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Heat operator with pure soliton potential: Properties of Jost and dual Jost solutions

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Properties of Jost and dual Jost solutions of the heat equation, $\Phi(x,\mathbf{k})$ and $\Psi(x,\mathbf{k})$, in the case of a pure solitonic potential are studied in detail. We describe their analytical properties on the spectral parameter $\mathbf{k}$ and their asymptotic behavior on the x-plane and we show that the values of $e^{-qx}\Phi(x,\mathbf{k})$ and the residues of $e^{qx}\Psi(x,\mathbf{k})$ at special discrete values of $\mathbf{k}$ are bounded functions of x in a polygonal region of the q-plane. Correspondingly, we deduce that the extended version $L(q)$ of the heat operator with a pure solitonic potential has left and right annihilators for q belonging to these polygonal regions. © *2011 American Institute of Physics*. [doi:10.1063/1.3621715]

I. INTRODUCTION

The Kadomtsev–Petviashvili equation in its version called KPII,

$$(u_t - 6uu_{x_1} + u_{x_1x_1x_1})_{x_1} = -3u_{x_2x_2}, \tag{1.1}$$

where $u = u(x,t)$, $x = (x_1, x_2)$, and the subscripts x_1, x_2, and t denote partial derivatives, is a $(2+1)$-dimensional generalization of the celebrated Korteweg–de Vries (KdV) equation. There are two nonequivalent versions of the KP equations, corresponding to the two choices $\pm$ for the sign in the rhs of (1.1), which are, respectively, referred to as KPI and KPII. The KP equations, originally derived as a model for small-amplitude, long-wavelength, weakly two-dimensional waves in a weakly dispersive medium,[1] have been known to be integrable since the beginning of the 1970s[2,3] and can be considered prototypical $(2+1)$-dimensional integrable equations.

The KPII equation is integrable via its association with the operator

$$\mathcal{L}(x,\partial_x) = -\partial_{x_2} + \partial_{x_1}^2 - u(x), \tag{1.2}$$

which defines the well-known equation of heat conduction, or heat equation for short, since it can be expressed as the compatibility condition $[\mathcal{L},\mathcal{T}] = 0$ of the Lax pair $\mathcal{L}$ and $\mathcal{T}$, where $\mathcal{T}$ is given by

$$\mathcal{T}(x,\partial_x,\partial_t) = \partial_t + 4\partial_{x_1}^3 - 6u\partial_{x_1} - 3u_{x_1} - 3\partial_{x_1}^{-1}u_{x_2}. \tag{1.3}$$

The spectral theory of the operator (1.2) was developed in Refs. 4–7 in the case of a real potential $u(x)$ rapidly decaying at spatial infinity, which, however, is not the most interesting case, since the KPII equation was just proposed in Ref. 1 in order to deal with a two-dimensional weak transverse perturbation of the one-soliton solution of the KdV. In fact, if $u_1(t,x_1)$ obeys KdV, then $u(t,x_1,x_2) = u_1(t,x_1+\mu x_2 - 3\mu^2 t)$ solves KPII for an arbitrary constant $\mu\in\mathbb{R}$. In particular, KPII admits a one-soliton solution of the form

$$u(x,t) = -\frac{(\kappa_1-\kappa_2)^2}{2}\operatorname{sech}^2\left[\frac{\kappa_1-\kappa_2}{2}x_1 + \frac{\kappa_1^2-\kappa_2^2}{2}x_2 - 2(\kappa_1^3-\kappa_2^3)t\right], \tag{1.4}$$

where κ_1 and κ_2 are arbitrary real constants.

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A spectral theory of the heat operator (1.2) that also includes solitons has to be built. In the case of a potential $u(x) \equiv u_0(x)$ rapidly decaying at spatial infinity, according to Refs. 4–7, the main tools in building the spectral theory of the operator (3.1), as in the one-dimensional case, are the integral equations whose solutions define the related Jost solutions. However, if the potential $u(x)$ does not decay at spatial infinity, as is the case when line-soliton solutions are considered, the integral equations of the decaying case are ill-defined, and one needs a more general approach. In solving the analogous problem for the nonstationary Schrödinger operator, associated with the KPI equation, the extended resolvent approach was introduced.[8] Accordingly, a spectral theory of the KPII equation that also includes solitons has to be investigated using the resolvent approach. In this framework it was possible to develop the inverse scattering transform for a solution describing one soliton on a generic background[9] and to study the existence of the (extended) resolvent for (some) multisoliton solutions.[10]

However, the general case of N-solitons is still open. Following,[8] the first step in building the inverse scattering for this case lies in building a Green's function $G(x, x', \mathbf{k})$ of the heat operator (1.2) corresponding to the pure N-soliton potential $u(x)$, such that

$$e^{i\,\mathbf{k}(x_1'-x_1)+\mathbf{k}^2(x_2'-x_2)}G(x, x', \mathbf{k}) \tag{1.5}$$

is bounded for any x and any $\mathbf{k} \in \mathbb{C}$. We expect this Green's function to be bilinear in the Jost solutions $\Phi(x, \mathbf{k})$ and $\Psi(x, \mathbf{k})$ of the heat operator and its dual and we, therefore, first need to study the general properties of these solutions. Once this study is performed and the Green's function G is obtained, the Jost solution $\widetilde{\Phi}$ for a potential given as a sum of two terms,

$$\widetilde{u}(x) = u(x) + u'(x), \tag{1.6}$$

where $u(x)$ is a pure N-soliton potential and $u'(x)$ is an arbitrary bidimensional decaying smooth perturbation can be derived as a solution of the integral equation

$$\widetilde{\Phi} = \Phi + Gu'\widetilde{\Phi}, \tag{1.7}$$

which is well defined since it has a well behaving kernel Gu' thanks to u'. Analogously, for the Jost solution $\widetilde{\Psi}$ of the dual heat operator. Then, having these building blocks, $\widetilde{\Phi}$ and $\widetilde{\Psi}$, at hand, one can assemble the inverse scattering theory.

The paper in organized as follows. In Sec. II, we consider the multisoliton potential obtained in its general form in Ref. 11 and showing N_b "incoming" rays and N_a "outgoing" rays at large spaces. We reformulate this potential in terms of τ-functions as in the review paper[12] and in Ref. 13. In Sec. III, we consider the Jost solutions of the heat operator and its dual, already obtained in Ref. 11, but they are also expressed in terms of τ-functions as in Ref. 13. In view of getting the asymptotic behavior of the Jost solutions, the asymptotic behavior at large x of the multisoliton potential, studied in detail in Ref. 13, is reviewed in Sec. II, showing that the round angle at the origin can be divided into $\mathcal{N} = N_a + N_b$ angular sectors, such that the potential on their bordering rays has a constant soliton-like behavior, while the potential along directions inside the sectors has an exponentially decaying behavior, which is explicitly given. In Sec. IV by using the results obtained for the potential, the asymptotic behavior at large x of the Jost and dual Jost solutions and their values or, accordingly, residues at the discrete points of the spectrum are also obtained. In Sec. V by using the results obtained in Sec. IV, we show that the values of $e^{-qx}\Phi(x, \mathbf{k})$ and the residues of $e^{qx}\Psi(x, \mathbf{k})$ at special discrete values of $\mathbf{k}$ are bounded functions of x in a polygonal region of the q-plane included inside the parabola $q_2 = q_1^2$. Correspondingly, we conclude that the extended version $L(q)$ of the heat operator with a pure solitonic potential has left and right annihilators for q belonging to these polygonal regions. It follows that the extended resolvent $M(q)$, inverse to $L(q)$, does not exist for q belonging to these polygons. However, the Green's function $G(x, x', \mathbf{k})$, needed, as indicated above, for building the inverse scattering theory in the case of KPII solutions with solitons, can be obtained by a reduction procedure involving q values outside the parabola.

II. MULTISOLITON POTENTIALS AND THEIR PROPERTIES

A. Main notation

Soliton potentials are labeled by the two numbers (topological charges), N_a and N_b, that obey the condition

$$N_a, N_b \geq 1. \tag{2.1}$$

Let

$$\mathcal{N} = N_a + N_b, \tag{2.2}$$

so that $\mathcal{N} \geq 2$. We introduce the $\mathcal{N}$ real parameters

$$\kappa_1 < \kappa_2 < \cdots < \kappa_{\mathcal{N}}, \tag{2.3}$$

and the functions

$$K_n(x) = \kappa_n x_1 + \kappa_n^2 x_2, \quad n = 1, \ldots, \mathcal{N}. \tag{2.4}$$

Let

$$e^{K(x)} = \operatorname{diag}\{e^{K_n(x)}\}_{n=1}^{\mathcal{N}} \tag{2.5}$$

be a diagonal $\mathcal{N} \times \mathcal{N}$ matrix, let $\mathcal{D}$ be a $\mathcal{N} \times N_b$ constant matrix with at least two nonzero maximal minors, and let $\mathcal{V}$ be an "incomplete Vandermonde matrix," i.e., the $N_b \times \mathcal{N}$ matrix

$$\mathcal{V} = \begin{pmatrix} 1 & \ldots & 1 \\ \vdots & & \vdots \\ \kappa_1^{N_b-1} & \ldots & \kappa_{\mathcal{N}}^{N_b-1} \end{pmatrix}. \tag{2.6}$$

Then, the soliton potential is given by

$$u(x) = -2\partial_{x_1}^2 \log \tau(x), \tag{2.7}$$

where the τ-function can be expressed as

$$\tau(x) = \det\big(\mathcal{V} e^{K(x)} \mathcal{D}\big). \tag{2.8}$$

See the review paper,[12] the references therein and Ref. 13, where the same notation has been used.

There exists a dual representation for the potential in terms of the τ-function (see Refs. 10, 13, and 14),

$$\tau'(x) = \det\left(\mathcal{D}' e^{-K(x)} \gamma \mathcal{V}'\right), \tag{2.9}$$

where $\mathcal{D}'$ is a constant $N_a \times \mathcal{N}$ matrix that, such as the matrix $\mathcal{D}$, has at least two nonzero maximal minors and that is orthogonal to the matrix $\mathcal{D}$ in the sense that

$$\mathcal{D}'\mathcal{D} = 0, \tag{2.10}$$

where the zero in the rhs is a $N_a \times N_b$ matrix, $\mathcal{V}'$ is the $\mathcal{N} \times N_a$ matrix

$$\mathcal{V}' = \begin{pmatrix} 1 & \ldots & \kappa_1^{N_a-1} \\ \vdots & & \vdots \\ 1 & \ldots & \kappa_{\mathcal{N}}^{N_a-1} \end{pmatrix}, \tag{2.11}$$

and γ is the constant, diagonal, real $\mathcal{N} \times \mathcal{N}$ matrix

$$\gamma = \operatorname{diag}\{\gamma_n\}_{n=1}^{\mathcal{N}}, \qquad \gamma_n = \prod_{\substack{n'=1 \\ n' \neq n}}^{\mathcal{N}} (\kappa_n - \kappa_{n'})^{-1}. \tag{2.12}$$

In order to study the properties of the potential and the Jost solutions, it is convenient to use an explicit representation for the determinants. By using the Binet–Cauchy formula for the determinant of a product of matrices, we get

$$\tau(x)=\sum_{1\le n_1<n_2<\cdots<n_{N_b}\le\mathcal{N}} f_{n_1,\dots,n_{N_b}}\prod_{l=1}^{N_b}e^{K_{n_l}(x)} \tag{2.13}$$

with

$$f_{n_1,n_2,\dots,n_{N_b}}=V(\kappa_{n_1},\dots,\kappa_{n_{N_b}})\mathcal{D}(n_1,\dots,n_{N_b}), \tag{2.14}$$

where we used the notation

$$V(\kappa_1,\dots,\kappa_{\mathcal{N}})=\prod_{1\le m<n\le\mathcal{N}}(\kappa_n-\kappa_m) \tag{2.15}$$

for the Vandermonde determinant and

$$\mathcal{D}(n_1,\dots,n_{N_b})=\det\begin{pmatrix}\mathcal{D}_{n_1,1} & \dots & \mathcal{D}_{n_1,N_b}\\ \vdots & & \vdots\\ \mathcal{D}_{n_{N_b},1} & \dots & \mathcal{D}_{n_{N_b},N_b}\end{pmatrix}, \tag{2.16}$$

for the maximal minors of the matrix $\mathcal{D}$. Notice that the coefficients $f_{n_1,n_2,\dots,n_{N_b}}$ are invariant under permutations of the indices.

