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Efficient Algorithms for the Recognition of Topologically Conjugate Gradient-like Diffeomorhisms

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Abstract—It is well known that the topological classification of structurally stable flows on surfaces as well as the topological classification of some multidimensional gradient-like systems can be reduced to a combinatorial problem of distinguishing graphs up to isomorphism. The isomorphism problem of general graphs obviously can be solved by a standard enumeration algorithm. However, an efficient algorithm (i. e., polynomial in the number of vertices) has not yet been developed for it, and the problem has not been proved to be intractable (i. e., NP-complete). We give polynomial-time algorithms for recognition of the corresponding graphs for two gradient-like systems. Moreover, we present efficient algorithms for determining the orientability and the genus of the ambient surface. This result, in particular, sheds light on the classification of configurations that arise from simple, point-source potential-field models in efforts to determine the nature of the quiet-Sun magnetic field.

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INTRODUCTION

In 1937 Andronov and Pontryagin [3] introduced the concept of *roughness* of a system of differential equations defined in a bounded part of the plane: the system is rough if it retains its qualitative properties under small perturbations of the right-hand side. They proved that such a system is rough if and only if it is characterized by the following properties:

1) the set of equilibrium points and of limit cycles is finite, all its elements are hyperbolic;

2) there are no separatrices going from a saddle equilibrium point to a saddle equilibrium point.

The above properties are also known as the criterion of roughness of flows on the two-dimensional sphere. The principal difficulty in generalization of this result for the case of arbitrary orientable surfaces of positive genus is the possibility of new types of motion — nonclosed recurrent trajectories. It follows from Maier's 1939 paper [19] that there are no such trajectories for rough flows without equilibrium states on the two-dimensional torus. In 1959 Peixoto [24] introduced the concept of *structural stability* of flows to generalize the concept of roughness. We recall that a flow

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f^t is *structurally stable* if for any sufficiently close flow g^t there exists a homeomorphism h sending trajectories of the system g^t to trajectories of the system f^t. The original definition of a rough flow involved the additional requirement for the homeomorphism h to be C^0-close to the identity map. Peixoto [24] proved that the concept of the roughness and that of the structural stability are equivalent for flows on the 2-sphere. In [25] and [26] Peixoto proved that the above conditions 1) and 2) together with the condition

3) all ω- and α-limit sets are contained in the union of equilibrium points and limit cycles

are necessary and sufficient for the structural stability of a flow on an arbitrary orientable closed (compact and without boundary) surface, and he also showed that such flows are dense in the space of all C^1-flows.

An immediate generalization of the properties of rough flows on orientable surfaces leads to *Morse – Smale* systems (continuous and discrete). The non-wandering set of such a system consists of finitely many fixed points and periodic orbits, each of which is hyperbolic, while for any two distinct non-wandering points p, q their stable and unstable manifolds W^s_p and W^u_q intersect transversally. Morse – Smale systems are named after Smale's 1960 paper [28], where he introduced flows with the above properties (on manifolds of dimension greater than 2) and proved that they satisfy inequalities similar to Morse inequalities[1)]. Later Smale and Palis showed that Morse – Smale systems are structurally stable ([22, 23]). However, already in 1961, having constructed a structurally stable diffeomorphism on the two-dimensional sphere $\mathbb{S}^2$ with infinitely many periodic points, Smale [29] proved that such systems do not exhaust the class of all rough systems. This diffeomorphism is known now as the Smale horseshoe. Nevertheless, Morse – Smale systems have great value both in applications (because they adequately describe any regular stable processes) and in studying the topology of the phase space (because of the deep interrelation between the dynamics of these systems and the space where they are given — *ambient manifold*).

The key problem in studying dynamical systems is the determination of the set of complete topological invariants, that is, the properties of a system which uniquely determine the decomposition of the phase space into trajectories up to topological equivalence (conjugacy). Recall that two flows f^t and f'^t (or two diffeomorphisms f and f') on an n-manifold M^n are said to be *topologically equivalent* (*topologically conjugate*) if there exists a homeomorphism $h : M^n \to M^n$ that carries trajectories of f^t to trajectories of f'^t (such that $f'h = hf$). The topological classification of dynamical systems occupies a special place in the qualitative theory of differential equations, in addition to the direct use of the topological invariants obtained, an important information is the discovery of new dynamic effects. This problem has a rich history.

The equivalence class of Morse – Smale flows on the circle is uniquely determined by the number of its fixed points. For cascades on the circle Maier [19] found in 1939 a complete topological invariant consisting of a triple of numbers: the number of periodic orbits, their periods, and the so-called ordinal number. In 1955 Leontovich and Maier [18] introduced a complete topological invariant, called a scheme of a flow, for flows with finitely many singular trajectories on the two-dimensional sphere. This scheme contains a description of the singular trajectories (the equilibrium states, the periodic orbits, the separatrices of the saddle equilibrium states) and their relative positions. In 1971 Peixoto [27] generalized the Leontovich – Maier scheme for Morse – Smale flows on an arbitrary surface as the directed graph whose vertices are in a one-to-one correspondence with the fixed points and the closed trajectories of the flow, and whose edges correspond to the connected components of the invariant manifolds of the fixed points and the closed trajectories. Peixoto proved that the isomorphism class of such a directed graph is the complete topological invariant for the class of Morse – Smale systems on surfaces (where the isomorphisms preserve specially chosen subgraphs)[2)].

[1)] For a flow on a closed n-manifold M^n with properties 1)–3) let a_q be the number of unstable manifolds of dimension q associated to singular points and b_q the number of unstable manifolds of dimension q associated to closed orbits and let $N_q = a_q + b_q + b_{q+1}$. Smale proved that $N_0 \geqslant \beta_0, N_1 - N_0 \geqslant \beta_1 - \beta_0, N_2 - N_1 + N_0 \geqslant \beta_2 - \beta_1 + \beta_0, \ldots, \sum\limits_{k=0}^{n}(-1)^k N_k = (-1)^n \chi$, where β_k is the k-th Betti number and χ is the Euler characteristic of M^n. This result contains the classical theorem of Morse for a function f with nondegenerate critical points.

Morse – Smale flows (cascades) on manifolds of dimension $n \geqslant 3$ ($n \geqslant 2$), in contrast to lower-dimensional systems, feature a new type of motion due to the possibility of (*heteroclinic*) intersections of the invariant manifolds of distinct saddle points. Afraimovich and Shil'nikov [1] proved that the restriction of Morse – Smale flows to the closure of the set of heteroclinic trajectories is conjugate to a suspension over a topological Markov chain. Nevertheless, an invariant similar to Peixoto's graph is enough to describe the complete topological invariant for a wide subset of such systems [5, 6].

In Section 2.1 of this paper we show that a directed graph and a three-color graph are the complete topological invariants for gradient-like diffeomorphisms on different manifolds. In Section 3 we show how gradient-like systems on the 2-sphere appear naturally in the quiet-Sun magnetic field. In Section 2 we discuss the problem of distinguishing between graphs. An algorithm for solving such a problem is said to be *efficient* if it takes a polynomial time in the number of vertices of the graph[3)]. The complexity status of the isomorphism problem of general graphs is not proved to be polynomial or not and the problem of recognition of directed graphs is still open as well. We achieve the efficiency of the algorithms due to the following facts: three-color graphs are not graphs of general type, they possess specific combinatorial properties, their main property being trivalency; the directed graphs of the considered diffeomorphisms of the n-sphere can be embedded into the 2-sphere; the directed graphs of gradient-like diffeomorphisms are embeddable into a surface. The main results of the paper are Theorems 8, 9, 10, 11, and 12, in which we present efficient algorithms for recognition of the two directed graphs for diffeomorphisms on the n-sphere and orientable surfaces as well as for three-color graphs for diffeomorphisms on an arbitrary 2-manifold and determine the orientability of the ambient surface and its genus.

