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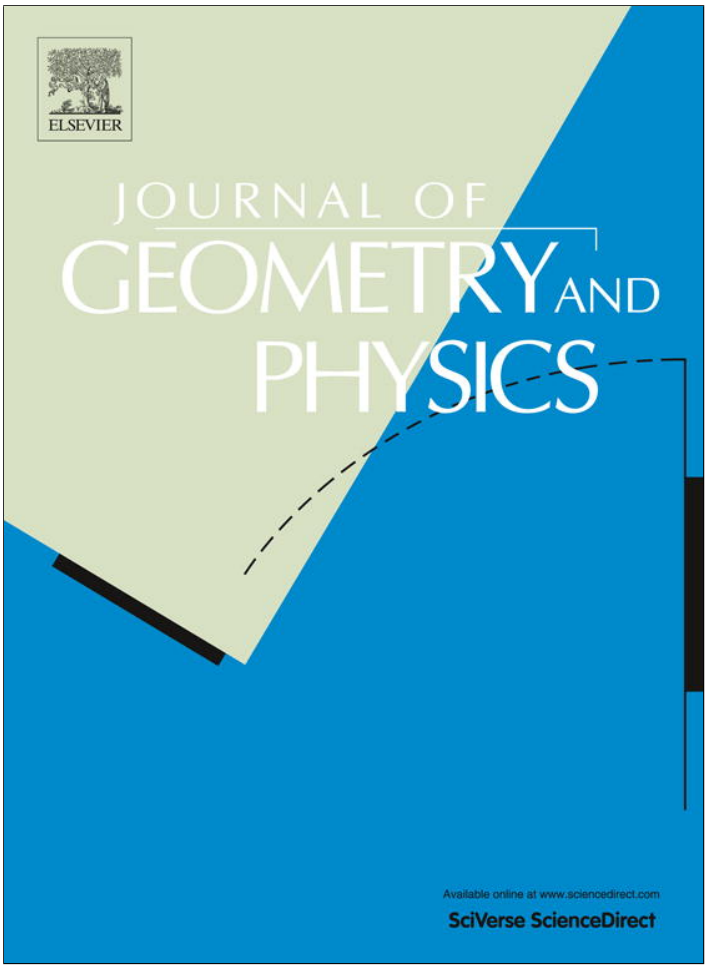

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Journal of Geometry and Physics 73 (2013) 243–251

Contents lists available at SciVerse ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

Cardy–Frobenius extension of the algebra of cut-and-join operators

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ARTICLE INFO

Article history:
Received 16 January 2013
Received in revised form 5 June 2013
Accepted 11 June 2013
Available online 25 June 2013

Keywords:
Open–closed topological theory
Hurwitz theory
Cardy–Frobenius algebra

ABSTRACT

Motivated by the algebraic open–closed string models, we introduce and discuss an infinite-dimensional counterpart of the open–closed Hurwitz theory describing branching coverings generated both by the compact oriented surfaces and by the foam surfaces. We manifestly construct the corresponding infinite-dimensional equipped Cardy–Frobenius algebra, with the closed and open sectors being represented by the conjugation classes of permutations and of pairs of permutations, i.e. by the algebras of Young diagrams and of bipartite graphs respectively.

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1. The main result

1.1. Introduction

It is an old idea to formulate the open–closed string theory in purely algebraic terms. This would allow one to consider much simpler examples of string phenomena and involve basic mathematical constructions in the string theory framework. Hence there arises the idea of formulating a set of axioms which have to be satisfied by an algebra of operators creating states, both on the boundary and in the bulk (world-sheet), in any two-dimensional open–closed topological field theory. This set of axioms was formulated and described in [1–4].

Having this axiomatic description in hand, one can look for simple examples that satisfy these axioms. Particular examples of this kind are characterized by a finite-dimensional space of states and are in one-to-one correspondence with the Cardy–Frobenius algebras [3]. They are closely related to the representation theory [5]. However, string theory requires instead a Hilbert space of states and, therefore, one would need to extend the examples to infinite-dimensional spaces of states. This is what we do here: we make such an extension and describe an example of a Cardy–Frobenius algebra of this kind. It consists of differential operators, which is quite natural for string state creating operators.

1.2. The Cardy–Frobenius algebra

We start by recalling the definition of the (finite-dimensional) equipped Cardy–Frobenius algebra following [3,6].

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http://dx.doi.org/10.1016/j.geomphys.2013.06.007

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A *Frobenius pair* is a set (C, l^C) which consists of a finite-dimensional associative algebra C with an identity element and a linear functional $l^C : C \to \mathbb{C}$ such that the bilinear form $(c_1, c_2)_C = l^C(c_1c_2)$ which it generates is non-degenerate.

The *Casimir* of the Frobenius pair (C, l^C) is the element $K_C = \sum_{i=1}^n F^{ij}e_ie_j \in C$, where $\{e_1, \ldots, e_n\}$ is a basis of the space C and $\{F^{ij}\}$ is the matrix inverse of $(e_i, e_j)_C$. For an involutive anti-automorphism $\star : C \to C$ we define the *twisted Casimir* $K_C^\star = \sum_{i=1}^n F^{ij}e_ie_j^\star \in C$.

Given the Frobenius pairs (A, l^A), (B, l^B) and the linear operator $\phi : A \to B$, denote as $\phi^* : B \to A$ the linear operator determined by the condition $(\phi^*(b), a)_A = (b, \phi(a))_B$.

A *Cardy–Frobenius algebra* is a set $((A, l^A), (B, l^B), \phi)$ which consists of:

(1) Frobenius pairs (A, l^A) and (B, l^B), where the algebra A is commutative;
(2) a homomorphism of the algebras $\phi : A \to B$ such that the image $\phi(A)$ belongs to the center of B and $(\phi^*(b'), \phi^*(b''))_A = \mathrm{tr}K_{b'b''}$, where the operator $K_{b'b''} : B \to B$ is defined as $K_{b'b''}(b) = b'bb''$.

An *equipped Cardy–Frobenius algebra* is a set $((A, l^A), (B, l^B), \phi, U, \star)$ which consists of:

(1) a Cardy–Frobenius algebra $((A, l^A), (B, l^B), \phi)$;
(2) involutive anti-automorphisms $\star : A \to A$ and $\star : B \to B$ such that $l^A(x^\star) = l^A(x)$, $l^B(x^\star) = l^B(x)$, $\phi(x^\star) = \phi(x)^\star$ where $x^\star = \star(x)$ for $x \in A \bigcup B$;
(3) an element $U \in A$ such that $U^2 = K_A^\star$ and $\phi(U) = K_B^\star$.

Any Cardy–Frobenius algebra with semi-simple A and B can be equipped in multiple ways [3].

