Circles and Quadratic Maps Between Spheres

Vladlen Timorin *

Department of Mathematics, University of Toronto 100 St. George street, Toronto ON M5S3G3 Canada

E-mail address: vtimorin@math.toronto.edu

Abstract. Consider an analytic map from a neighborhood of 0 in a vector space to a Euclidean space. Suppose that this map takes all germs of vector lines at 0 to germs of circles. Such map is called rounding. Two roundings are equivalent if they take the same lines to the same circles. We prove that any rounding whose differential at 0 has rank at least 2 is equivalent to a fractional quadratic rounding. The latter gives rise to a quadratic map between spheres. Results of P. Yiu on quadratic maps between spheres have some interesting implications concerning roundings.

Introduction

By a circle in a Euclidean space we mean an honest Euclidean circle, or a straight line, or a point.

Fix a vector space structure on $\mathbb{R}^m$ and a Euclidean structure on $\mathbb{R}^n$. Consider a germ of an analytic map $\Phi : (\mathbb{R}^m, 0) \to (\mathbb{R}^n, 0)$. We say that Φ is a *rounding* if it takes all germs of lines passing through 0 to germs of circles. By the *rank of a rounding* we mean the rank of its first differential at 0.

Throughout this paper, we will always assume that the rank of a rounding is at least 2.

Roundings of ranks 1 and 0 are also interesting but their study requires different methods.

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Say that two roundings are *equivalent* if they send the germs of the same lines to the germs of the same circles. We are mostly interested in description of roundings up to the equivalence.

Roundings from $(\mathbb{R}^2, 0)$ to $(\mathbb{R}^2, 0)$ were described by A. Khovanskii in [1]. The problem was motivated by nomography (see [2]). All roundings of the above type are equivalent to local Möbius transformations. In other words, all circles in the image eventually meet somewhere outside the origin. Izadi [3] proved the same for roundings from $(\mathbb{R}^3, 0)$ to $(\mathbb{R}^3, 0)$ with an invertible first differential at 0.

It turns out that in dimension 4 this is wrong. The simplest counterexample is a complex projective transformation (with respect to an identification $\mathbb{R}^4 = \mathbb{C}^2$ such that the multiplication by i is an orthogonal operator). It takes all real lines to circles but it is nowhere equivalent (as a rounding) to a Möbius transformation. Nevertheless, a simple description of all roundings from $(\mathbb{R}^4, 0)$ to $(\mathbb{R}^4, 0)$ is possible [4]. There are 2 natural roundings from $(\mathrm{Im}(\mathbb{H}) \times \mathbb{H}, 0)$ to $(\mathbb{H}, 0)$ where $\mathbb{H}$ is the skew-field of quaternions and $\mathrm{Im}(\mathbb{H})$ is the set of all purely imaginary quaternions. The first rounding sends (x, y) to $(1 + x)^{-1}y$ and the second to $y(1 + x)^{-1}$. Any rounding from $(\mathbb{R}^4, 0)$ to $(\mathbb{R}^4, 0)$ admits an equivalent rounding obtained from one of these 2 roundings by composing it with an $\mathbb{R}$-linear map $\mathbb{R}^4 \to \mathrm{Im}(\mathbb{H}) \times \mathbb{H}$.

In this paper, we prove that any rounding is equivalent to some fractional quadratic rounding. In Section 3, we give a definition of a degenerate rounding and show that any such rounding goes through a linear projection to a smaller space. A nondegenerate quadratic rounding gives rise to a quadratic map between spheres. Thus the description of nondegenerate roundings is reduced to description of quadratic maps between spheres. Yiu's results [5, 6] lead to some interesting consequences regarding roundings.

Namely, a nondegenerate rounding from $(\mathbb{R}^m, 0)$ to $(\mathbb{R}^n, 0)$ exists if and only if n is bigger than a certain function of m (introduced by Yiu) which is very easy to compute.

The paper is organized as follows. We introduce a complexification of the notion of circle in Section 1. With its help, in Section 2, we establish a crucial algebraic property of the Taylor expansion of a rounding. Section 3 contains the proof of our main theorem: any rounding is equivalent to a fractional quadratic rounding. Closely related with fractional quadratic roundings are Hurwitz multiplications and quadratic maps between spheres which are briefly discussed in Sections 4 and 5. Results of Yiu [6] on quadratic maps between spheres are used in Section 5 to describe possible dimensions m and n for which there is a nondegenerate rounding from $(\mathbb{R}^m, 0)$ to $(\mathbb{R}^n, 0)$.

