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MARINE PHYSICS

Transformation of Internal Waves over an Uneven Bottom: Analytical Results

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Abstract—The transformation of internal waves over the oceanic shelf of variable depth is studied analytically within a linear theory of a two-layer flow. It is shown that, at a specific character of depth variation, the internal wave propagates without reflection from the slope even if it is sufficiently steep. The properties of such progressive waves are studied—their form and the current structure in the upper and lower layers. The transformation of the wave propagating from the open ocean, where the depth is assumed to be constant, is considered. It is shown that the wave is transformed at the shelf edge and does not change its form in the course of time during its further propagation over the shelf. The height and form of the internal wave are calculated at the interface of the transition of the two-layer flow into the one-layer flow. Applications of the developed analytical theory to the estimation of internal wave transformation over a real shelf are discussed.

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INTRODUCTION

The transformation of the internal wave field observed over the shelf is now actively modeled within different models of the stratified ocean from weakly nonlinear to fully linear models. We shall present here the latest reviews and books on this problem [16, 20, 26]. The role of the analytical models developed for the idealized conditions is sufficiently important because it makes it possible to study the effects in a wide range of parameter variation and deeply understand the physics of the processes. However, the number of such models in the theory of internal waves is extremely small. All of them are related to the vertical propagation of a monochromatic wave in the ocean with nonhomogeneous distribution of the buoyancy frequency or to the description of the mode structure of internal waves [3, 6, 11]. In the case of the propagation of internal wave packets over an uneven bottom, approximate analytical expressions are obtained on the basis of the asymptotic methods in the case of a smoothly varying bottom and slow variation in the vertical stratification of density and currents in the horizontal direction [7, 10, 27]. Such methods can also be applied to the analysis of the transformation of a solitary weakly nonlinear wave (soliton) in the coastal zone [10, 15, 19, 21, 22]. Approximate solutions are known for the transformation of an internal wave in a two-layer flow with a step at the bottom [17, 23]. Here we present one exact analytical solution in the model of internal wave transformation over uneven bottom in a two-layer ocean, which shows the possibility of nonreflecting wave propagation over a slope even if it is sufficiently steep. Such a model is also useful for testing the newly developed numerical models of internal waves. First, we shall describe the structure of the wave field over the variable bottom of a special shape, which we shall call the "nonreflecting bottom" (Section 2). Then, we shall consider the transformation of this field by the waves arriving from the open ocean and its transformation in the transition zone when a two-layer flow becomes a one-layer flow (Section 3). The results are summarized in the Conclusions.

1. PROGRESSIVE WAVES IN A TWO-LAYER OCEAN OF VARIABLE DEPTH

Here we consider the propagation of waves in a two-layer ocean of variable depth (Fig. 1). Using the rigid lid approximation at the free surface and the Boussinesq approximation, we write linearized equations for shallow water [6, 7, 8, 12]:

$$h_1 u_1 + h_2(x) u_2 = 0, \tag{1}$$

$$\frac{\partial(u_2 - u_1)}{\partial t} + g'\frac{\partial\eta}{\partial x} = 0, \tag{2}$$

$$\frac{\partial\eta}{\partial t} + \frac{\partial}{\partial x}[h_2(x)u_2] = 0, \tag{3}$$

where η is the displacement of the interface between fluids of different densities, u_1 and u_2 are average velocities of the flows in the upper and lower layers, h_1 and $h_2(x)$ are the depths of the upper and lower layers, and $g' = g(\rho_2 - \rho_1)/\rho_1$ is the reduced acceleration due to gravity, We exclude the flow velocity in the upper layer from (1). Then Eq. (2) is written as

$$\frac{\partial u_2}{\partial t} + g' \frac{h_1}{h_1 + h_2} \frac{\partial \eta}{\partial x} = 0, \tag{4}$$

and system of equations (3) and (4) becomes closed. These equations are reduced to the wave equation for the displacement of the interface boundary,

$$\frac{\partial^2 \eta}{\partial t^2} - \frac{\partial}{\partial x}\left[c^2(x)\frac{\partial \eta}{\partial x}\right] = 0, \tag{5}$$

where

$$c^2 = g' \frac{h_1 h_2}{h_1 + h_2} \tag{6}$$

determines the velocity of long waves' propagation in a two-layer ocean.