The commutative Frobenius pairs are in one-to-one correspondence [7] with closed topological field theories in the meaning of [8]. The Cardy–Frobenius algebras are in one-to-one correspondence [3] with open–closed topological field theories in the meaning of [2,1,4]. The equipped Cardy–Frobenius algebras are in one-to-one correspondence [3] with Klein topological field theories, i.e. with those defined on non-oriented Riemann surfaces [3].

Each real representation of a finite group generates a semi-simple equipped Cardy–Frobenius algebra [5]. Some of them describe the Hurwitz numbers of finite-sheeted coverings [6,9]. There exists a complete classification of the semi-simple equipped Cardy–Frobenius algebras [3].

For an extension of our definitions to the infinite-dimensional case we use the direct *Cartesian product* $C = \prod_{\gamma\in\mathfrak{C}} C_\gamma$ of the finite-dimensional algebras $\{C_\gamma | i = 1, \ldots, \infty\}$, which consists of all infinite sequences $\{c_1, c_2, \ldots, |c_i \in C_i\}$ with operations by coordinates.

In the definition of an equipped Cardy–Frobenius algebra one needs to invert matrices. Therefore, one needs additional accuracy when dealing with the infinite-dimensional case. We additionally require that such algebras can be presented as a Cartesian product of finite-dimensional algebras $A = \prod_{\gamma\in\mathfrak{C}} A_\gamma$, $B = \prod_{\gamma\in\mathfrak{C}} B_\gamma$ and, instead of functionals on A and B, we consider the families of functionals $l^A = \{l^A_\gamma : A_\gamma \to \mathbb{C}\}$, $l^B = \{l^B_\gamma : B_\gamma \to \mathbb{C}\}$ such that:

(1) (A_γ, l^A_γ) and (B_γ, l^B_γ) are Frobenius pairs;
(2) $\phi(A_\gamma) \in B_\gamma$, and restrictions ϕ_γ of the homomorphism ϕ on A_γ give rise to the Cardy–Frobenius algebras $((A_\gamma, l^A_\gamma), (B_\gamma, l^B_\gamma), \phi_\gamma)$;
(3) the involution $\star$ preserves the subalgebras A_γ, B_γ and, along with the projections U_γ of the element $U \in A$ onto A_γ, gives rise to the equipped Cardy–Frobenius algebras $((A_\gamma, l^A_\gamma), (B_\gamma, l^B_\gamma), \phi_\gamma, U_\gamma, \star)$.

No restrictions on the set $\mathfrak{C}$ are required; however, we consider below only $\mathfrak{C} = \{1, 2, \ldots\}$.

1.3. Cut-and-join operators

Let us construct an equipped Cardy–Frobenius algebra that consists of differential operators on the space of functions of infinitely many variables $\{X_{ij} | i, j = 1, \ldots\}$.

The role of algebra A is played by the algebra W of cut-and-join operators $W(\Delta)$ [10–13].

We recall a construction of these latter. Consider differential operators of the form $D_{ab} = \sum_{e=1}^\infty X_{ae}\frac{\partial}{\partial X_{be}}$ and associate with the Young diagram $\Delta = [\mu_1, \mu_2, \ldots, \mu_k]$ with line lengths $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k$ the numbers $m_j = m_j(\Delta) = |\{i | \mu_i = j\}|$ and $\kappa(\Delta) = (|\mathrm{Aut}(\Delta)|)^{-1} = (\prod_j m_j! j^{m_j})^{-1}$. First, restrict the set of $\{X_{ij}\}$ to finitely many variables: $\{X_{ij} | i, j = 1, \ldots, N\}$. Multiplication of operators is given by the rule

$$D_{ab}D_{cd} = \sum_{1\leq e_1,e_2\leq N} X_{ae_1}X_{ce_2}\frac{\partial}{\partial X_{be_1}}\frac{\partial}{\partial X_{de_2}} + \delta_{bc}\sum_{e\in\mathbb{N}} X_{ae}\frac{\partial}{\partial X_{de}}.$$

Commutation relations for the operators D_{ab} coincide with the commutation relations for the generators of the matrix algebra. Hence, the operators D_{ab} give rise to the algebra U_N naturally isomorphic to the universal enveloping algebra $U(gl(N))$. In the limit $N \to \infty$ there emerges the algebra U_∞ of the formal differential operators which are finite or countable sums of operators of the form

$$:D_{a_1b_1}\cdots D_{a_nb_n}: := \sum_{e_1,\ldots,e_n\in\mathbb{N}} X_{a_1e_1}\cdots X_{a_ne_n}\frac{\partial}{\partial X_{b_1e_1}}\cdots\frac{\partial}{\partial X_{b_ne_n}}.$$

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Then, one associates with the Young diagram Δ *the cut-and-join operator* $W(\Delta) = \kappa(\Delta) : \prod_j (\mathrm{tr} D^j)^{m_j} :$, where D is the infinite-dimensional matrix with elements $D_{ab} = \sum_{e=1}^{\infty} X_{ae} \frac{\partial}{\partial X_{be}}$. We denote the product of operators as $\circ$. The properties of these operators $W(\Delta)$ differ a lot from those of their finite-dimensional counterparts [14].

Denote as W_n the vector space generated by the cut-and-join operators of degree n. The product $\circ$ of operators continued to infinite formal sums $w_1 + w_2 + w_3 + \cdots$ with $w_n \in W_n$ generates an associative commutative algebra W of formal series of differential operators.

1.4. Graph-operators

The cut-and-join operators are related (see [11]) to the Hurwitz numbers of branching coverings generated by the compact oriented surfaces [7]. The role of algebra B is played by the algebra V of graph-operators related to the Hurwitz numbers of branching coverings generated by the foam surfaces [9].

We define a *bipartite graph of degree n* as a graph with n edges and vertices, forming two ordered groups $L = L(\Gamma)$, $R = R(\Gamma)$, such that the edges $E = E(\Gamma)$ connect the vertices from different groups. Denote by $L(e) \in L$ and $R(e) \in R$ the endings of $e \in E$. Isomorphisms are homomorphisms of graphs preserving the partition of vertices into groups and ordering in the groups. Hereafter, we do not make any difference between isomorphic graphs.

We call a graph *simple* if all its connected components are graphs with two vertices. We name as *the standard extension* of the graph Γ any graph obtained from Γ by adding simple connected components and/or by adding any number of edges to any pair of vertices connected with an edge in Γ. Denote as $\mathcal{E}_n(\Gamma)$ the set of all standard extensions of degree n of the graph Γ. We put $\sigma_n(\Gamma) = \sum_{\hat{\Gamma} \in \mathcal{E}_n(\Gamma)} \frac{|\mathrm{Aut}(\hat{\Gamma})|}{|\mathrm{Aut}(\hat{\Gamma} \setminus \Gamma)||\mathrm{Aut}(\Gamma)|} \hat{\Gamma}$ for $n \geq |\Gamma|$ and $\sigma_n(\Gamma) = 0$ for $n < |\Gamma|$; see [15].