1 Complex circles

Let $\mathbb{R}^n$ be a Euclidean space with the Euclidean inner product $\langle \cdot, \cdot \rangle$. Consider the complexification $\mathbb{C}^n$ of $\mathbb{R}^n$ and extend the inner product to it by bilinearity. The extended complex bilinear inner product will be also denoted by $\langle \cdot, \cdot \rangle$.

A *complex circle* in $\mathbb{C}^n$ is either a line, or the intersection of an affine 2-plane with a *complex sphere*, i.e. a quadratic hypersurface $\{x \in \mathbb{C}^n | \langle x, x \rangle = \langle a, x \rangle + b\}$ where $a \in \mathbb{C}^n$ and $b \in \mathbb{C}$.

Proposition 1.1 *A complex circle is a smooth curve, or a pair of intersecting lines, or a plane.*

PROOF. Consider a complex circle $C = P \cap S$ where P is a plane and S is a complex sphere. If P does not belong to the tangent hyperplane to S at any point of C then the intersection of P and S is transverse, hence C is smooth. Now suppose that P belongs to the tangent hyperplane to S at a point $x \in C$. The intersection $S \cap T_x S$ is a quadratic cone centered at the point x which lies in P. If we now intersect this quadratic cone with P then we get either the whole plane P or a pair of intersecting (possibly coincident) lines from P. $\square$

The *null cone* (isotropic cone) $\{x \in \mathbb{C}^n | \langle x, x \rangle = 0\}$ will be denoted by $\mathcal{N}$.

Proposition 1.2 *Consider a germ of holomorphic curve $\gamma : (\mathbb{C}, 0) \to (\mathbb{C}^n, 0)$ whose image lies in a complex circle. Suppose that $\gamma'(0)$ is nonzero and belongs to $\mathcal{N}$. Then the linear span of the image of γ belongs entirely to $\mathcal{N}$.*

PROOF. Consider the complex circle C containing the image of γ. If C is a line or a pair of lines then the statement is obvious (the curve γ cannot switch from one line to the other).

Suppose that C is the intersection of some plane and a sphere S. Since S contains the origin, its equation has the form

$$\langle x, x \rangle = \langle a, x \rangle, \quad a \in \mathbb{C}^n.$$

The curve γ belongs to S. Therefore, $\langle \gamma, \gamma \rangle = \langle a, \gamma \rangle$ identically. Differentiating this at 0 we obtain that $\langle a, v \rangle = 0$ where $v = \gamma'(0)$. It follows that the whole vector line spanned by v belongs to S and hence to the circle C containing the image of γ.

It follows that C is a 2-plane P. Then P must belong entirely to $\mathcal{N}$. Indeed, if the restriction of the inner product to P is nontrivial then in some coordinates (x_1, x_2) on P it is given by $x_1^2 + x_2^2$ or x_1^2. In both cases a point with sufficiently large real x_1 and a real x_2 would not be in S. $\square$

Proposition 1.3 *A set $X \subset \mathbb{R}^n$ lies in a complex circle if and only if it lies in a real circle.*

PROOF. Suppose that $X \subset \mathbb{R}^n$ lies in a complex circle C. If C is a complex line then $C \cap \mathbb{R}^n$ is a real line containing X. Otherwise $C = P \cap S$ where P is a complex plane and S is a complex sphere. Then X belongs to the real plane $P \cap \mathbb{R}^n$. Let S be given by equation

$$\langle x, x\rangle = \langle a, x\rangle + b, \quad a \in \mathbb{C}^n,\ b \in \mathbb{C}.$$

The real part of this equation gives a real sphere in $\mathbb{R}^n$. Thus X lies in the intersection of a real plane with a real sphere which is a real circle.

