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MARINE PHYSICS

Reflection of a Long Wave from an Underwater Slope

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Abstract—Reflection of long sea waves from and underwater slope described by a power law is studied within the shallow water theory. The slope connects with the flat bottom. This model allows us to estimate the roles of a pointwise reflection from the inflection point of the bottom profile and distributed reflection at the underwater slope. The case in which the underwater slope is described by the so-called nonreflecting beach ($h(x) \sim x^{4/3}$, where h is the depth of the basin and x is the coordinate) and the wave is reflected only from the inflection line (pointwise reflection) is specially considered. The reflection and refraction coefficients over the bottom topography were calculated, and it was shown that the sum of the squared absolute values of these values differs from unity for all profiles except the nonreflecting one. This difference is related to the distributed repeated reflections (resonances) over the underwater slope that lead to the differences in the wave height from the known Green's law.

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Investigation of the transformation of sea waves in the coastal zone is a classical and well-developed oceanographic problem [7–8, 10, 18]. The real coastal topography, even in the case of a solitary wave approach, leads to a very complex pattern of wave motions (seiche oscillations, resonances, alongshore waves), which is currently well modeled using modern computers within the shallow wave theory and its nonlinear dispersion generalizations. The analytical solutions important for the understanding of the wave physics were obtained mainly for the simplest forms of the plane slope. They are used for the demonstration of the main peculiarities of the wave transformation. For example, the well-known Green's law ($H \sim h^{-1/4}$, where H is the wave height and h is the depth) is obtained in the case of the smooth depth variation, when we can neglect reflection from the slope. Intensification of the wave over smooth slopes can be sometimes very significant. In the other limiting case of a stepwise depth variation (steps), the reflection is sufficiently strong and, if the depth jump is high, the wave can completely reflect from the step. It was shown recently that monotonous bottom profiles of the special form exist where reflection is absent even if the bottom slope is significant [4, 13–14]. In the transition zone with arbitrary variable depth, the effects of the distributed and pointwise reflection from the slope play comparable roles and, generally, it is impossible to separate them. These effects lead to both variations in the wave amplitude and its form. Here, we shall present the model of the long wave transformation over the bottom slope described by a power law, which connects with the flat bottom. This model allows us to see the influence of the underwater slope curvature on the characteristics of wave reflection and refraction at the boundary with the shelf that has an inflection point and compare the role of the pointwise and distributed reflections.

Let us consider the simplest geometry of the coastal zone when the shelf of a constant depth h_0 connects at $x = 0$ with the zone of the monotonous depth increase (underwater slope), which is described by the following relation:

$$h(x) = h_0\left(1 + \frac{x}{L}\right)^b, \quad x > 0, \quad b > 0, \qquad (1)$$

where parameters L and b characterize the steepness of the slope (Fig. 1). At point $x = 0$, there is an inflection of depth, which is characterized by inclination tan $\tan(\theta) = dh/dx = bh_0/L$. For simplicity, we shall not connect the zone of the variable depth with the region of constant depth on the right (on the open ocean side) assuming the finite value of the transition zone to separate in the pure form the effects of reflection only from one point at the bottom. We can naturally assume that the wave arrives from the open ocean; however, owing to the causality principle, which is valid in the linear problems, the locations of the source and the receiver can be exchanged and the opposite process considered in which the wave arrives from the shallow zone; this is mathematically simpler. Such wave transformation during the transition from the shallow water to the deep ocean also has a physical sense and can be considered, for example, for a tsunami when the wave reflected from the coast propagates to the open ocean.

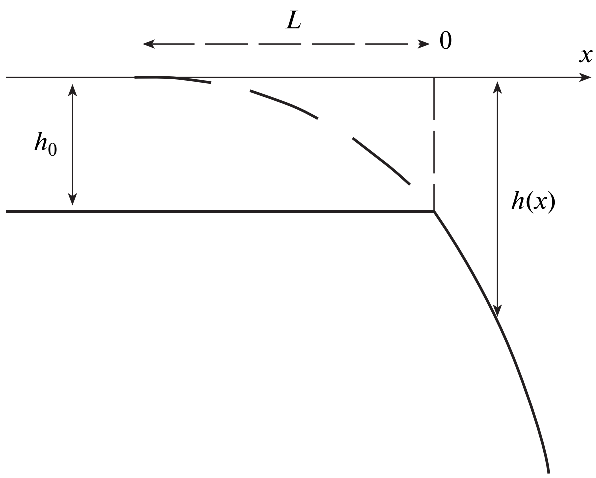

Fig. 1. Geometry of the coastal zone.

