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Boris Brodsky, Henry Penikas, Irina Safaryan

COPULA STRUCTURAL SHIFT IDENTIFICATION

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Boris E. Brodsky[1], Henry I. Penikas[2], Irina A. Safaryan[3]

COPULA STRUCTURAL SHIFT IDENTIFICATION

This paper aims at presenting the research results of revealing a structural shift in copula-models of multivariate time-series. A nonparametric method of structural shift identification and estimation is used. The asymptotical characteristics (the probabilities of the I-type and II-type errors, and the probability of the estimation error) of the proposed method are analyzed. The simulation method verification results for Clayton and Gumbel copulas are presented and discussed. The empirical part of the paper is devoted to structural shift identification for multivariate time series of interest rates for Euro-, US Dollar- and Ruble-zones. The empirical application provides strong evidence of the efficiency for the proposed method of structural shift identification.

Key words: Copula, structural shift, Kolmogorov-Smirnov statistics, interest rates.

JEL Classification: C14, C46.

1. Introduction

Let us take a continuous random vector $\mathbf{X} = \{X_1, \ldots, X_d\}$ with the joint cumulative distribution function (d.f.) marked as V and the marginal distribution functions of its components $F_1, \ldots, F_d$. The copula for the joint d.f. V then can be written as follows:

$$V(x_1, \ldots, x_d) = G(F_1(x_1), \ldots, F_d(x_d)),$$

where G is the only continuous cumulative d.f. which has univariate marginals equally distributed on $[0,1]$.

Copula belongs to unknown G-type function:

$$\mathsf{A} = \{G_\theta : \theta \in \Theta\},$$

where Θ - is an open set in R^p space.

The two most well-known books containing detailed descriptions of parametric copula families are those of Harry Joe [Joe (1997)] and Roger Nelsen [Nelsen (2006)]. Copulas are often of use in empirical applications in modern actuarial calculations, econometrics and hydrology (see for example [Frees and Valdez (1998)], [Cui and Sun (2004)], [Genest and Favre (2007))]. Nevertheless, they are increasingly applied to solving financial and risk-management tasks (e.g. [Cherubini et al. (2004)], [McNeil et al. (2005)]).

[1] Central Economics and Mathematics Department. National Research University Higher School of Economics. Applied Macroeconomics Department. Professor.

[2] National Research University Higher School of Economics. Mathematical Economics and Econometrics Department (Senior Lecturer), International Laboratory of Decision Choice and Analysis (Junior Research Fellow).

[3] Institute for Informatics and Automation Problems of the National Academy of Sciences of the Republic of Armenia; Research Fellow.

This paper aims at the problem of structural shift identification in copula models. Non-stationary copulas in discrete time are analyzed for their structural stability (structural parameters may change abruptly at some unknown time). This problem is of vital importance, because the majority of real financial time series are unstable and subject to structural shifts (the notorious example is the world financial crisis of 2007 – 2009 which revealed the inadequacies of most of copula models used in financial calculations). We argue that structural shift identification in copula models is of vital importance for furthering empirical research of financial markets.

Recent scientific findings made in the field of copula modeling can be generally classified into two principal groups:

(1) Papers devoted to the estimating and goodness-of-fit testing of parametric copula models (e.g. for Gaussian copulas [(Malevergne and Sornette (2003))], for Clayton copulas [(Shih, (1998)], [Glidden, (1999)]; [Cui and Sun, (2004)]); and

(2) Articles on non-parametric methods of goodness-of-fit testing of the copula-models, including blanket tests (e.g. [Genest et al. (2006)], [Breymann et al. (2003)], [Dobric and Schmid (2005)], [Junker and May (2005)]).

This paper proposes a non-parametric way of a change-point (an instant of a structural shift) estimation in copula models. The exact problem statement is formulated below.