In the other direction, the statement is obvious. $\square$

Proposition 1.4 *A set $X \subset \mathbb{C}^n$ lies in a complex circle passing through 0 if and only if the set of points $(x, \langle x, x\rangle)$, $x \in X$ spans at most 2-dimensional subspace of $\mathbb{C}^{n+1}$.*

PROOF. Assume that X lies in a complex circle passing through 0. If it lies on a vector line L then all vectors of the form $(x, \langle x, x\rangle)$ are linear combinations of $(a, 0)$ and $(0, 1)$ where a is any point of L.

If X does not belong to a single line then it lies in some sphere S containing the origin. The restriction of $\langle\cdot,\cdot\rangle$ to S equals to some linear function restricted to S. In an orthonormal coordinate system we have

$$\langle x, x\rangle = \lambda_1 x_1 + \ldots + \lambda_n x_n$$

where x_i are the coordinates of any point x from X. The coefficients λ_i are independent of x. Therefore the system of vectors $(x, \langle x, x\rangle)_{x\in X}$ has the same rank as the set X. But a circle is a plane curve so X has rank 2.

The proof in the opposite direction is a simple reversion of the above argument. $\square$

Proposition 1.5 *Let Φ be an analytic map from a neighborhood of 0 in $\mathbb{C}^m$ to $\mathbb{C}^n$. Take a vector $v \in \mathbb{R}^m$. The Φ-image of the germ of the line spanned by v lies in a complex circle if and only if the set of vectors $(\partial_v^k \Phi(0), \partial_v^k\langle\Phi, \Phi\rangle(0))_{k\in\mathbb{N}}$ spans at most 2-dimensional subspace of $\mathbb{C}^{n+1}$. Here ∂_v is the Lie differentiation along v.*

This follows immediately from Proposition 1.4.

2 Roundings

In this Section, we obtain an algebraic constrain on the 2-jet of a rounding.

Theorem 2.1 *Let $\Phi : (\mathbb{R}^m, 0) \to (\mathbb{R}^n, 0)$ be a rounding and A its first differential at 0. Then $\langle A, \Phi\rangle$ and $\langle \Phi, \Phi\rangle$ are divisible by $\langle A, A\rangle$ in the class of formal power series.*

Proof. Since Φ is analytic, it admits an analytic continuation to a neighborhood of 0 in $\mathbb{C}^m$.

By Proposition 1.5 we have that for any $v \in \mathbb{R}^m$ the set of vectors $(\partial_v^k \Phi(0), \partial_v^k \langle \Phi, \Phi\rangle(0))_{k\in\mathbb{N}}$ spans at most 2-dimensional subspace of $\mathbb{R}^{n+1}$. But this is an algebraic condition on the coefficients of Φ. Thus it holds for all $v \in \mathbb{C}^m$. Using Proposition 1.5 again we conclude that the image of the germ at 0 of any complex vector line from $\mathbb{C}^m$ lies in some complex circle.

Suppose that a point $x \in \mathbb{C}^m$ is such that $\langle A(x), A(x)\rangle = 0$. Denote by Φ_k the power series of Φ at 0 truncated at degree k. Note that the linear span of the Φ-image of the line spanned by x contains all $\Phi_k(x)$. Then by Proposition 1.2 the linear span of $\Phi_k(x)$ and $A(x)$ belongs entirely to $\mathcal{N}$ for any k. Thus

$$\langle A(x), \Phi_k(x)\rangle = \langle \Phi_k(x), \Phi_k(x)\rangle = 0.$$

This holds for all x satisfying the condition $\langle A(x), A(x)\rangle = 0$. Since A has rank at least 2, this condition defines an irreducible hypersurface Γ. Any polynomial vanishing on Γ is divisible by $\langle A, A\rangle$, the equation of Γ. Therefore for any k the polynomials $\langle A, \Phi_k\rangle$ and $\langle \Phi_k, \Phi_k\rangle$ are divisible by $\langle A, A\rangle$. The theorem now follows. □

Theorem 2.1 has 2 important corollaries:

Corollary 2.2 *Let Φ be a rounding. Denote by A its linear part and by B its quadratic part. Then both polynomials $\langle A, B\rangle$ and $\langle B, B\rangle$ are divisible by $\langle A, A\rangle$.*

Corollary 2.3 *Let Φ be a rounding with the first differential A. Then the kernel of A maps to 0 under Φ.*

Now we are going to establish a criterion of the equivalence of 2 roundings.

