# Hyperkähler SYZ conjecture and semipositive line bundles

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**Abstract**
Let $M$ be a compact, holomorphic symplectic Kähler manifold, and $L$ a non-trivial line bundle admitting a metric of semi-positive curvature. We show that some power of $L$ is effective. This result is related to the hyperkähler SYZ conjecture, which states that such a manifold admits a holomorphic Lagrangian fibration, if $L$ is not big.

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## 1 Introduction

### 1.1 Lagrangian fibrations on hyperkähler manifolds

Let $M$ be a compact, Kähler, holomorphically symplectic manifold. By Calabi-Yau theorem, such a manifold admits a hyperkähler metric, which is Ricci-flat and necessarily unique in its Kähler class. Throughout this paper, we shall use "hyperkähler" as "compact, Kähler, holomorphically symplectic".

Using Bochner vanishing and Berger's classification of irreducible holonomy, one proves Bogomolov's decomposition theorem ([Bes], [Bo1]), stating that any compact hyperkähler manifold has a finite covering $\widetilde{M} \cong T \times M_1 \times ... \times M_n$, where $T$ is a hyperkähler torus, and all $M_i$ are holomorphically symplectic manifolds with

$$H^1(M_i) = 0, \quad H^{2,0}(M_i) \cong \mathbb{C} \tag{1.1}$$

By Cheeger-Gromoll theorem, a compact Ricci-flat manifold with $H^1(M_i) = 0$ has finite fundamental group. An easy argument involving the Bochner vanishing would immediately imply that a hyperkaehler manifold with finite fundamental group is in fact simply connected.

A simply connected hyperkähler manifold satisfying $H^{2,0}(M) = \mathbb{C}$ is called **simple**.

In [Ma1], D. Matsushita proved the following theorem (see also [H1]).

**Theorem 1.1:** ([Ma1]) Let $M$ be a simple hyperkähler manifold, and $\pi : M \longrightarrow X$ a surjective holomorphic map from $M$ onto a Kähler variety $X$. Assume that $0 < \dim X < \dim M$. Then $\dim X = \frac{1}{2} \dim M$, and the fibers of $\pi$ are Lagrangian subvarieties of $M$.

■

Such a map is called **a holomorphic Lagrangian fibration**. A (real) Lagrangian subvariety $S$ of an $n$-dimensional Calabi-Yau manifold is called **special Lagrangian** if a holomorphic $(n, 0)$-form, restricted to $S$, is proportional to its Riemannian volume.

From Calabi-Yau theorem it follows immediately that a compact, holomorphically symplectic Kähler manifold admits a triple of complex structures $I, J, K$, satisfying quaternionic relations (see [Bes]). A complex Lagrangian subvariety of $(M, I)$ is special Lagrangian with respect to $J$, which is clear from the linear algebra. Therefore, Matsushita's theorem (Theorem 1.1) gives a way to produce special Lagrangian fibrations on hyperkähler manifolds.

Holomorphic Lagrangian fibrations are important in Mirror Symmetry. In [SYZ], Strominger, Yau and Zaslow conjectured that Mirror Symmetry of Calabi-Yau manifolds comes from real Lagrangian fibrations. From arguments making sense within the framework of string theory, it occurs that any Calabi-Yau manifold which admits Mirror Symmetry must apparently admit a special Lagrangian fibration, and the dual fibrations should correspond to the mirror dual Calabi-Yau manifolds.

The Strominger-Yau-Zaslow conjecture remains a mystery even now. For a current survey of SYZ-conjecture, please see [G].

Examples of special Lagrangian fibrations are very rare; indeed, all known examples are derived from holomorphic Lagrangian fibrations on K3, torus, or other hyperkähler manifolds.

Existence of holomorphic Lagrangian fibrations on hyperkähler manifolds is predicted by the Strominger-Yau-Zaslow interpretation of mirror symmetry. In the weakest form, the hyperkähler SYZ conjecture is stated as follows.

**Conjecture 1.2:** Let $M$ be a hyperkähler manifold. Then $M$ can be deformed to a hyperkähler manifold admitting a holomorphic Lagrangian fibration.

For a more precise form of a hyperkähler SYZ conjecture, see Subsection 1.3. The hyperkahler SYZ conjecture was stated by several people: [Bo2] (unpublished), [HT] (Conjecture 3.8 and Remark 3.12), [Saw], [H1] (Section 21.4). In [Bo3], the conjecture was attributed to an unpublished collaboration of F. Bogomolov and A. Tyurin around 1985. In algebraic geometry, a version of this conjecture is sometimes called *an abundance conjecture*; see Remark 1.8.

## 1.2 Bogomolov-Beauville-Fujiki form on hyperkähler manifolds

Let $M$ be a simple hyperkähler manifold, $\dim_{\mathbb{C}} M = 2n$, $\omega$ its Kähler form, and $q : H^2(M) \times H^2(M) \longrightarrow \mathbb{R}$ a symmetric form on $H^2(M)$ defined by the formula

$$q(\eta_1, \eta_2) := \int_X \omega^{2n-2} \wedge \eta_1 \wedge \eta_2 - \frac{2n-2}{(2n-1)} \frac{\int_X \omega^{2n-1}\eta_1 \cdot \int_X \omega^{2n-1}\eta_2}{\int_M \omega^{2n}} \tag{1.2}$$

It is well known (see e. g. [Bea], [V1], Theorem 6.1, or [H1], 23.5, Exercise 30), that this form is, up to a constant multiplier, independent from the choice of complex and Kähler structure on $M$, in its deformation class. The form $q$ is called **the Bogomolov-Beauville-Fujiki form of** $M$.