In what follows, it is convenient to consider indices $n, n_1,\dots,n_{N_b}$ of parameters κ_n, coefficients $f_{n_1,n_2,\dots,n_{N_b}}$, etc., to be running mod $\mathcal{N}$ throughout $\mathbb{Z}$, i.e., let

$$n\to n\,(\mathrm{mod}\ \mathcal{N}). \tag{2.17}$$

From (2.13), it follows directly that condition

$$f_{n_1,\dots,n_{N_b}}\ge 0\quad\text{for all } 1\le n_1<n_2<\cdots<n_{N_b}\le\mathcal{N} \tag{2.18}$$

is sufficient (see Ref. 15) for the regularity of the potential $u(x)$, i.e., for the absence of zeros of $\tau(x)$ on the x-plane. Thanks to (2.3), (2.14), and (2.15), this condition is equivalent to the condition that all maximal minors of the matrix $\mathcal{D}$ are non-negative. In Ref. 13, it was mentioned that condition (2.18) is also necessary for the regularity of a potential under evolution with respect to an arbitrary number of higher times of the KP hierarchy. In Ref. 15, it was suggested to decompose soliton solutions of KPII into subclasses, associated with the Schubert cells on Grassmanian. It is proved there that condition (2.18) is necessary for the regularity of the all solutions associated with a cell. However, the problem of finding necessary conditions for the regularity of the multisoliton solution under evolution with respect to KPII only is still open.

Under condition (2.17) and thanks to (2.3), we have the following lemma (see Ref. 16),which we use below.

Lemma 2.1: Let

$$g_{l,m,n}=(\kappa_l-\kappa_m)(\kappa_n-\kappa_{n+N_b})(\kappa_l+\kappa_m-\kappa_n-\kappa_{n+N_b}). \tag{2.19}$$

Then

$$g_{l,m,n}\ge 0\quad\textit{for any}\quad n\in\mathbb{Z},\quad l=n,\dots,n+N_b,\quad m=n+N_b,\dots,\mathcal{N}+n, \tag{2.20}$$

and the equality takes place only when l and m independently take the values n or $n+N_b$ mod $\mathcal{N}$.

Matrices $\mathcal{D}$ and $\mathcal{D}'$ have rather interesting properties (see Refs. 13 and 16). In particular, the products $\mathcal{D}^\dagger\mathcal{D}$ and $\mathcal{D}'\mathcal{D}'^\dagger$, where $\dagger$ denotes Hermitian conjugation of matrices (in fact, transposition here), are positive and then invertible, and so there exist matrices (see Ref. 17),

$$(\mathcal{D})^{(-1)}=(\mathcal{D}^\dagger\mathcal{D})^{-1}\mathcal{D}^\dagger,\qquad (\mathcal{D}')^{(-1)}=\mathcal{D}'^\dagger(\mathcal{D}'\mathcal{D}'^\dagger)^{-1}, \tag{2.21}$$

that are, respectively, the left inverse of the matrix $\mathcal{D}$ and the right inverse of the matrix $\mathcal{D}'$, i.e.,

$$\big(\mathcal{D}\big)^{(-1)}\mathcal{D} = E_{N_b}, \qquad \mathcal{D}'\big(\mathcal{D}'\big)^{(-1)} = E_{N_a}, \tag{2.22}$$

where E_{N_a} and E_{N_b} are the $N_a \times N_a$ and $N_b \times N_b$ identity matrices. Products of these matrices in the opposite order give the real self-adjoint $\mathcal{N} \times \mathcal{N}$ matrices

$$P = \mathcal{D}\big(\mathcal{D}\big)^{(-1)} = \mathcal{D}(\mathcal{D}^{\dagger}\mathcal{D})^{-1}\big(\mathcal{D}\big)^{\dagger}, \tag{2.23}$$

$$P' = \big(\mathcal{D}'\big)^{(-1)}\mathcal{D}' = \big(\mathcal{D}'\big)^{\dagger}(\mathcal{D}'\mathcal{D}'^{\dagger})^{-1}\mathcal{D}', \tag{2.24}$$

which are orthogonal projectors, i.e.,

$$P^2 = P, \qquad (P')^2 = P', \qquad PP' = 0 = P'P, \tag{2.25}$$

and complementary in the sense that

$$P + P' = E_{\mathcal{N}}. \tag{2.26}$$

The matrices $\mathcal{D}$ and $\mathcal{D}'$ are not in one-to-one correspondence with the potential $u(x)$. Indeed, by (2.7), (2.8), and (2.9) one gets that the potential is invariant under the substitutions

$$\mathcal{D} \to \mathcal{D}v, \qquad \mathcal{D}' \to v'\mathcal{D}', \tag{2.27}$$

where v and v' are, respectively, arbitrary constant, nonsingular $N_b \times N_b$ and $N_a \times N_a$ matrices. Thus under condition (2.3), the (N_a, N_b)-soliton potential is parameterized by the point of the Grassmanian $\mathrm{Gr}_{N_b,\mathcal{N}}$ if representation (2.8) is used or by the point of the Grassmanian $\mathrm{Gr}_{N_a,\mathcal{N}}$ if representation (2.9) is used.

B. Asymptotic behavior of the τ-function and potential $u(x)$

Here, we briefly report the main asymptotic properties of the function $\tau(x)$ and potential $u(x)$ at large space (see Ref. 16 for details), which are necessary to know in order to derive the asymptotic properties of the Jost and dual Jost solutions and their discrete values.

The asymptotic of $\tau(x)$ is determined by the interrelations between the functions $K_n(x)$ for different n, that is, due to (2.4), by the differences

$$K_l(x) - K_m(x) = (\kappa_l - \kappa_m)(x_1 + (\kappa_l + \kappa_m)x_2) \qquad \text{for any } l, m \in \mathbb{Z}, \tag{2.28}$$

which are linear with respect to the space variables. Then, the asymptotic behavior must have a sectorial structure on the x-plane, and in order to describe this structure we divide the x-plane into sectors.

We introduce rays r_n on the x-plane given by

$$r_n = \{x : x_1 + (\kappa_{n+N_b} + \kappa_n)x_2 = 0,\ (\kappa_{n+N_b} - \kappa_n)x_2 < 0), \qquad n = 1, \ldots, \mathcal{N}. \tag{2.29}$$

They intersect the corresponding lines $x_2 = \pm 1$ at the points

$$\begin{aligned} &(\kappa_{n+N_b} + \kappa_n, -1) \qquad \text{for } n = 1, \ldots, N_a, \\ &(-\kappa_{n+N_b} - \kappa_n, 1) \qquad \text{for } n = N_a + 1, \ldots, \mathcal{N}. \end{aligned} \tag{2.30}$$

Therefore, as n increases from $n = 1$ to $n = N_a$, the ray rotates anticlockwise in the lower half x-plane, crosses the positive part of the x_1 axis in coming to $n = N_a + 1$ and then rotates anticlockwise in the upper half x-plane up to $n = \mathcal{N}$, and finally crossing the negative part of the x_1 axis for $n = \mathcal{N} + 1$ comes back to the ray $r_1(x)$ thanks to (2.17) (see Fig. 1).

Let us assume that some rays, e.g., r_m and r_n, where for definiteness $m < n$, are parallel. By (2.29), this means that $\kappa_{m+N_b} + \kappa_m = \kappa_{n+N_b} + \kappa_n$, i.e., that $\kappa_n - \kappa_m = \kappa_{m+N_b} - \kappa_{n+N_b}$, where the lhs is positive thanks to (2.3). Then, because of (2.17), it is easy to see that the rhs can be positive only if $1 \le m \le N_a < n \le \mathcal{N}$, and hence by (2.30), the ray r_m is in the bottom half-plane and r_n is in the upper one. We see that for generic values of the κ_n and when $N_a \neq N_b$, the rays in the upper and

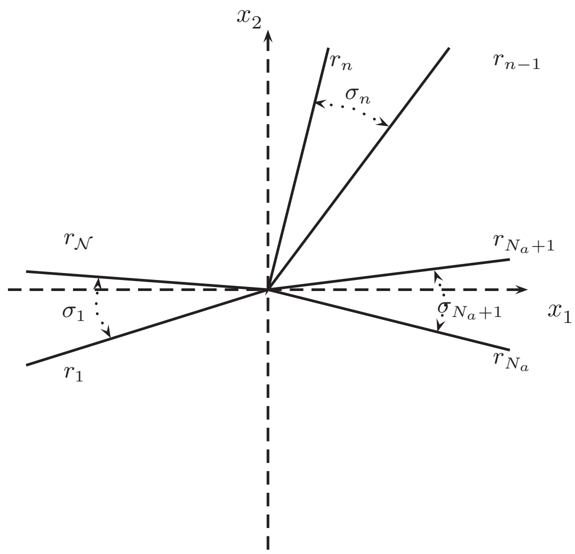

FIG. 1. Sector σ_n corresponds to $N_a + 2 \leq n \leq \mathcal{N}$.

bottom half-planes cannot be parallel, while this is possible for a special choice of the κ_n. In contrast, in the case $N_a = N_b$, all pairs r_n and r_{n+N_a}, $n = 1, \ldots, N_a$, and only these pairs give parallel rays. In the special case $N_a = N_b = 1$, we get two rays producing the straight line $x_1 + (\kappa_1 + \kappa_2)x_2 = 0$ that divides the x-plane into two half-planes.

In the same way, we introduce sectors σ_n, which are subsets of the x-plane characterized as

$$\sigma_n = \{x : K_{n-1}(x) < K_{n+N_b-1}(x) \text{ and } K_n(x) > K_{n+N_b}(x)\} \quad \text{for } n = 1, \ldots, \mathcal{N}. \tag{2.31}$$

From (2.28) and the discussion above, it follows that the σ_n are sharp (for $\mathcal{N} > 2$) angular sectors with vertices at the origin of the coordinates bounded from the right (looking from the origin) by the ray r_{n-1} and from the left (looking from the origin) by the ray r_n and that the sectors σ_n are ordered anticlockwise as n increases, starting "from the left" with the sector σ_1, which includes the negative part of the x_1 axis, and then with the sectors σ_n ($n = 2, \ldots, N_a$) in the bottom half-plane, the sector σ_{N_a+1} "to the right," that includes the positive part of the x_1 axis, and the sectors σ_n ($n = N_a + 2, \ldots, \mathcal{N}$) on the upper half-plane, finishing with the sector $\sigma_{\mathcal{N}}$ adjacent to the sector σ_1, in this way covering the whole x-plane with the exception of the bordering rays r_n (see Fig. 1).

Therefore, the sectors σ_n define an $\mathcal{N}$-fold discretization of the round angle at the origin, with the integer n (mod $\mathcal{N}$) playing the role of a discrete angular variable. It is clear that in the study of the asymptotic behavior of the τ-function we consider the x-plane a vector space, since the finite part of x is irrelevant when $x \to \infty$.

For determining the directions of the rays r_n and sectors σ_n, we introduce the vectors

$$y_n = (\kappa_{n+N_b}^2 - \kappa_n^2, \kappa_n - \kappa_{n+N_b}), \qquad n = 1, \ldots, \mathcal{N}, \tag{2.32}$$

and use them for giving the following definition.

Definition 2.1: We say that $x \to \infty$ along the ray r_n, if $x \to \infty$ and there exists $\alpha \to +\infty$ such that $x - \alpha y_n$ is bounded. This will be denoted by $x \xrightarrow{r_n} \infty$.

We say that $x \to \infty$ in the sector σ_n, $n = 1, \ldots, \mathcal{N}$, if $x \to \infty$ and there exist $\alpha \to +\infty$ and $\beta \to +\infty$, such that $x - \alpha y_{n-1} - \beta y_n$ is bounded. This will be denoted by $x \xrightarrow{\sigma_n} \infty$.