A multitude of such reflections (from Sri Lanka, the Maldives and Seychelles Islands, and the Mascarene Reef) were recorded during the Indonesia tsunami in 2004 [15–16]).

We assume that the surface waves are long everywhere (this assumption, which is formally not valid at large depths, will not influence our conclusions, because the wave field in infinity becomes vanishingly small) and linear so that the linear theory in shallow water can be used for their description. In the case of a monochromatic wave with frequency ω, it is reduced to an ordinary differential equation,

$$\frac{d}{dx}\left[h(x)\frac{d\eta}{dx}\right] + \frac{\omega^2}{g}\eta = 0, \qquad (2)$$

where η is the elevation of the water surface and g is the acceleration due to gravity. We assume that the wave propagates from the left to the inflection point, then, in the zone of the flat bottom ($x < 0$), the solution of Eq. (2) is the sum of the incident and reflected waves:

$$\eta_1(x, t) = A_i \exp[i\omega(t - x/c_0)] + A_r \exp[i\omega(t + x/c_0)], \qquad (3)$$

where $c_0 = \sqrt{gh_0}$ is the velocity of long waves over the flat bottom, t is the time, and A_i and A_r are the amplitudes of the incident and reflected waves, respectively. In the region ($x > 0$), the solution of Eq. (2) for the bottom profile described by (1) can also be found explicitly as the Bessel functions

$$\eta_2(x, t) = \frac{A_{tr}\left(1 + \frac{x}{L}\right)^{\alpha} H_{\nu}^{(2)}\left(\frac{2kL}{2-b}\left[1 + \frac{x}{L}\right]^{\gamma}\right)}{H_{\nu}^{(2)}\left(\frac{2kL}{2-b}\right)} \exp(i\omega t), \qquad (4)$$

where

$$\alpha = \frac{1-b}{2}, \quad \gamma = \frac{2-b}{2}, \quad \nu = \frac{1-b}{2-b}, \quad k = \frac{\omega}{\sqrt{gh_0}}, \qquad (5)$$

Here, $H^{(2)}$ is the Hunkel function of the second kind [3] and A_{tr} determines the height (amplitude) of the wave immediately near the inflection. We actually used the boundary condition of the wave in infinity (the Sommerfeld radiation condition) to select solution (4) so that solution (4) at a large distance from the inflection point has the form of a progressive wave in the direction to the right and it is described by an approximate relation

$$\eta_2(x, t) = \frac{A_{tr}\sqrt{\frac{2-b}{\pi kL}}\left(1 + \frac{x}{L}\right)^{-b/4} \exp\left[i\left(\omega t - \frac{2kL}{2-b}\left[1 + \frac{x}{L}\right]^{\gamma} + \frac{\pi\nu}{2} + \frac{\pi}{4}\right)\right]}{H_{\nu}^{(2)}\left(\frac{2kL}{2-b}\right)}. \qquad (6)$$

It is clear that the asymptotic solution can be obtained using the method of slowly varying amplitudes (WKB). In particular, the wave amplitude in (6) is proportional to $h^{-1/4}$ (the Green's law) and the phase multiplier in (6) is the wave phase proportional to the time of the wave propagation over the bottom topography, i.e., $\int dx/[gh(x)]^{1/2}$.

In it important to note that the wave solution (4) exists only at $b < 2$, which is assumed everywhere (in the opposite case, the solution decays exponentially due to a strong depth increase, which destructs the shallow water approximation).