2. Problem Statement and Proposed Solution

We start from the selection $\{\mathbf{X}_1,\dots,\mathbf{X}_N\}$ of independent R^d-dimensional vectors with the cumulative d.f. $V_1,...,V_N$.

Suppose there exist two alternatives. The null hypothesis H_0 (cf. (2) below) is that the copula remains the same, that is $G_1 = G_2$. The alternative is that the copula changes after some instant $m = [\theta N]$. Here we suppose that all the marginal d.f.s $F_1,...,F_d$ remain unchanged. To summarize, the joint d.f. $V_i(x_1,...,x_d)$ at each time *i* can be represented by the following system (1):

$$V_i(x_1,...,x_d) = \begin{cases} G_1(F_1(x_1),...,F_d(x_d)), & 1 \le i \le m, \\ G_2(F_1(x_1),...,F_d(x_d)), & m < i \le n. \end{cases} \qquad (1)$$

Thus we want to test the following null hypothesis (2):

$$H_0 : G_1 = G_2 \qquad (2)$$

Given H_0 is rejected we are interested in estimating the instant m of copula structural shift. In other words, we are testing whether there exists a structural shift in a pattern of comovement of observed vector components. The goal is to propose a method having:

(1) I[st] ("false alarm") and II[nd] ("false calmness") type estimation errors relatively small (tending to zero with increasing sample size N);

(2) the change-point parameter estimate $\hat{\theta}_N$ to be consistent, that is tending to the true value of θ given the increase of the sample size N.

The proposed method is based on the non-parametric approach. Let us take empirical copula-processes so that for every $l = 1, \dots, N-1$ we have the following:

$$\begin{aligned} D_l(u) &= \frac{1}{l}\sum_{i=1}^{l} I(U_{i,l} \le u) = \frac{1}{l}\sum_{i=1}^{l}\prod_{j=1}^{d} I(U_{ij,l} \le u_j) \\ D_{N-l}(u) &= \frac{1}{N-l}\sum_{i=l+1}^{N} I(U_{i,N-l} \le u) = \frac{1}{N-l}\sum_{i=l+1}^{N}\prod_{j=1}^{d} I(U_{ij,N-l} \le u_j), \end{aligned} \tag{3}$$

where $U_{i,l} = (U_{i1,l}, \dots, U_{id,l})$ and for every $j = [1, \dots, d]$

$$\begin{aligned} U_{ij,l} &= \frac{l}{l+1} F_{j,l}(X_{ij}) = rank(X_{ij})/(l+1), \quad 1 \le i \le l, \\ U_{ij,N-l} &= rank(X_{ij})/(N-l+1), \quad l+1 \le i \le N. \end{aligned} \tag{4}$$

Aiming at a step-by-step structural shift identification we fix the constant N and use the following statistics as a modification of the Kolmogorov-Smirnov test:

$$\Psi_{l,N-l}(u) = (D_l(u) - D_{N-l}(u))\sqrt{l(N-l)}/N \tag{5}$$

and

$$T_N = \max_{[\beta N] \le l \le [(1-\beta)N]} \sup_u |\Psi_{l,N-l}(u)|. \tag{6}$$

Therefore we arrive at the following estimate (7) of an unknown change-point:

$$\hat{m}_N \in \arg\max_{[\beta N] \le l \le [(1-\beta)N]} \left(\sup_u |\Psi_{l,N-l}(u)| \right) , \tag{7}$$

Then the structural shift parameter estimate will be as follows $\hat{\theta}_N = \hat{m}_N / N$.