Lemma 2.4 *If 2 roundings have the same 2-jets then they are equivalent.*

Proof. Let $\Phi, \Psi : (\mathbb{R}^m, 0) \to (\mathbb{R}^n, 0)$ be analytic roundings with the same 2-jet $A+B$ where A is the linear part and B is the quadratic part. It is clear that if a line from $\mathbb{R}^m$ does not belong to the kernel of A then it goes to the same circle under both maps. Indeed, a circle is determined by its velocity and acceleration at 0 with respect to any parameterization such that the velocity at 0 does not vanish. Corollary 2.3 concludes the proof. $\Box$

We need the following technical fact:

Lemma 2.5 *Consider a linear map $A : \mathbb{R}^m \to \mathbb{R}^n$ of rank at least 2 and a quadratic (resp., linear) homogeneous map $B : \mathbb{R}^m \to \mathbb{R}^n$ such that B is everywhere parallel to A. Then $B = l \cdot A$ for some linear (resp., constant) function $l : \mathbb{R}^m \to \mathbb{R}$.*

Proof. We will work out only the case when B is quadratic. The linear case is only easier. The function $l = B/A$ is defined on the complement to the kernel of A. The parallelogram equality for B reads

$$l(x+y)A(x+y) + l(x-y)A(x-y) = 2(l(x)A(x) + l(y)A(y))$$

where x and y are vectors from $\mathbb{R}^m$ such that none of the vectors x, y, $x+y$ and $x-y$ lies in the kernel of A.

Suppose that $A(x)$ and $A(y)$ are linearly independent. Equating the coefficients with $A(x)$ in the parallelogram equality we obtain $l(x+y) + l(x-y) = 2l(x)$. Put $u = x+y$ and $v = x-y$. Then $l(u+v) = l(u) + l(v)$. This holds for almost all u and v. Therefore l extends to a linear function. $\Box$

Proposition 2.6 *Roundings $\Phi = A+B+\cdots$ and $\Phi' = A'+B'+\cdots$ are equivalent if and only if $A' = \lambda A$ and $B' = \lambda^2 B + lA$ where λ is a real number and l is a linear functional.*

Proof. First suppose that A' and B' are related with A and B as above. Composing Φ with the local diffeomorphism $x \mapsto \lambda x + l(x)x$ which preserves all germs of vector lines we obtain a rounding equivalent to Φ and having the 2-jet $A' + B'$. By Lemma 2.4 this rounding is equivalent to Φ'. Thus Φ and Φ' are equivalent.

Now suppose that roundings Φ and Φ' map the same lines to the same circles. The same circles have the same tangent vectors, hence $A' || A$ everywhere. By Lemma 2.5 we have $A' = \lambda A$ for a real constant λ. The same circles have the same centers. Hence the orthogonal projection of B' to the orthogonal complement of A is the same as that of $\lambda^2 B$. Thus the quadratic map $B' - \lambda^2 B$ is everywhere parallel to A. The Proposition now follows from Lemma 2.5. $\Box$

3 Fractional quadratic maps

Let F be a quadratic map from $\mathbb{R}^m$ to $\mathbb{R}^n$ and Q a quadratic function on $\mathbb{R}^m$ (both are not necessarily homogeneous). The map F/Q is called *a fractional quadratic map*. It is defined on the complement to the zero level of Q. Nonetheless, it will be referred to as a fractional quadratic map from $\mathbb{R}^m$ to $\mathbb{R}^n$.

A map from an open subset U of $\mathbb{R}^m$ to $\mathbb{R}^n$ is said to take all lines to circles if any germ of line contained in U goes to a germ of circle under this map.

Proposition 3.1 *Let $F : \mathbb{R}^m \to \mathbb{R}^n$ be a quadratic map such that the polynomial $\langle F, F\rangle$ is divisible by some quadratic function Q. Then the fractional quadratic map F/Q takes all lines to circles.*

PROOF. Introduce an orthonormal basis in $\mathbb{R}^n$. Denote the components of F with respect to this basis by $F_1, \ldots, F_n$.

Take an arbitrary line L from $\mathbb{R}^m$ not lying in the zero level of Q. The functions $F_1, F_2, \ldots, F_n$ and Q restricted to L span a subspace V of the space of all quadratic polynomials on L. It is easily seen that V is naturally isomorphic to the space of affine functions on the image of L under F/Q.