Associate with the monomial $x = X_{a_1 b_1} \ldots X_{a_n b_n}$ of degree n the bipartite graph $\Gamma(x)$ with edges $\{E_1, \ldots, E_n\}$ such that the edges E_i and E_j have a common left (right) vertex if and only if $a_i = a_j$ ($b_i = b_j$), a_i and b_i being positive integers and the left group (L) is ordered to be the first, the right one (R) to be the second. Now associate with the graph Γ the *graph-variable* $X_\Gamma = \frac{1}{|\mathrm{Aut}(\Gamma)|} \sum x$, where the (formal) sum runs over all monomials x such that $\Gamma(x) = \Gamma$. Denote as X_n the vector space generated by the graph-variables of degree n.

Associate with the operator $D =: D_{a_1 b_1} \ldots D_{a_n b_n}$: the bipartite graph $\Gamma(\mathcal{D})$ with edges $\{E_1, \ldots, E_m\}$ such that the edges E_i and E_j have a common left (right) vertex if and only if $a_i = a_j$ ($b_i = b_j$). Now associate with the graph Γ the operator $V[\Gamma] = \frac{1}{|\mathrm{Aut}(\Gamma)|} \sum \mathcal{D}$, where the sum runs over all operators $\mathcal{D}$ such that $\Gamma(D) = \Gamma$. We call such operators *graph-operators*.

Let us give an action of the graph-operator of degree n on the graph-variables of the same degree. The usual action of the graph-operator on the graph-variable results in a linear combination of the graph-variables with (generally) infinite coefficients. Therefore, the correct definition of differentiation requires a regularization. To construct it, consider, along with the (full) graph-operator and graph-variable $V[\Gamma]$, $X_{[\Gamma']}$, the restricted graph-operator $V^N[\Gamma]$ and graph-variable $X^N_{[\Gamma']}$ defined similarly to the full ones, but with the infinite set of variables $\{X_{ij} | i,\ j = 1, 2, \ldots\}$ replaced with the finite one $\{X_{ij} | i,\ j = 1, \ldots, N\}$.

We define the action of the graph-operator $V^N[\Gamma]$ on the graph-variable $X^N_{[\Gamma']}$ as the usual action of the differential operator multiplied with $\frac{(N - |R(\Gamma)|)!}{N!}$. One can easily see that $V^N[\Gamma](X^N_{[\Gamma']})$ is a linear combination of the restricted graph-variables $X^N_{[\Gamma'']}$. Moreover, the coefficients of this linear combination are the same at any $N > |E(\Gamma)|$.

We define $V[\Gamma](X_{[\Gamma']}) = \lim_{N \to \infty} V^N[\Gamma](X^N_{[\Gamma']})$ for $|\Gamma| = |\Gamma'|$ and continue it naturally to $V[\Gamma](X_{[\Gamma']}) = 0$ for $|\Gamma| > |\Gamma'|$ and $V[\Gamma](X_{[\Gamma']}) = V[\sigma_{|\Gamma'|}(\Gamma)](X_{[\Gamma']})$ for $|\Gamma| < |\Gamma'|$.

Denote as V_n the vector space generated by the graph-operators of degree n and as V the vector space of the formal differential operators $v_1 + v_2 + v_3 + \cdots$, where $v_n \in V_n$. Define on V the operation $\circ$, requiring that the operator $V[\Gamma_1] \circ V[\Gamma_2]$ acts on all the graph-variables $X_{[\Gamma]}$ as $V[\Gamma_1](V[\Gamma_2](X_{[\Gamma]}))$.

1.5. The Cardy–Frobenius algebra of differential operators

The cut-and-join operators $W(\Delta)$ act on the space of graph-variables $X_{[\Gamma]}$ via the usual differentiation. Define the homomorphism of algebras $f : W \to V$, requiring that the operator $f(w)$ acts on all the graph-variables in the same way as the operator w (below we prove that such an operator exists).

The involution $D_{ab} \leftrightarrow D_{ab}$ gives rise to trivial automorphisms $\star : W \to W$ and involutive automorphisms $\star : V \to V$.

The Young diagrams of degree n are naturally identified with a basis at the center of the group algebra of the symmetric group S_n. The sum U_n of squares of all the elements of the symmetric group that act without fixed points belongs to the same center. This allows one to associate an operator $r_n \in W_n$ with the element U_n. Now put $R = \sum_{i=1}^{\infty} r_n$.

Our main result is the following:

Theorem 1.1. *There exist decompositions* $W = \prod_{\gamma \in \mathfrak{C}} W_\gamma$, $V = \prod_{\gamma \in \mathfrak{C}} V_\gamma$ *and the family of functionals* $l^W = \{l^W_\gamma : W_\gamma \to \mathbb{C}\}$, $l^V = \{l^V_\gamma : V_\gamma \to \mathbb{C}\}$ *such that the set* $((W, l^W), (V, l^V), f, R, \star)$ *forms an equipped Cardy–Frobenius algebra.*

The constructed algebra is a counterpart of the algebra of cut-and-join operators for the foam-coverings (simplest examples of such operators are constructed in [16]). We discuss it in detail elsewhere.

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2. Finite-dimensional algebras of Young diagrams and bipartite graphs

2.1. Young diagrams

First, we recall the standard facts that we will need. Denote as $|\mathfrak{M}|$ the number of elements in the finite set $\mathfrak{M}$ and as S_n the symmetric group that acts by permutations on the set $\mathfrak{M}$ with $|\mathfrak{M}| = n$. The permutation $\sigma \in S_n$ generates the subgroup $\langle\sigma\rangle$, with its action dividing $\mathfrak{M}$ into orbits $\mathfrak{M}_1, \ldots, \mathfrak{M}_k$. The set of numbers $|\mathfrak{M}_1|, \ldots, |\mathfrak{M}_k|$ is called *cyclic type of the permutation* σ. It produces the Young diagram $\Delta(\sigma) = [|\mathfrak{M}_1|, \ldots, |\mathfrak{M}_k|]$ of degree n. Permutations are conjugated in S_n if and only if they have the same cyclic type.

Linear combinations of permutations from S_n form the group algebra $G_n = G(S_n)$. We denote multiplication in this algebra as "$*$". Associate with each Young diagram Δ the sum $G_n(\Delta) \in G_n$ of all permutations of the cyclic type Δ. These sums (which, for the sake of brevity, we denote with the same symbol Δ as the corresponding Young diagram) form a basis of the *algebra of the conjugation classes* $A_n \subset G_n$ coinciding with the center of the algebra G_n.

Example 2.1. $[1] * [1] = [1]$, $[2] * [2] = [11]$, $[11] * [11] = [11]$, $[3] * [3] = 2[111]$.