1. GRAPHS IN GRADIENT-LIKE DYNAMICS

1.1. Directed Graphs for Gradient-like Diffeomorphisms on Surfaces

Let M^2 be a two-dimensional closed orientable manifold of any genus and let $G_+(M^2) = G_+$ be the class of all orientation-preserving gradient-like diffeomorphisms of M^2. This means that a diffeomorphism $f \in G_+$ has a finite hyperbolic nonwandering set and the invariant manifolds of different saddle points are disjoint. In this case a complete topological invariant for f is a directed graph. It is similar to Peixoto's graph for the structurally stable flows on surfaces, which is a generalization of the Leontovich – Maier scheme. This invariant reduces the problem of topological classification of diffeomorphisms to the combinatorial problem.

Recall that a *directed graph* is a set of vertices and directed edges (or ordered pairs of vertices). The two vertices of an edge are said to be *incident* to it.

Definition 1. *Two directed graphs are said to be* isomorphic *if there is a one-to-one correspondence between the sets of their vertices and edges that preserves the direction of edges.*

Let $f \in G_+$ and let σ be a saddle point of f of period m_σ. Denote by ρ_σ the *type of orientation of* σ, which is 1 if the diffeomorphism $f^{m_\sigma}|_{W^u_\sigma}$ preserves orientation and -1 in the opposite case. Let l^s_σ (l^u_σ) be a stable (unstable) separatrix of σ, i.e., l^s_σ (l^u_σ) is a connected component of $W^s_\sigma \setminus \sigma$ ($W^u_\sigma \setminus \sigma$). Since l^s_σ (l^u_σ) does not intersect the unstable (stable) manifold of any saddle point, there exists a sink ω (respectively, a source α) such that $cl(l^u_\sigma) = l^u_\sigma \cup \sigma \cup \omega$ (respectively, $cl(l^s_\sigma) = l^s_\sigma \cup \sigma \cup \alpha$); see, for example, [13], Lemma 3.2.1. For $\delta \in \{s, u\}$ we define the direction of the separatrix l^δ_σ to be towards the saddle point if $\delta = s$ and from the saddle point if $\delta = u$.

Definition 2. *We say the directed graph Γ_f to be the graph of the diffeomorphism $f \in G_+$ if*

1) *the vertices of the graph Γ_f correspond to the periodic points of the nonwandering set Ω_f; we equip the vertex corresponding to the saddle periodic point σ with the value ρ_σ;*

2) *the directed edges of the graph Γ_f correspond to the directed separatrices of the saddle points.*

[2)] In [21] Oshemkov and Sharko pointed out a certain inaccuracy concerning the Peixoto invariant due to the fact that the isomorphism of the graphs does not distinguish types of the decompositions into trajectories for the domain bounded by two periodic orbits.

[3)] The notion of an effectively solvable problem was suggested by A. Cobham and it means that a computational problem can be feasibly computed on some device only if it can be computed in time bounded by a polynomial in a parameter representing the length of input data [9].

Let π denote a one-to-one correspondence between the vertices and the edges of the graph Γ_f and the set of periodic points and separatrices of the diffeomorphism f. The diffeomorphism f induces an automorphism of the graph Γ_f, which we denote by P_f. Let $\Gamma_f,\ \Gamma_{f'}$ be the graphs of diffeomorphisms $f \in G_+,\ f' \in G'_+$.

The existence of an isomorphism of graphs Γ_f and $\Gamma_{f'}$ conjugating the automorphisms P_f and $P_{f'}$ is necessary for the topological conjugacy of the diffeomorphisms f and f'. Nevertheless, the existence of an isomorphism of the graphs is generally not sufficient for the conjugacy even if all the periodic points are fixed and each separatrix is invariant. For example, the diffeomorphisms f and f' with phase portraits shown in Fig. 1 have isomorphic graphs, but they are not topologically conjugate. To see it, notice that any conjugating homeomorphism necessarily carries the basin of the sink ω of the diffeomorphism f into the basin of the sink ω' of the diffeomorphism f', but such a homeomorphism cannot be extended to the entire sphere in such a way as to carry the invariant manifolds of the saddle points of f into the invariant manifolds of the saddle points of f'. Thus, the graph must contain more information.

Let ω be a sink of a diffeomorphism $f \in G_+$ and let L_ω be a subset of the manifold M^2 that consists of the separatrices which have ω in their closures. Then there is a smooth 2-disk B_ω such that $\omega \in B_\omega$ and each separatrix $l \subset L_\omega$ intersects ∂B_ω at a unique point; see, for example, [13], Proposition 2.1.3. For the vertex w corresponding to the periodic sink point ω let E_w denote the set of edges of the graph Γ_f incident to w. Let N_w denote the cardinality of the set E_w. We enumerate the edges of the set E_w in the following way. First we pick in the basin of the sink ω a 2-disk B_ω and set $c_\omega = \partial B_\omega$. We define a pair of vectors $(\vec{\tau}, \vec{n})$ at some point of the curve c_ω in such a way that the vector $\vec{n}$ is directed inside the disk B_ω, the vector $\vec{\tau}$ is tangent to the curve c_ω and induces a counterclockwise orientation on c_ω with respect to B_ω (we call this orientation positive). Enumerate the edges from E_w: $e_1, \dots, e_{N_w}$ in accordance with the order of the corresponding separatrices as we move along c_ω starting from some point on c_ω. This numeration of the edges of the set E_w is said to be *compatible* with the embedding of the separatrices.

Definition 3. *The graph Γ_f is said to be* equipped *if each vertex w is numbered with respect to the numeration of the edges of the set E_w and the numeration is compatible with the embedding of the separatrices. We denote such a graph by Γ^*_f.*

Let Γ^*_f and $\Gamma^*_{f'}$ be equipped graphs of diffeomorphisms f and f', respectively, and let Γ^*_f and $\Gamma^*_{f'}$ be isomorphic by an isomorphism ξ. Let a vertex w of the graph Γ^*_f correspond to a sink, then the isomorphism ξ induces the permutation $\Theta_{w,w'},\ w' = \xi(w)$ on the set $1, \dots, N$ (where $N = N_w = N_{w'}$) by $\Theta_{w,w'}(i) = j \Leftrightarrow \xi(e_i) = e'_j$.

Definition 4. *Two equipped graphs Γ^*_f, $\Gamma^*_{f'}$ of diffeomorphisms f, f' are said to be* isomorphic *if there is an isomorphism ξ of the graphs Γ_f, $\Gamma_{f'}$ such that*

1) *ξ sends the vertices into the vertices and preserves the values of the vertices corresponding to the saddle periodic points; it sends the edges into the edges and preserves their direction;*

2) *the permutation $\Theta_{w,w'}$ induced by ξ is a power of a cyclic permutation*[4] *for each vertex w corresponding to a sink;*

3) *$P_{f'} = \xi P_f \xi^{-1}$.*

The equipped graph Γ^*_f of a diffeomorphism $f \in G_+$ is the topological invariant up to isomorphism.