If the subspace V is one-dimensional, then $(F/Q)(L)$ is just a point. If V is 2-dimensional, then $(F/Q)(L)$ belongs to a line. Finally, if V is 3-dimensional (i.e. it contains all quadratic polynomials on L) then $(F/Q)(L)$ is a plane curve.

Consider the last case. The ratio $\langle F, F\rangle/Q$ is quadratic. Its restriction to L belongs to V. Therefore $\langle F, F\rangle/Q$ is a linear combination of functions F_i and Q on L, and $\langle F, F\rangle/Q^2$ is a linear combination of F_i/Q and 1 on L. This means that on the image of L the square of the Euclidean norm equals to some affine function. Hence the image of L lies in some sphere. A plane curve which lies in a sphere is necessarily a circle. $\Box$

The main result of this Section is a strong converse statement to Theorem 2.1:

Theorem 3.2 *Let A be a linear homogeneous map and B a quadratic homogeneous map from $\mathbb{R}^m$ to $\mathbb{R}^n$ such that both polynomials $\langle A, B\rangle$ and $\langle B, B\rangle$ are divisible by $\langle A, A\rangle$. Then the map*

$$\Psi = \frac{A + B - 2pA}{1 - 2p + q}, \qquad p = \frac{\langle A, B\rangle}{\langle A, A\rangle},\ q = \frac{\langle B, B\rangle}{\langle A, A\rangle},$$

is a fractional quadratic rounding whose 2-jet at 0 is $A + B$. Moreover, Ψ takes all lines to circles, not only those passing through 0.

PROOF. By our assumptions, p and q are polynomials, the first is linear and the second is quadratic.

It is readily seen that $\Psi = A + B$ plus higher order terms. We also claim that Ψ rounds all germs of lines. Indeed, this follows from Proposition 3.1 since the square of the norm of $A + (B - 2pA)$ equals to $(1 - 2p + q)\langle A, A\rangle$. $\square$

Corollary 3.3 *Any rounding* $\Phi = A + B + \cdots$ *is equivalent to the fractional quadratic rounding of the form*

$$\frac{A + B - 2pA}{1 - 2p + q}$$

where $p = \langle A, B\rangle / \langle A, A\rangle$ *and* $q = \langle B, B\rangle / \langle A, A\rangle$.

This follows from Theorem 3.2 and Lemma 2.4.

A rounding $\Phi = A + B + \cdots$ is said to be *degenerate* if there is a point $x_0 \in \mathbb{R}^m$ such that $A(x_0) = 0$ and $q(x_0) = p^2(x_0)$ where p and q are as above. This definition may look artificial but it is explained by the following

Lemma 3.4 *A rounding which is equivalent to a degenerate rounding is itself degenerate. Any degenerate rounding* $\Phi = A + B + \cdots$ *is equivalent to a rounding* $\Phi' = A' + B' + \cdots$ *such that the kernels of* A' *and* B' *intersect nontrivially.*

PROOF. Let $\Phi = A + B + \cdots$ be a degenerate rounding. Assume that a rounding $\Phi' = A' + B' + \cdots$ is equivalent to Φ. Then by Lemma 2.6 we have $A' = \lambda A$ and $B' = \lambda^2 B + lA$ where λ is a number and l is a linear function. The polynomials p and q of these 2 roundings are related as follows:

$$p' = \lambda p + \frac{l}{\lambda}, \quad q' = \lambda^2 q + 2lp + \frac{l^2}{\lambda^2}.$$

If in some point $A = 0$ then $A' = 0$ in the same point. If in some point $q = p^2$ then $q' = p'^2$. This proves the first part of the Lemma.