Notice that, for $x \xrightarrow{r_n} \infty$,

$$K_n(x) - K_{n+N_b}(x) \quad \text{is bounded and} \qquad (\kappa_{n+N_b} - \kappa_n)x_2 \to -\infty, \tag{2.33}$$

and, for $x \xrightarrow{\sigma_n} \infty$,

$$K_{n+N_b-1}(x) - K_{n-1}(x) \to +\infty, \qquad K_n(x) - K_{n+N_b}(x) \to +\infty, \tag{2.34}$$

as follows directly from the definition. In fact, we have a more general statement.

Lemma 2.2: Let $N_a, N_b \geq 1$ and $n \in \mathbb{Z}$ be arbitrary. Then

1. *if $x \xrightarrow{r_n} \infty$, we have that $K_l(x) - K_m(x) \to +\infty$ or is bounded for any $l = n, \ldots, n + N_b$ and $m = n + N_b, \ldots, \mathcal{N} + n$, where the boundedness takes place if and only if $(l, m) = (n, n)$, $(n, n + N_b)$, $(n + N_b, n)$, $(n + N_b, n + N_b)$;*
2. *if $x \xrightarrow{\sigma_n} \infty$, we have that $K_l(x) - K_m(x) \to +\infty$ for any $l = n, \ldots, n + N_b - 1$ and $m = n + N_b, \ldots, \mathcal{N} + n - 1$;*

where summation of indices is always understood mod $\mathcal{N}$.

Thanks to Lemma 2.2 above, the asymptotics of the τ-function is given by the following Theorem (see Ref. 16).

Theorem 2.3: *If condition (2.18) is satisfied, the asymptotic expansion of $\tau(x)$ for $x \to \infty$ for any $n \in \mathbb{Z}$ is given by*

$$x \xrightarrow{r_n} \infty: \quad \tau(x) = \left(z_n + z_{n+1} e^{K_{N_b+n}(x) - K_n(x)} + o(1)\right) \exp\left(\sum_{j=n}^{n+N_b-1} K_j(x)\right), \tag{2.35}$$

$$x \xrightarrow{\sigma_n} \infty: \quad \tau(x) = (z_n + o(1)) \exp\left(\sum_{l=n}^{n+N_b-1} K_l(x)\right), \tag{2.36}$$

where the notation

$$z_n = f_{n,n+1,\ldots,n+N_b-1} \equiv V(\kappa_n, \ldots, \kappa_{n+N_b-1}) \mathcal{D}(n, \ldots, n + N_b - 1) \tag{2.37}$$

was introduced for any $n = 1, \ldots, \mathcal{N}$.

As we see from (2.35) and (2.36), the leading asymptotic term along the ray direction r_n is given by the sum of the leading terms obtained for $x \xrightarrow{\sigma_n} \infty$ and $x \xrightarrow{\sigma_{n+1}} \infty$. Since the exponential factor cancels out when this expansion (2.35) of $\tau(x)$ is inserted in (2.7), the factor in parenthesis gives the ray behavior of the potential $u(x)$ at infinity. Explicitly, taking (2.17) into account, we get N_a asymptotic rays in the bottom half-plane: $x_2 \to -\infty$, $x_1 + (\kappa_n + \kappa_{n+N_b})x_2$ bounded. Along these rays, the potential behaves as

$$u(x) = -2\partial_{x_1}^2 \log\left(z_n + z_{n+1} e^{K_{n+N_b}(x) - K_n(x)}\right), \quad n = 1, \ldots, N_a. \tag{2.38}$$

In the same way, the potential $u(x)$ has N_b asymptotic rays in the upper half-plane: $x_2 \to +\infty$, $x_1 + (\kappa_n + \kappa_{n+N_a})x_2$ bounded. Along these rays, it behaves as

$$u(x) = -2\partial_{x_1}^2 \log\left(z_{n+N_a} + z_{n+N_a+1} e^{K_n(x) - K_{n+N_a}(x)}\right), \quad n = 1, \ldots, N_b. \tag{2.39}$$

The asymptotic behavior inside the sectors σ_n is given by (2.36), where the only x-dependent term is the exponential factor. Taking its linear dependence on x and (2.18) into account, we get that $u(x)$ decays (exponentially) in all directions inside all sectors, i.e., on the whole x-plane with exception of the rays $r_1, \ldots, r_{\mathcal{N}}$.

Thanks to condition (2.17), the coefficients z_n defined in (2.37) are a special subset of the coefficients $f_{n_1,\ldots,n_{N_b}}$ in (2.14). Theorem 2.3 shows that the behavior of the potential at large x is determined by the coefficients z_n only. Therefore, the condition

$$z_n > 0, \qquad n = 1, \ldots, \mathcal{N}, \tag{2.40}$$

is a sufficient condition for the regularity of the potential at large x. If the stronger condition (2.18) is not satisfied, singularities can appear, but only in a finite region of the x-plane.

Taking the special role played by coefficients z_n into account, let us introduce the matrices

$$v_n = \begin{pmatrix} \mathcal{D}_{n,1}, & \dots, & \mathcal{D}_{n,N_b} \\ \dots & \dots & \dots \\ \mathcal{D}_{n+N_b-1,1}, & \dots, & \mathcal{D}_{n+N_b-1,N_b} \end{pmatrix}, \qquad n = 1, \dots, \mathcal{N}, \tag{2.41}$$

and hence by (2.16), $\det v_n = \mathcal{D}(n, \dots, n+N_b-1)$, and by (2.37)

$$z_n = V(\kappa_{n,\dots,\kappa_{n+N_b-1}}) \det v_n, \tag{2.42}$$

which is different from zero thanks to (2.40). Then, we can perform the special permutation

$$\pi_n(l) = N_a + 1 + l - n, \quad l = 1, \dots, \mathcal{N}, \tag{2.43}$$

which shifts the matrix v_n in the bottom part of the matrix $\mathcal{D}$. Thus, we get the representations

$$\mathcal{D} = \pi_n \begin{pmatrix} d_{(n)} \\ E_{N_b} \end{pmatrix} v_n, \qquad \mathcal{D}' = v_n'(E_{N_a}, -d_{(n)})\pi_n^\dagger, \tag{2.44}$$

where the $N_a \times N_a$ matrix v_n' is defined by analogy. Here, π_n denotes the matrix of permutation (2.43), $\pi_n^\dagger$ is its Hermitially conjugate, and E_{N_a} and E_{N_b} are the $N_a \times N_a$ and $N_b \times N_b$ unity matrices. The $N_a \times N_b$ rectangular submatrix $d_{(n)}$ is defined by any of the equalities in (2.44), and it is obvious that matrices $\mathcal{D}$ and $\mathcal{D}'$ obey (2.10) thanks to their block structure. Representation (2.44) is convenient for studying the asymptotic properties of the Jost solutions.

III. JOST SOLUTIONS

Since the heat operator is not self-dual, one must simultaneously consider its dual $\mathcal{L}^d(x, \partial_x) = \partial_{x_2} + \partial_{x_1}^2 - u(x)$ and then introduce the Jost solution $\Phi(x, \mathbf{k})$ and the dual Jost solution $\Psi(x, \mathbf{k})$ obeying the equations

$$\mathcal{L}(x, \partial_x)\Phi(x, \mathbf{k}) = 0, \qquad \mathcal{L}^d(x, \partial_x)\Psi(x, \mathbf{k}) = 0, \tag{3.1}$$

where $\mathbf{k}$ is an arbitrary complex variable, playing the role of a spectral parameter. The reality of the potential $u(x)$, that we always assume here, is equivalent to the conjugation properties

$$\overline{\Phi(x, \mathbf{k})} = \Phi(x, -\overline{\mathbf{k}}), \qquad \overline{\Psi(x, \mathbf{k})} = \Psi(x, -\overline{\mathbf{k}}). \tag{3.2}$$

The τ-function representations for the Jost solutions were derived in Ref. 13 and are given as

$$\Phi(x, \mathbf{k}) = \frac{\tau_\Phi(x, \mathbf{k})}{\tau(x)} e^{-i\mathbf{k}x_1 - \mathbf{k}^2 x_2}, \qquad \Psi(x, \mathbf{k}) = \frac{\tau_\Psi(x, \mathbf{k})}{\tau(x)} e^{i\mathbf{k}x_1 + \mathbf{k}^2 x_2} \tag{3.3}$$

with

$$\tau_\Phi(x, \mathbf{k}) = \det\bigl(\mathcal{V}e^{K(x)}(\kappa + i\,\mathbf{k})\mathcal{D}\bigr), \qquad \tau_\Psi(x, \mathbf{k}) = \det\bigl(\mathcal{V}e^{K(x)}(\kappa + i\,\mathbf{k})^{-1}\mathcal{D}\bigr), \tag{3.4}$$

where $\kappa + i\,\mathbf{k}$ denotes the diagonal $\mathcal{N} \times \mathcal{N}$ matrix

$$\kappa + i\,\mathbf{k} = \operatorname{diag}\{\kappa_1 + i\,\mathbf{k}, \dots, \kappa_\mathcal{N} + i\,\mathbf{k}\} \tag{3.5}$$

and analogously for the matrix $(\kappa + i\,\mathbf{k})^{-1}$. By these definitions, $e^{i\mathbf{k}x_1 + \mathbf{k}^2 x_2}\Phi(x, \mathbf{k})$ is a polynomial with respect to $\mathbf{k}$ of degree N_b and $e^{-i\mathbf{k}x_1 - \mathbf{k}^2 x_2}\Psi(x, \mathbf{k})$ is a meromorphic function of $\mathbf{k}$ that becomes a polynomial of degree N_a after multiplication by $\prod_{n=1}^{\mathcal{N}}(\kappa_n + i\,\mathbf{k})$. In other words, $\Phi(x, \mathbf{k})$ is an entire function of $\mathbf{k}$ and $\Psi(x, \mathbf{k})$ a meromorphic function with poles at points $\mathbf{k} = i\kappa_n$, $n = 1, \dots, \mathcal{N}$. Introducing the discrete values of $\Phi(x, \mathbf{k})$ at these points as an $\mathcal{N}$ row

$$\Phi(x, i\kappa) = \{\Phi(x, i\kappa_1), \dots, \Phi(x, i\kappa_\mathcal{N})\}, \tag{3.6}$$

and the residues of $\Psi(x, \mathbf{k})$ at these points

$$\Psi_{\kappa_n}(x) = \operatorname*{res}_{\mathbf{k}=i\kappa_n} \Psi(x, \mathbf{k}) \tag{3.7}$$

as an $\mathcal{N}$-column

$$\Psi_{\kappa}(x) = \{\Psi_{\kappa_1}(x), \ldots, \Psi_{\kappa_{\mathcal{N}}}(x)\}^{\mathrm{T}}, \tag{3.8}$$

we have the relations (see Ref. 13),

$$\Phi(x, i\kappa)\mathcal{D} = 0, \qquad \mathcal{D}'\Psi_{\kappa}(x) = 0. \tag{3.9}$$

Thanks to (2.23) and (2.24) relations (3.9) can be written equivalently in the form $\Phi(x, i\kappa)P = 0$, $P'\Psi_{\kappa}(x) = 0$, and hence by (2.26) and the last equality in (2.25),

$$\Phi(x, i\kappa) = \Phi(x, i\kappa)(P + P') = \Phi(x, i\kappa)P', \quad \Psi_{\kappa}(x) = (P + P')\Psi_{\kappa}(x) = P\Psi_{\kappa}(x), \tag{3.10}$$

and then by (2.25) for any x and x',

$$\Phi(x, i\kappa)\Psi_{\kappa}(x') \equiv \sum_{n=1}^{\mathcal{N}} \Phi(x, i\kappa_n)\Psi_{\kappa_n}(x') = 0. \tag{3.11}$$

This means that the product $\Phi(x, \mathbf{k})\Psi(x', \mathbf{k})$ of the Jost solutions obeys the well-known Hirota bilinear identity[18] for the Beiker–Akhiezer solutions, if the contour of integration surrounds all points $\mathbf{k} = i\kappa_n$.