Theorem 1 ([13], Theorem 3.2.1). *Diffeomorphisms $f \in G_+, f' \in G'_+$ are topologically conjugate if and only if their equipped graphs are isomorphic.*

[4] It can be directly checked that the property of the permutation is a power of a cyclic permutation is independent of the choice of the curves c_ω and $c_{\omega'}$.

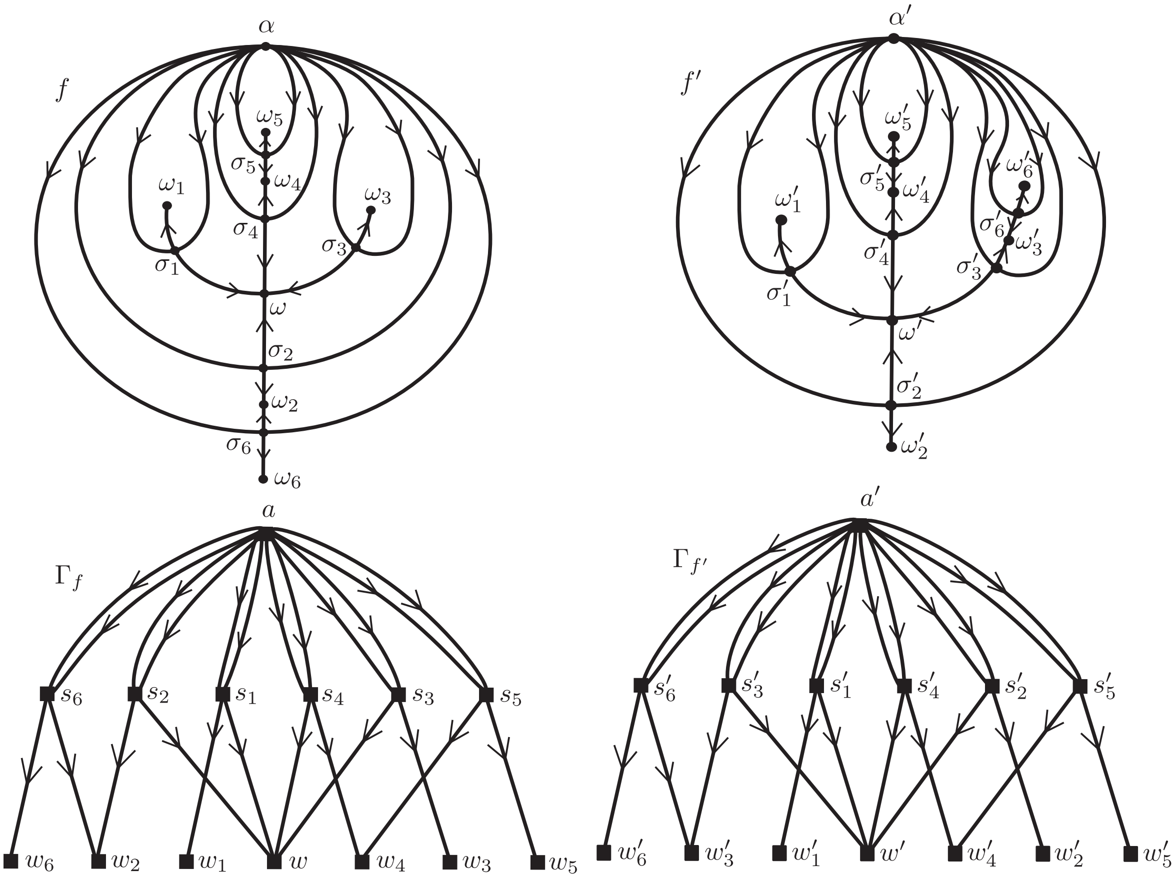

Fig. 1. The diffeomorphisms $f, f' : \mathbb{S}^2 \to \mathbb{S}^2$ have isomorphic graphs, but they are not topologically conjugate as their equipped graphs are not isomorphic (see Theorem 1 and explanations below).

Consider Fig. 1 once again and suppose that the vertex w (w') of the graph corresponds to the sink point ω (ω') and the vertex a (a') of the graph corresponds to the source point α (α'). We renumber the separatrices $l_1^u, l_2^u, l_3^u, l_4^u$ $(l_1'^u, l_2'^u, l_3'^u, l_4'^u)$ of the saddle points belonging to the stable manifold of the point ω (ω') in accordance with the positive orientation on a closed curve around ω (ω') and denote by $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ $(\sigma_1', \sigma_2', \sigma_3', \sigma_4')$ the saddle points that satisfy the condition that σ_i (σ_i') belongs to the closure of l_i^u $(l_i'^u)$. Introduce a numeration on the set E_ω (E'_ω) compatible with the embedding of the separatrices. As we have already noted, the graphs Γ_f and $\Gamma_{f'}$ are isomorphic. There are exactly two isomorphisms of these graphs: the isomorphism ξ_1 of the natural identification of the graph Γ_f with the graph $\Gamma_{f'}$ and the isomorphism ξ_2 that is the composition of the natural identification and the reflection with respect to the axis $a'w'$. One can check directly that the isomorphism ξ_1 induces the permutation $\Theta_{w,w'} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix}$ and the isomorphism ξ_2 induces the permutation $\Theta_{w,w'} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$. Neither permutation is a power of a cyclic permutation and the equipped graphs Γ_f^*, $\Gamma_{f'}^*$ are not isomorphic according to Definition 4.

1.2. Three-color Graphs for Gradient-like Diffeomorphisms of Surfaces

We consider the class $G(M^2) = G$ of gradient-like diffeomorphisms defined on a surface M^2 (not necessarily orientable).

Let $f \in G$. Then $\Omega_f = \Omega^0_f \cup \Omega^1_f \cup \Omega^2_f$, where Ω^0_f, Ω^1_f, Ω^2_f are the sets of the sink points, the saddle points, and the source points of the diffeomorphism f, respectively. Throughout the rest of the paper, we assume that f has at least one saddle point[5)].

We remove from the surface M^2 the closure of the union of stable manifolds and unstable manifolds of all saddle points of the diffeomorphism f and let $\tilde{M}$ denote the resulting set, that is, $\tilde{M} = M^2 \setminus (\Omega^0_f \cup W^u_{\Omega^1_f} \cup W^s_{\Omega^1_f} \cup \Omega^2_f)$. The set $\tilde{M}$ is represented in the form of a union of domains (*cells*) homeomorphic to the open two-dimensional disc such that the boundary of each of these cells has one of the forms depicted by boldface lines in Fig. 2, and it contains exactly one source, one sink, one or two saddle points, and some of their separatrices.