To prove the second part choose $\lambda = 1$, $l = -p$. Then $p' = 0$, $q' = q - p^2$. Now assume that in a point $x_0 \in \mathbb{R}^m$ we have $A(x_0) = 0$ and $q(x_0) = p^2(x_0)$. Then $q'(x_0) = 0$, hence x_0 is a zero of B' of order greater than 1, so it lies in the kernel of B'. Thus the point x_0 lies in the kernels of both A' and B'. $\square$

We may concentrate on nondegenerate roundings only, due to the following

Proposition 3.5 *Let $\Phi : (\mathbb{R}^m, 0) \to (\mathbb{R}^n, 0)$ be a degenerate rounding. Then there is a projection π from $\mathbb{R}^m$ to a smaller space $\mathbb{R}^k$, $k < m$, and a nondegenerate fractional quadratic rounding rounding $\Psi : (\mathbb{R}^k, 0) \to (\mathbb{R}^n, 0)$ such that $\Psi \circ \pi$ is equivalent to Φ.*

PROOF. By Lemma 3.4 we can assume that the intersection K of the kernels of A and B is a nontrivial subspace of $\mathbb{R}^m$. Denote by π the natural projection of $\mathbb{R}^m$ to the quotient $\mathbb{R}^k = \mathbb{R}^m/K$. Then $A = \tilde{A} \circ \pi$ and $B = \tilde{B} \circ \pi$ where the maps $\tilde{A}$ and $\tilde{B}$ from $\mathbb{R}^k$ to $\mathbb{R}^n$ are linear and quadratic, respectively. Both polynomials $\langle \tilde{A}, \tilde{B} \rangle$ and $\langle \tilde{B}, \tilde{B} \rangle$ are divisible by $\langle \tilde{A}, \tilde{A} \rangle$.

By Theorem 3.2 there is a fractional quadratic rounding $\Psi : (\mathbb{R}^k, 0) \to (\mathbb{R}^n, 0)$ with the 2-jet $\tilde{A} + \tilde{B}$. Hence $\Psi \circ \pi$ is a fractional quadratic rounding with the 2-jet $A + B$. By Lemma 2.4 it is equivalent to Φ. $\square$

4 Hurwitz multiplications

Hurwitz in 1898 posed the following problem which is still unsolved: find all relations of the form

$$(x_1^2 + \cdots + x_r^2)(y_1^2 + \cdots + y_s^2) = z_1^2 + \cdots + z_n^2$$

where $z_1, \ldots, z_n$ are bilinear functions of the indeterminates $x_1, \ldots, x_r$ and $y_1, \ldots, y_s$. A relation above is represented by a bilinear map $f : \mathbb{R}^r \times \mathbb{R}^s \to \mathbb{R}^n$ such that for all $x \in \mathbb{R}^r$ and $y \in \mathbb{R}^s$ we have $|f(x, y)| = |x| \cdot |y|$. Such maps are called *normed pairings* or *Hurwitz multiplications* of size $[r, s, n]$.

Many examples of Hurwitz multiplications are known. The most familiar are the multiplications of real numbers, complex numbers, quaternions and octonions. These are Hurwitz multiplications of sizes $[1, 1, 1]$, $[2, 2, 2]$, $[4, 4, 4]$ and $[8, 8, 8]$, respectively. As Hurwitz proved in 1898 [7], these are the only possible Hurwitz multiplications of size $[n, n, n]$ up to orthogonal transformations. Later on, he managed to describe all normed pairings of size $[r, n, n]$, see [8]. The same result was independently obtained by Radon [9].

Recall that the Clifford algebra $\mathrm{Cliff}(k)$ is the associative algebra generated over reals by r elements $e_1, \ldots, e_k$ satisfying the relations

$$e_i^2 = -1, \qquad e_i e_j + e_j e_i = 0 \quad (i \neq j).$$

A linear representation of $\mathrm{Cliff}(k)$ in a Euclidean space $\mathbb{R}^n$ is called *compatible with the Euclidean structure* if all generators e_i act as orthogonal operators. Any finite dimensional representation of a Clifford algebra is compatible with a suitable Euclidean structure on the space of representation. The result of Hurwitz and Radon is as follows:

Theorem 4.1 *Suppose that $f : \mathbb{R}^r \times \mathbb{R}^n \to \mathbb{R}^n$ is a Hurwitz multiplication. Then there is a representation ϕ of* Cliff$(r-1)$ *in $\mathbb{R}^n$ compatible with the Euclidean structure and such that*

$$f(x, y) = \phi(x_0 + x_1 e_1 + \cdots + x_{r-1} e_{r-1}) A(y)$$

where A is a linear conformal transformation and $x_0, \ldots, x_{r-1}$ are coordinates of x in some orthonormal basis.