In the class of the power law profiles of depth variation considered here, we can distinguish one case ($b = 4/3$) in which the wave propagates over variable bottom topography without reflection (nonreflecting beach). In this case, the strict solution of (4) is written in the elementary functions and actually coincides with the asymptotic solution (6):

$$\eta(x, t) = A_{tr}\left[\frac{h_0}{h(x)}\right]^{1/4} \exp\left[i\omega\left(t - \int\frac{dx}{\sqrt{gh(x)}}\right)\right]. \qquad (7)$$

The existence of such a nonreflecting beach for monochromatic waves has been known for a long time [9–11]. Mathematically, the initial wave equation for such beaches with variable coefficients is reduced to a wave equation with constant coefficients [12–14]; there-

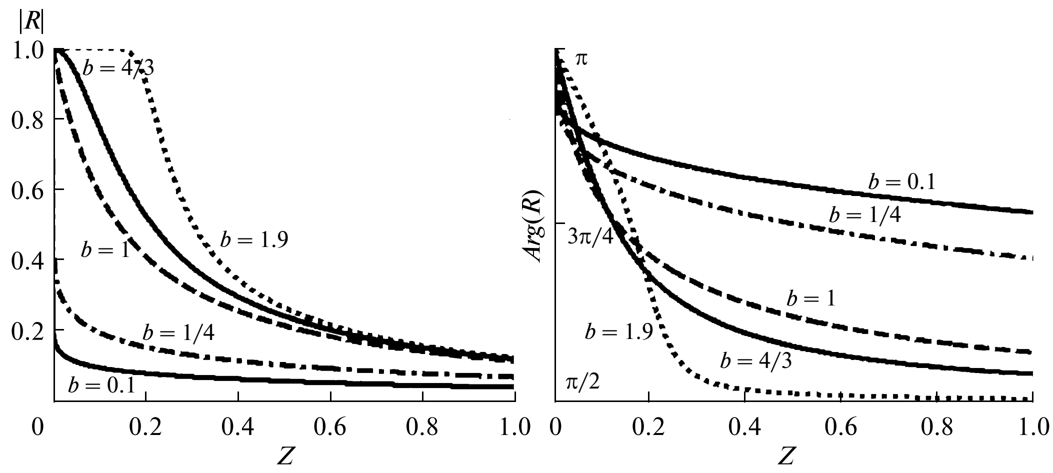

Fig. 2. Amplitude and phase components of the reflection coefficient for different slopes.

fore, its solution can also be obtained for the more general type of sea waves. It was shown recently [13—14], that the form of a progressive wave over an underwater slope should have alternating sign. It is worth noting that the wave over a nonreflecting beach propagates without reflection even if the slope of the bottom is steep enough so that the Green's law deduced for a smoothly varying depth (on a wave length scale) is valid here for any arbitrary value of the angle. If the real bottom topography is approximated by a set of nonreflecting beaches, all reflections become concentrated at the points where profiles are sewed, which allows us to reduce the problem of the wave propagation over complex topography to a system of algebraic equations for the amplitude and phase of the wave. The accuracy of such an approach can be estimated by comparing the obtained solutions for the bottom profiles of different form, which we are doing in this work using the example of reflection from an underwater slope.

We sew together solutions (3) and (4) at the inflection point using the standard boundary conditions of the water level and discharge (current velocity) continuity and obtain the sought expression for the complex coefficient of wave reflection from an underwater slope:

$$R = \frac{A_r}{A_i} = \frac{H_\nu^{(2)}\left(\frac{2bZ}{2-b}\right) - iH_{\nu-1}^{(2)}\left(\frac{2bZ}{2-b}\right)}{H_\nu^{(2)}\left(\frac{2bZ}{2-b}\right) + iH_{\nu-1}^{(2)}\left(\frac{2bZ}{2-b}\right)}, \tag{8}$$

where

$$Z = \frac{kh_0}{\tan(\theta)}. \tag{9}$$

Parameter kh_0 is always small within the shallow water theory, while the bottom slope can vary in the wide limits; therefore, argument Z also changes in wide limits. Relation (8) allows us to estimate the total reflection of the wave from the slope not separating it on the reflection from the slope (distributed reflection) and reflection from the depth inflection point (pointwise reflection).