An equation of type (5) is the main equation in the theory of waves in nonhomogeneous media not only for internal waves [1, 2]. In the general case, it describes various processes of wave propagation, reflection, scattering, and formation of resonance effects. It has been noted for a long time in mathematical physics that, under specific conditions, Eq. (5) can be reduced to a wave equation with constant coefficients (see, for example, [25]), which rather simplifies its solution. Recently, an interpretation was given for solutions as nonreflecting waves in a nonhomogeneous medium that provide the possibility of the energy transfer over long distances without scattering. These phenomena were considered for the surface waves in the ocean [5, 9]. Actually, solutions of the same class were also found for vertical propagation of internal waves in the ocean with nonhomogeneous distribution of the buoyancy frequency [3, 6, 11, 18]. In this case the internal wave energy can spread into the deep layers of the ocean. Here we shall discuss the possibility of the existence of nonreflecting progressive internal waves in a two-layer ocean of variable depth. Let us reproduce briefly the procedure of obtaining such solutions within Eq. (5); see [18]. It is based on the presentation of the solution in the following form:

$$\eta(t, x) = B(x)\Phi(t, \tau), \tag{7}$$

We shall see below that here the new variables have the sense of amplitude $B(x)$ and phase: the time of propagation $\tau(x)$. Then Eq. (5) is reduced to an equation of the Klein–Gordon type with variable coefficients for the new unknown function $\Phi(t, \tau)$:

$$B\left[\frac{\partial^2 \Phi}{\partial t^2} - c^2\left(\frac{d\tau}{dx}\right)^2 \frac{\partial^2 \Phi}{\partial \tau^2}\right] - \left[c^2 \frac{dB}{dx}\frac{d\tau}{dx} + \frac{d}{dx}\left(c^2 B \frac{d\tau}{dx}\right)\right]\frac{\partial \Phi}{\partial \tau} - \frac{d}{dx}\left(c^2 \frac{dB}{dx}\right)\Phi = 0. \tag{8}$$

Special requirements should be satisfied to make the coefficients of Eq. (8) constant (then its wave solutions have a clear physical sense). The first of them is obvious from the structure of the wave operator in square brackets. It leads to the following definition of phase:

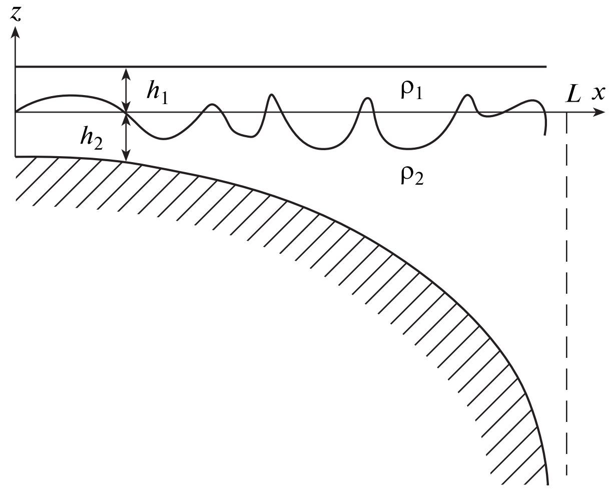

Fig. 1. Geometry of the problem.

$$\tau(x) = \int \frac{dx}{c(x)}, \tag{9}$$

Here we selected the plus sign for definiteness. The second parenthesis in (8) should turn to zero providing nongrowing spatial solutions, which leads to the following condition:

$$B(x) = \frac{\text{const}}{\sqrt{c(x)}}. \tag{10}$$

Finally, the coefficient at the last term in (8) should be proportional to B if we want to transform it to an equation with constant coefficients. Here we shall consider only one specific case, in which the last additive in (8) vanishes, which, with (10) taken into account, leads to the following dependence of $c(x)$:

$$c(x) = \tilde{c}_0 (x/L)^{2/3}, \quad \tilde{c}_0 = \sqrt{g' h_1}, \tag{11}$$

where L characterizes the width of the shelf. In this case, Eq. (8) is reduced to the wave equation with constant coefficients

$$\frac{\partial^2 \Phi}{\partial t^2} - \frac{\partial^2 \Phi}{\partial \tau^2} = 0. \tag{12}$$

Equation (12) is well studied in mathematical physics, and it is easy to formulate and solve the Cauchy problem for it. In particular, the solutions of Eq. (12) can be presented as a sum of the two waves propagating to the opposite sides (without the boundary conditions yet being taken into account), so that solution (7) of the initial wave equation with variable coefficients describes the progressive waves, which exist despite the velocity variation with distance. This

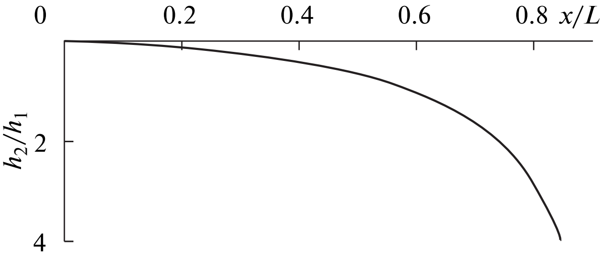

Fig. 2. Depth of the lower layer satisfying relation (13).

is precisely why we call these waves nonreflecting waves, keeping in mind that they reflect not at the underwater slope but only at its boundaries. The objectives of our research are the physical properties of such solutions in the theory of internal waves.