Let us introduce

$$\varphi(x) = \Phi(x, i\kappa)\mathcal{D}'^{(-1)}, \qquad \psi(x) = \mathcal{D}^{(-1)}\Psi_{\kappa}(x), \tag{3.12}$$

where notation (2.21) was used. By this definition, we have

$$\varphi(x) = \bigl(\varphi_1(x), \ldots, \varphi_{N_a}(x)\bigr), \quad N_a\text{-row}, \tag{3.13}$$

$$\psi(x) = \bigl(\psi_1(x), \ldots, \psi_{N_b}(x)\bigr)^{\mathrm{T}}, \quad N_b\text{-column}, \tag{3.14}$$

and, by (2.23) and (2.24), we get the representations

$$\Phi(x, i\kappa) = \varphi(x)\mathcal{D}', \qquad \Psi_{\kappa} = \mathcal{D}\psi(x). \tag{3.15}$$

Thus, we constructed N_a solutions and N_b dual solutions that parameterize discrete values of the Jost and dual Jost solutions (which have $\mathcal{N}$ components). It is clear that $\varphi(x)$ and $\psi(x)$ are not invariant with respect to the redefinition of the matrices $\mathcal{D}$ and $\mathcal{D}'$ mentioned in (2.27), while, in contrast, the combinations $\mathcal{D}v_n^{-1}$ and $v_n'^{-1}\mathcal{D}'$ are invariant with respect to transformation (2.27), as follows from (2.44).

It is necessary to mention that, with the above Definitions (3.3), the Jost solutions at large $\mathbf{k}$ have the asymptotics

$$\lim_{\mathbf{k}\to\infty}(i\,\mathbf{k})^{-N_b}e^{i\,\mathbf{k}x_1+\mathbf{k}^2x_2}\Phi(x, \mathbf{k}) = 1, \qquad \lim_{\mathbf{k}\to\infty}(i\,\mathbf{k})^{N_b}e^{-i\,\mathbf{k}x_1-\mathbf{k}^2x_2}\Psi(x, \mathbf{k}) = 1, \tag{3.16}$$

and the potential is reconstructed as

$$\begin{aligned} u(x) &= -2\lim_{\mathbf{k}\to\infty}(i\,\mathbf{k})^{-N_b+1}\partial_{x_1}\bigl(e^{i\,\mathbf{k}x_1+\mathbf{k}^2x_2}\Phi(x, \mathbf{k})\bigr) \\ &\equiv 2\lim_{\mathbf{k}\to\infty}(i\,\mathbf{k})^{N_b+1}\partial_{x_1}\bigl(e^{-i\,\mathbf{k}x_1-\mathbf{k}^2x_2}\Psi(x, \mathbf{k})\bigr). \end{aligned} \tag{3.17}$$

IV. ASYMPTOTICS OF THE JOST SOLUTIONS

A. Asymptotics of the Jost solutions $\Phi(x, k)$ and $\Psi(x, k)$ for a generic $\mathbf{k} \in \mathbb{C}$

Since, as was already noted in Ref. 13, the functions $\tau_{\Phi}(x, \mathbf{k})$ and $\tau_{\Psi}(x, \mathbf{k})$ defined in (3.4) can be obtained from the function $\tau(x)$ by means of the respective special Miwa shifts[18] $e^{K(x)} \to e^{K(x)}(\kappa + i\,\mathbf{k})$ and $e^{K(x)} \to e^{K(x)}(\kappa + i\,\mathbf{k})^{-1}$, their asymptotic behavior follows trivially from Theorem 2.3

for $\tau(x)$, if $\mathbf{k} \neq i\kappa_n$ for all n. Therefore, from Eq. (3.3), we get that the Jost solutions for $x \xrightarrow{\sigma_n} \infty$ inside an arbitrary angular sector σ_n, $n = 1, \ldots, \mathcal{N}$, have the following asymptotic behaviors:

$$\Phi(x, \mathbf{k}) = e^{-i\,\mathbf{k}x_1 - \mathbf{k}^2 x_2} \prod_{j=n}^{n+N_b-1} (\kappa_j + i\,\mathbf{k}) + \ldots, \tag{4.1}$$

$$\Psi(x, \mathbf{k}) = e^{i\,\mathbf{k}x_1 + \mathbf{k}^2 x_2} \prod_{j=n}^{n+N_b-1} (\kappa_j + i\,\mathbf{k})^{-1} + \ldots, \tag{4.2}$$

where summation of indices is mod $\mathcal{N}$ and the dots denote weaker terms. The asymptotic behavior along the ray $x \xrightarrow{r_n} \infty$ also easily follows from Theorem 2.3, and we omit it here since, in what follows, we are interested only in the exponential behavior of the leading terms, while we have two leading terms with the same asymptotic behavior in the ray directions (see (2.35)). To emphasize this, we introduce the "closed" sectors $\overline{\sigma_n}$. In other words we introduce the following.

Definition 4.1: We say that $x \to \infty$ along the direction of the closed sector $\overline{\sigma_n}$, $n = 1, \ldots, \mathcal{N}$, if $x \to \infty$ and there exist non-negative α and β, such that $x - \alpha y_{n-1} - \beta y_n$ is bounded when $\alpha + \beta \to \infty$. This will be denoted by $x \xrightarrow{\overline{\sigma_n}} \infty$.

According to this definition, asymptotics (4.1) and (4.2) are valid for $x \xrightarrow{\overline{\sigma_n}} \infty$.

In what follows, it is enough to study the asymptotic behavior of only one of the Jost solutions, $\Phi(x, \mathbf{k})$ or $\Psi(x, \mathbf{k})$. Indeed, thanks to (3.3), the dual representations for τ in (2.8) and (2.9), and the analogous ones for τ_Φ and τ_Ψ (see Remark 5.1 in Ref. 13), one can derive that the Jost solutions $\Phi(x, \mathbf{k})$ and $\Psi(x, \mathbf{k})$ are related by the equation

$$\Psi(x, \mathbf{k}) = \left[\Phi(-x, \mathbf{k})\right] \prod_{n=1}^{\mathcal{N}} (\kappa_n + i\,\mathbf{k})^{-1}, \tag{4.3}$$

where the square brackets in the rhs means that in $\Phi(-x, \mathbf{k})$, as defined in (3.4), one has to perform the substitutions

$$N_a \leftrightarrow N_b, \qquad \mathcal{D} \to \gamma \mathcal{D}'^{\mathrm{T}}, \tag{4.4}$$

where the matrices $\mathcal{D}'$ and γ are defined in (2.10) and (2.12) and the superscript T denotes matrix transposition. It is easy to see that asymptotics (4.1) and (4.2) have this property, if one notices that substitution (4.4) thanks to Definition 2.1 leads to the sector relations

$$[\sigma_n] = \sigma_{n+N_a}, \qquad [\sigma_{n+N_a}] = \sigma_n, \tag{4.5}$$

and hence the asymptotic behavior of $\Psi(x, \mathbf{k})$ in the sector σ_n is given by the asymptotic behavior of $\Phi(-x, \mathbf{k})$ in the sector σ_{n+N_a}, once the transformations (4.4) are performed.

When $\mathbf{k} = i\kappa_m$ for some m, some terms in $\tau_\Phi(x, \mathbf{k})$ (see (3.4)) are absent. In the case when such terms are coefficients of the leading exponents, we get zero asymptotics, and the exact behavior of the Jost solution cannot be derived from (4.1). The same is valid for residues of $\Psi(x, \mathbf{k})$. Thus, we have to consider asymptotic behavior at these values of $\mathbf{k}$ separately.

B. Asymptotics of the discrete values of the Jost solutions

In order to find the asymptotic behavior of the discrete values of the Jost solutions, we notice that according to (3.3) and (3.4), and Definition (2.4), they equal

$$\Phi(x, i\kappa_m) = \frac{\tau_\Phi(x, i\kappa_m)}{\tau(x)} e^{K_m(x)}, \qquad \Psi_{\kappa_m}(x) = \frac{\operatorname*{res}_{\mathbf{k}=i\kappa_m} \tau_\Psi(x, \mathbf{k})}{\tau(x)} e^{-K_m(x)}, \tag{4.6}$$

where $m = 1, \dots, \mathcal{N}$. We consider the asymptotic of $\Phi(x, i\kappa_m)$, since the behavior of $\Psi_{\kappa_m}(x)$ can be obtained from (4.3), which for these discrete values means

$$\Psi_{\kappa_m}(x) = i(-1)^{\mathcal{N}}\gamma_m\big[\Phi(-x, i\kappa_m)\big], \tag{4.7}$$

where we used notation (2.12) and the square brackets denotes operation (4.4).

Let us fix some $n = 1, \dots, \mathcal{N}$, and let $x \xrightarrow{\overline{\sigma_n}} \infty$. From (4.1), we have

$$\Phi(x, i\kappa_m) = \begin{cases} a_m^{(n)}(x), & m = n, \dots, n + N_b - 1, \\ p_m^{(n)} e^{K_m(x)}, & m = n + N_b, \dots, n + \mathcal{N} - 1, \end{cases} + \dots, \tag{4.8}$$

where the constants $p_m^{(n)}$ equal

$$p_m^{(n)} = \prod_{j=n}^{n+N_b-1} (\kappa_j - \kappa_m) \tag{4.9}$$

and the functions $a_m^{(n)}(x)$, which give the leading asymptotic behavior for the corresponding values of m, should be determined (thanks to (4.1) we know only that they tend to zero when $x \xrightarrow{\overline{\sigma_n}} \infty$).

Using the permutation introduced in (2.43), we write

$$\begin{aligned} \Phi(x, i\kappa)\pi_n &\equiv \big(\Phi(x, i\kappa_{m+n+N_b-1})\big)_{m=1}^{\mathcal{N}} \\ &= \Big(p_{n+N_b}^{(n)} e^{K_{n+N_b}(x)}, \dots, p_{n+\mathcal{N}-1}^{(n)} e^{K_{n+\mathcal{N}-1}(x)}, a_n^{(n)}, \dots, a_{n+N_b-1}^{(n)}\Big) + \dots. \end{aligned} \tag{4.10}$$

Now by (2.44), from (3.9), we get

$$a_m^{(n)} = -\sum_{l=1}^{N_a} p_{n+l+N_b-1}^{(n)} e^{K_{n+l+N_b-1}(x)} (d_{(n)})_{l,m-n+1}, \quad m = n, \dots, n + N_b - 1,$$

where we used that the matrices v_n are invertible. Inserting these relations in (4.10) and again using (2.44), we get

$$\Phi(x, i\kappa) = \Big(p_{n+N_b}^{(n)} e^{K_{n+N_b}(x)}, \dots, p_{n+\mathcal{N}-1}^{(n)} e^{K_{n+\mathcal{N}-1}(x)}\Big) v_n'^{-1} \mathcal{D}' + \dots, \tag{4.11}$$

$$\Psi_\kappa(x) = -i\mathcal{D} v_n^{-1} \Big(\big(p_n^{(n)}\big)^{-1} e^{-K_n(x)}, \dots, \big(p_{n+N_b-1}^{(n)}\big)^{-1} e^{-K_{n+N_b-1}(x)}\Big)^{\mathrm{T}} + \dots, \tag{4.12}$$

where the second equality is derived by means of (4.7), (2.44), and (4.9). Thanks to (3.12), this means that for $x \xrightarrow{\overline{\sigma_n}} \infty$, we proved that

$$\varphi(x) = \Big(p_{n+N_b}^{(n)} e^{K_{n+N_b}(x)}, \dots, p_{n+\mathcal{N}-1}^{(n)} e^{K_{n+\mathcal{N}-1}(x)}\Big) v_n'^{-1} + \dots, \tag{4.13}$$

$$\psi(x) = -i v_n^{-1} \Big(\big(p_n^{(n)}\big)^{-1} e^{-K_n(x)}, \dots, \big(p_{n+N_b-1}^{(n)}\big)^{-1} e^{-K_{n+N_b-1}(x)}\Big)^{\mathrm{T}} + \dots. \tag{4.14}$$

The asymptotic values of $\varphi(x)$ and $\psi(x)$ depend on the choice of the matrices $\mathcal{D}$ and $\mathcal{D}'$ (cf. discussion after (2.43)), while asymptotics (4.11) and (4.12) are invariant. In fact, we can give a more detailed description of this asymptotic behavior. For this let us divide the sectors σ_n in two subsectors

$$\begin{aligned} \sigma_n' &= \{x : K_{n+N_b-1}(x) > K_{n-1}(x) \text{ and } K_{n-1}(x) > K_{n+N_b}(x)\}, \\ \sigma_n'' &= \{x : K_{n+N_b}(x) > K_{n-1}(x) \text{ and } K_n(x) > K_{n+N_b}(x)\}. \end{aligned} \tag{4.15}$$

It is easy to see that

$$\sigma_n = \sigma_n' \cup \sigma_n'' \cup r_n', \tag{4.16}$$

where r_n' is the ray belonging to the sector σ_n,

$$r_n' = \{x : K_{n-1}(x) = K_{n+N_b}(x) \text{ and } (\kappa_{n+N_b} - \kappa_{n-1})x_2 < 0\}. \tag{4.17}$$

By analogy with (2.32), we can determine it by means of the directing vector

$$y'_n = (\kappa^2_{n+N_b} - \kappa^2_{n-1}, \kappa_{n-1} - \kappa_{n+N_b}). \tag{4.18}$$

In the case $N_a = 1$, i.e., $N_b = \mathcal{N} - 1$ by (2.2), these definitions are senseless as follows from (2.17). So, we will consider this case below separately. Now by analogy with Definition 2.1, we introduce the following.