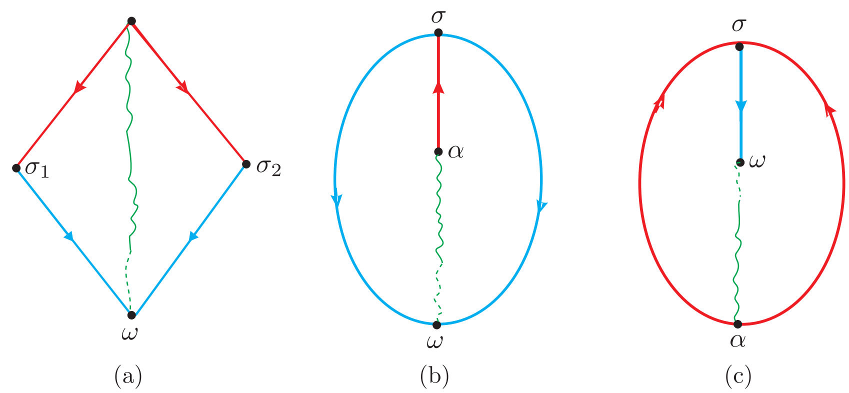

Fig. 2. Types of cells.

Let A be any cell from the set $\tilde{M}$, and let α and ω be the source and sink contained in its boundary. A simple curve $\tau \subset A$ whose boundary points are the source α and the sink ω is called a *t-curve* (see Fig. 2). Let $\mathcal{T}$ denote a set which is invariant under the diffeomorphism f and which consists of t-curves taken one from each cell.

Any connected component of the set $M_\Delta = \tilde{M} \setminus \mathcal{T}$ is called *a triangular area*. Let Δ_f denote the set of all triangular domains of diffeomorphism f. The boundary of every triangular domain $\delta \in \Delta_f$ contains three periodic points: a source α, a saddle σ, and a sink ω. It contains also the stable separatrix l^s_σ (called the *s-curve*) with α and σ as boundary points, the unstable separatrix l^u_σ (called the *u-curve*) with ω and σ as boundary points and a curve τ (a t-curve) with α and ω as boundary points (see Fig. 3). *A triangular domain* is bounded by s-, u- and t-curves. We say that two triangular areas *have a common side* if this side belongs to the closures of both domains. *The period of the triangular domain* δ is defined to be the least positive integer $k \in \mathbb{N}$, such that $f^k(\delta) = \delta$.

In order to introduce a combinatorial topological invariant of the diffeomorphism $f \in G$, we recall the following definitions.

A *finite graph* is an ordered pair (B, E), such that the following conditions hold: B is a nonempty finite set of *vertices*; E is a set of pairs of vertices called *edges*.

If a graph contains an edge $e = (a, b)$, then each of the vertices a and b is said to be *incident to the edge* e and the vertices a and b are said to be connected by the edge e.

A path in a graph is a finite sequence of its vertices and edges of the form: $b_0, (b_0, b_1), b_1, \cdots, b_{i-1}, (b_{i-1}, b_i), b_i, \cdots b_{k-1}, (b_{k-1}, b_k), b_k$, $k \geqslant 1$. The number k is called *the length of the path*, it is equal to the number of edges involved in the path.

A graph is said to be *connected* if any two of its vertices can be connected by a path.

A cycle of length k, $k \in \mathbb{N}$ in a graph is a finite subset of vertices and edges of the form $\{b_0, (b_0, b_1), b_1, \cdots, b_{i-1}, (b_{i-1}, b_i), b_i, \cdots b_{k-1}, (b_{k-1}, b_0)\}$. *A simple cycle* is a cycle all of whose vertices and edges are pairwise distinct.

[5)] If a Morse–Smale diffeomorphism $f : M^n \to M^n$ has no saddle points, then its nonwandering set consists of one source and one sink. All source-sink diffeomorphisms are topologically conjugate; the proof of this fact is given, for example, in [13] (Theorem 2.2.1).

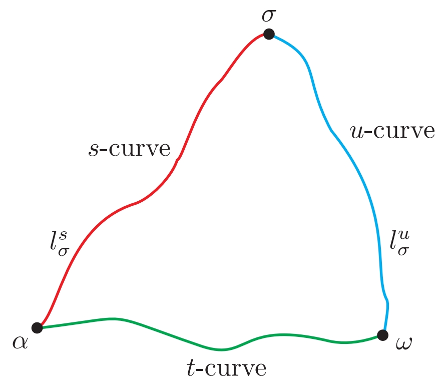

Fig. 3. Triangular domain.

Definition 5. *A graph T is called a three-color graph if:*

1) *the set of edges of T is a union of three subsets each of which consists of edges of the same color (the colors of the edges in the different subsets are different; we denote these colors by the letters s, t, u and, for brevity, refer to these edges as s- , t-, u-edges);*

2) *each vertex of T is incident to exactly three edges of different colors;*

3) *the graph contains no cycles of length one.*

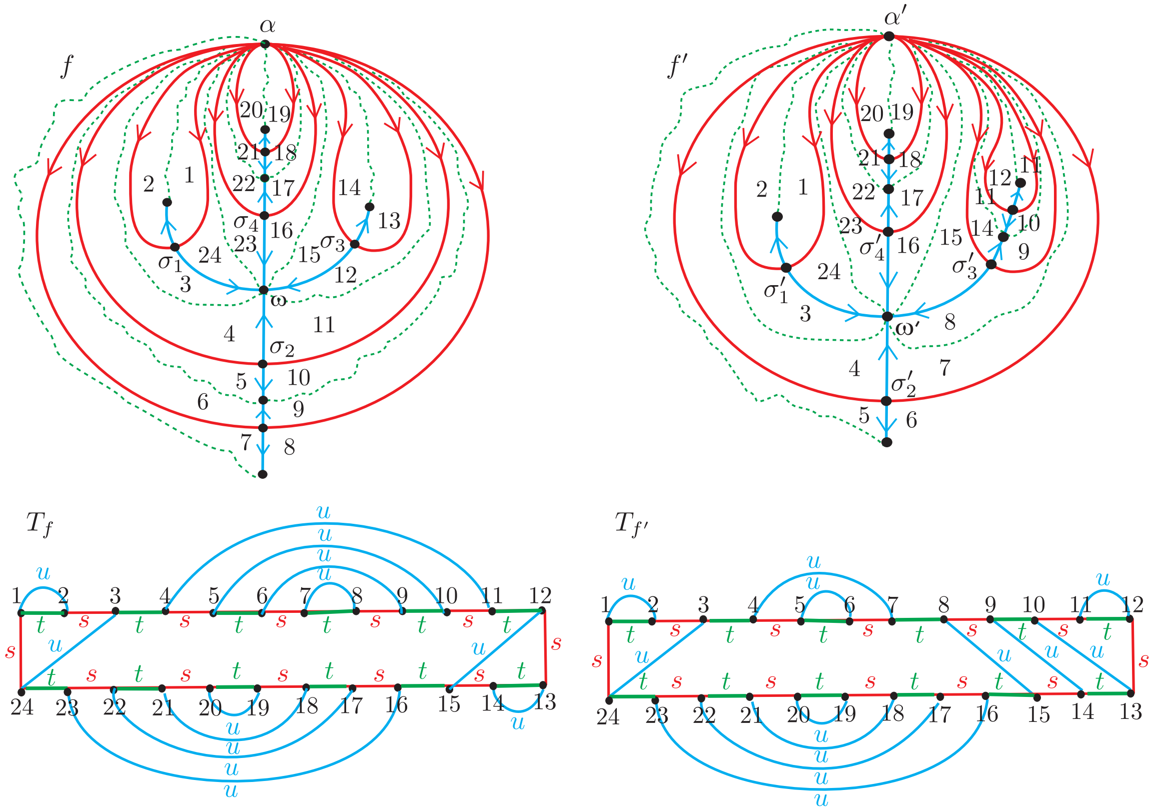

Fig. 4. Nonisomorphic three-color graphs $T_f, T_{f'}$ for nonconjugated diffeomorphisms $f, f' \in G$.