First of all, we consider the types of bottom topography, where nonreflecting propagation of progressive waves is possible. Using (6) and (11), we find the depth of the lower layer,

$$h_2(x) = h_1\frac{(x/L)^{4/3}}{1-(x/L)^{4/3}}. \tag{13}$$

The form of a nonreflecting slope is shown in Fig. 2. Formally, function $h_2(x)$ is defined only in the interval $0 < x < L$. At point $x = 0$, the fluid becomes homogeneous and, if we consider that, in the region $x < 0$, the depth of the basin coincides with the depth of the upper layer h_1 is or less than it, then internal waves cannot propagate in the region $x < 0$. Actually, point $x = 0$ is the edge for the surface waves: internal waves approaching this point can either reflect or break (we shall discuss this problem below). At the right boundary ($x = L$), the depth of the basin increases sharply, although the velocity of the propagation of internal waves remains finite and tends to $\tilde{c}_0$ (within the shallow water theory). It is clear that sewing of the bottom profile with a more realistic depth profile is needed in this region (or we have to reject the shallow water approximation). This will be discussed in Section 3.

Let us now describe the structure of the progressive waves propagating over the bottom topography (13). The displacement of the interface boundary in a progressive wave (propagating to the left for definiteness) is

$$\eta(t,x) = B(x)\Phi[t+\tau(x)],$$
$$B(x) = \frac{\text{const}}{\sqrt{c(x)}} \sim \frac{1}{x^{1/3}}, \tag{14}$$

where function $\Phi(t)$ should describe a sufficiently short pulse with rapid decay at the ends providing localization of the wave within interval $0 < x < L$. The temporal form of the internal wave at different spatial point remains constant (only its amplitude and phase change); however, its spatial structure changes in time. As seen from (14), the amplitude of the wave infinitely grows when the wave approaches the edge ($x = 0$). Observations show that internal waves can either break at the slopes or completely reflect from them [13, 14, 24]. In the first case, nonlinear effects should be taken into account and, here, the capabilities of the analytical solutions are restricted. In the second case, similarly to the long surface waves [4], the problem can be solved under specific conditions in the linear approximation. Precisely this case ill be considered below. When the wave approaches the other boundary ($x = L$), its amplitude remains finite. However, a change in the bottom topography actually occurs near this boundary because the depth never approaches infinity and the transformation of the wave in this region should be studied separately (Section 3).

The current induced by the progressive wave in the lower layer is found from (4), and the current in the upper layer is found from (1):

$$u_1(x,t) = -B(x)\frac{c(x)}{h_1}U(\xi,x),$$
$$u_2(x,t) = B(x)\frac{c(x)}{h_2(x)}U(\xi,x), \tag{15}$$

$$U(\xi,x) = \Phi(\xi) - \frac{1}{2}\frac{dc(x)}{dx}\int\Phi(\xi)d\xi,$$
$$\xi = t+\tau(x). \tag{16}$$

The expressions for the current velocities contain two additives: the first of them repeats the form of the displacement wave, and the second is determined by the integral of this function. As a result, the form of the velocity wave, unlike the displacement wave, changes in time at different points. We note that in the vicinity of the edge ($x = 0$), the current velocity in the upper layer remains finite, while in the lower layer it increases infinitely inversely proportional to the depth of the lower layer. At the same time, the water discharge in each layer remains finite and the total discharge is zero. At the second boundary ($x = L$), the velocity field remains finite, and the current velocity in the lower layer decreases owing to the unlimited increase in the thickness of the layer.

Formally, the progressive displacement wave should change its sign in time to provide the conversion of integral (16) at large times. However, in reality, the internal wave propagates through the shelf zone with a specific law of depth variation (13) during a finite time, so that the wave field is limited.