The sum U_n of squares of elements of the group S_n belongs to the algebra A_n. Denote as $I_n^A : A_n \to \mathbb{C}$ a linear functional equal to $\frac{1}{n!}$ on the Young diagram with all lines having unit length and equal to zero otherwise. Denote as $\star : A_n \to A_n$ the identity map $a \mapsto a^\star = a$.

2.2. Bipartite graphs

Describe following [17,6] the operation of multiplication $*$ on the vector space B_n generated by the set of classes of isomorphisms of the bipartite graphs of degree n.

Let (L, E, R) and (L', E', R') be a pair of bipartite graphs with n edges. Denote as $\mathrm{Hom}(R, L')$ the set of maps $\chi : R \to L'$ which preserve the vertex valences. Every such map is associated with the bipartite graph (R, E_χ, L') whose edges connect only the vertices $\tilde{v}$ and $\chi(\tilde{v})$, where $\tilde{v} \in R$, the number of edges connecting $\tilde{v}$ and $\chi(\tilde{v})$ being equal to the valence of the vertex $\tilde{v}$.

To define a multiplication of graphs (L, E, R) and (L', E', R') we connect a set M_χ of *consistent* bijections $E \leftrightarrow E'$ with $\chi \in \mathrm{Hom}(R, L')$. We describe any consistent bijection from M_χ as a subset $F \subset E \times E'$ with the properties:

(1) the restrictions on F of the natural projections $E \times E' \to E$ and $E \times E' \to E'$ form the bijection between F and E, E';
(2) $\chi(R(e)) = L(e')$ for any $(e, e') \in F$.

Associate with the set $F \in M_\chi$ the bipartite graph $(L, \overline{F}, R')$ whose edges are the pairs of edges $(e, e') \in F$ glued at the points $R(e)$ and $L(e')$. Denote as $\mathrm{Aut}_F(L, \overline{F}, R') \subset \mathrm{Aut}(L, \overline{F}, R')$ the subgroup which consists of the automorphisms inducing on the set E the automorphisms of the graph (L, E, R).

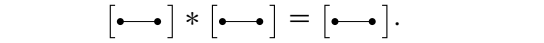
Let us now construct the map $B_n \times B_n \to B_n$ by putting

$$[(L, E, R)][(L', E', R')] = \sum_{\chi \in \mathrm{Hom}(R, L')} \sum_{F \in M_\chi} \frac{|\mathrm{Aut}((R, E_\chi, L'))|}{|\mathrm{Aut}_F((L, \overline{F}, R'))|} [(L, \overline{F}, R')].$$

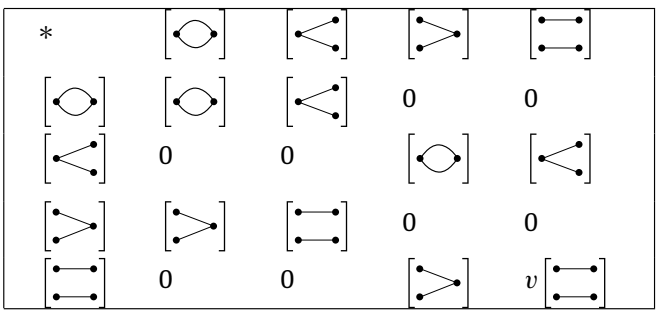
Continuing this by linearity, one obtains a binary operation which transforms B_n into an algebra.

This operation has a simple geometrical meaning. The contribution to the product $[(L, E, R)] * [(L', E', R')]$ is given by the valence-preserving identifications of vertices from R and L'. As a result of such an identification, there emerges "a special graph" with vertices $L \cup R'$ and edges that intersect on "the set of singularities" $R = L'$. We name as a product a linear combination of "resolutions" of these singularities, i.e. of (generating the bipartite graph) pairwise gluings of edges from (L, E, R) and (L', E', R') coming to the common vertex.

The algebra B_1 looks as follows:



The multiplication table for B_2 looks as follows:



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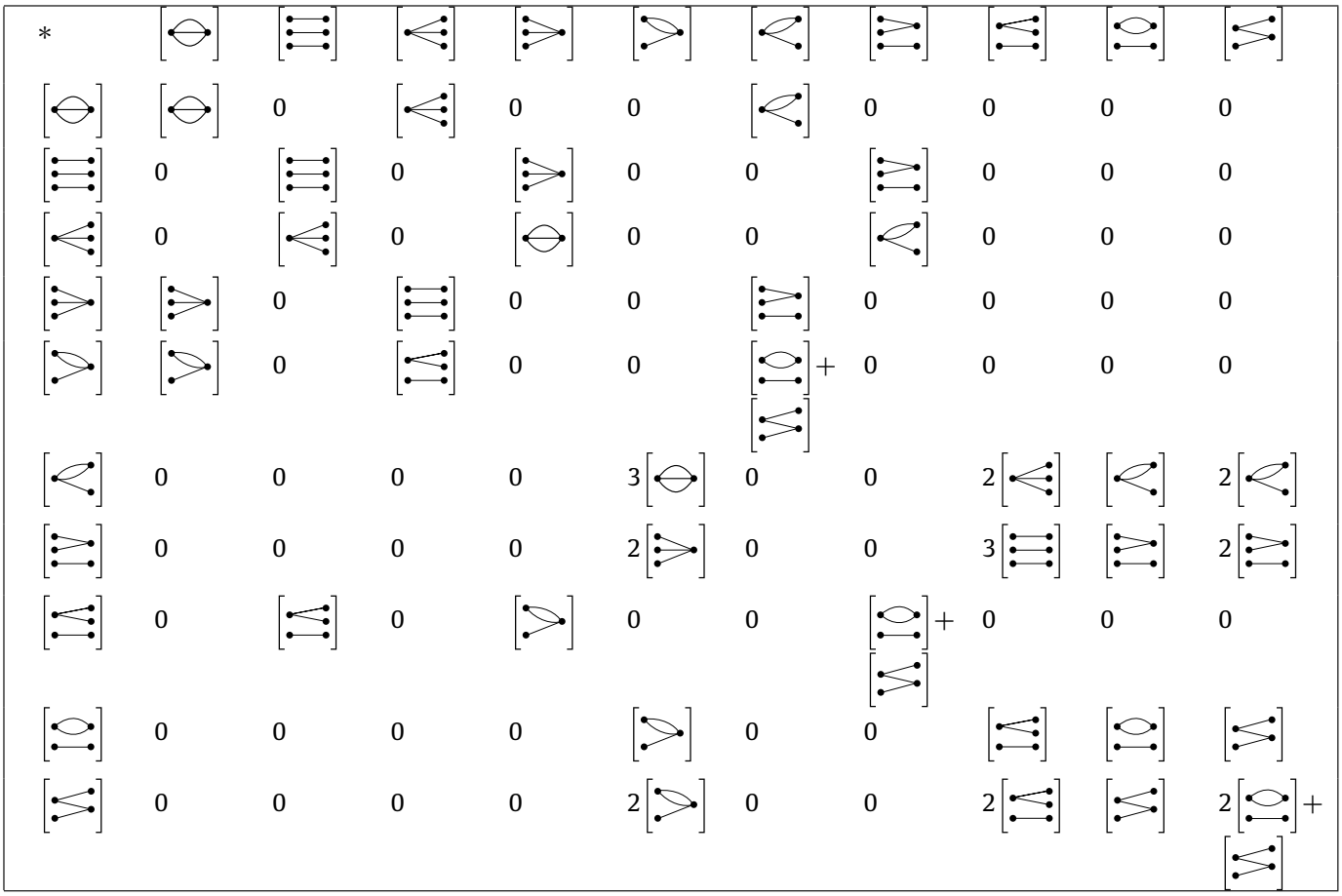
Example 2.2. The multiplication table for B_3 is