A simple cycle of a three-color graph T is called a two-color cycle of type su, tu or st if it contains edges of exactly two colors s and u, t and u or s and t. It follows immediately from the definition of a three-color graph that the length of any two-color cycle is an even number (since the colors of the edges strictly alternate), and the relation on the set of vertices belonging to a two-color cycle of a certain type is the equivalence relation.

A one-to-one map P of the graph T onto itself taking vertices to vertices, preserving the relations of incidence and the color (that is, a vertex incident to s-, t- and u-edges goes to a vertex incident to an edge of the same color) is called the *automorphism of the graph* T. In what follows, the symbol (T, P) means the graph T equipped with the automorphism P.

Two three-color graphs (T, P) and (T', P') are said to be *isomorphic* if there exists a one-to-one correspondence ξ between the sets of their vertices which preserve the relations of incidence and the color, as well as the conjugating automorphisms P and P' (that is, $P' = \xi P \xi^{-1}$).

An automorphism P of a three-color graph T is said to be *periodic* of period $m \in \mathbb{N}$ if $P^m(a) = a$ and $P^\mu(a) \neq a$ for positive integers $\mu < m$ for any vertex a of the graph T.

We construct a three-color graph T_f, corresponding to a diffeomorphism $f \in G$ as follows (see Fig. 4):

1) the vertices of T_f are in a one-to-one correspondence with the triangular domains of the sets Δ;

2) two vertices of the graph are incident to an edge of color s, t and u if the corresponding triangular domains have a common s, t and u-curve.

Let B_f denote the set of vertices of the graph T_f. Since the sides of any triangular domain are colored in different colors, the edges of three different colors come together at the vertex corresponding to the triangular domain. Since any side of a triangular domain is adjacent to exactly two different triangular domains, the graph T_f has no cycles of length 1. Thus, the graph T_f satisfies the definition of the three-color graph. Let $\pi_f : \Delta_f \to B_f$ denote a one-to-one map of the set of triangular domains of the diffeomorphism f into the set of vertices of the graph T_f. The diffeomorphism f induces the automorphism $P_f = \pi_f f \pi_f^{-1}$ on the set of vertices of the graph T_f. By construction, three-color graphs obtained from different partitions into triangular domains (depending on the choice of t-curves) are isomorphic.

Theorem 2 ([15], Theorem 1). *For diffeomorphisms f, f' of class G to be topologically conjugate, it is necessary and sufficient that their graphs (T_f, P_f) and $(T_{f'}, P_{f'})$ are isomorphic.*

In order to solve the realization problem in the class of G, we introduce the following notion.

Definition 6. *We say that a three-color graph (T, P) is admissible if it has the following properties:*

1) *the graph T is connected;*

2) *the length of any su-cycle of T is equal to 4;*

3) *the automorphism P is periodic.*

Lemma 1 ([15], Lemma 1.1). *Let $f \in G$. Then the three-color graph (T_f, P_f) is admissible.*

Theorem 3 ([15], Theorem 2). *Let (T, P) be an admissible three-color graph. Then there exists a diffeomorphism $f : M^2 \to M^2$ of class G for which the graph (T_f, P_f) is isomorphic to (T, P). Furthermore,*

i) the Euler characteristic of the surface M^2 is calculated by the formula

$\chi(M^2) = \nu_0 - \nu_1 + \nu_2,$

where ν_0, ν_1, ν_2 is the number of all tu-, su-, st-cycles of the graph T, respectively;

ii) the surface M^2 is orientable if and only if all cycles of T have even length.

1.3. Directed Graphs for Gradient-like Diffeomorphisms with One Source on the k-sphere $k > 3$

In dimension $k = 3$ two diffeomorphisms with only one saddle point each may be topologically nonconjugate even though they have the same directed graph. For instance, this follows from the studies by C. Bonatti and V. Grines [7].

We consider a class of Morse – Smale diffeomorphisms defined on orientable manifolds of dimension greater than 3, for which it is proven that the complete topological invariant is again the directed graph endowed with an automorphism.

The proof of this result is essentially based on recent results from the topology of manifolds of higher dimensions that generally cannot be applied in dimension 3.

Let $G_1(M^k) = G_1$ be the class of orientation-preserving Morse – Smale diffeomorphisms defined on a connected closed smooth oriented manifold M^k of dimension $k \geqslant 4$ such that any $f \in G_1$ satisfies the following conditions:

1) the unstable manifold of any saddle periodic point of the nonwandering set Ω_f is one-dimensional;

2) the unstable manifolds and the stable manifolds of different saddle periodic points of Ω_f do not intersect.

Theorem 4 ([12], Proposition 1.2). *For any diffeomorphism $f \in G_1$, $k > 4$ the nonwandering set $\Omega(f)$ consists of exactly one repelling point, $m > 1$ saddle points, and $m + 1$ attracting points; the ambient manifold M^k is the sphere $\mathbb{S}^k$.*

As in Section 1.1, we define the graph Γ_f of a diffeomorphism f (see Fig. 5) as a directed graph whose set of vertices is isomorphic to the set of nonwandering points Ω_f of the diffeomorphism f and the set of edges is isomorphic to the set of separatrices of the saddle periodic points. The restriction of the diffeomorphism f to the nonwandering set Ω_f induces an automorphism on the vertex set of the graph Γ_f. Denote this automorphism by P_f.

Theorem 5 ([12], Theorem 2.1). *Diffeomorphisms $f, f' \subset G_1(M^k)$ $(k \geqslant 4)$ are topologically conjugate if and only if there exists an isomorphism η of the graphs Γ_f, $\Gamma_{f'}$ which preserves the orientation of the edges and such that $P_{f'} = \eta P_f \eta^{-1}$.*

Recall that the necessary and sufficient condition for two gradient-like diffeomorphisms on surfaces to be conjugate (Theorem 1) is that their equipped graphs are isomorphic. Notice that this requirement is stronger than the requirement of the existence of an isomorphism of the graphs in Theorem 1.

We call a connected directed graph Γ an *admissible graph* if its vertex set Γ^0 can be represented as a union of three nonempty disjoint subsets $\Gamma^0_1 = \{a^1_1\}$, $\Gamma^0_2 = \{a^1_2, \ldots, a^m_2\}$, $\Gamma^0_3 = \{a^1_3, \ldots, a^{m+1}_3\}$ such that:

a) for any $i \in \{1, \ldots, m\}$ the vertex a^i_2 is incident to exactly three edges one of which connects the vertex a^i_2 with the vertex a^1_1 and the two other edges connect the vertex a^i_2 with different vertices from the set Γ^0_3;

b) there are no edges that connect the vertices from the set Γ^0_3 with each other; nor are there edges that connect the vertex a^1_1 with vertices from the set Γ^0_3;

c) for any $j \in \{1, \ldots, m\}$ the edge (a^1_1, a^j_2) is oriented from a^1_1 to a^j_2;

d) for any pair of $i \in \{1, \ldots, m\}$, $j \in 1, \ldots, m + 1$ such that the vertices a^i_2, a^j_3 are adjacent, the edge (a^i_2, a^j_3) is oriented on a^i_2 to a^j_3;

e) the graph $\Gamma \setminus a^1_1$ is connected.