Usually, the Bogomolov-Beauville-Fujiki form is defined as

$$\begin{aligned} q(\eta, \eta) := (n/2) \int_X \eta \wedge \eta \wedge \Omega^{n-1} \wedge \overline{\Omega}^{n-1} - \\ - (1-n) \left( \int_X \eta \wedge \Omega^{n-1} \wedge \overline{\Omega}^{n} \right) \left( \int_X \eta \wedge \Omega^{n} \wedge \overline{\Omega}^{n-1} \right) \end{aligned}$$

where $\Omega$ is the holomorphic symplectic form. This definition (up to a constant multiplier) is equivalent to the one given above (*loc. cit.*).

The Bogomolov-Beauville-Fujiki form largely determines the structure of cohomology of $M$, as implied by the following theorem.

**Theorem 1.3:** (see [V1], [V2]) Let $M$ be a simple hyperkähler manifold, $\dim_{\mathbb{C}} M = 2n$, and $A^{2*} \subset H^{2*}(M)$ the subalgebra in cohomology generated by $H^2(M)$. Then

**(i)** The natural action of $SO(H^2(M), q)$ on $H^2(M)$ can be extended to a multiplicative action on $A^{2*}$.

**(ii)** As an $SO(H^2(M), q)$-representation, $A^{2i}$ is isomorphic to the symmetric power $\mathrm{Sym}^i(H^2(M))$ for $i \leqslant n$ and to $\mathrm{Sym}^{2n-i}(H^2(M))$ for $i \geqslant n$.

**(iii)** The properties (i)-(ii) determine the algebra structure on $A^{2*}$ uniquely.

**(iv)** The automorphism group of $A^{2*}$ is isomorphic to $\mathbb{R}^* \times SO(H^2(M), q)$.

■

**Remark 1.4:** From Theorem 1.3 (iv), it follows that the Bogomolov--Beauville-Fujiki form is uniquely, up to a constant, determined by topology of $M$.

The following theorem was proven by A. Fujiki in [F]. It follows immediately from the explicit description of $A^{2*} \subset H^*(M)$ given in Theorem 1.3. This theorem is also sometimes used as a definition of Bogomolov-Beauville-Fujiki form.

**Theorem 1.5:** (Fujiki's formula) Let $M$ be a simple hyperkähler manifold, $\dim_{\mathbb{C}} M = 2n$, and $a \in H^2(M)$ a non-zero cohomology class. Then

$$\int_M a^{2n} = q(a,a)^n \lambda,$$

for some constant $\lambda$ determined by the choice of Bogomolov-Beauville-Fujiki form.

■

From Fujiki's formula and Theorem 1.3, Matsushita's theorem (Theorem 1.1) follows quite easily. Indeed, consider a surjective holomorphic map $M \xrightarrow{\pi} X$, with $X$ Kähler, $0 < \dim X < \dim M$, and let $\omega_X$ be a Kähler class of $X$. Then $(\pi^*\omega_X)^{\dim_{\mathbb{C}} X} = \pi^* \operatorname{Vol}_X$ is a non-zero positive closed form, hence its class in $H^*(M)$ is non-zero. Since $\omega_X^{\dim_{\mathbb{C}} X+1}$ vanishes identically, $(\pi^*\omega_X)^{\dim_{\mathbb{C}} X+1} = 0$. By Fujiki's formula, $q(\pi^*\omega_X, \pi^*\omega_X) = 0$. By Theorem 1.3, then, $(\pi^*\omega_X)^n \neq 0$, where $2n = \dim_{\mathbb{C}} M$, and $(\pi^*\omega_X)^{n+1} = 0$. This gives $n = \dim_{\mathbb{C}} X$.

## 1.3 Semipositive line bundles and effectivity

**Definition 1.6:** Let $L$ be a line bundle on a compact complex manifold $M$. Then $L$ is called **semiample** if there exists a holomorphic map $\pi : M \longrightarrow X$ to a projective variety $X$, and $L^N \cong \pi^*(\mathcal{O}(1))$, for some $N > 0$.

Recall that a cohomology class $\eta \in H^{1,1}(M)$ on a Kähler manifold is called **nef** if $\eta$ belongs to a closure of the Kähler cone of $M$. A line bundle $L$ on $M$ is called **nef** if $c_1(L)$ is nef.

From the above argument we obtain that holomorphic Lagrangian fibrations are associated with cohomology classes $\eta \in H^2(M)$, $q(\eta, \eta) = 0$. Now we can state the hyperkähler SYZ conjecture in its strongest, most precise form. From now on, we shall always abbreviate $q(c_1(L), c_1(L))$ as $q(L, L)$.

**Conjecture 1.7:** Let $M$ be simple hyperkähler manifold, and $L$ a non-trivial nef bundle on $M$, with $q(L, L) = 0$. Then $L$ is semiample.