Definition 4.2: We say that $x \to \infty$ *in the closed subsector* $\overline{\sigma'_n}$ *(notation* $x \xrightarrow{\overline{\sigma'_n}} \infty$*) or in the closed subsector* $\overline{\sigma''_n}$ *(notation* $x \xrightarrow{\overline{\sigma''_n}} \infty$*),* $n = 1, \ldots \mathcal{N}$*, if there exist non-negative* α *and* β*, such that* $\alpha + \beta \to +\infty$ *and*

$$\text{for} \qquad x \xrightarrow{\overline{\sigma'_n}} \infty: \qquad x - \alpha y_{n-1} - \beta y'_n \text{ is bounded,}$$

$$\text{for} \qquad x \xrightarrow{\overline{\sigma''_n}} \infty: \qquad x - \alpha y'_n - \beta y_n \text{ is bounded.}$$

Then, we have the following lemma.

Lemma 4.1: 1. If $x \xrightarrow{\overline{\sigma'_n}} \infty$*, then* $K_{n-1}(x) > K_m(x)$ *for any* $m = n + N_b, \ldots, n + \mathcal{N} - 2$ *and* $K_m(x) > K_{n+N_b-1}(x)$ *for any* $m = n, \ldots, n + N_b - 2$*;*

2. if $x \xrightarrow{\overline{\sigma''_n}} \infty$*, then* $K_{n+N_b}(x) > K_m(x)$ *for any* $m = n + N_b + 1, \ldots, n + \mathcal{N} - 1$ *and* $K_n(x) > K_m(x)$ *for any* $m = n + 1, \ldots, n + N_b - 1$*.*

Proof: Thanks to Definition 4.2, $K_{n-1}(x) - K_m(x) = \alpha g_{n-1,m,n-1} + \beta \widetilde{g}_{n-1,m,n-1}$ and $K_m(x) - K_{n+N_b-1}(x) = \alpha g_{m,n-1,n-1} + \beta \widetilde{g}_{m,n+N_b-1,n-1}$, where $g_{l,m,n}$ is defined in (2.19) and $\widetilde{g}_{l,m,n}$ is $g_{l,m,n}$ with the substitution $N_b \to N_b + 1$. Then the first statement follows by Lemma 2.1. In the same way, we get $K_{n+N_b}(x) - K_m(x) = \alpha \widetilde{g}_{n-1,m,n-1} + \beta g_{n,m,n}$ and $K_m(x) - K_n(x) = \alpha \widetilde{g}_{m,n,n-1} + \beta g_{m,n,n}$, and the second statement again results from Lemma 2.1. ■

Thanks to this lemma, from (4.11), we get

$$\Phi(x, i\kappa_m) = p^{(n)}_{n-1} \left({v'_n}^{-1} \mathcal{D}' \right)_{N_a,m} e^{K_{n-1}(x)} + \ldots, \qquad x \xrightarrow{\overline{\sigma'_n}} \infty, \tag{4.19}$$

$$\Phi(x, i\kappa_m) = p^{(n)}_{n+N_b} \left({v'_n}^{-1} \mathcal{D}' \right)_{1,m} e^{K_{n+N_b}(x)} + \ldots, \qquad x \xrightarrow{\overline{\sigma''_n}} \infty, \tag{4.20}$$

for any $n = 1, \ldots, \mathcal{N}$ and $m = n, \ldots, n + N_b - 1$. In the exceptional case $N_a = 1$, it is easy to see that, thanks to (2.17), both these equalities coincide, and we can use either of them for the asymptotic behavior in the whole sector $\overline{\sigma_n}$. Correspondingly, relations for the asymptotic values of $\Psi_{\kappa_m}(x)$ follow from (4.7) and (4.12), where it is necessary to take into account that images of subsectors (4.15) are different from σ'_{n+N_a} and σ''_{n+N_a} (cf. (4.5)). Exactly, we get a decomposition of sectors σ_n different from (4.16),

$$\overline{\sigma_n} = \overline{[\sigma'_{n+N_a}] \cup [\sigma''_{n+N_a}]}, \tag{4.21}$$

where the square brackets by (4.4) denotes substitution $N_a \leftrightarrow N_b$. Hence, by (4.15),

$$\begin{aligned} [\sigma'_{n+N_a}] &= \{x : K_{n+N_b-1}(x) > K_{n-1}(x) \text{ and } K_n(x) > K_{n+N_b-1}(x)\}, \\ [\sigma''_n] &= \{x : K_{n+N_b-1}(x) > K_n(x) \text{ and } K_n(x) > K_{n+N_b}(x)\}. \end{aligned} \tag{4.22}$$

The same remark as above must be given in the case $N_b = 1$.

In the following, we need the asymptotics of $\Phi(x, i\kappa_m)$ and $\Psi_{\kappa_m}(x)$ for fixed m in dependence on n, while in the formulas above we have its asymptotic for fixed n in dependence on running

m. These asymptotics can be easily obtained from (4.19) and (4.20), and analogous formulas for $\Psi_{\kappa_m}(x)$, and we can state the following theorem.

Theorem 4.2: *The leading asymptotic behavior of the discrete values of the Jost solutions* $\Phi(x,\mathbf{k})$ *and* $\Psi(x,\mathbf{k})$ *is given by*

$$\Phi(x, i\kappa_m) = \begin{cases} p_m^{(n)} e^{K_m(x)}, & x \xrightarrow{\overline{\sigma_n}} \infty, \quad n \in [m+1, m+N_a], \\ p_{n-1}^{(n)} \left(v_n'^{\,-1}\mathcal{D}'\right)_{N_a,m} e^{K_{n-1}(x)}, & x \xrightarrow{\overline{\sigma_n'}} \infty, \quad n \in [m+N_a+1, m+\mathcal{N}], \\ p_{n+N_b}^{(n)} \left(v_n'^{\,-1}\mathcal{D}'\right)_{1,m} e^{K_{n+N_b}(x)}, & x \xrightarrow{\overline{\sigma_n''}} \infty, \quad n \in [m+N_a+1, m+\mathcal{N}], \end{cases} \tag{4.23}$$

$$+\ldots,$$

$$i\Psi_{\kappa_m}(x) = \begin{cases} (p_m^{(n)})^{-1} e^{-K_m(x)}, & x \xrightarrow{\overline{\sigma_n}} \infty, \quad n \in [m+N_a+1, m+\mathcal{N}], \\ \dfrac{\left(\mathcal{D}v_n{}^{-1}\right)_{mN_b}}{p_{n+N_b-1}^{(n)}} e^{-K_{n+N_b-1}(x)}, & x \xrightarrow{\overline{[\sigma_{n+N_a}']}} \infty, \quad n \in [m+1, m+N_a], \\ \dfrac{\left(\mathcal{D}v_n{}^{-1}\right)_{m1}}{p_{n-1}^{(n)}} e^{-K_n(x)}, & x \xrightarrow{\overline{[\sigma_{n+N_a}'']}} \infty, \quad n \in [m+1, m+N_a], \end{cases} \tag{4.24}$$

$$+\ldots.$$

We see that for $n = m+1, \ldots, m+N_a$, the Jost solution $\Phi(x, i\kappa_m)$ has the same asymptotics along the directions on the x-plane obtained by considering the union of the corresponding sectors $\overline{\sigma_n}$, that is, thanks to (2.31), along the directions

$$\bigcup_{n=m+1}^{m+N_a} \overline{\sigma_n} = \begin{cases} \{K_{m+N_a}(x) - K_m(x) > -\infty\} \cap \{K_{m+N_b}(x) - K_m(x) > -\infty\}, & N_a < N_b, \\ \{K_{m+N_a}(x) - K_m(x) > -\infty\}, & N_a = N_b, \\ \mathbb{R}^2 \setminus \{K_m(x) - K_{m+N_a}(x) > -\infty\} \cap \{K_m(x) - K_{m+N_b}(x) > -\infty\}, & N_a > N_b, \end{cases} \tag{4.25}$$

where, say, the condition $K_{m+N_a}(x) - K_m(x) > -\infty$ means that $x \to \infty$ in such a way that this difference is bounded from below. Geometrically, the set in (4.25) essentially depends on the relation between N_a and N_b. Thus, in the first line, we have a sector with a sharp angle at the vertex (being the intersection of two half-planes); in the second line, a half-plane and in the third line, a sector with a blunt angle at the vertex.

V. ANNIHILATORS

In the theory of the one-dimensional Sturm–Liouville operator, it is well known that the discrete value of the Jost solution corresponding to the one-soliton potential (i.e., the potential in (1.4) with $\kappa_1 = -\kappa_2$) decays exponentially on the x_1 axis as $e^{-|x_1\kappa_1|}$. The Jost solution of the heat equation, obviously, cannot have an analogous property on the whole x-plane because of the x_2-dependence of the exponential factor (see, e.g., (4.6)).

In order to find a hint for a proper formulation in the case of the heat equation, let us consider the trivial example of the one-soliton potential, i.e., the case $N_a = N_b = 1$ and arbitrary κ_1 and κ_2 obeying (2.3). By (2.13) and (2.14), we have $\tau(x) = de^{K_1(x)} + e^{K_2(x)}$, where d is a constant, that thanks to (2.7) gives (1.4). Then from (4.6), we have

$$\Phi(x, i\kappa_2) = -d\Phi(x, i\kappa_1) = \frac{d(\kappa_1 - \kappa_2)e^{K_1(x)+K_2(x)}}{de^{K_1(x)} + e^{K_2(x)}}, \tag{5.1}$$

and, for the asymptotics of the discrete value of the Jost solution, we have

$$\Phi(x, i\kappa_1) \cong \Phi(x, i\kappa_2) \cong \begin{cases} e^{K_2(x)}, & x \xrightarrow{\overline{\sigma_1}} \infty, \\ e^{K_1(x)}, & x \xrightarrow{\overline{\sigma_2}} \infty, \end{cases} \tag{5.2}$$

where closed sectors $\overline{\sigma_1}$ and $\overline{\sigma_2}$ are given by

$$\begin{aligned} \overline{\sigma_1} &= \{K_1(x) - K_2(x) > -\infty\} \equiv \{x_1 + (\kappa_1 + \kappa_2)x_2 < +\infty\}, \\ \overline{\sigma_2} &= \{K_2(x) - K_1(x) > -\infty\} \equiv \{x_1 + (\kappa_1 + \kappa_2)x_2 > -\infty\}, \end{aligned} \tag{5.3}$$

where we understand inequalities here as in Definition (4.1). Now let the scalar product of the two-dimensional vectors $x = (x_1, x_2)$ and $q = (q_1, q_2)$ be denoted by

$$qx = q_1x_1 + q_2x_2, \tag{5.4}$$

and let

$$q_{mn} = q_2 - (\kappa_m + \kappa_n)q_1 + \kappa_m\kappa_n \tag{5.5}$$

for any $m, n \in \mathbb{Z}$. Then, thanks to (5.2), for the asymptotic behavior of the functions $e^{-qx}\Phi(x, i\kappa_1)$ and $e^{-qx}\Phi(x, i\kappa_2)$, we get

$$\begin{aligned} e^{-qx}\Phi(x, i\kappa_1) &\cong e^{-qx}\Phi(x, i\kappa_2) \\ &\cong \begin{cases} \exp[-(q_1 - \kappa_2)(x_1 + (\kappa_1 + \kappa_2)x_2 - q_{12}x_2], & x \xrightarrow{\overline{\sigma_1}} \infty, \\ \exp[-(q_1 - \kappa_1)(x_1 + (\kappa_1 + \kappa_2)x_2 - q_{12}x_2], & x \xrightarrow{\overline{\sigma_2}} \infty, \end{cases} \end{aligned} \tag{5.6}$$

and we deduce that they are bounded for all x if and only if $q_{12} = 0$ and $\kappa_1 < q_1 < \kappa_2$, this on the segment of the line $q_{12} = 0$ on the q-plane cutoff by the parabola $q_2 = q_1^2$ (see Fig. 2).