Denote by $\mathbf{\Gamma}$ the set of all admissible graphs.

Lemma 2 ([12], Lemma 3.1). *If f is a diffeomorphism of class $G_1(M^k)$, then $\Gamma_f \in \mathbf{\Gamma}$.*

The following theorem states the existence of a representative in each class of topological conjugacy of the diffeomorphisms of G_1, i.e., Theorem 6 is the realization theorem.

Theorem 6 ([12], Theorem 3.1). *Let P be an arbitrary automorphism of a graph $\Gamma \in \mathbf{\Gamma}$ that preserves the direction of the edges. Then there exists a diffeomorphism $f \in G_1$ such that the graph Γ_f is isomorphic to the graph Γ by the isomorphism $\zeta : \Gamma_f \to \Gamma$ such that $P = \zeta P_f \zeta^{-1}$.*

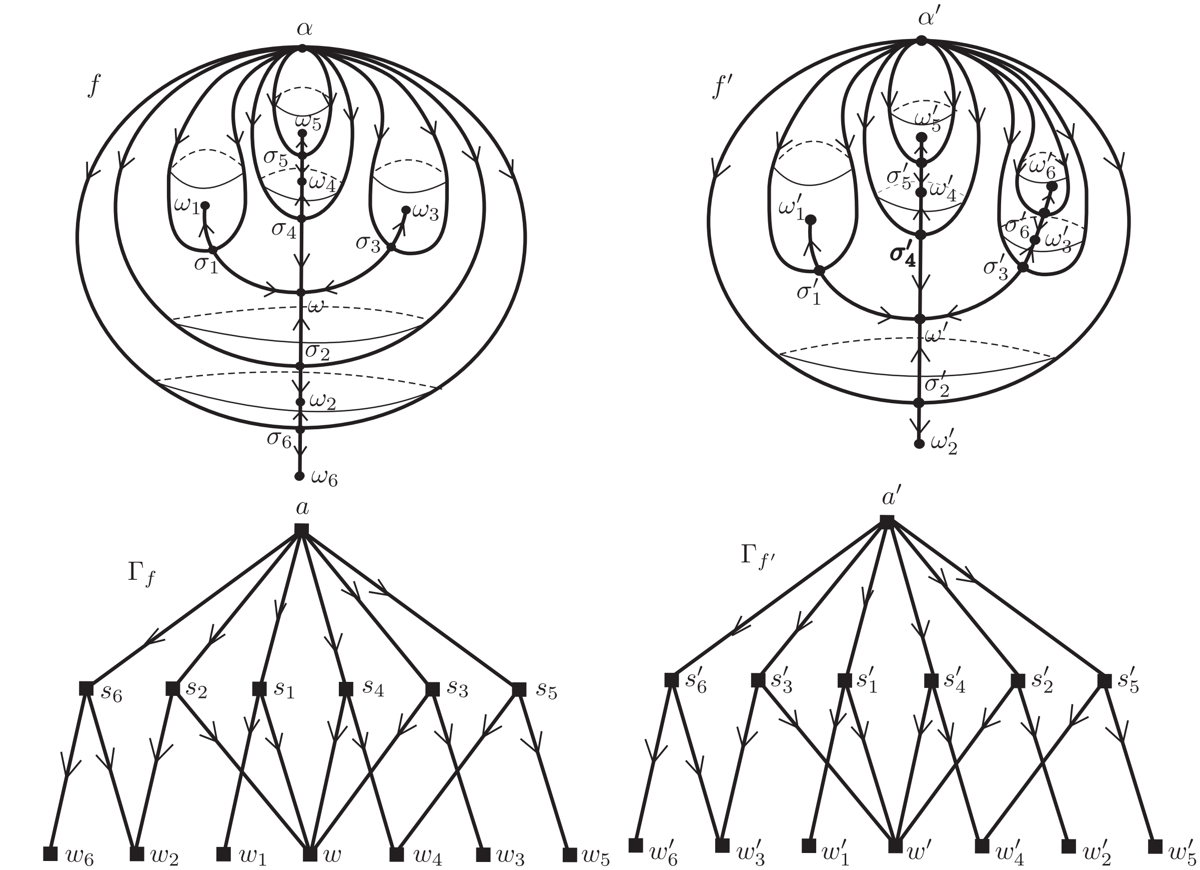

Fig. 5. Diffeomorphisms $f, f' \in G_1$ have the isomorphic graphs $\Gamma_f, \Gamma_{f'}$ and are topologically conjugate due to Theorem 5 below.

2. EFFICIENT ALGORITHMS FOR THE RECOGNITION OF GRAPHS

The complexity status of the isomorphism problem of general graphs is still unknown. That is, it has not yet been proved to be polynomial or NP-complete (NP-completeness is a standard of intractability, see [11]). But the directed graph for a gradient-like diffeomorphism on a surface or on a k-sphere and the three-color graph are not graphs of general type as they possess specific combinatorial properties which are used to develop efficient algorithms for their recognition.

2.1. Three-color Graphs

Recall that two gradient-like diffeomorphisms of M^2 are topologically conjugate if and only if their three-color graphs are isomorphic (Theorem 2). A. A. Oshemkov and V. V. Sharko in [21] proposed an algorithm for recognition of three-color graphs, but it was enumerative, as it uses an exhaustive enumeration of all permutations of all su-cycles. Hence, it cannot be efficient, as it takes too much time. More precisely, the algorithm can run in exponential time on the number of vertices of given three-color graphs. The goal of the present section is to give an efficient (i.e., polynomial in the parameters) algorithm for recognition (up to isomorphism) of the two three-color graphs T_f and $T_{f'}$ of some diffeomorhisms $f, f' \in G(M^2)$. The main property of the graphs $T_f, T_{f'}$ is their trivalency.

Definition 7. *A graph is said to be trivalent if each of its vertices is adjacent to at most three other vertices.*

Definition 8. *A graph is said to be simple if it is an undirected graph without loops and without multiple edges.*

The following propositions state that the isomorphism problem is polynomial for simple trivalent graphs.

Proposition 1 ([10]). *An isomorphism of two simple n-vertex trivalent graphs can be recognized in $O(n^3 \log(n))$ time.*

Proposition 2 ([16]). *An isomorphism of two n-vertex planar (i.e., embeddable into a zero genus manifold) simple graphs can be recognized in $O(n)$ time.*

Unfortunately, the propositions above cannot be immediately applied to three-color graphs, because some of them are not simple (see, for example, Fig. 4). Nevertheless, by subdividing edges one can reduce the isomorphism problem for three-color graphs to the isomorphism problem for trivalent simple graphs.