Notice that the numerical dimension of $L$ is equal to $n = \frac{1}{2} \dim_{\mathbb{C}} M$. By Kawamata's theorem ([K]), semiampleness of $L$ is implied by $\kappa(L) = \nu(L)$, where $\kappa$ is Kodaira dimension of $L$, and $\nu(L)$ is its numerical dimension. Therefore, to prove Conjecture 1.7 it suffices to show that $H^0(L^x)$ grows as $x^n$ as $x$ tends to $\infty$.

**Remark 1.8:** When $L$ is the canonical bundle of a manifold, the equality $\kappa(L) = \nu(L)$ is sometimes called "the abundance conjecture" (see e.g. [DPS2], 2.7.2). This assertion is equivalent to the canonical bundle being semiample ([K]).

In [Ma2], D. Matsushita much advanced this argument. From his results it follows that $\kappa(L) = \nu(L)$ holds if the union of all closed 1-dimensional subvarieties $C \subset M$ such that $L\Big|_C = \mathcal{O}_C$ is Zariski dense in $M$.

If Conjecture 1.7 is true, any nef bundle with $q(L, L) = 0$ should admit a smooth metric with semi-positive curvature. The implications of semipositivity of $L$ seem to be of independent interest.

Recall that a holomorphic line bundle is called **effective** if it admits a non-trivial holomorphic section, and **$\mathbb{Q}$-effective**, if its positive tensor power admits a section. The main result of the present paper is the following theorem.

**Theorem 1.9:** Let $M$ be a simple hyperkähler manifold, and $L$ a non-trivial nef bundle on $M$, with $q(L, L) = 0$. Assume that $L$ admits a smooth metric with semi-positive curvature. Then $L$ is $\mathbb{Q}$-effective.

Theorem 1.9 is proven as follows. Using a version of Kodaira-Nakano argument, we construct an embedding of $H^i(L)$ to a space of $L$-valued holomorphic differential forms. This result is known in the literature as *hard Lefschetz theorem with coefficients in a bundle* (Section 2). The holomorphic Euler characteristic of a line bundle $L$, denoted as $\chi(L)$, is a polynomial on $q(L, L)$ with coefficients which depend only on the Chern classes of $M$, as shown by Fujiki ([F], 4.12). Therefore, $\chi(L) = \chi(\mathcal{O}_M) = n+1$ ([Bes]). This implies existence of non-trivial $L^N$-valued holomorphic differential forms on $M$, for any $N > 0$ (Corollary 2.4).

To prove Theorem 1.9, it remains to deduce that $H^0(L^N) \neq 0$ from existence of $L$-valued holomorphic differential forms (Theorem 3.1). This is done in Section 3. We use results of Huybrechts and Boucksom on duality of pseudo-effective cone and the modified nef cone on a hyperkähler manifold. Recall that cohomology class $\eta \in H^{1,1}(M)$ is called **pseudo-effective** if it can be represented by a closed, positive current. The pseudo-effective classes form a closed cone in $H^{1,1}(M)$. A **modified nef** cone is a closure of the union of all classes $\eta \in H^{1,1}(M)$, such that for some birational morphism $\widetilde{M} \longrightarrow M$, $\varphi^*\eta$ is nef. In the literature, the modified nef cone is often called *the movable cone* (this terminology was introduced by Kawamata) and its interior the *birational Kähler cone.* In [H2] and [Bou], Huybrechts and Boucksom prove that a dual cone (under the Bogomolov-Beauville-Fujiki pairing) to the pseudo-effective cone is the is the modified nef cone of $M$.

Using this duality, and stability of the tangent vector bundle, we prove that for any coherent subsheaf $F \subset \mathfrak{T}$, the class $-c_1(F)$ is pseudo-effective, where $\mathfrak{T}$ is some tensor power of $TM$ (Theorem 3.11).

This result, together with Boucksom's divisorial Zariski decomposition ([Bou]), is used to show that any $L$-valued holomorphic differential form on $M$ is non-zero in codimension 2, unless $L$ is $\mathbb{Q}$-effective (Proposition 3.15). Then (unless $L^N$ is effective) the above construction produces infinitely many sections $s_i \in L^N \otimes \Omega^p M$, all non-vanishing in codimension 2. Taking the determinant bundle $D$ of a sheaf generated by all $s_i$, and using the codimension-2 non-vanishing of $s_i$, we obtain that $D \cong L^N$, for some $N > 0$. By construction, $D$ has non-zero holomorphic sections. This proves effectivity of $L^N$, for some $N > 0$.