This trivial result has an unexpected generalization in the case $N_a \neq N_b$. Precisely, for $N_b > N_a \geq 1$, the set of the q-plane where the discrete values $\Phi(x, i\kappa_m)$ of the Jost solution multiplied by the exponent e^{-qx} are bounded becomes a polygon. As regards the dual Jost solution, the functions $e^{qx}\Psi_{\kappa_m}(x)$ are bounded inside a polygon for $N_a > N_b \geq 1$.

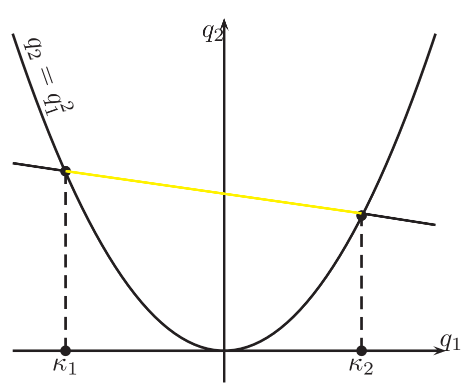

FIG. 2. (Color online) One-dimensional case.

We then consider the case $N_b > N_a \geq 1$ and introduce the angular sectors

$$\rho_k(q) = \theta(\widetilde{q}_{k,k-N_a})\theta(\widetilde{q}_{k+N_a,k}), \tag{5.7}$$

where the θ is step functions and

$$\widetilde{q}_{mn} = (\kappa_m - \kappa_n)q_{mn}, \tag{5.8}$$

and we call

$$Q_m = (\kappa_m, \kappa_m^2), \qquad Q_n = (\kappa_n, \kappa_n^2), \tag{5.9}$$

the points of intersection of the line $\widetilde{q}_{mn} = 0$ with the parabola $q_2 = q_1^2$ in the q-plane.

The angular sector $\rho_k(q)$ has a vertex on the parabola at the point Q_k. The ray bordering the sector from the left (looking from the vertex inside the sector) crosses the parabola at the point Q_{k-N_a} and the ray bordering the sector from the right (looking from the vertex inside the sector) crosses the parabola at the point Q_{k+N_a}. If we add the point at infinity to the parabola and, then consider it a closed curve, as k increases, the sector moves along the parabola in the anticlockwise direction.

Then, we can prove the following lemmas.

Lemma 5.1: The region of the q-plane defined by the characteristic function

$$P_m(q) = \prod_{k=m+N_a}^{m+N_b} \rho_k(q) \tag{5.10}$$

is a polygon included in the parabolic region $q_2 \geq q_1^2$ with a vertex at the point Q_m. More precisely, this polygon is included in the polygon with the vertices $Q_m, Q_{m+1}, \ldots, Q_{m+N_b}$ belonging to the parabola, coincides with this polygon in the case $N_a = 1$ and belongs to the strip

$$\min_{m\leq l\leq m+N_b} \kappa_l \leq q_1 \leq \max_{m\leq l\leq m+N_b} \kappa_l. \tag{5.11}$$

Proof: The angular sector ρ_k at $k = m + N_a$ has the left ray (looking from the vertex inside the sector) crossing the parabola at the point Q_m. As k increases, the sector rotates along the parabola in the anticlockwise direction up to the sector ρ_{m+N_a}, which has the right (looking from the vertex inside the sector) crossing the parabola just at the point Q_m. Therefore, the sectors ρ_k for k running in the interval $k = m + N_a, \ldots m + N_b$ cover a common region, which is a polygon inside the parabola with a vertex at Q_m. For $N_a = 1$, each sector ρ_k has a right ray coinciding with the left ray of the sector ρ_{k+1} and the sides of the polygon are, therefore, just the intersections of the rays bordering the angular sectors ρ_k with the region inside the parabola and the vertices of the polygon are the points $Q_m, Q_{m+1}, \ldots, Q_{m+N_b}$ belonging to the parabola. For $N_a > 1$, the point Q_{k+N_a}, the intersection of the right ray of the sector ρ_k with the parabola, lies to the left of the left ray of the sector ρ_{k+1} and, therefore, does not belong to the polygon, and we deduce that the polygon has only one vertex belonging to the parabola, precisely, the point Q_m, and that it belongs to the polygon with the vertices $Q_m, Q_{m+1}, \ldots, Q_{m+N_b}$. In addition, we can deduce that the polygon defined by (5.10) belongs to the strip in (5.11). ■

Lemma 5.2: The polygon defined by the characteristic function

$$\epsilon_m(q) = \prod_{k=m}^{m+N_b} \theta(\widetilde{q}_{k+N_a,k}) \tag{5.12}$$

satisfies the equation

$$\epsilon_m(q) = P_m(q) \begin{cases} 1, & N_b \geq 2N_a - 1, \\ \displaystyle\prod_{k=m+N_b-N_a+1}^{m+N_a-1} \theta(\widetilde{q}_{k+N_a,k}), & N_b < 2N_a - 1, \end{cases} \tag{5.13}$$

and, therefore, belongs to the polygon $P_m(q)$ considered in Lemma 5.1 and is included in the parabolic region $q_2 \geq q_1^2$ of the q-plane. Moreover, the polygon defined by (5.12) is not void.

Proof: Thanks to the equality $\prod_{k=m}^{m+N_b} \theta(\widetilde{q}_{k+N_a,k}) = \prod_{k=m+N_a}^{m+\mathcal{N}} \theta(\widetilde{q}_{k,k-N_a})$ obtained by shifting $k \to k - N_a$, we get that

$$\prod_{k=m}^{m+N_b} \theta(\widetilde{q}_{k+N_a,k}) = \left(\prod_{k=m}^{m+N_b} \theta(\widetilde{q}_{k+N_a,k})\right) \prod_{k=m+N_a}^{m+\mathcal{N}} \theta(\widetilde{q}_{k,k-N_a}),$$

as θ of the second factor are all present in the first one. Then, factoring $P_m(q)$ from the rhs and shifting the running index k for the remaining θ in the two corresponding products for $k \to k - N_a$ and $k \to k + N_a$, we get

$$\epsilon_m(q) = P_m(q) \left(\prod_{k=m+N_a}^{m+2N_a-1} \theta(\widetilde{q}_{k,k-N_a})\right) \prod_{k=m+N_b-N_a+1}^{m+N_b} \theta(\widetilde{q}_{k+N_a,k}). \tag{5.14}$$

If $N_b \geq 2N_a - 1$, the θ in the two products in the rhs are already present in $P_m(q)$, and we then get the first equality in (5.13). If $N_b < 2N_a - 1$, we have

$$\prod_{k=m+N_a}^{m+2N_a-1} \theta(\widetilde{q}_{k,k-N_a}) = \prod_{k=m+N_a}^{m+N_b} \theta(\widetilde{q}_{k,k-N_a}) \prod_{k=m+N_b+1}^{m+2N_a-1} \theta(\widetilde{q}_{k,k-N_a}), \tag{5.15}$$

$$\prod_{k=m+N_b-N_a+1}^{m+N_b-1} \theta(\widetilde{q}_{k+N_a,k}) = \left(\prod_{k=m+N_b-N_a+1}^{m+N_a-1} \theta(\widetilde{q}_{k+N_a,k})\right) \prod_{k=m+N_a}^{m+N_b} \theta(\widetilde{q}_{k+N_a,k}), \tag{5.16}$$

and then recalling the expression for $P_m(q)$ in (5.10) and (5.7) and noticing that the second product in (5.15) and the first product in (5.16) coincide, we get the second equality in (5.13).

In order to prove that the polygon defined by (5.12) is not void, it is sufficient to prove that the product in the second line of the rhs of (5.13) equals 1 at the point Q_m. In fact, from Definitions (2.19) and (5.8), we have that $\widetilde{q}_{k+N_a,k}(Q_m) = -g_{k+N_a,m,k+N_a}$, which, thanks to Lemma 2.1, is greater or equal to zero only for $m = k, \dots, k + N_a \pmod{\mathcal{N}}$, which is impossible for $k = m + N_b - N_a + 1, \dots, m + N_a - 1$ and $N_b < 2N_a - 1$. ■

Lemma 5.3: The characteristic functions $\epsilon_m(q)$ introduced in Lemma 5.2 coincide with the characteristic function

$$\widetilde{\epsilon}_m(q) = \prod_{k=m}^{m+N_b} \theta(\widetilde{q}_{k+N_a,k}) \prod_{k=m}^{m+N_b-1} \theta(\widetilde{q}_{l+N_a,l+1}). \tag{5.17}$$

Proof: It follows if we notice that the parameters κ_{N_a+1}, κ_l, and κ_{l+1} for any value of l turn out to be ordered only in one of the two following ways: $\kappa_{N_a+1} < \kappa_l < \kappa_{l+1}$ or $\kappa_l < \kappa_{l+1} < \kappa_{N_a+1}$ with the only exception of the case $l = \mathcal{N} \pmod{\mathcal{N}}$, where $\kappa_{\mathcal{N}+1} < \kappa_{N_a+1} < \kappa_{\mathcal{N}}$. Taking into account that q is inside the parabola $q_2 = q_1^2$, we get that for any value of l, $\theta(\widetilde{q}_{l+N_a,l})\theta(\widetilde{q}_{l+N_a,l+1}) = \theta(\widetilde{q}_{l+N_a,l})$. ■

Now, we can prove the following theorem.

Theorem 5.4: *1. In the case $N_a < N_b$, the function $e^{-qx}\Phi(x, i\kappa_m)$ for any $m = 1, \dots, \mathcal{N}$ is exponentially decreasing with respect to x for any q inside the polygon defined by the characteristic function*

$$\varepsilon_m(q) = \prod_{k=m}^{m+N_b} \theta(\widetilde{q}_{k+N_a,k}), \tag{5.18}$$

while there exists no such domain on the q-plane for $N_a \geq N_b$.

2. In the case $N_a > N_b$, the function $e^{qx}\Psi_{\kappa_m}(x)$ for any $m = 1, \ldots, \mathcal{N}$ is exponentially decreasing with respect to x for any q inside the polygon defined by the characteristic function

$$\varepsilon_m(q) = \prod_{k=m}^{m+N_b} \theta(\widetilde{q}_{k,k+N_a}), \tag{5.19}$$

while there exists no such domain on the q-plane for $N_a \leq N_b$.