2. TRANSFORMATION OF AN INTERNAL WAVE AT THE BOUNDARIES OF THE NONREFLECTING BOUNDARY

We already noted that progressive waves over a nonreflecting bottom exist only in a limited interval, so that it is important to investigate their generation by waves propagating from the open ocean. In a sketch of a nonreflecting depth profile over slope (13), it should be sewed with the flat bottom at a point, which we denote here as $x = x_0$ (the geometry of the problem is shown in Fig. 3).

A limited wave field over a nonreflecting slope has a form of a standing wave (first, we consider the monochromatic processes):

$$\eta(x, t) = A_t\sqrt{\frac{c_0}{c(x)}}[e^{i\omega(t+\tau)} - e^{i\omega(t-\tau)}] = 2iA_t\sqrt{\frac{c_0}{c(x)}}\sin(\omega\tau)\exp(i\omega t), \quad (17)$$

where

$$\tau(x) = \int_0^x \frac{dy}{c(y)} = \frac{3x_0}{c_0}\sqrt{\frac{c(x)}{c_0}}. \quad (18)$$

Here A_t is the amplitude of the wave that propagated through point $x = x_0$ and c_0 is the velocity of the internal wave at the shelf edge x_0 ($c_0 = \tilde{c}_0\,(x_0/L)^{2/3}$).

In the region of the constant depth, we have a standard superposition,

$$\eta(x, t) = A_i\exp\left[i\omega\left(t + \frac{x - x_0}{c_0}\right)\right] + A_r\exp\left[i\omega\left(t - \frac{x - x_0}{c_0}\right)\right]. \quad (19)$$

The amplitude of the reflected A_i and transformed A_t waves are found from the sewing condition at the shelf edge (at point x_0). This is the continuity of displacement (pressure) and its spatial derivative (discharge). The obtained algebraic system is easily solved; in particular, the amplitude of the transformed wave is

$$A_t = A_i\left[\frac{1}{\exp(i\omega\tau_0) - \dfrac{\sin(\omega\tau_0)}{\omega\tau_0}}\right]. \quad (20)$$

It is determined by only one parameter $\omega\tau_0$, where τ_0 is the propagation time of the internal wave from the shelf edge to the water edge;

$$\tau_0 = \int_0^{x_0}\frac{dx}{c(x)} = \frac{3x_0}{c_0}. \quad (21)$$

This parameter has a clear physical sense as the ratio of the wave runtime over the slope to the period of the wave. As was expected, short waves ($\omega\tau_0 \gg 1$) almost completely propagate over the slope, while long waves ($\omega\tau_0 \ll 1$) are completely reflected from the slope. Using the spectral coefficients of reflection and transformation, it is possible to calculate the wave field in the vicinity of the shelf break. We are interested here only in the transformed wave, which will further propagate to the edge as a progressive wave. At the shelf edge, we use the inverse Fourier–Laplace transformation for the wave and obtain the following Fredholm equation of the second kind:

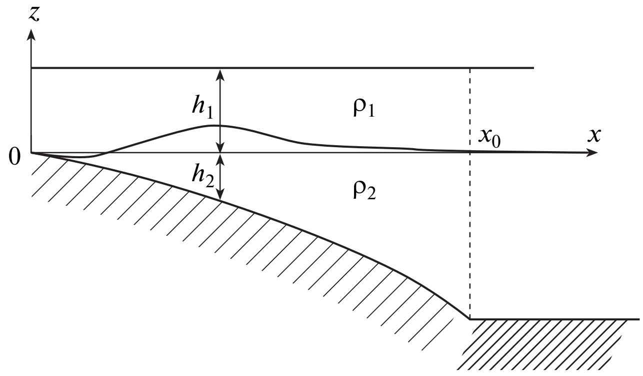

Fig. 3. Geometry of the problem: nonreflecting slope is sewed with the shelf break and the edge.

$$\eta_t(t) - \frac{1}{2\tau_0}\int_{-\tau_0}^{\tau_0}\eta_t(\xi + t - \tau_0)d\xi = \eta_i(t). \quad (22)$$

We present here an approximate analytical solution of Eq. (22) for the internal wave, which has the form of the Korteweg de Vries soliton:

$$\eta_i(t) = A\mathrm{sech}^2\left(\frac{t}{T}\right), \quad (23)$$

where T is the characteristic duration of the soliton. Then in the case $T \ll \tau_0$, we substitute in the integral η_t for η_i and obtain

$$\eta_t = A\left\{\mathrm{sech}^2\left(\frac{t}{T}\right) + \frac{T}{2\tau_0}\left[\tanh\left(\frac{t}{T}\right) - \tanh\left(\frac{t - 2\tau_0}{T}\right)\right]\right\} \quad (24)$$

As was expected, the short pulse practically completely propagates to the slope and almost does not change its form. A positive tail follows the soliton after the transformation at the inflection point, which ends at the moment of the arrival of the wave reflected from the inflection point. The wave form is presented in Fig. 4 for $\tau_0 = 8T$.