In the case of sewing of a nonreflecting profile with the region of constant depth the reflection coefficient is determined by means of elementary functions (the same expression can be obtained from (8) at $b = 4/3$)

$$R = \frac{A_r}{A_i} = -\frac{1}{1+8iZ}, \tag{10}$$

and it determines the pointwise reflection from the point of the depth inflection [4]. Let us separate the amplitude and phase parts of the reflection coefficient

$$|R| = \frac{1}{\sqrt{(8Z)^2+1}}, \quad \mathrm{Arg}(R) = -\arctan(8Z) + \pi. \tag{11}$$

In the limiting case of a very steep slope $\theta \longrightarrow \longrightarrow \pi/2$ ($Z \longrightarrow 0$), the wave completely reflects with the change of the sign (polarity), but when the inflection is very small the reflection of the wave energy is almost absent.

The reflection coefficients of the wave from a slope for different profiles of the bottom topography calculated from general relation (8) are shown in Fig. 2. We see that in the general case the total reflection characterized by the absolute value R can be either greater or smaller than the pointwise reflection at the nonreflecting beach ($b = 4/3$). In particular, if the bottom profile is steeper (for example, $h \sim x^{1.9}$), the wave would more strongly reflect from the slope than in the case of a nonreflecting beach at the same values of the inflection angle and wavelength (frequency); and if the bottom profile is more flat as at $h \sim x$ or $h \sim x^{0.25}$, the reflection would be smaller. Thus, the distributed reflection can be either added to the pointwise reflection (at steep slopes) or it can be subtracted from it (at flat

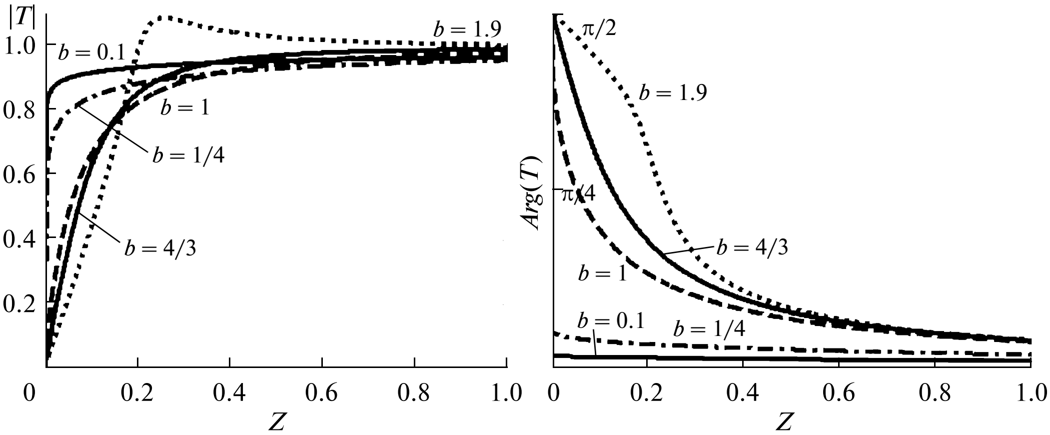

Fig. 3. Amplitude and phase components of the transformation coefficient for different slopes.

slopes). In the limiting case ($Z \longrightarrow 0$), when the depth increases sharply, the wave reflects completely from any profile, and when $Z \longrightarrow \infty$ (very flat slope) the reflection is almost absent as was expected.

The phase of the reflected wave changes in the range ($\pi/2$, π); this means that at least the leading part of the wave changes its polarity when reflected from the underwater slope. In principle, the change in the wave polarity during the reflection from the large depth zone was known earlier. It is easily obtained analytically for the case of a stepwise depth change [8]. This limiting case is obtained from relation (8) when $Z \longrightarrow 0$ and the phase tends to π. Naturally, this does not depend on the details of the underwater slope (exponent b). In the other limiting case of a gradual slope ($Z \longrightarrow \infty$), asymptotic $\pi/2$ is valid for the phase. It is also seen from Fig. 2 that, when exponent b changes, the phase does not change in the same manner at large and small values of Z, which leads to the crossings of the phase curves in the transition region.