3. These polygons are all included in the parabolic region $q_2 \geq q_1^2$.

Proof: In (4.23) and (4.25), we have seen that the geometry of the sector $\overline{\Sigma_m} = \bigcup_{n=m+1}^{m+N_a} \overline{\sigma_n}$, where $\Phi(x, i\kappa_m)$ has asymptotic behavior $e^{K_m(x)}$, essentially depends on the relation between N_a and N_b. Then, we first let $N_a < N_b$ and consider the directing vectors of the rays bordering the sectors $\overline{\Sigma_m}$, $\overline{\sigma'_n}$, and $\overline{\sigma''_n}$ ($n = m + N_a + 1, \ldots, m + \mathcal{N}$), where the product $e^{-qx}\Phi(x, i\kappa_m)$, according to (4.23) and (4.25), has the respective asymptotic behaviors $e^{K_m(x)-qx}$, $e^{K_{n-1}(x)-qx}$, and $e^{K_{n+N_b}(x)-qx}$. For $\overline{\Sigma_m}$, they are y_m and y_{m+N_a}; for $\overline{\sigma'_n}$, they are y_{n-1} and y'_n; and for $\overline{\sigma''_n}$, they are y_n and y'_n (see (2.32) and (4.18)). According to Definition 2.1, we say that $x \to \infty$ in the sector $\overline{\Sigma_m}$, if $x - \alpha y_m - \beta y_{m+N_a}$ is bounded while $\alpha^2 + \beta^2 \to \infty$ and both α and β are bounded from below. On the other hand, it is easy to see that

$$K_m(x) - qx\big|_{x=\alpha y_m+\beta y_{m+N_a}} = \alpha\widetilde{q}_{m+N_b,m} - \beta\widetilde{q}_{m+N_a,m}, \tag{5.20}$$

and we deduce that $e^{-qx}\Phi(x, i\kappa_m)$ is bounded in the sector $\overline{\Sigma_m}$ if and only if $\widetilde{q}_{m+N_b,m} \leq 0$ and $\widetilde{q}_{m+N_a,m} \geq 0$. By Definition 4.2, we say that $x \to \infty$ in the sectors $\overline{\sigma'_n}$ and $\overline{\sigma''_n}$, if $x - \alpha y_{n-1} - \beta y'_n$ and $x - \alpha y'_n - \beta y_n$ are, respectively, bounded while $\alpha^2 + \beta^2 \to \infty$ and both α and β are bounded from below. Since we have

$$K_{n-1}(x) - qx\big|_{x=\alpha y_{n-1}+\beta y'_n} = \alpha\widetilde{q}_{n+N_b-1,n-1} + \beta\widetilde{q}_{n+N_b,n-1} \tag{5.21}$$

and

$$K_{n+N_b}(x) - qx\big|_{x=\alpha y'_n+\beta y_n} = \alpha\widetilde{q}_{n+N_b,n-1} + \beta\widetilde{q}_{n+N_b,n}, \tag{5.22}$$

we conclude that $e^{-qx}\Phi(x, i\kappa_m)$ is bounded in the sectors $\overline{\sigma'_n}$ and $\overline{\sigma''_n}$ if and only if $\widetilde{q}_{n+N_b-1,n-1}$, $\widetilde{q}_{n+N_b,n-1}$, and $\widetilde{q}_{n+N_b,n}$ are less than or equal to zero.

Thus, the condition that the product $e^{-qx}\Phi(x, i\kappa_m)$ decays asymptotically in any direction on the x-plane is equivalent to the condition that q belongs to the set on the q-plane given by characteristic function

$$\widetilde{\epsilon}_m(q) = \prod_{k=m}^{m+N_b} \theta(\widetilde{q}_{k+N_a,k}) \prod_{k=l}^{m+N_b-1} \theta(\widetilde{q}_{l+N_a,l+1}), \tag{5.23}$$

where we took into account that the indices are defined mod $\mathcal{N}$ and that $\widetilde{q}_{m,n} = -\widetilde{q}_{n,m}$ (see (5.5) and (5.8)).

Thanks to the four lemmas above, the first part of statement 1 of the theorem for $N_a < N_b$ is proved.

In the case $N_a = N_b$, by (2.32), we have that $y_{m+N_a} = -y_m$, $m = 1, \ldots, N_a$. Thus, it is enough to consider the asymptotic behavior along the ray r_m (see Definition 2.1). We choose y_m as the directing vector, and by (5.20) for $\beta = 0$, we have to consider the limits in both directions, $\alpha \to \pm\infty$. It is clear that the rhs of (5.20), independently of the sign of $\widetilde{q}_{m,m+N_b}$, cannot decay in both these directions, and this rhs is bounded only if $\widetilde{q}_{m,m+N_b} = 0$, which, thanks to (5.5) and (5.8), gives a line and not a domain on the q-plane. Analogously, in the case $N_b < N_a$, the directing vectors of the sector in the third line of (4.25) are the same as in (5.20), but the conditions on the behavior of α and β are different. Together with the limit $\alpha \to +\infty$, $\beta \to +\infty$, we have to take sectors $\alpha \to +\infty$, $\beta \to -\infty$, and $\alpha \to -\infty$, $\beta \to +\infty$ into account. Thus, we see that also in this case, it is impossible to give conditions on $\widetilde{q}_{m,m+N_b}$ and $\widetilde{q}_{m,m+N_a}$ that always make the limit of the rhs of (5.20) equal to $-\infty$. This proves statement 1 of the theorem. Statement 2 for $e^{qx}\Psi_{\kappa_m}(x)$ follows from (4.7). In

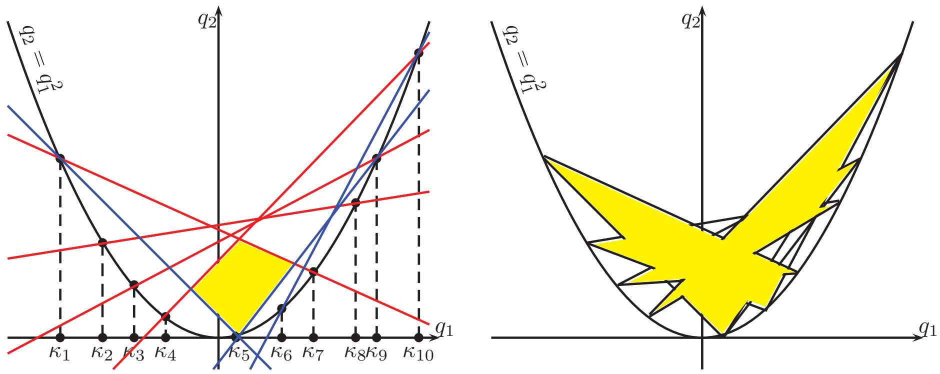

FIG. 3. (Color online) $N_a = 4$, $N_b = 6$, $\mathcal{N} = 10$, polygon for $m = 5$ and union of polygons for $m = 1, \ldots, 10$.

particular, the existence of the polygon only in the case $N_b < N_a$ results from above and (4.4). The third statement of the theorem is proved in Lemma 5.2. ■

Using (2.3), (2.17), and (5.8), we can rewrite (5.18) more explicitly,

$$\varepsilon_m(q) = \begin{cases} \prod_{k=m}^{N_b} \theta(q_{k,k+N_a}) \prod_{k=1}^{m} \theta(-q_{k,k+N_b}), & 1 \leq m \leq N_a, \\ \prod_{k=m}^{N_b} \theta(q_{k,k+N_a}) \prod_{k=1}^{N_a} \theta(-q_{k,k+N_b}) \prod_{k=N_a+1}^{m} \theta(q_{k,k-N_a}), & N_a + 1 \leq m \leq N_b, \\ \prod_{k=m-N_b}^{N_a} \theta(-q_{k,k+N_b}) \prod_{k=1}^{m-N_a} \theta(q_{k,k+N_a}), & N_b + 1 \leq m \leq \mathcal{N}. \end{cases} \tag{5.24}$$

For the potential $u(x)$ with $N_a = 4$ and $N_b = 6$, Fig. 3 shows the polygon with characteristic function $\varepsilon_m(q)$ on the q-plane in the case $m = 5$ and the union of all polygons for $m = 1, \ldots, 10$.

VI. CONCLUDING REMARKS

Theorem 5.4 shows that the extended L-operator

$$L(x, x'; q) = \{-\partial_{x_2} - q_2 + (\partial_{x_1} + q_1)^2\}\delta(x - x') - u(x)\delta(x - x') \tag{6.1}$$

is not invertible in the case, where $u(x)$ is an (N_a, N_b)-soliton potential. It has right annihilators if $N_a < N_b$ and left ones if $N_b < N_a$.

In fact, in the case $N_a < N_b$, if we introduce an operator H_m with the kernel

$$H_m(x, x'; q) = \varepsilon_m(q)e^{-qx}\Phi(x, \kappa_m)\psi(x'; q), \tag{6.2}$$

where $\psi(x'; q)$ is any arbitrary self-adjoint function bounded in x', we have

$$L(q)H_m(q) = 0. \tag{6.3}$$

Analogously, for the case $N_b < N_a$, we have

$$K_m(q)L(q) = 0 \tag{6.4}$$

with

$$K_m(x, x'; q) = \varepsilon_m(q)\varphi(x; q)e^{qx'}\Psi_{\kappa_m}(x'), \tag{6.5}$$

where $\varphi(x; q)$ is any arbitrary self-adjoint function bounded in x inside the polygon defined by $\varepsilon_m(q)$.

We note that the total Green's function $G(x, x', \mathbf{k})$ of the operator $\mathcal{L}$ can be defined (see Refs. 9 and 11) as the value of the kernel $M(x, x'; q)$ at $q_1 = \mathbf{k}_{\Im}$, $q_2 = \mathbf{k}_{\Im}^2 - \mathbf{k}_{\Re}^2$ for a complex spectral parameter $\mathbf{k} = \mathbf{k}_{\Re} + i\,\mathbf{k}_{\Im}$. These values of q lie outside the parabola $q_2 = q_1^2$ and, therefore, outside the polygon and touch it only at the vertices. The Green's function then exists for any $\mathbf{k}$ but it is singular at the points $\mathbf{k} = i\kappa_m$ corresponding to the vertices of the polygon touching the parabola.

ACKNOWLEDGMENTS

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APPENDIX: REGULARITY OF POTENTIAL AND JOST SOLUTIONS

We defined the Jost solutions $\Phi(x, \mathbf{k})$ in a symmetric form in (3.3), so that

$$\chi(x, \mathbf{k}) \equiv \Phi(x, \mathbf{k}) e^{i\,\mathbf{k} x_1 + \mathbf{k}^2 x_2} \tag{A1}$$

became a polynomial in the spectral parameter $\mathbf{k}$ with the higher power term given by $(i\,\mathbf{k})^{N_b}$. Therefore, it can be determined by giving its values at N_b points. If these points are chosen to be $\mathbf{k} = i\kappa_n$ with n belonging to the subset $J = \{n_1, \cdots, n_{N_b}\}$ of numbers in the interval $\{1, \ldots, \mathcal{N}\}$, these values can be determined by a combined use of the analyticity properties of $\chi(x, \mathbf{k})$ and the condition $\Phi(i\kappa)\mathcal{D} = 0$ in (3.9).

Let

$$\Delta(\mathbf{k}) = \prod_{n \in J} (i\,\mathbf{k} + \kappa_n). \tag{A2}$$

Then the ratio $\chi(x, \mathbf{k})\Delta^{-1}(\mathbf{k})$ is normalized to 1 at infinity, and the poles at the points $\mathbf{k} = i\kappa_n$ with $n \in J$ are the only departures from analyticity. We therefore have

$$\frac{\chi(x, \mathbf{k})}{\Delta(\mathbf{k})} = 1 + \sum_{n \in J} \frac{\chi(x, i\kappa_n)}{(\mathbf{k} - i\kappa_n)\Delta'(i\kappa_n)}. \tag{A3}$$

This gives

$$\Phi(x, i\kappa_m) = \Phi(x, i\kappa_m), \quad m \in J, \tag{A4}$$

$$\Phi(x, i\kappa_m) = \Delta(i\kappa_m) e^{K_m(x)} \left(1 - i \sum_{n \in J} \frac{\chi(x, i\kappa_n)}{(\kappa_m - \kappa_n)\Delta'(i\kappa_n)}\right), \quad m \notin J, \tag{A5}$$

Then the condition (3.9) is equivalent to

$$\sum_{m \in J} \chi(i\kappa_m) \left[e^{K_m} \mathcal{D}_{ml} - i \sum_{n \notin J} \frac{\Delta(i\kappa_n) e^{K_n}}{(\kappa_n - \kappa_m)\Delta'(i\kappa_m)} \mathcal{D}_{nl} \right] = -\sum_{n \notin J} \Delta(i\kappa_n) e^{K_n} \mathcal{D}_{nl}. \tag{A6}$$

Comparing this with (4.6) for the τ-function (up to the standard similarity property), we get

$$\tau(x) = \det \left(\mathcal{D}_{ml} - i \sum_{n \notin J} \frac{\Delta(i\kappa_n) e^{K_n - K_m}}{(\kappa_n - \kappa_m)\Delta'(i\kappa_m)} \mathcal{D}_{nl} \right)_{\substack{m \in J \\ l = 1 \ldots, N_b}}. \tag{A7}$$

By inserting this τ-function in (2.7), we get the soliton solution, and different choices of the set J correspond to different equivalent formulation, of the soliton solution, in accordance with the result presented in Ref. 13.