The SYZ-type problem was treated by Campana, Oguiso and Peternell in [COP], who proved that a hyperkaehler manifold of complex dimension 4 is either algebraic, has no meromorphic functions, or admits a holomorphic Lagrangian fibration. Using an argument based on hard Lefschetz theorem with coefficients in a bundle, they also proved the following result. Let $M$ be a hyperkaehler manifold of complex dimension $\geqslant 4$ admitting a nef bundle $L$ with $q(L,L) = 0$. Then $M$ admits complex subvarieties of dimension at least 2. This result can be deduced from Theorem 4.1 (ii). Indeed, suppose that $M$ has no divisors, $\dim_{\mathbb{C}} M = 2n > 2$. Then, by Theorem 4.1 (ii), for any singular metric on $L$, its curvature current $\Theta_L$ must have non-trivial Lelong sets $\{S_i\}$, with Lelong numbers $c_i$ and codimensions $d_i$. For an appropriate choice of a singular metric on $L$, we may obtain $\Theta_L$ as a weak limit of Kaehler forms $\omega_i$. Then $\Theta_L^{d_i}$ makes sense as a limit of $\omega_i^{d_i}$, and the Lelong numbers of $\Theta_L^{d_i}$ at $S_i$ are bounded from below by $c_i^{d_i}$ ([D2], Corollary 2.19). Therefore, by Siu's decomposition theorem, $\Theta_L^{d_i} \geqslant c_i^{d_i} V_{S_i}$. Since $\Theta_L^{d_i}$ is cohomologous to 0 for $d_i > n$, and this is a positive current, $\Theta_L^{d_i} = 0$ for $d_i > n$. Then $d_i \leqslant n$.

We obtained that any Lelong set for such an $L$ has codimension $\leqslant n$.

# 2 Cohomology of semipositive line bundles and Hard Lefschetz theorem

The "Hard Lefschetz theorem with coefficients in a bundle" was rediscovered several times during the 1990-ies: see [E], [T] and [Mo]. We refer to [DPS1], where the multiplier ideal version of this result is stated and proven.

**Definition 2.1:** Let $M$ be a complex manifold, and $L$ a holomorphic Hermitian line bundle. We say that $L$ is **semipositive** if its curvature is a positive (but not necessarily positive definite) $(1,1)$-form.

Let $K$ denote the canonical bundle of $M$. The standard proof of Kodaira-Nakano theorem can be used to show that for any positive bundle $L$, one has $H^i(L \otimes K) = 0$ for all $i > 0$, can be generalized for semipositive bundles. In semipositive case, we obtain that any non-zero cohomology class $\eta \in$

$H^i(L \otimes K) = 0$ corresponds to a non-zero holomorphic $(K \otimes L)$-valued $i$-form on $M$.

**Theorem 2.2:** ([DPS1], Theorem 2.1.1) Let $(M, I, \omega)$ be a compact Kähler manifold, $\dim_{\mathbb{C}} M = n$, $K$ its canonical bundle, and $L$ a holomorphic line bundle on $M$ equipped with a singular Hermitian metric $h$. Assume that the curvature $\Theta$ of $L$ is a positive current on $M$, and denote by $\mathcal{I}(h)$ the corresponding multiplier ideal (Section 4). Then the wedge multiplication operator $\eta \longrightarrow \omega^i \wedge \eta$ induces a surjective map

$$H^0(\Omega^{n-i} M \otimes L \otimes \mathcal{I}(h)) \xrightarrow{\omega^i \wedge \cdot} H^i(K \otimes L \otimes \mathcal{I}(h)).$$

■

When $L$ is a semipositive line bundle, its multiplier ideals are all trivial, and we can use Theorem 2.2 to obtain a result about holomorpic forms with values in $L$.

**Corollary 2.3:** Let $(M, I, \omega)$ be a compact Kähler manifold with trivial canonical bundle, $\dim_{\mathbb{C}} M = n$, $L$ a semipositive line bundle on $M$, and $\Theta$ its curvature. Then $\dim H^i(L) \leqslant \dim H^0(L \otimes \Omega^{n-i} M)$

■

Corollary 2.3 immediately leads to some interesting results about semipositive bundles on hyperkähler manifolds.

**Corollary 2.4:** Let $(M, I)$ be a simple hyperkähler manifold, $\dim_{\mathbb{C}} M = 2n$, and $L$ a non-trivial nef bundle which satisfies $q(L, L) = 0$. Assume that $L$ admits a Hermitian metric with semipositive curvature form $\Theta$. Then

**(i)** $H^i(L) = 0$, for all $i > n$.

**(ii)** $\dim H^i(L) \leqslant \dim H^0(L \otimes \Omega^{n-i} M)$

**(iii)** $\sum \dim H^0(L \otimes \Omega^{n-i} M) \geqslant n + 1$.

**Proof:** Corollary 2.4 (i) follows from [V3], Theorem 1.7, and Proposition 6.4. Corollary 2.4 (ii) is a restatement of Corollary 2.3. Corollary 2.4 (ii) follows from [F], 4.12. Indeed, as Fujiki has shown, the holomorphic Euler characteristic $\chi(L)$ of $L$ is expressed through $q(L, L)$. Therefore, it is equal to the holomorphic Euler characteristic $\chi(\mathcal{O}_M)$ of the trivial sheaf

$\mathcal{O}_M$. However, $\chi(\mathcal{O}_M) = n+1$, as follows from Bochner's vanishing theorem (see e.g. [Bes]). ■

**Remark 2.5:** In [DPS2], Theorem 2.2 was used to obtain a weak form of Abundance Conjecture for manifolds $M$ with pseudo-effective canonical bundle $K_M$ admitting a singular metric with algebraic singularities. It was shown here (Theorem 2.7.3) that either such a manifold admits a non-trivial holomorphic differential form, or $H^0(\Omega^*(M) \otimes K_M^{\otimes m}$ is non-zero for infinitely many $m > 0$.