The sum $\epsilon_n^B = \sum \frac{\Gamma}{|\mathrm{Aut}(\Gamma)|}$ over the set of simple graphs Γ of degree n is the identity element of algebra B_n. Denote as $l_n^B : B \to \mathbb{C}$ the linear functional which is equal to $\frac{1}{|\mathrm{Aut}(\Gamma)|}$ on the simple graphs Γ and vanishes on all other graphs.

The involution $(L, E, R) \mapsto (L, E, R)^* = (R, E, L)$ induces an anti-isomorphism $* : B_n \to B_n$ (i.e. $(ab)^* = b^*a^*$).

2.3. Homomorphisms of algebras

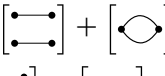
Let $\mathfrak{N}$ be the set of all partitions of the set $\mathfrak{M}$ into non-empty subsets. Consider a vector space H_n over $\mathbb{C}$ with a basis $\mathfrak{N}$.

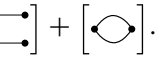
Construct following [6] a homomorphism $\varrho : B_n \to \mathrm{End}(H_n)$. To this end, it suffices to determine the image $\varrho(\Gamma)$ of the graph $\Gamma = (L, E, R)$. The vertices from the sets L and R give rise to partitions of the set of edges E into subsets σ_L, σ_R. The bijection $\chi : E \to \mathfrak{M}$ maps them to the partitions $\sigma_L^\chi, \sigma_R^\chi \in \mathfrak{N}$. Denote as $\varrho_\chi(\Gamma) \in \mathrm{End}(H_n)$ the endomorphism mapping to zero all the partitions from $\mathfrak{N}$ not equal to σ_R^χ, and $\varrho_\chi(\sigma_R^\chi) = \sigma_L^\chi$. Denote as $\varrho(\Gamma) \in \mathrm{End}(H_n)$ the sum of the endomorphisms ϱ_χ over all bijections $\chi : E \to \mathfrak{M}$. The map $\varrho : B_n \to \mathrm{End}(H_n)$ is an injective homomorphism of algebras (Lemmas 2.3, 2.4 from [6]).

The natural action of the group S_n on $\mathfrak{M}$ gives rise to a natural action of the group S_n on $\mathfrak{N}$ and, hence, a homomorphism of the group algebra $\tilde{\phi}_n : G_n \to \mathrm{End}(H_n)$. The image $\chi(E) \subset \mathrm{End}(H_n)$ consists of all endomorphisms commuting with $\tilde{\phi}_n(G_n)$ and, in particular, $\tilde{\phi}_n(A_n) \subset \chi(B_n)$. This allows one to define the homomorphism $\phi_n = \varrho^{-1}\tilde{\phi}_n : A_n \to B_n$:

Example 2.3.

$\phi_1[1] =$

$\phi_2[1, 1] =$ 

$\phi_2[2] =$  .

Theorem 2.1 (*[6]*). *The set* $((A_n, l_n^A), (B_n, l_n^B), \phi_n, U_n, \star)$ *forms an equipped Cardy–Frobenius algebra.*

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3. Proof of the main theorem

3.1. Restricted multiplication

First consider a few examples of graph-operators and graph-variables and the action of the graph-operators on the graph-variables.

Example 3.1.

$$X_{\left[\bullet\!-\!\bullet\right]} = \sum_{a,b} X_{ab} \qquad X_{\left[\begin{smallmatrix}\bullet\!-\!\bullet\\ \bullet\!-\!\bullet\end{smallmatrix}\right]} = \frac{1}{2}\sum_{\substack{a_1\neq a_2\\ b_1\neq b_2}}^{N} X_{a_1b_1}X_{a_2b_2}$$

$$X_{\left[\begin{smallmatrix}\bullet\!-\!\bullet\\ \bullet\end{smallmatrix}\right]} = \frac{1}{2}\sum_{\substack{a\\ b\neq c}} X_{ab}X_{ac} \qquad X_{\left[\begin{smallmatrix}\bullet\!-\!\bullet\\ \bullet\end{smallmatrix}\right]} = \frac{1}{2}\sum_{\substack{a\neq b\\ c}} X_{ac}X_{bc} \qquad X_{\left[\bullet\!\frown\!\bullet\right]} = \frac{1}{2}\sum_{a,b=1} X_{ab}^2.$$

Example 3.2.

$$V\left[\bullet\!-\!\bullet\right] = \sum_{a,b} D_{ab} \qquad V\left[\begin{smallmatrix}\bullet\!-\!\bullet\\ \bullet\!-\!\bullet\end{smallmatrix}\right] = \frac{1}{2}\sum_{\substack{a_1\neq a_2\\ b_1\neq b_2}} :D_{a_1b_1}D_{a_2b_2}:$$

$$V\left[\begin{smallmatrix}\bullet\!-\!\bullet\\ \bullet\end{smallmatrix}\right] = \frac{1}{2}\sum_{\substack{a\\ b\neq c}} :D_{ab}D_{ac}: \qquad V\left[\begin{smallmatrix}\bullet\!-\!\bullet\\ \bullet\end{smallmatrix}\right] = \frac{1}{2}\sum_{\substack{a\neq b\\ c}} :D_{ac}D_{bc}: \qquad V\left[\bullet\!\frown\!\bullet\right] = \frac{1}{2}\sum_{a,b=1} :D_{ab}^2:.$$

Example 3.3.