Similarly to (8) a general expression for the coefficient of wave transformation over the slope can be found, which we define as

$$T = \frac{A_{tr}}{A_i} = \frac{2H_\nu^{(2)}\left(\frac{2bZ}{2-b}\right)}{H_\nu^{(2)}\left(\frac{2bZ}{2-b}\right) + iH_{\nu-1}^{(2)}\left(\frac{2bZ}{2-b}\right)}, \quad (12)$$

so that this coefficient characterizes the wave amplitude immediately after the inflection point. The wave field here is formed due to the effects of repeated reflections at the underwater slope. Beyond the inflection point, the wave is described by relation (4) and in the general case its amplitude as function of depth differs from the known Green's relation (we shall discuss this below). Here, we also separate the nonreflecting beach ($b = 4/3$). In this case, the wave amplitude exactly satisfies the Green's law [4, 14]. We give the relation for the coefficient of wave transformation at the inflection point to a nonreflecting profile,

$$T = \frac{A_{tr}}{A_i} = \frac{8iZ}{1+8iZ}. \quad (13)$$

Naturally, expression (14) can be deduced from (13) at $b = 4/3$. At very steep slopes ($Z \longrightarrow 0$), the wave almost does not propagate to the zone of the variable depth, while, at smooth ($Z \longrightarrow \infty$) slopes, it always propagates without change in polarity.

The coefficient of the wave transformation through the inflection point for different profiles of the bottom calculated from relation (13) is shown in Fig. 3. In the general case, the wave that passed the point is weak if the slope is sufficiently steep ($Z \longrightarrow 0$). If the slope is gentle ($Z \longrightarrow \infty$), the entire wave energy is transferred to the slope. It is interesting to note that, at steep slopes, the wave phase is equal to $\pi/2$, which, in principle, should lead to the appearance of a negative tail in the case of positive momentum incidence. At gentle slopes, the wave phase does not differ from the phase of the incident wave. These conclusions are valid for the slopes with arbitrary values of exponent b. The influence of the slope form is manifested in the character of the curves in Fig. 3. Unlike Fig. 2 for the coefficient of reflection where the phase lines crossed, here, the phase changes monotonously when exponent b changes; however this non-monotonous change is manifested in the amplitude component.

The new fact revealed here is the nonmonotonous character of the transformation coefficient as function of the slope angle variation if exponent $b > 4/3$ so that it exceeds unity at some angles (values of nondimensional parameter Z). It is known that, in the case of the pointwise reflection effect (for example, when the wave passes over a step or over a zone of variable depth), the sum of the squared absolute values of the reflection and transformation coefficients is equal to unity, which reflects the wave energy balance at both sides of the transition zone. In the case of the variable

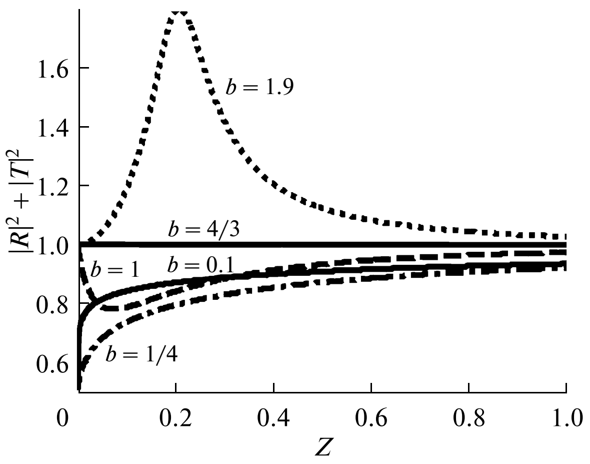

Fig. 4. Sum of the squared amplitude components of the reflection and transformation coefficients for different slopes.