Definition 9. *To k-subdivide an edge $e = (a, b)$ means to delete e from the graph and then to add new vertices $c_1, c_2, \ldots, c_k$ and new edges $(a, c_1), (c_1, c_2), \ldots, (c_{k-1}, c_k), (c_k, b)$.*

Theorem 7. *An isomorphism of two three-color n-vertex graphs can be recognized in $O(n^3 \log(n))$ time.*

Proof. For the two three-color (s,t,u) n-vertex graphs Γ_1 and Γ_2 one can construct the simple graphs $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ as follows. We 1-subdivide all u-edges, 2-subdivide all s-edges and 3-subdivide all t-edges of Γ_1 and of Γ_2. Clearly, if Γ_1 and Γ_2 are isomorphic, then $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ are isomorphic as well. Inversely, let the graphs $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ be isomorphic by an isomorphism ξ. Let γ be a path in $\tilde{\Gamma}_1$ connecting two vertices v_1 and v_2 and let it have i internal (i.e., having exactly two neighbors) vertices $w_1, \ldots, w_i$. Then $\xi(\gamma)$ is a path in $\tilde{\Gamma}_2$ connecting two vertices $\xi(v_1)$ and $\xi(v_2)$, and it has i internal vertices as well, hence Γ_1 and Γ_2 are isomorphic. So, Γ_1 and Γ_2 are isomorphic if and only if $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ are isomorphic. It is easy to see that Γ_1 and Γ_2 have $\frac{n}{2}$ edges of each of three types. Hence, $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ have $4n = O(n)$ vertices. Moreover, $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ are trivalent. Therefore, by Proposition 1, an isomorphism of $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ can be recognized in $O(n^3 \log(n))$ time. $\square$

The following theorem immediately follows from Theorem 7.

Theorem 8. *An isomorphism of the two three-color n-vertex graphs $T_f, T_{f'}$ of diffeomorphisms $f, f' \in G(M^2)$ can be recognized in $O(n^3 \log(n))$ time.*

Notice that T_f and $T_{f'}$ for $f, f' \in G(M^2)$ can be embedded into M^2. Moreover, a k-subdivision of an edge preserves the embeddability for each k. Hence, the following theorem follows from the proof of Theorem 7 and Proposition 2.

Theorem 9. *An isomorphism of the two three-color n-vertex graphs $T_f, T_{f'}$ of diffeomorphisms $f, f' \in G(\mathbb{S}^2)$ can be recognized in $O(n)$ time.*

To recognize the orientability and genus of M^2 from the graph T_f for a diffeomorphism $f \in G(M^2)$, we need the following information.

Definition 10. *A simple graph is bipartite if the set of its vertices can be partitioned into two parts such that there is no edge incident to two vertices in the same part.*

Proposition 3 (König theorem, [17]). *A simple graph is bipartite if and only if it does not contain odd cycles.*

Proposition 4 ([2]). *Bipartiteness can be recognized in $O(n + m)$ time for any simple graph with n vertices and m edges.*

Proposition 5 ([2]). *The number of connected components of a given simple graph with n vertices and m edges can be calculated in $O(n + m)$ time.*

Theorem 10. *For an arbitrary given M^2 and a diffeomorphism $f \in G(M^2)$ the orientability and the genus of M^2 can be determined in linear time on the number of vertices of the three-color graph T_f.*

Proof. Let M^2 be an arbitrary manifold (orientable or not), let f be an arbitrary diffeomorphism in $G(M^2)$, and let T_f be its three-color graph. By Theorem 3 the manifold M^2 is orientable if and only if T_f has no odd cycles. To recognize the orientability of M^2, we 2-subdivide each edge of T_f. The resultant graph is bipartite if and only if T_f has no odd cycles. Hence, by Propositions 3 and 4, the orientability of M^2 can be checked in linear time on the number of vertices of T_f.

By Theorem 3, $\chi(M^2) = 2 - 2g(M^2) = \nu_0 - \nu_1 + \nu_2$, where $\chi(M^2)$ and $g(M^2)$ are the Euler characteristic and the genus of M^2, ν_0, ν_1, ν_2 are the numbers of all *tu*-, *su*-, *st*-cycles of the graph T_f, respectively. Deleting all s-edges from T_f produces a graph whose connected components are the *tu*-cycles of T_f. The same is true for t-edges and *su*-cycles, u-edges and *st*-cycles. Hence, by Proposition 5, the numbers of *tu*-, *su*- and *st*-cycles of T_f can be computed in linear time on the number of the vertices of T_f and consequently $\chi(M^2)$ and $g(M^2)$ as well. □

2.2. Directed Graphs

Let f and f' be arbitrary orientation-preserving Morse-Smale diffeomorphisms from the class $G_1(\mathbb{S}^k), k > 3$ with the directed graphs Γ_f and $\Gamma_{f'}$. To recognize the graphs, we give the following definition.

Definition 11. *To 2^*-subdivide an edge (a, b) means to delete it from the graph and to add new vertices x, y, z and edges $(a, x), (x, y), (y, b), (y, z)$.*

Theorem 11. *For each fixed k, an isomorphism of the two n-vertex directed graphs of diffeomorphisms in $G_1(\mathbb{S}^k)$ can be recognized in $O(n)$ time.*

Proof. By Theorem 5, $f, f' \in G_1(\mathbb{S}^k)$ are conjugate if and only if the graphs Γ_f and $\Gamma_{f'}$ are isomorphic. If we 2^*-subdivide each directed edge of Γ_f and $\Gamma_{f'}$, we get the graphs $\tilde{\Gamma}_f$ and $\tilde{\Gamma}_{f'}$. It follows from [12] that Γ_f and Γ_2 can be embedded into the 2-sphere $\mathbb{S}^2$. Then $\tilde{\Gamma}_f$ and $\tilde{\Gamma}_{f'}$ are simple planar graphs. The graphs $\tilde{\Gamma}_f$ and $\tilde{\Gamma}_{f'}$ are isomorphic whenever Γ_f and $\Gamma_{f'}$ are isomorphic. The graph Γ_f ($\Gamma_{f'}$) can be uniquely reconstructed from the graph $\tilde{\Gamma}_f$ ($\tilde{\Gamma}_{f'}$). Indeed, a vertex of Γ_f (Γ_f) corresponds to a source if and only if all its neighbors are vertices of degree two; a vertex of Γ_f (Γ_f) corresponds to a sink if and only if all its neighbors are vertices of degree three. Hence, $\tilde{\Gamma}_f$ and $\tilde{\Gamma}_{f'}$ are isomorphic if and only if Γ_f and $\Gamma_{f'}$ are isomorphic. Due to Proposition 2, to recognize whether $\tilde{\Gamma}_f$ and $\tilde{\Gamma}_{f'}$ are isomorphic or not, one can apply a linear-time algorithm. □

Isomorphism of the directed graphs $\Gamma_f, \Gamma_{f'}$ is a necessary and sufficient condition for topological conjugation of two diffeomorphisms in $f, f' \in G_+(M^2)$. To recognize the graphs, we give the following fact.

Proposition 6 ([20]). *An algorithm for recognition of two simple n-vertex graphs embeddable into a surface of genus g has the complexity $O(n^{O(g)})$.*

As the genus of the directed graphs Γ_f of a diffeomorphism $f \in G_+(M^2)$ is at most $g(M^2)$, we can prove the following theorem similarly to the proof of Theorem 2.11.