$$V\left[\bullet\!-\!\bullet\right](X_{\left[\bullet\!-\!\bullet\right]}) = \lim_{N\to\infty}\frac{(N-1)!}{N!}\sum_{a,b}^{N} D_{ab}\left(\sum_{a',b'}^{N} X_{a'b'}\right) = X_{\left[\bullet\!-\!\bullet\right]}$$

$$V\left[\begin{smallmatrix}\bullet\!-\!\bullet\\ \bullet\!-\!\bullet\end{smallmatrix}\right](X_{\left[\begin{smallmatrix}\bullet\!-\!\bullet\\ \bullet\!-\!\bullet\end{smallmatrix}\right]}) = \lim_{N\to\infty}\frac{(N-2)!}{N!}\frac{1}{2}\sum_{\substack{a_1\neq a_2\\ b_1\neq b_2}}^{N} :D_{a_1b_1}D_{a_2b_2}:\left(\frac{1}{2}\sum_{\substack{a'_1\neq a'_2\\ b'_1\neq b'_2}}^{N} X_{a'_1b'_1}X_{a'_2b'_2}\right) = X_{\left[\begin{smallmatrix}\bullet\!-\!\bullet\\ \bullet\!-\!\bullet\end{smallmatrix}\right]}$$

$$V\left[\bullet\!\frown\!\bullet\right](X_{\left[\bullet\!\frown\!\bullet\right]}) = \lim_{N\to\infty}\frac{(N-1)!}{N!}\frac{1}{2}\sum_{a,b=1}^{N} :D_{ab}^2:\left(\frac{1}{2}\sum_{a',b'=1}^{N} X_{a'b'}^2\right) = X_{\left[\bullet\!\frown\!\bullet\right]}.$$

The restricted product $\Gamma_1 * \Gamma_2$ of the graph-operators Γ_1 and Γ_2 of degree n is called the operator $\mathrm{pr}_n(\Gamma_1\Gamma_2)$, i.e. the projection of the regularized product of differential operators $V[\Gamma_1]V[\Gamma_2] = \lim_{N\to\infty} V[\Gamma_1]^N V[\Gamma_2]^N$ on the subspace of operators of degree n. This operation transforms V_n into an associative algebra.

Let us continue the correspondence between the graphs and endomorphisms described in the previous subsection to an isomorphism of the vector spaces $\theta_n : V_n \to \varrho(B_n) \subset \mathrm{End}(H_n)$.

Lemma 3.1. *The map θ_n is an isomorphism of algebras, $V[\Gamma](X_{[\Gamma']}) = X_{[\Gamma*\Gamma']}$ and $V[\Gamma] * V[\Gamma'] = V_{[\Gamma*\Gamma']}$.*

Proof. Consider the monomial $D =: D_{a_1b_1}\ldots D_{a_nb_n} :$, where $\Gamma(D) = \Gamma$ and the monomial $X[\Gamma'] = X_{c_1d_1}\ldots X_{c_nd_n}$, where $\Gamma(X) = \Gamma'$. Then, $D(X)$ is a linear combination of all monomials of the form $X_{s_1d_1}\ldots X_{s_nd_n}$ such that the set of numbers $\{s_1, \ldots, s_n\}$ coincides with the set of numbers $\{a_1, \ldots, a_n\}$ and $\Gamma(X_{s_1d_1}\ldots X_{s_nd_n})$ is one of the summands in the expansion of the product $\Gamma * \Gamma'$. Thus, $V[\Gamma](X_{[\Gamma']}) = X_{[\Gamma*\Gamma']}$. Hence, we have the homomorphicity of the map θ_n and $V[\Gamma] * V[\Gamma'] = V_{[\Gamma*\Gamma']}$. □

Denote as $l_n^V : l_n^V \to \mathbb{C}$ the linear functional such that $l_n^V(V[\Gamma]) = l_n^B(\Gamma)$. From Lemma 3.1 there follows:

Theorem 3.1. *The correspondence $\Gamma \mapsto V[\Gamma]$ gives rise to an isomorphism of Frobenius pairs $\mathcal{V}_n : (V_n, l_n^V) \to (B_n, l_n^B)$.*

3.2. The algebra of polygraph-operators

The direct product of algebras V_i forms the algebra $V_\uparrow = \prod_{i=1}^{\infty} V_i$; we call its elements *polygraph-operators*. Thus, a polygraph-operator is an infinite sequence $v = (v_1, v_2, \ldots)$, where $v_i \in V_i$ with the componentwise restricted product.

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Denote as X_n the linear space generated by the graph-variables of degree n. A *polygraph-variable* is an infinite sequence $x = (x_1, x_2, \ldots)$, where $x_i \in X_i$. Denote as $X_\uparrow = \prod_{i=1}^{\infty} X_i$ the vector space generated by the polygraph-variables. Let the action of the polygraph-operator on the polygraph-variable be given by the formula $v(x) = (v_1, v_2, \ldots)(x_1, x_2, \ldots) = (v_1(x_1), v_2(x_2), \ldots)$. Then $(v' * v'') = v'(v''(x))$.

The correspondence $V[\Gamma] \mapsto V[\rho_n(\Gamma)]$ gives rise to a homomorphism $\sigma_n : V_m \to V_n$ of vector spaces. The set of these homomorphisms gives rise to a homomorphism of vector spaces $\sigma_\uparrow : V_m \to V_\uparrow$, which, in turn, gives rise to a homomorphism of vector spaces $\sigma_\uparrow : V \to V_\uparrow$.

Theorem 3.2. *The map $\sigma_\uparrow : V \to V_\uparrow$ is an isomorphism of algebras.*

Proof. It follows immediately from the definitions that $\mathrm{pr}_n(\sigma_\uparrow(V[\Gamma])) = \sigma_n(V[\Gamma])$. Let $x_n \in X_n$ and let $x = \sigma_\uparrow(x_n)$ be the graph-variable with all zero components except the nth, with the nth component equal to x_n. Then $\sigma_\uparrow(V[\Gamma_1] \circ V[\Gamma_2])(x) = \sigma_\uparrow((V[\Gamma_1] \circ V[\Gamma_2])(x_n)) = \sigma_\uparrow(V[\Gamma_1](V[\Gamma_2](x_n))) = \sigma_\uparrow(V[\Gamma_1])(\sigma_\uparrow(V[\Gamma_2])(x)) = (\sigma_\uparrow(V[\Gamma_1])) * (\sigma_\uparrow(V[\Gamma_2]))(x)$. Thus, $\sigma_\uparrow(V[\Gamma_1] \circ V[\Gamma_2]) = \sigma_\uparrow(V[\Gamma_1]) * \sigma_\uparrow(V[\Gamma_2])$. Monomorphicity of the homomorphism $\sigma_\uparrow$ is immediate. Epimorphicity follows from Theorem 3.1. □

Example 3.4.