In the course of time, as the wave propagates over the slope, its form does not change, but the amplitude increases when the wave approaches the water edge according to relation (14). Of course, the wave form in the space changes in time (see Fig. 5, in which the transformation of the soliton is shown without taking into account a weak positive tail). As the wave approaches the water edge, its form becomes asymmetrical and the front becomes steeper.

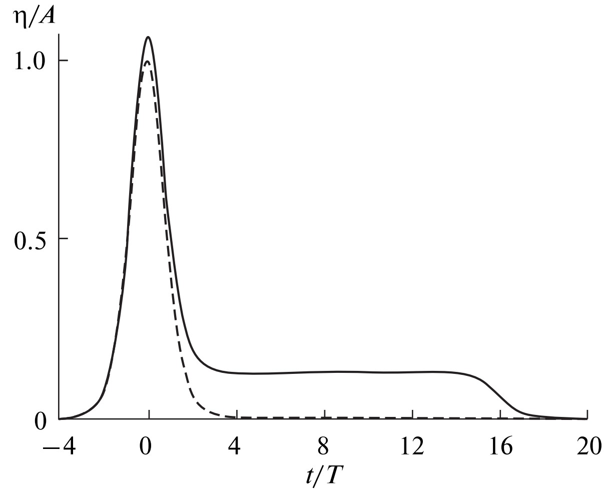

Fig. 4. Incident (dashed line) and transformed (solid line) waves at the shelf break.

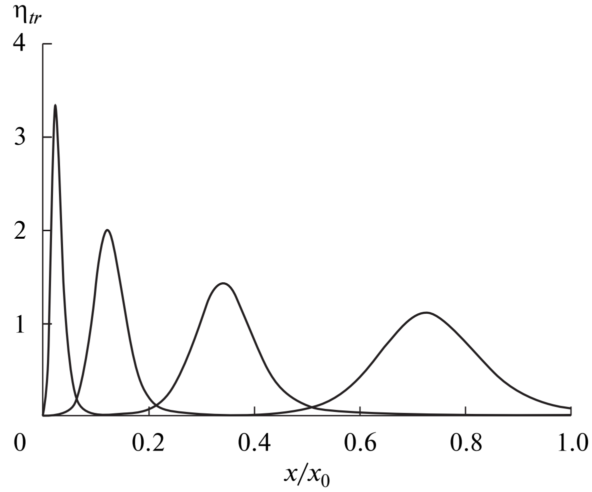

Fig. 5. Wave transformation at the nonreflecting slope ($\tau_0 = 4T$).

Point $x = 0$ corresponds to the edge where the two-layer fluid becomes a one-layer fluid. According to (17), the wave is limited at the edge and the displacement of the interface boundary between two fluids at this point can be determined analytically through the form of the transformed wave at the shelf edge:

$$\eta(x = 0, t) = 2\tau_0 \frac{d\eta_{tr}(t - \tau_0)}{dt}. \tag{25}$$

A similar relation can also be obtained for the runup of the surface waves over a nonreflecting beach [9]. Thus, the maximum displacement of the interface boundary between the two media is proportional to the ratio of the arrival time to the wave period. For the short waves it can be sufficiently high. It is clear, however, that the waves of large amplitudes would most likely break over the slope, which is frequently observed [13, 14, 22]. The investigation of the breaking conditions for internal waves over the slope is quite a difficult problem that has yet to be solved. At the same time, the scenario of the wave transformation without reflection at the slope seems quite realistic.

CONCLUSIONS

It was shown that progressive internal waves can exist in a two-layer nonhomogeneous ocean if the depth variation law is specific even if the bottom slope is steep. Such progressive waves exist only in a limited shelf region without reflection from the slope. Their form does not change in time, but they deform in space, becoming steeper at the coast. The process of their generation by the waves propagating from the open ocean was investigated. In the general case, the wave form transforms at the shelf edge. However, if the wave is sufficiently short (compared with the arrival time to the coast), it propagates over the slope with only slight deformation. If the initial amplitude of the internal wave is quite small, its runup can occur without breaking, and then the characteristics of the runup can be calculated analytically. The importance of the analytical solutions obtained here is related to the demonstration of the possibility of strong wave amplification over the slope even if it is not smooth. The other application of this theory is the fact that it can be used to test the numerical models of internal wave transformation in the ocean.

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