# 3 Stability and $L$-valued holomorphic forms

The main result of this section is the following theorem, proven in Subsection 3.4.

**Theorem 3.1:** Let $M$ be a compact hyperkähler manifold, and $L$ a nef line bundle satisfying $q(L, L) = 0$. Assume that $M$ admits a non-trivial $L^k$-valued holomorphic differential form, for infinite number of $k \in \mathbb{Z}^{>0}$.[1] Then $L$ is $\mathbb{Q}$-effective.

## 3.1 Stability and Yang-Mills connections

We remind some standard facts about the Kobayashi-Hitchin correspondence, proven by Donaldson and Uhlenbeck-Yau ([UY], [LT])

**Definition 3.2:** Let $F$ be a coherent sheaf over an $n$-dimensional compact Kähler manifold $M$. We define **the degree** $\deg(F)$ as

$$\deg(F) = \int_M \frac{c_1(F) \wedge \omega^{n-1}}{vol(M)}$$

and $\operatorname{slope}(F)$ as

$$\operatorname{slope}(F) = \frac{1}{\operatorname{rank}(F)} \cdot \deg(F).$$

Let $F$ be a torsion-free coherent sheaf on $M$ and $F' \subset F$ a proper subsheaf. Then $F'$ is called **destabilizing subsheaf** if $\operatorname{slope}(F') \geqslant \operatorname{slope}(F)$

[1] This would follow immediately from $\chi(L^k) = n+1$, if $L$ admits a semi-positive metric, by the result of Fujiki which ismentioned in Corollary 2.4.

A coherent sheaf $F$ is called **stable** [2] if it has no destabilizing subsheaves. A coherent sheaf $F$ is called **polystable** if it is a direct sum of stable sheaves of the same slope.

**Definition 3.3:** Let $B$ be a holomorphic Hermitian bundle over a Kähler manifold $M$, $\nabla$ its Chern connection, and $\Theta \in \Lambda^{1,1} \otimes End(B)$ its curvature. The Hermitian metric on $B$ and the connection $\nabla$ defined by this metric are called **Yang-Mills** if

$$\Lambda(\Theta) = constant \cdot \mathrm{Id}\Big|_B ,$$

where $\Lambda$ is a Hodge operator and $\mathrm{Id}\Big|_B$ is the identity endomorphism which is a section of $End(B)$.

**Theorem 3.4:** (Uhlenbeck-Yau) Let B be an indecomposable holomorphic bundle over a compact Kähler manifold. Then $B$ admits a Hermitian Yang-Mills connection if and only if it is stable. Moreover, the Yang-Mills connection is unique, if it exists.

**Proof:** [UY]. ■

**Remark 3.5:** Any tensor power of a Yang-Mills bundle is again Yang-Mills. This implies that a tensor power of a polystable bundle is again polystable. Notice that this result follows from Theorem 3.4.

**Remark 3.6:** Given a Kähler-Einstein manifold (e.g. a Calabi-Yau, or a hyperkähler manifold), its tangent bundle is manifestly Yang-Mills (the curvature condition $\mathrm{Ric}(M) = const$ is equivalent to the Yang-Mills condition, as follows from a trivial linear-algebraic argument; see [Bes] for details). Therefore, $TM$ is polystable, for any Calabi-Yau manifold.

### 3.2 The birational Kähler cone

**Definition 3.7:** ([H2], see also [Bou]) Let $(M, \omega)$ be a compact Kähler manifold, and $\{(M_\alpha, \varphi_\alpha)\}$ the set of all compact manifolds equipped with a birational morphism $\varphi_\alpha : M_\alpha \longrightarrow M$. Let $\mathcal{MN}(M) \subset H^{1,1}(M)$ be the closure of the set of all classes $\eta \in H^{1,1}(M)$ such that for some $(M_\alpha, \varphi_\alpha)$, the pullback $\varphi_\alpha^* \eta$ is a Kähler class on $M_\alpha$. The set $\mathcal{MN}(M)$ is called **the modified nef cone**, and its interior part **the birational Kähler cone** of $M$, or **the modified Kähler cone**.

[2] In the sense of Mumford-Takemoto

**Definition 3.8:** Let $M$ be a compact Kähler manifold, and $\eta \in H^{1,1}_{\mathbb{R}}(M)$ a real $(1,1)$-class which can be represented by a positive, closed (1,1)-current. Then $\eta$ is called **pseudoeffective**.

**Theorem 3.9:** Let $M$ be a compact hyperkähler manifold, and $\mathcal{MN}(M) \subset H^{1,1}_{\mathbb{R}}(M)$ its birational nef cone. Denote by $\mathcal{P}(M) \subset H^{1,1}_{\mathbb{R}}(M)$ its pseudoeffective cone. Then $\mathcal{P}(M)$ is dual to $\mathcal{MN}(M)$ with respect to Bogomolov-Beauville-Fujiki pairing.

