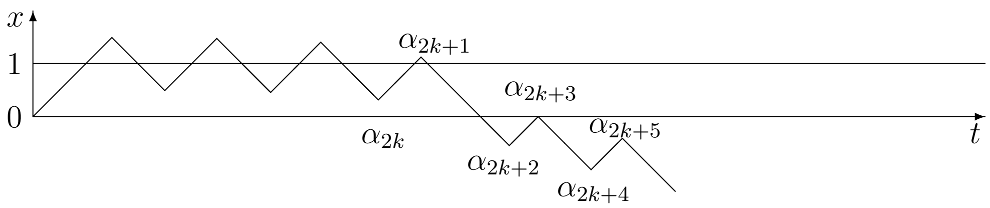

Fig. 3.

If $\theta_k < \tau < \tau_{k+1}$, the result depends on the sign of α_{2k+2}. From this we receive the last set of critical points

$$\frac{2^{2k+2} - 1}{3} \zeta_k - 2^{2k+1} + 1 = 0,$$

or $\zeta_k = 3 \cdot (2 \cdot 4^k - 1)/(4^{k+1} - 1)$.

If $\theta_k < \tau < \zeta_k$, then we can verify by direct calculating, that $\alpha_{2k+5} = \tau - 2 < 0$. So the solution after switching in points α_{2k+4}, α_{2k+5} and α_{2k+6} gets to the beginning of it's period after $2k + 6$ switchings (Fig. 4).

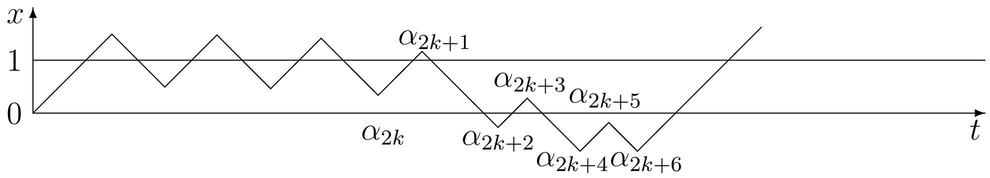

Fig. 4.

If $\tau = \zeta_k$. Then we can observe, that $\alpha_{2k+3} = \tau - 1$. It means, that the solution further behave as if $t \in [\beta_1, \beta_{2k+3}]$, and after that, having in all $2k+2+2k+2+1 = 4k+5$ switchings, goes to $-\infty$ (Fig. 5).

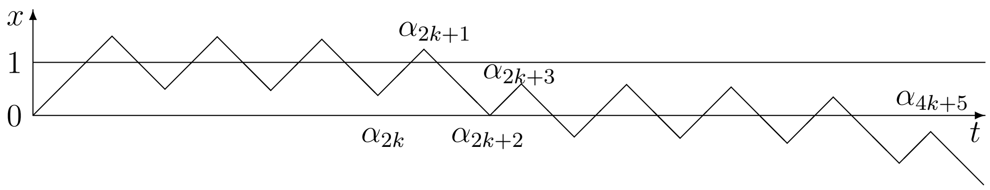

Fig. 5.

And finally if $\zeta_k < \tau < \tau_{k+1}$, then because in this case $\alpha_{2k+2} > 0$ and $\alpha_{2k+3} < 1$, there is only one another switching, after which the solution gets to the beginning of it's period after $2k+4$ switchings (Fig. 6). This ends proof of Theorem 1.

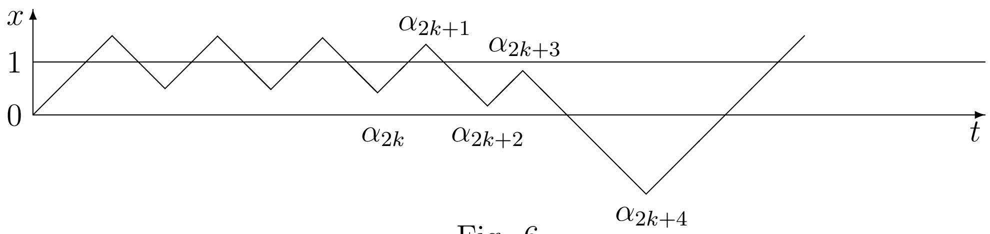

Fig. 6.

□

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