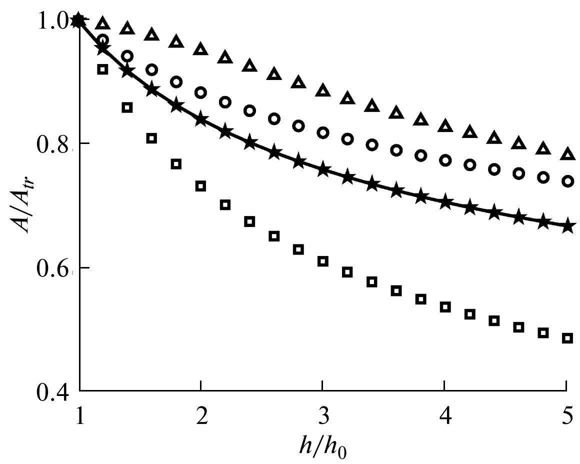

Fig. 5. Variation in the amplitude of the transformed wave versus distance from the inflection point; triangles correspond to the profile with b = 1/4, circles correspond to a flat slope (b = 1), asterisks correspond to a nonreflecting profile (b = 4/3), and squares correspond to the profile with b = 1.9; the straight line corresponds to the Green's law (Z = 0.1).

depth, the wave energy would not be determined by its amplitude only as over the constant depth. The calculated sum of the squared absolute values of the reflection and transformation coefficients for the bottom profiles described by a power law is shown in Fig. 4. We see, that in the case of a nonreflecting beach (b = 4/3), this sum is equal to unity emphasizing the pointwise character of the reflection and transformation of the wave. The distortions from unity are found for all other profiles; the distortions can make the sum either smaller or greater than unity.

It was already noted above that in the general case, solution (4) does not correspond to the wave that passed the region in the pure form due to the distributed reflection along the slope described by a power law and amplitude A_{tr} has no simple physical sense in the energy balance equation. It is our opinion that the nonmonotonous character of the curves in Fig. 4 is related to the resonance phenomenon, caused by a complex pattern of the repeated wave reflection over the slope beyond the edge. It was already mentioned in the beginning of the paper that, at large distances from the inflection point, the wave amplitude satisfies the Green's law for an arbitrary profile and the wave is progressive. In the case of nonreflecting beach, when the transformed wave is progressive at any distance from the inflection point, its amplitude also satisfies the Green's law (Fig. 5). For all the rest profiles, the deviations to both sides from this law are observed (Fig. 5). The differences to the greater values are observed for the profiles when $b < 4/3$ and the sum of the squared absolute values of the transformation and reflection coefficients exceeds unity (Fig. 4); visa versa, the amplitude of the wave decreases with the depth faster than the Green's law if the sum of the squared absolute values of the transformation and reflection coefficients is less than unity. These differences are related to the distributed reflection from the slope, its value, and phase.

Thus, even for the simple geometrical forms of the underwater slope connected with the shelf of constant depth the distributed reflection competes with the pointwise reflection at the inflection point. In the pure form, the pointwise reflection at the inflection point can be separated only for a nonreflecting beach (b = 4/3). In the general case, their ratio is characterized by the deviations from the Green's law in the amplitude of the transformed wave and in the value of the sum of the squared absolute values of the transformation and reflection coefficients at the inflection point, which differs from unity. In this sense, the natural expectations that the greatest variation in the height should occur at nonreflecting beaches can be weakened by the presence of the inflection point where pointwise reflection appears comparable with the distributed reflection at the underwater slope. At the same time, the results presented here demonstrate only slight difference in the reflection and transformation coefficients for the flat (b = 1) and nonreflecting (b = 4/3) slopes. The wave field for the flat slope is described by the Bessel functions, while elementary functions describe the wave field for the nonreflecting beach. The authors of [17] suggested using the approximation of the bottom topography by intervals of the constant slope angle for the calculations of the wave field. We think that application of the nonreflecting bottom profile even at individual segments can simplify the calculations and give an even clearer presentation of the physical results rather than application of segments with the constant slope angles.

In conclusion, we note that these results can also be applied for the other types of oceanic waves, in particular, for the internal waves propagating into the deep ocean [2, 6], as well as for the acoustic waves in a heterogeneous medium [1, 5]. The existence of a solution of type (7) [1–2, 5–6] was found for them, although their correlation with the propagating (nonreflecting) waves remained unnoticed.

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