$$\sigma_\uparrow(V\left[\bullet\!\!-\!\!\bullet\right] \circ V\left[\bullet\!\!-\!\!\bullet\right]) = \sigma_\uparrow(V\left[\bullet\!\!-\!\!\bullet\right] + 2V\left[\begin{smallmatrix}\bullet\!\!-\!\!\bullet\\ \bullet\!\!-\!\!\bullet\end{smallmatrix}\right]) = (V\left[\bullet\!\!-\!\!\bullet\right], 4V\left[\begin{smallmatrix}\bullet\!\!-\!\!\bullet\\ \bullet\!\!-\!\!\bullet\end{smallmatrix}\right], \ldots)$$

$$\sigma_\uparrow(V\left[\bullet\!\!-\!\!\bullet\right]) * \sigma_\uparrow(V\left[\bullet\!\!-\!\!\bullet\right]) = (V\left[\bullet\!\!-\!\!\bullet\right], 2V\left[\begin{smallmatrix}\bullet\!\!-\!\!\bullet\\ \bullet\!\!-\!\!\bullet\end{smallmatrix}\right], \ldots) * (V\left[\bullet\!\!-\!\!\bullet\right], 2V\left[\begin{smallmatrix}\bullet\!\!-\!\!\bullet\\ \bullet\!\!-\!\!\bullet\end{smallmatrix}\right], \ldots) = (V\left[\bullet\!\!-\!\!\bullet\right], 4V\left[\begin{smallmatrix}\bullet\!\!-\!\!\bullet\\ \bullet\!\!-\!\!\bullet\end{smallmatrix}\right], \ldots).$$

3.3. The algebra of cut-and-join operators

We start with a few examples of cut-and-join operators and their products.

Example 3.5. $D_e = W([1]) = \sum_{a\in\mathbb{N}} : D_{aa} : W([11]) = \frac{1}{2}\sum_{a,b\in\mathbb{N}} : D_{aa}D_{bb} : W([2]) = \frac{1}{2}\sum_{a,b\in\mathbb{N}} : D_{ab}D_{ba} : W([21]) = \frac{1}{2}\sum_{a,b,c\in\mathbb{N}} : D_{ab}D_{ba}D_{cc} : W([3]) = \frac{1}{2}\sum_{a,b,c\in\mathbb{N}} : D_{ab}D_{bc}D_{ca} :$.

Example 3.6. $W[1] \circ W[1] = \sum_{a,b=1}^{N} D_{aa}D_{bb} = \sum_{a,b}^{N} : D_{aa}D_{bb} : + \sum_a D_{aa} = 2W[11] + W[1]$

$W[1] \circ W[2] = W[2] \circ W[1] = 2W[2] + W[21]$,

$W[1] \circ W[11] = W[11] \circ W[1] = 2W[11] + 3W[111]$,

$W[2] \circ W[2] = W[11] + 3W[3] + 2W[22]$,

$W[2] \circ W[11] = W[11] \circ W[2] = W[2] + 2W[21] + W[211]$,

$W[11] \circ W[11] = W[11] + 6W[111] + 6W[1111]$.

Denote as W_n the vector space generated by the differential operators of the form $W(\Delta)$, where $|\Delta| = n$. Introduce on W_n the structure of an associative algebra, defining *restricted multiplication* $*$ with the equality $w^1 * w^2 = \mathrm{pr}_n(w^1 \circ w^2)$for $w^i \in W_n$. Denote as $W_\uparrow = \prod_{n=1}^{\infty} W_n$ the direct product of algebras W_n with the restricted multiplication $*$.

Define a linear operator $\rho_n : W_m \to W_n$, assuming that $\rho_n(W) = 0$ for $n < m$ and $\rho_n(W[\Delta]) = \frac{k!}{m!(k-m)!} : D_e^{(n-m)}W[\Delta] :$, where k is the number of unit lines in the Young diagram Δ, for $n \geq m$. The correspondence $W \mapsto \rho_n(W)$ gives rise to a homomorphism of vector spaces $\rho_n : W_m \to W_n$. The set of homomorphisms $\rho_n : W_m \to W_n$ gives rise to a homomorphism of vector spaces $\rho_\uparrow : W_m \to W_\uparrow$. Continue the operator $\rho_\uparrow : W_m \to W_\uparrow$ up to the linear operator $\rho_\uparrow : W \to W_\uparrow$.

Example 3.7.

$\rho_\uparrow(W[1]) = (W[1], 2W[11], 3W[111], \ldots)$,

$\rho_\uparrow(W[2]) = (W[2], W[21], W[211], \ldots)$,

$\rho_\uparrow(W[11]) = (W[11], 3W[111], 4W[1111], \ldots)$,

$\rho_\uparrow(W[3]) = (W[3], W[31], W[311], \ldots)$,

$\rho_\uparrow(W[21]) = (W[21], 2W[211], 3W[2111], \ldots)$,

$\rho_\uparrow(W[111]) = (W[111], 4W[1111], 5W[1111], \ldots)$.

Theorem 3.3. *The map $\rho_\uparrow : W \to W_\uparrow$ is an isomorphism of algebras.*

Proof. It follows immediately from the definitions that $\mathrm{pr}_n(\rho_\uparrow(W(\Delta))) = \rho_n(W(\Delta))$ and $\mathrm{pr}_n(\rho_\uparrow(W(\Delta_1) \circ W(\Delta_2))) = \mathrm{pr}_n(\rho_\uparrow(\rho_n(W(\Delta_1)) \circ \rho_n(W(\Delta_2)))) = \mathrm{pr}_n(\rho_\uparrow(\rho_n(W(\Delta_1)) * \rho_n(\Delta_2))) = \rho_n(W(\Delta_1)) * \rho_n(W(\Delta_2)) = \mathrm{pr}_n(\rho_\uparrow(W(\Delta_1)) * \rho_\uparrow(W(\Delta_2)))$. Thus, $\rho_\uparrow(W(\Delta_1) \circ W(\Delta_2)) = \rho_\uparrow(W(\Delta_1)) * \rho_\uparrow(W(\Delta_2))$. Monomorphicity of the homomorphism $\rho_\uparrow$ is immediate. Epimorphicity follows from the equality $W_\uparrow = \sum_{n=1}^{\infty} \rho_\uparrow(W_n)$. □

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Example 3.8.

$$\begin{aligned}\rho_\uparrow(W([1]\circ W([1]))) &= \rho_\uparrow(W([1]+2W([11]))) = \rho_\uparrow(W([1])) + 2\rho_\uparrow(W([11]))\\ &= (W[1], 2W[11], 3W[111], \ldots) + 2(W[11], 3W[111], 4W[1111], \ldots)\\ &= (W[1], 4W[11], 9W[111], \ldots).\end{aligned}$$

On the other hand, $\rho_\uparrow(W([1])) * \rho_\uparrow(W([1])) = \rho_\uparrow(W([1])) * \rho_\uparrow(W([1])) = (W[1], 2W[11], 3W[111], \ldots) * (W[1], 2W[11], 3W[111], \ldots) = (W[1], 4W[11], 9W[111], \ldots)$.