The largest r for which there is a representation of Cliff$(r-1)$ in $\mathbb{R}^n$ is denoted by $\rho(n)$ and is called the *Hurwitz–Radon function* of n. The Hurwitz–Radon theorem implies that for any normed pairing of size $[r, n, n]$ we have $r \leqslant \rho(n)$. The Hurwitz–Radon function may be computed explicitly due to the result of É. Cartan [10] who classified all Clifford algebras and their representations in 1908. See also [11]. Let $n = 2^s u$ where u is odd. If $s = 4a + b$, $0 \leqslant b \leqslant 3$, then $\rho(n) = 8a + 2^b$.

Proposition 4.2 *Let $f : \mathbb{R}^r \times \mathbb{R}^s \to \mathbb{R}^n$ be a Hurwitz multiplication. Denote by Q the quadratic form on $\mathbb{R}^m = \mathbb{R}^r \oplus \mathbb{R}^n$ obtained as the composition of the projection to $\mathbb{R}^r$ and the Euclidean form on $\mathbb{R}^r$. Then $\langle f, f \rangle$ is divisible by Q, hence f/Q takes all lines to circles.*

This Proposition follows immediately from definitions and Proposition 3.1. Thus a complete description of all fractional quadratic maps taking lines to circles should contain a solution to the Hurwitz general problem.

A bilinear map $f : \mathbb{R}^r \times \mathbb{R}^s \to \mathbb{R}^n$ is called *nonsingular* if from $f(x, y) = 0$ it follows that $x = 0$ or $y = 0$. Clearly, any Hurwitz multiplication is a nonsingular bilinear map. Possible sizes of nonsingular bilinear maps are restricted by the following theorem due to Stiefel [12] and Hopf [13].

Theorem 4.3 *If there is a nonsingular bilinear map of size $[r, s, n]$ then the binomial coefficient $\binom{n}{k}$ is even whenever $n - r < k < s$.*

This is a topological theorem. It uses the ring structure in the cohomology of projective spaces.

5 Quadratic maps between spheres

By a *Euclidean sphere* S^n we mean the set of all vectors in $\mathbb{R}^{n+1}$ with unit Euclidean length. A map $f : S^m \to S^n$ between Euclidean spheres is called *quadratic* if it extends to a quadratic homogeneous map from $\mathbb{R}^{m+1}$ to $\mathbb{R}^{n+1}$. This extension must satisfy the condition $\langle f(x), f(x)\rangle = \langle x, x\rangle^2$. By a *great circle* in S^m we mean a circle obtained as the intersection of S^m with a vector 2-plane. The following simple but very important statement is proved in [5]:

Proposition 5.1 *Any quadratic map $f : S^m \to S^n$ takes great circles to circles.*

The next proposition is verified by a simple direct computation:

Proposition 5.2 *Consider a homogeneous quadratic map $F : \mathbb{R}^{m+1} \to \mathbb{R}^n$ such that $\langle F, F\rangle = Q_1 \cdot Q_2$ where Q_1 and Q_2 are quadratic forms. Suppose that the kernels of Q_1 and Q_2 intersect trivially. If we endow $\mathbb{R}^{m+1}$ with the Euclidean form $Q_1 + Q_2$ then the quadratic map $(2F, Q_1 - Q_2)$ takes the unit sphere $S^m \subset \mathbb{R}^{m+1}$ to the unit sphere $S^n \subset \mathbb{R}^{n+1}$.*

The most examples of quadratic maps between spheres come from Hurwitz multiplications. Consider a Hurwitz multiplication $f : \mathbb{R}^r \times \mathbb{R}^s \to \mathbb{R}^n$. Then the *Hopf map*

$$H_f(x, y) = (2f(x, y), \langle x, x\rangle - \langle y, y\rangle)$$

takes S^{r+s-1} to S^n.