**Proof:** [H2], Proposition 4.7, [Bou], Proposition 4.4. ■

**Remark 3.10:** Let $M$ be a hyperkähler manifold, and $M_\alpha$ another hyperkähler manifold, which is birationally equivalent to $M$. It is well-known that there is a natural isomorphism $H^{1,1}(M) \cong H^{1,1}(M_\alpha)$. In [H2], Huybrechts defined the birational Kähler cone as a inner part of a cone obtained as a closure of a union of all Kähler cones $\mathcal{K}(M_\alpha)$, for all hyperkähler manifolds $M_\alpha$ birationally equivalent to $M$. This definition is equivalent to the one given above, as shown in [Bou].

**Theorem 3.11:** Let $M$ be a compact hyperkähler manifold, $\mathfrak{T}$ a tensor power of a tangent bundle (such as a bundle of holomorphic forms), and $E \subset \mathfrak{T}$ a coherent subsheaf of $\mathfrak{T}$. Then the class $-c_1(E) \in H^{1,1}_{\mathbb{R}}(M)$ is pseudoeffective.

**Remark 3.12:** In [CP], Theorem 0.3, Campana and Peternell prove that for any projective manifold $X$, and any surjective map $(\Omega^1 X)^{\otimes m} \longrightarrow S$, with $S$ a torsion-free sheaf, $\det S$ is pseudo-effective, unless $X$ is uniruled. This general (and beautiful) result easily implies Theorem 3.11, if $M$ is projective. However, its proof is quite difficult, and does not work for non-algebraic Kähler manifolds.

**Proof of Theorem 3.11:** Since $M$ is hyperkähler, $TM$ is a Yang-Mills bundle (Remark 3.6), of slope 0. Therefore, its tensor power $\mathfrak{T}$ is also a Yang-Mills bundle. Since a Yang-Mills bundle is polystable, we have

$$\int_M c_1(E) \wedge \omega^{\dim_{\mathbb{C}} M - 1} \leqslant 0, \tag{3.1}$$

for any Kähler form $\omega$ on $M$. Using the formula (1.2), we can express the integral (3.1) in terms of the Bogomolov-Beauville-Fujiki form, obtaining that $\int_M c_1(E) \wedge \omega^{\dim_{\mathbb{C}} M-1}$ is proportoinal to $q(c_1(E), \omega)$, with positive coefficient. Therefore, (3.1) holds if and only if $-c_1(E)$ lies in the dual Kähler cone $\mathcal{K}^*(M)$.

Consider a hyperkähler manifold $M_\alpha$ which is birationally equivalent to $M$, let $M_1 \subset M \times M_\alpha$ be the correspondence defining this birational equivalence, and $\pi, \sigma : M_1 \longrightarrow M, M_\alpha$ the corresponding projection maps. Since the canonical class of $M$, $M_\alpha$ is trivial, the birational equivalence $M \xrightarrow{\varphi_\alpha} M_\alpha$ is an isomorphism outside of codimension 2. This allows one to identify $H^2(M)$ and $H^2(M_\alpha)$. Let $Z \subset M_\alpha$ be a set where $\varphi_\alpha$ is not an isomorphism. Outside of $Z$, the sheaf $\sigma_*\pi^*\mathfrak{T}$ is isomorphic to a similar tensor power $\mathfrak{T}_\alpha$ on $M_\alpha$. Therefore, the reflexization $(\sigma_*\pi^*\mathfrak{T})^{**}$ is naturally isomorphic to $\mathfrak{T}_\alpha$.

Consider the sheaf $E_\alpha := (\sigma_*\pi^*E)^{**}$. Outside of $Z$, $\sigma_*\pi^*E$ is naturally embedded to $\sigma_*\pi^*\mathfrak{T}$. Therefore, the corresponding map of reflexizations is also injective, and we may consider $E_\alpha$ as a subsheaf of $\mathfrak{T}_\alpha$. Since $\varphi_\alpha$ is an isomorphism outside of codimension 2, $c_1(E) = c_1(E_\alpha)$. Applying (3.1) again, we find that $-c_1(E)$ lies in the dual Kähler cone $\mathcal{K}^*(M_\alpha)$. We have shown that

$$-c_1(E) \in \bigcap_\alpha \mathcal{K}^*(M_\alpha),$$

for all hyperkähler birational modifications $M_\alpha$ of $M$. From Remark 3.10, we obtain that $\bigcap_\alpha \mathcal{K}^*(M_\alpha)$ is the dual cone to the birational Kähler cone of $M$. However, the birational Kähler cone is dual to the pseudoeffective cone, as follows from Theorem 3.9. We have shown that $-c_1(E)$ is pseudoeffective. Theorem 3.11 is proven. ∎

## 3.3 Zariski decomposition and $L$-valued holomorphic forms

The following easy lemma directly follows from the Hodge index theorem

**Lemma 3.13:** Let $M$ be a compact hyperkähler manifold, $\eta \in H^{1,1}(M)$ a nef class satisfying $q(\eta, \eta) = 0$, and $\nu \in H^{1,1}(M)$ a class satisfying $q(\eta, \nu) = 0$ and $q(\nu, \nu) \geqslant 0$. Then $\eta$ is proportional to $\nu$.