Example 3.9.

$$\begin{aligned}\rho_\uparrow(W([2]\circ W([2]))) &= \rho_\uparrow(W[11] + 3W[3] + 2W[22])\\ &= \rho_\uparrow(W[11]) + 3\rho_\uparrow(W[3]) + 2\rho_\uparrow(W[22])\\ &= (W[11], 3W[111], 6W[1111], \ldots) + 3(W[3], W[31], W[311], \ldots)\\ &\quad + 2(W[22], W[221], W[2211], \ldots)\\ &= (W[11], 3(W[3] + W[111]), 3W[31] + 2W[22] + 6W[1111], \ldots).\end{aligned}$$

On the other hand, $\rho_\uparrow(W([2])) * \rho_\uparrow(W([2])) = \rho^W_\uparrow(W[2]) * \rho^W_\uparrow(W[2]) = (W[2], W[21], W[211], \ldots) * (W[2], W[21], W[211], \ldots) = (W[2] * W[2], W[21] * W[21], W[211] * W[211], \ldots) = (W[11], 3(W[3] + W[111]), 3W[31] + 2W[22] + 6W[1111], \ldots)$.

Denote as $l^W_n : l^W_n \to \mathbb{C}$ the linear functional such that $l^W_n(W[\Delta]) = l^A_n(\Delta)$. Then, as proved in [11, Lemma 4.4]:

Theorem 3.4. *The correspondence $\Delta \mapsto W[\Delta]$ gives rise to an isomorphism of the Frobenius pairs $\mathcal{W}_n : (A_n, l^A_n) \to (W_n, l^W_n)$.*

3.4. Cardy–Frobenius structures

Define a linear operator $f_n : W_n \to V_n$ by the equality $f_n(w) = \mathrm{pr}_n(f(w))$. The set of these gives rise to a linear operator $f_\uparrow : W_\uparrow \to V_\uparrow$.

There follow a few examples of the action of the cut-and-join operators on the graph-variables.

Example 3.10.

$$W[1](X_{\left[\bullet\!\!-\!\!\bullet\right]}) = \sum_a D_{aa}\left(\sum_{a,b} X_{ab}\right) = X_{\left[\bullet\!\!-\!\!\bullet\right]}$$

$$W[1](X_{\left[\begin{smallmatrix}\bullet\!\!-\!\!\bullet\\ \bullet\!\!-\!\!\bullet\end{smallmatrix}\right]}) = \frac{1}{2}W[1,1](X_{\left[\begin{smallmatrix}\bullet\!\!-\!\!\bullet\\ \bullet\!\!-\!\!\bullet\end{smallmatrix}\right]}) = \frac{1}{4} : \sum_a D_{aa}\sum_b D_{bb} : \left(\frac{1}{2}\sum_{\substack{a_1\neq a_2\\ b_1\neq b_2}} X_{a_1b_1}X_{a_2b_2}\right) = \frac{1}{2}X_{\left[\begin{smallmatrix}\bullet\!\!-\!\!\bullet\\ \bullet\!\!-\!\!\bullet\end{smallmatrix}\right]}$$

$$W[1](X_{\left[\bullet\!\!\frown\!\!\bullet\right]}) = \frac{1}{2}W[1,1](X_{\left[\bullet\!\!\frown\!\!\bullet\right]}) = \frac{1}{4} : \sum_a D_{aa}\sum_b D_{bb} : \left(\frac{1}{2}\sum_{a,b} X^2_{ab}\right) = \frac{1}{2}X_{\left[\bullet\!\!\frown\!\!\bullet\right]}$$

$$W[2](X_{\left[\begin{smallmatrix}\bullet\!\!-\!\!\bullet\\ \bullet\!\!-\!\!\bullet\end{smallmatrix}\right]}) = \frac{1}{2} : \sum_{a,b} D_{ab}D_{ba} : \left(\frac{1}{2}\sum_{\substack{a_1\neq a_2\\ b_1\neq b_2}} X_{a_1b_1}X_{a_2b_2}\right) = X_{\left[\begin{smallmatrix}\bullet\!\!-\!\!\bullet\\ \bullet\!\!-\!\!\bullet\end{smallmatrix}\right]}$$

$$W[2](X_{\left[\bullet\!\!\frown\!\!\bullet\right]}) = \frac{1}{2} : \sum_{a,b} D_{ab}D_{ba} : \left(\frac{1}{2}\sum_{a,b} X^2_{ab}\right) = X_{\left[\bullet\!\!\frown\!\!\bullet\right]}.$$

Lemma 3.2. $\rho_\uparrow f = f_\uparrow \rho_\uparrow$.

Proof. Consider the operator $w = W[\Delta]$ and the graph-variable x of degree n. Then, $(f_\uparrow(\rho_\uparrow(w)))(x) = (f_n(w)(x))_\uparrow = (\rho_\uparrow f(w))(x)$, where $(f_n(w)(x))_\uparrow$ is the polygraph-variable with the single non-zero component $f_n(w)(x) \in X_n$. □

Lemma 3.3. *If $n = |\Delta|$, then $f_n(W[\Delta]) = V[\phi_n(\Delta)]$.*

Proof. The operator $W[\Delta]$ can be represented in the form $W[\Delta] = \kappa(\Delta)\sum_{a_1,\ldots,a_n\in\mathbb{N}} : D_{a_1a_{\sigma(1)}}\cdots D_{a_na_{\sigma(n)}} :$, where $\sigma \in S_n$ is a permutation of the cyclic type Δ. In accordance with the definition, the operator $V[\phi_n(\Delta)]$ has the same form. □

Theorem 3.5. *The set $((W, l^W), (V, l^V), f, R, \star)$ forms an equipped Cardy–Frobenius algebra.*

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Proof. Due to Theorems 3.2 and 3.3 and Lemma 3.2, it suffices to prove that the isomorphisms $\mathcal{W}_n$ and $\mathcal{V}_n$ give rise to isomorphisms of Cardy–Frobenius algebras $((A_n, l_n^A), (B_n, l_n^B), \phi_n, U_n, \star)$ and $((W_n, l_n^W), (V_n, l_n^V), f_n, R_n, \star)$. Homomorphicity of the Frobenius pairs is proved in Theorems 3.1 and 3.4. The relation $f_n\mathcal{W}_n = \mathcal{V}_n\phi_n$ follows from Lemma 3.3. The remaining requirements follow immediately from the definitions. □

Acknowledgments

S.N. is grateful to MPIM and IHES for kind hospitality and support.

Our work was partly supported by the Ministry of Education and Science of the Russian Federation under contract 8498, by the Russian Federation Government Grant No. 2010-220-01-077, by research grant No. 12-01-0122 ("The National Research University Higher School of Economics", Academic Fund Program in 2013–2014), by NSh-3349.2012.2 (A. Mir. and A. Mor.) and 4850.2012.1 (S.N.), by RFBR grants 10-02-00509 (A. Mir. and S.N.), 10-02-00499 (A. Mor.) and by the joint grants 11-02-90453-Ukr, 12-02-92108-Yaf-a, 11-01-92612-Royal Society.

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