Yiu [6] described all pairs of positive integers m, n such that there is a non-constant quadratic map from S^m to S^n. Namely, $n \geq \kappa(m)$ where the Yiu function κ is defined recurrently as follows:

$$\kappa(2^t + m) = \begin{cases} 2^t, & 0 \leqslant m < \rho(2^t) \\ 2^t + \kappa(m), & \rho(2^t) \leqslant m < 2^t \end{cases}$$

Let $f : S^m \to S^n$ be any map between Euclidean spheres. Denote by ϕ a map from an open subset U of $\mathbb{R}^m$ to $\mathbb{R}^n$ obtained as the composition of

- an affine embedding of $U \subseteq \mathbb{R}^m$ into $\mathbb{R}^{m+1}$,
- the central projection to $S^m \subset \mathbb{R}^{m+1}$,
- the map $f : S^m \to S^n$,

- a stereographic projection of S^n to some hyperplane H in $\mathbb{R}^{n+1}$,
- a Euclidean identification of H with $\mathbb{R}^n$.

Then we say that ϕ *goes through* f.

Theorem 5.3 *Any nondegenerate rounding $\Phi : (\mathbb{R}^m, 0) \to (\mathbb{R}^n, 0)$ has an equivalent rounding that goes through a quadratic map between spheres S^m and S^n.*

Proof. By Corollary 3.3 there is a rounding Ψ equivalent to Φ that extends to a fractional quadratic map F/Q where F is a quadratic map, Q is a quadratic form and $\langle F, F\rangle$ is divisible by Q. Extend $\mathbb{R}^m$ to $\mathbb{R}^{m+1}$ by adding an extra coordinate t. Let $\tilde{F}$ and $\tilde{Q}$ be homogeneous quadratic map and homogeneous quadratic form, respectively, that restrict to F and Q on the hyperplane $t = 1$.

Consider the map $\tilde{F}/\tilde{Q} : \mathbb{R}^{m+1} \to \mathbb{R}^n$. By Proposition 3.1 it sends all lines to circles. Now compose this map with the inverse stereographic projection from $\mathbb{R}^n$ to the unit sphere in $\mathbb{R}^{n+1}$. We obtain the map

$$f' : \mathbb{R}^{m+1} \to \mathbb{R}^{n+1}, \quad f' = \left(\frac{2\tilde{F}}{\tilde{Q}_1 + \tilde{Q}_2}, \frac{\tilde{Q}_1 - \tilde{Q}_2}{\tilde{Q}_1 + \tilde{Q}_2} \right), \quad f'(\mathbb{R}^{m+1}) \subseteq S^n$$

where $\tilde{Q}_1$ and $\tilde{Q}_2$ are quadratic forms such that $\tilde{F} = \tilde{Q}_1\tilde{Q}_2$.

If $\tilde{Q}_1 + \tilde{Q}_2$ is nondegenerate then it is a Euclidean form. Introduce the Euclidean structure on $\mathbb{R}^{m+1}$ by means of this form. Then the quadratic map $f = (2\tilde{F}, \tilde{Q}_1 - \tilde{Q}_2)$ from Proposition 5.2 takes the unit sphere S^m in $\mathbb{R}^{m+1}$ to the unit sphere S^n in $\mathbb{R}^{n+1}$ and coincides with f' on S^m. Thus Ψ goes through f.

It remains to verify that $\tilde{Q}_1 + \tilde{Q}_2$ is indeed nondegenerate. Using the explicit construction of F and Q we can write

$$\tilde{Q}_1 + \tilde{Q}_2 = t^2 - 2pt + q + \langle A, A\rangle = (q - p^2) + (p - t)^2 + \langle A, A\rangle.$$

Here $A + B$ is the 2-jet of Φ (and of Ψ), $p = \langle A, B\rangle/\langle A, A\rangle$ and $q = \langle B, B\rangle/\langle A, A\rangle$. If in some point $x_0 \in \mathbb{R}^m$ the form $\tilde{Q}_1 + \tilde{Q}_2$ vanishes then $A(x_0) = 0$, $q(x_0) = p^2(x_0)$, and hence Ψ is degenerate. Contradiction. $\square$

In [15] we found a simple condition on a rounding which guarantees that it goes through the Hopf map associated with a Clifford algebra representation.

Theorem 5.3 combined with results of Yiu leads to the following

Theorem 5.4 *There exists a nondegenerate rounding $\Phi : (\mathbb{R}^m, 0) \to (\mathbb{R}^n, 0)$ if and only if $n \geq \kappa(m)$ where κ is the Yiu function.*

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