**Proof:** Suppose $\eta$ is not proportional to $\nu$. Let $\nu = k\eta + \nu'$, where $\nu'$ is orthogonal to $\eta$, and $W \subset H^{1,1}_{\mathbb{R}}(M)$ be a 2-dimensional subspace generated by $\nu', \eta$. By the Hodge index theorem, the form $q$ on $H^{1,1}_{\mathbb{R}}(M)$ has signature $(+, -, -, -, -, ...)$ ([Bea]), hence $q(\nu', \nu') < 0$. However, $q(\nu, \nu) = q(\nu', \nu')$, because $q(\eta, \nu') = q(\eta, \nu) = 0$. We obtained a contradiction, proving Lemma 3.13. ∎

**Remark 3.14:** In the sequel, this lemma is applied to the following situation: $\eta$ is a nef class, satisfying $q(\eta, \eta) = 0$, and $\nu$ a modified nef class. Then $q(\eta, \nu) = 0$ implies that $\eta$ is proportional to $\nu$.

**Proposition 3.15:** Let $M$ be a compact hyperkähler manifold, $L$ a nef line bundle satisfying $q(L, L) = 0$, $\mathfrak{T}$ some tensor power of a tangent bundle, and $\gamma \in \mathfrak{T} \otimes L$ a non-zero holomorphic section. Consider the zero divisor $D$ of $\gamma$ (the sum of all divisorial components of the zero set of $\gamma$ with appropriate multiplicities). Assume that $L$ is not $\mathbb{Q}$-effective. Then $D$ is trivial.

**Proof:** Let $L_0$ be a rank 1 subsheaf of $\mathfrak{T}$ generated by $\gamma \otimes L^{-1}$. By Theorem 3.11, $\nu := -c_1(L_0)$ is pseudoeffective. Clearly, $[D] = c_1(L \otimes L_0)$. Therefore, $c_1(L) = [D] + \nu$. To prove Proposition 3.15 we are going to show that $\nu$ is proportional to $c_1(L)$.

Since $c_1(L)$ is a limit of Kähler classes, we have $q(L, D) \geqslant 0$ and $q(L, \nu) \geqslant 0$. Since $0 = q(L, L) = q(L, D) + q(L, \nu)$, this gives

$$q(L, D) = q(L, \nu) = 0.$$

In [Bou], Proposition 3.10, S. Boucksom has constructed *the Zariski decomposition* for pseudoeffective classes, showing that any pseudoeffective class $\nu$ can be decomposed as $\nu = \nu_0 + \sum \lambda_i [D_i]$, where $\lambda_i$ are positive numbers, $D_i$ exceptional divisors, and $\nu_0$ is a modified nef class. On a hyperkähler manifold, the numbers $\lambda_i$ are rational, if $\nu$ is a rational class ([Bou], Corollary 4.11).

Since $\eta := c_1(L)$ is nef, it is obtained as a limit of Kähler classes, hence $q(\eta, D_i) \geqslant 0$, and $q(\eta, \nu_0) \geqslant 0$. Therefore, $q(L, \nu) = 0$ implies that $q(\eta, \nu_0) = 0$ and $q(\eta, D_i) = 0$. By Lemma 3.13, a modified nef class $\nu_0$ which satisfies $q(\eta, \nu_0) = 0$ is proportional to $\eta$ (see Remark 3.14). Therefore, $\nu = \lambda c_1(L) + \sum \lambda_i [D_i]$, where $\lambda \geqslant 0$. We obtain

$$c_1(L) = \lambda c_1(L) + \sum \lambda_i [D_i] + [D]. \tag{3.2}$$

From (3.2), we immediately infer that $(1 - \lambda)^{-1} c_1(L)$ is effective, unless $\lambda = 1$, $[D]$ is trivial and all $\lambda_i$ vanish. ■

**Remark 3.16:** Using the terminology known from algebraic geometry, Lemma 3.13 can be rephrased by saying that a nef class $\eta \in H^{1,1}(M)$ which satisfies $q(\eta, \eta) = 0$ generates an extremal ray in the nef cone. Then Proposition 3.15 would follow from already known arguments (see also *e. g.* [CP], Corollary 1.12).

### 3.4 $L$-valued holomorphic forms on hyperkähler manifolds

Now we can prove Theorem 3.1. Let $M$ be a compact hyperkähler manifold, and $L$ a nef line bundle satisfying $q(L,L) = 0$. Assume that $M$ admits a non-trivial $L^k$-valued holomorphic differential form, for infinite number of $k \in \mathbb{Z}^{>0}$. We have to show that $L^{\otimes N}$ is effective, for some $N > 0$.

Suppose that $L^{\otimes k}$ is never effective. Then, by Proposition 3.15, any non-zero section of $\mathfrak{T} \otimes L^{\otimes k}$ is non-zero outside of codimension 1.

Let $E = \bigoplus_i \Omega^i M$ be the bundle of all differential forms, and $E_k \subset E$ its subsheaf generated by global sections of $E \otimes L^{\otimes i}$, $i = 1, ..., k$. Since $E_1 \subset E_2 \subset ...$, this sequence stabilizes. Let $E_\infty \subset E$ be its limit, $\operatorname{rk} E_\infty = r$. Choose an $r$-tuple $\gamma_1 \in E \otimes L^{\otimes i_1}, ..., \gamma_r \in E \otimes L^{\otimes i_r}$, of linearly independent sections of $E_\infty$. Then the top exterior product $\gamma_1 \wedge ... \wedge \gamma_r$ is a section of $\det E_\infty \otimes L^{\otimes I}$, where $I = \sum_{k=1}^r i_k$. The rank 1 sheaf $\det E_\infty \subset \Lambda^r E$ is a subsheaf in a tensor bundle

$$\Lambda^r E = \Lambda^r \left( \bigoplus_i \Omega^i M \right),$$

hence any section of $\det E_\infty \otimes L^{\otimes I}$ is non-zero in codimension 2 (Proposition 3.15). Therefore, $(\det E_\infty)^* \cong L^{\otimes I}$.