We want to get a regularity condition for the potential equivalent to (2.18), but involving only the $N_a \times N_b$ submatrix $d_{(n)}$ introduced in (2.44) and not the full $\mathcal{N} \times N_b$ matrix $\mathcal{D}$.

This, of course, can be obtained by purely algebraic methods. However, we want to get this result by using the freedom in expressing the $\tau(x)$ function offered by (A7), thus showing its usefulness.

To get this result, we consider the evolution with respect to a multivalue time describing the evolution of the entire KPII hierarchy. Then, the exponents in the different formulations of (A7) are independent, and the requirement that their coefficients are greater than or equal to zero in one formulation implies the same requirement in all other formulations.

We can then state the following theorem.

Theorem A.1: *Let $\mathcal{D}$ be a $\mathcal{N} \times N_b$ matrix of the form*

$$\mathcal{D} = \pi\mathcal{B} \qquad \textit{with } \mathcal{B} = \begin{pmatrix} d \\ E_b \end{pmatrix}, \tag{A8}$$

where d is a real $N_a \times N_b$ matrix and E_b is the unit $N_b \times N_b$ matrix and where the $\mathcal{N} \times \mathcal{N}$ matrix

$$\pi = \|\pi_{mn}\|_{m,n=1}^{\mathcal{N}}, \qquad \pi_{mn} = \delta_{\pi(m),n} \equiv \delta_{m,\pi^{-1}(n)}, \tag{A9}$$

performs a permutation of the $\mathcal{N}$ rows of the matrix on the right according to the transformation

$$\pi: \quad (1,\ldots,\mathcal{N}) \to (\pi(1),\ldots,\pi(\mathcal{N}). \tag{A10}$$

Then all the maximal minors of the matrix $\mathcal{D}$ are non-negative if and only if the $N_a \times N_b$ matrix

$$\widetilde{d}_{kr} = (-1)^{1+l_r} d_{\pi(j_{(N_a+1-k)}),l_r}, \qquad k = 1,\ldots,N_a, \quad r = 1,\ldots,N_b, \tag{A11}$$

where $\{j_1, j_2, \ldots, j_{N_a}\}$ is a permutation of $\{1, 2, \ldots, N_a\}$, such that

$$\pi^{-1}(j_1) < \pi^{-1}(j_2) < \cdots < \pi^{-1}(j_{N_a}), \tag{A12}$$

and $\{l_1, l_2, \ldots, l_{N_b}\}$ is a permutation of $\{1, 2, \ldots, N_b\}$, such that

$$\pi^{-1}(N_a + l_1) < \pi^{-1}(N_a + l_2) < \cdots < \pi^{-1}(N_a + l_{N_b}) \tag{A13}$$

is totally non-negative according to the definition in Ref. 19, i.e., has all its minors non-negative.

Proof: From (A8) and (A9), we have

$$\mathcal{D}_{ml} = \mathcal{B}_{\pi(m),l}. \tag{A14}$$

Now let J be the set

$$J = \{\pi^{-1}(N_a + 1), \ldots, \pi^{-1}(\mathcal{N})\}, \tag{A15}$$

and let

$$\mathcal{C}J = \{\pi^{-1}(1), \ldots, \pi^{-1}(N_a) \tag{A16}$$

be its complement in the set $\{1, \ldots, \mathcal{N}\}$.

Then, from (A7), we have

$$\begin{aligned}\sum_{l=1}^{N_b} \chi\left(i\kappa_{\pi^{-1}(N_a+l)}\right) &\left[e^{K_{\pi^{-1}(N_a+l)}}\delta_{l,l'} + i\sum_{j=1}^{N_a} \frac{\Delta\left(i\kappa_{\pi^{-1}(j)}\right)e^{K_{\pi^{-1}(j)}}}{\left(\kappa_{\pi^{-1}(N_a+l)} - \kappa_{\pi^{-1}(j)}\right)\Delta'\left(i\kappa_{\pi^{-1}(N_a+l)}\right)} d_{j,l'}\right] \\ &= -\sum_{j=1}^{N_a} \Delta\left(i\kappa_{\pi^{-1}(j)}\right) e^{K_{\kappa_{\pi^{-1}(j)}}} d_{j,l'}.\end{aligned} \tag{A17}$$

For the τ function, we therefore obtain

$$\tau(x) = \det\left(\delta_{l,l'} + i\sum_{j=1}^{N_a} \frac{\Delta(i\kappa_{\pi^{-1}(j)})e^{K_{\pi^{-1}(j)} - K_{\pi^{-1}(N_a+l)}}}{\left(\kappa_{\pi^{-1}(N_a+l)} - \kappa_{\pi^{-1}(j)}\right)\Delta'(i\kappa_{\pi^{-1}(N_a+l)})} d_{j,l'}\right)_{l,l'=1,\ldots,N_b}. \tag{A18}$$

Let us re-order the rows in the matrix in the determinant from $l = 1, 2, \ldots, N_b$ to a permutation $l_1, l_2, \ldots, l_{N_b}$, such that

$$\kappa_{\pi^{-1}(N_a+l_1)} < \kappa_{\pi^{-1}(N_a+l_2)} < \cdots < \kappa_{\pi^{-1}(N_a+l_{N_b})}, \tag{A19}$$

and let us re-order the sum over j from $j = 1, 2, \ldots, N_a$ to a permutation $j_1, j_2, \ldots, j_{N_a}$, such that

$$\kappa_{\pi^{-1}(j_1)} < \kappa_{\pi^{-1}(j_2)} < \cdots < \kappa_{\pi^{-1}(j_{N_a})}. \tag{A20}$$

Then, up to an unessential sign, the τ-function can be rewritten as

$$\tau(x) = \det\left(E_{r,l'} + \sum_{k=1}^{N_a}\left(\frac{ie^{-K_{\pi^{-1}(N_a+l_r)}}}{\Delta'\left(i\kappa_{\pi^{-1}(N_a+l_r)}\right)}\right)\Lambda_{rk}\left(\Delta\left(i\kappa_{\pi^{-1}(j_k)}\right)e^{K_{\pi^{-1}(j_k)}}\right)d_{j_k,l'}\right)_{r,l'=1,\ldots,N_b}, \tag{A21}$$

where

$$E_{r,l'} = \delta_{l_r,l'}, \tag{A22}$$

$$\Lambda_{rk} = \frac{1}{\kappa_{\pi^{-1}(N_a+l_r)} - \kappa_{\pi^{-1}(j_k)}}. \tag{A23}$$

If we multiply the matrix in the determinant by E^{-1} from the right, we get

$$\tau(x) = \det(E_{N_b} + T) \tag{A24}$$

up to a sign, where $(r, s = 1, \ldots, N_b)$,

$$T_{rs} = \sum_{k=1}^{N_a}\left(\frac{ie^{-K_{\pi^{-1}(N_a+l_r)}}}{\Delta'\left(i\kappa_{\pi^{-1}(N_a+l_r)}\right)}\right)\Lambda_{rk}\left(\Delta\left(i\kappa_{\pi^{-1}(j_k)}\right)e^{K_{\pi^{-1}(j_k)}}\right)d_{j_k,l_s}. \tag{A25}$$

By using the Binet–Cauchy formula repeatedly, we have

$$\tau(x) = \sum_{n=0}^{N_b}\sum_{1\le r_1<r_2<\cdots<r_n\le N_b} T\begin{pmatrix} r_1, r_2, \ldots, r_n \\ r_1, r_2, \ldots, r_n\end{pmatrix}, \tag{A26}$$

and then

$$\begin{aligned} T\begin{pmatrix} r_1, r_2, \ldots, r_n \\ r_1, r_2, \ldots, r_n\end{pmatrix} &= \sum_{1\le k_1<k_2<\cdots<k_n\le N_a}\left(\prod_{m=1}^{n}\frac{ie^{-K_{\pi^{-1}(N_a+l_{r_m})}}}{\Delta'\left(i\kappa_{\pi^{-1}(N_a+l_{r_m})}\right)}\right) \\ &\times\left(\prod_{m=1}^{n}\Delta\left(i\kappa_{\pi^{-1}(j_{k_m})}\right)e^{K_{\pi^{-1}(j_{k_m})}}\right)\Lambda\begin{pmatrix} r_1, r_2, \ldots, r_n \\ k_1, k_2, \ldots, k_n\end{pmatrix} d\begin{pmatrix} k_1, k_2, \ldots, k_n \\ r_1, r_2, \ldots, r_n\end{pmatrix}. \end{aligned} \tag{A27}$$

Now, if we require the coefficients of the exponents be non-negative, we get

$$\begin{aligned} &\det E\left(\prod_{m=1}^{n}\frac{i}{\Delta'\left(i\kappa_{\pi^{-1}(N_a+l_{r_m})}\right)}\right)\left(\prod_{m=1}^{n}\Delta\left(i\kappa_{\pi^{-1}(j_{k_m})}\right)\right) \\ &\quad\times\Lambda\begin{pmatrix} r_1, r_2, \ldots, r_n \\ k_1, k_2, \ldots, k_n\end{pmatrix} d\begin{pmatrix} k_1, k_2, \ldots, k_n \\ r_1, r_2, \ldots, r_n\end{pmatrix} \ge 0. \end{aligned} \tag{A28}$$

Since

$$\Lambda\begin{pmatrix} r_1, r_2, \ldots, r_n \\ k_1, k_2, \ldots, k_n\end{pmatrix} = \frac{\displaystyle\prod_{1\le i<m\le n}\left(\kappa_{\pi^{-1}(N_a+l_{r_i})} - \kappa_{\pi^{-1}(N_a+l_{r_m})}\right)\left(\kappa_{\pi^{-1}(j_{k_m})} - \kappa_{\pi^{-1}(j_{k_i})}\right)}{\displaystyle\prod_{i,m=1}^{n}\left(\kappa_{\pi^{-1}(N_a+l_{r_i})} - \kappa_{\pi^{-1}(j_{k_m})}\right)}, \tag{A29}$$

the denominator in (A29) cancels with $\prod_{m=1}^{n}\Delta(i\kappa_{\pi^{-1}(j_{k_m})})$ in (A28), and after reversing the order of the r_i, we have

$$\left(\prod_{m=1}^{n}\Delta(i\kappa_{\pi^{-1}(j_{k_m})})\right)\Lambda\begin{pmatrix} r_n, r_2, \dots, r_1 \\ k_1, k_2, \dots, k_n \end{pmatrix} > 0. \tag{A30}$$

Finally, since

$$-i\Delta'(i\kappa_{\pi^{-1}(N_a+l_{r_m})}) = (-1)^{l_{r_m}-1}\prod_{s=1, s\neq m}^{n}\left|-\kappa_{\pi^{-1}(N_a+l_{r_m})}+\kappa_{\pi^{-1}(N_a+l_{r_s})}\right|, \tag{A31}$$

we get the condition

$$(-1)^{\sum_{m=1}^{n}(1+l_m)} d\begin{pmatrix} k_n, k_{n-1}, \dots, k_1 \\ r_1, r_2, \dots, r_n \end{pmatrix} \geq 0. \tag{A32}$$

We conclude that the matrix

$$\widetilde{d}_{kr} = (-1)^{1+l_r} d_{\pi(j_{(N_a+1-k)}), l_r}, \qquad k = 1, \dots, N_a, \quad r = 1, \dots, N_b, \tag{A33}$$

is totally non-negative. ∎

The following corollary follows easily from this theorem.

Corollary A.1: An $N_a \times N_b$ matrix $\widetilde{d}$ is totally non-negative, i.e., all its minors are non-negative, if and only if all the maximal minors of the $\mathcal{N} \times N_b$ matrix

$$\mathcal{D} = \begin{pmatrix} d \\ E_b \end{pmatrix}, \tag{A34}$$

where

$$d_{jl} = (-1)^{l+1}\widetilde{d}_{N_a+1-j,l}, \qquad j = 1, \dots, N_a, \quad l = 1, \dots, N_b, \tag{A35}$$

and E_b is the unit $N_b \times N_b$ matrix, are non-negative. ∎

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