Theorem 12. *An isomorphism of the two directed graphs of two diffeomorphisms in $G_+(M^2)$ can be recognized in polynomial time.*

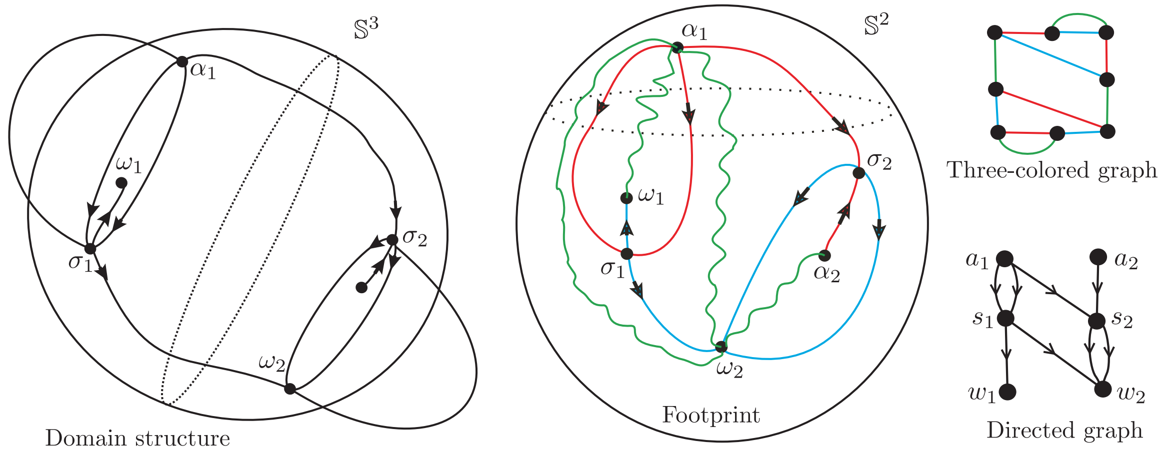

Fig. 6. Domain structure before reconnection.

3. GRAPHS IN MAGNETIC FIELDS

Many energetic processes that are observed in the Sun's corona are caused by the presence of the magnetic field. In the quiet Sun in particular, where magnetic flux pierces through the photospheric surface in many discrete fragments, the relentless motions of these fragments result in reconnection up in the corona, where the magnetic field is tangled up into complex regions of interconnecting flux. Thus, in order to capture something of the complexity of such fields, three-dimensional models need to include information concerning the topology of the magnetic field. The numerous magnetic fragments that populate the mixed polarity, quiet-Sun photosphere give rise to many interesting topological features in the corona. In light of this, much recent work has gone into classifying the configurations that arise from simple, point-source potential-field models in efforts to determine the nature of the quiet-Sun magnetic field [4, 8]. *Null points* are locations in 3D space where the magnetic field vanish, it has invariant *spine curves* and *fan surfaces* (*separatrix*). Separatrix surfaces divide the volume into distinct regions, called *domains*, within which all the field lines connect the same pair of sources. Thus, separatrix surfaces provide the borders between domains. Typically, photospheric magnetic fragments are (at least under optimal conditions) well-separated. This means that the magnetic fragment is a single point — *source*. For a model of the magnetic field with point sources the two-dimensional sphere $\mathbb{S}^2=\{(x,y,z,w)\in\mathbb{S}^3 \mid w=0\}$ in $\mathbb{S}^3=\{(x,y,z,w)\in\mathbb{R}^4 \mid x^2+y^2+z^2+w^2=1\}$ is used as a sphere of symmetry – *photosphere* (in which all the sources and nulls are placed), with the configuration in the region $\{(x,y,z,w)\in\mathbb{S}^3 \mid w<0\}$ being termed the mirror corona. Thus, we deal with gradient-like flow on $\mathbb{S}^3$ whose singular points belong to $\mathbb{S}^2$; among them the saddle points correspond to the null points and the sink (source) points correspond to the positive (negative) sources.

Separatrix surfaces divide the volume into different regions called *domains*. When two separatrix surfaces have intersection, they form a feature called a *separator*, which joins two oppositely signed null points. The appearance and disappearance of separators change the topology of domains splitting; such a situation is called *separator reconnection*, which is one of the major reconnection mechanisms. The null points and the separator are locations in the configuration where the magnetic energy tends to dissipate due to the creation of current sheets. The reconnection transfers magnetic flux across the separatrix surfaces from one region to another. The simplest case where a separator occurs is shown in Fig. 8. This topology is known as the *intersecting state*. In Fig. 7 we see a moment of reconnection, and Fig. 6 represents a situation before the intersecting state.

Together, the fan traces and spine field lines from all nulls in the system outline the boundaries of all domain *footprints* on the photosphere. Obviously, the topological structure of the domains is reduced to the topological structures of their footprints which are uniquely defined by the three-colored or directed graph. Thus, the classification of the magnetic field topology in the corona is reduced to the recognition of the graphs.

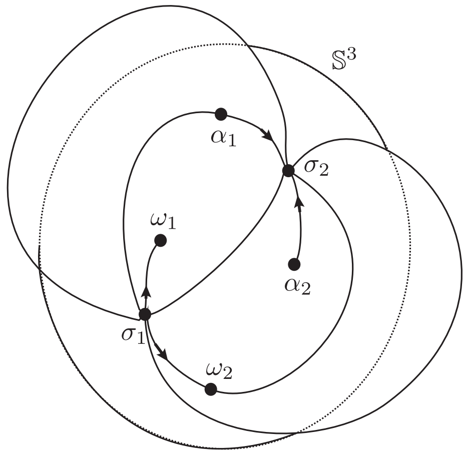

Fig. 7. The reconnection.

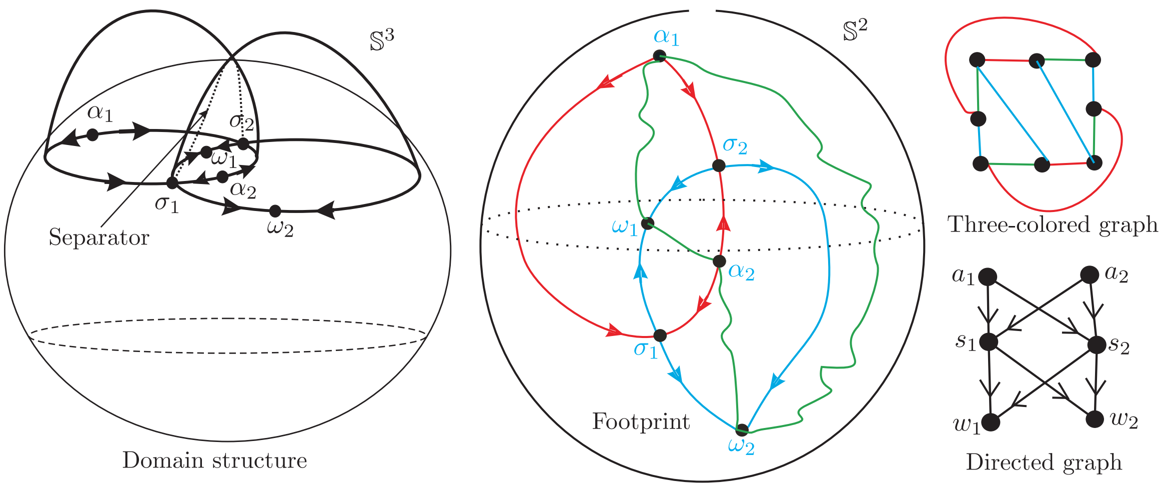

Fig. 8. Domain structure after reconnection.

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