There is infinite number of $\gamma_k \in E \otimes L^{\otimes i_k}$ to choose, hence for appropriate choice of $\{\gamma_k \in E \otimes L^{\otimes i_k}\}$, the number $I = \sum_{k=1}^r i_k$ can be chosen as big as we wish. Therefore, the isomorphism $(\det E_\infty)^* \cong L^{\otimes I}$ cannot hold for most choices of the set $\{\gamma_k\}$. We came to contradiction, proving effectivity of $L^{\otimes k}$ for some $k > 0$. Theorem 3.1 is proven. ■

## 4 Multiplier ideal sheaves

Let $\psi : M \longrightarrow [-\infty, \infty[$ be a plurisubharmonic function on a complex $n$-dimensional manifold $M$, and $Z := \psi^{-1}(-\infty)$. Recall that such a subset is called **a pluripolar set**. It is easy to check that a complement to a pluripolar set is open and dense.

By definition, a **singular metric** on a line bundle $L$ is a metric of form $h = h_0 e^{-2\psi}$, where $\psi$ is a locally integrable function, defined outside of a closed pluripolar set.

A function is called **quasi-plurisubharmonic** if it can be locally expressed as a sum of a smooth function and a plurisubharmonic function.

Let $L$ be a nef bundle on a compact Kähler manifold $M$. Then $c_1(L)$ is a limit of a Kähler classes $\{\omega_i\}$, which are uniformly bounded. Since the set of positive currents is relatively compact, the sequence $\{\omega_i\}$ has a limit $\Xi$, which is a closed, positive current on $M$, representing $c_1(L)$. Consider a smooth, closed form $\theta$, representing $c_1(L)$. Using $\partial\overline{\partial}$-lemma for currents, we may assume that $\Xi - \theta = \partial\overline{\partial}\psi$, where $\psi$ is a 0-current, that is, an $L^1$-integrable distribution. Clearly, $\psi$ is quasi-plurisubharmonic; in particular, $\psi$ is upper semi-continuous and locally bounded outside of a pluripolar set.

Let $h_0$ be a Hermitian metric on $L$ such that $\theta$ is its curvature (such a metric always exists by $\partial\overline{\partial}$-lemma; see e.g. [GH]), and $h := h_0 \cdot e^{-2\psi}$ the corresponding singular metric. The curvature of $h$ is equal to $\partial\overline{\partial}\psi + \theta = \Xi$. We have shown that any nef bundle admits a singular metric with positive current as its curvature.

Let $\mathcal{I}$ denote the corresponding *multiplier ideal sheaf*. It can be defined directly in terms of the function $\psi$, but for our purposes it is more convenient to define the tensor product $L \otimes \mathcal{I}$ directly as a sheaf of all sections of $L$ which are locally $L^2$-integrable in the singular metric $h$ defined above.

Assume now that $M$ is a compact Kähler manifold with trivial canonical bundle. By Theorem 2.2, there is a natural surjection

$$H^0(\Omega^{n-i} \otimes L \otimes \mathcal{I}) \longrightarrow H^i(L \otimes \mathcal{I}).$$

To show that $M$ admits $L$-valued holomorphic differential forms, it suffices to show that $H^i(L)$ is non-vanishing, for some $i$.

**Theorem 4.1:** Let $M$ be a simple hyperkähler manifold, $\dim_{\mathbb{C}} M = 2n$, and $L$ a nef bundle on $M$. Consider a singular metric on $L$ with its curvature a positive current, and let $\mathcal{I}(L^m)$ be the sheaf of $L^2$-integrable holomorphic sections of $L^m$.

**(i)** Assume that for infinitely many $m > 0$, $H^i(\mathcal{I}(L^m)) \neq 0$. Then $L$ is $\mathbb{Q}$-effective.

**(ii)** Assume that all Lelong numbers of $L$ vanish. Then $L$ is $\mathbb{Q}$-effective.

**Proof:** By the multiplier ideal version of Hard Lefschetz theorem (Theorem 2.2), $H^i(\mathcal{I}(L^m)) \neq 0$ implies existence of holomorphic $\mathcal{I}(L^m)$-valued differential forms on $M$. However, $\mathcal{I}(L^m)$ is by construction a subsheaf of $L^m$, hence any $\mathcal{I}(L^m)$-valued differential form can be considered as an $L^m$-valued differential form. Now, Theorem 3.1 implies that $L^N$ is effective. This proves Theorem 4.1 (i). Then, Theorem 4.1 (ii) follows, because in this case $\mathcal{I}(L^m) = L^m$, and the usual calculation (see Corollary 2.4) implies that $\chi(\mathcal{I}(L^m) = n + 1$. ∎

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