To verify whether the change-point found is a good estimate of the true structural shift point, we are using three performance measures listed below:

1) I[st] type error probability ("false alarm"):

$$\alpha_N = P_0\{T_N > C\}, \tag{8}$$

where $C > 0$ is the decision threshold that we accept in order to test the null hypothesis of structural shift absence;

2) II[nd] type error probability:

$$\delta_N = P_m\{T_N \le C\}. \tag{9}$$

3) The probability of estimation error: for $0 < \varepsilon < 1/2$ we use the following measure:

$$\gamma_N = P_m\{|\hat{\theta}_N - \theta| > \varepsilon\}. \tag{10}$$

3. Major Findings

To review, the major assumption is that $X_1, \dots, X_n$ are independent random d-dimensional vectors with continuous univariate marginal d.f. Then it follows that random variables $U_{i,l}$ being defined above in (3) – (4) are independent at different times $i = 1, \dots, l$. Besides, their distributions are the same under the null hypothesis meaning the absence of structural shift. The Cramer condition $E_0 \exp(tU_{i,l}) < \infty$ given $|t| < T$ for some $T > 0$ is still satisfied.

Theorem 1 provides an upper exponential estimate for the Ist type error probability of the proposed method.

Theorem 1.

$$\alpha_N \le L_1 \exp(-L_2 C^2 N), \qquad (11)$$

where L_1, L_2 are positive constants not dependent on N.

The proof of theorem 1 comes from the logic described below. Given continuous marginal distributions, it is true (cf. [Tsukahara (2005)]) that under the null hypothesis $H_0 : G_1 = G_2$ and $[\beta N] \le l \le [(1-\beta)N]$ we have the following:

$$\sqrt{l}(D_l(u) - G_1(u)) \to W_1(u),$$
$$\sqrt{N-l}(D_{N-l}(u) - G_1(u)) \to W_2(u),$$

where $W_1(\cdot), W_2(\cdot)$ are independent Wiener processes on $[0,1]^d$, and the symbol "$\to$" is used to signify weak convergence in $D[0,1]^d$ space as $N \to \infty$. Therefore it is true that:

$$(D_l(u) - D_{N-l}(u)) \frac{\sqrt{l(N-l)}}{N} \to \frac{1}{\sqrt{N}}((1-\frac{l}{N})^{1/2} W_1(u) - (\frac{l}{N})^{1/2} W_2(u))$$

Based on the result above and using the exponential estimates for the probability of the Wiener process intersecting the horizontal border we arrive at the theorem 1 result.

By analogy we obtain the exponential upper estimates for the IInd type error and the estimation of the probability of the error. The following theorem holds true:

Theorem 2.

Denote $\eta = \sup_u |G_1(u) - G_2(u)|$ and assume that $0 < C < \eta/4$. Let $d = \eta/4 - C$. Then the following is true:

$$\begin{aligned} \delta_N &\le L_1 \exp(-L_2 \min(d, d^2) N) \\ \gamma_N &\le C_1 \exp(-C_2 \min(\varepsilon, \varepsilon^2) N) \end{aligned}, \qquad (12)$$

where L_1, L_2, C_1, C_2 are positive constants not dependent on N.

Proof.

Below the main idea of the proof of theorem 2 is presented.

We start from the case of $[\beta N] \le l \le m$. Since $D_l(u) = \frac{1}{l}\sum_{i=1}^{l} I(U_{i,l} \le u)$, we can write

$$ED_l(u) = G_1(u).$$

Then we conclude that:

$$D_{N-l}(u) = \frac{1}{N-l}\left(\sum_{i=l+1}^{m} I(U_{i,N-l} \le u) + \sum_{i=m+1}^{N} I(U_{i,N-l} \le u)\right)$$

Therefore,

$$ED_{N-l}(u) = \frac{1}{N-l}((m-l)G_1(u) + (N-m)G_2(u)).$$

Then,

$$E(D_l(u) - D_{N-l}(u)) = G_1(u)(1 - \frac{m-l}{N-l}) - G_2(u)\frac{N-m}{N-l} = \frac{N-m}{N-l}(G_1(u) - G_2(u)).$$

Thus,

$$\max_{l \le m} \sup_u E\Psi_{l,N-l}(u) = \frac{\sqrt{m(N-m)}}{N} \sup_u |G_1(u) - G_2(u)|.$$

The case $m < l \le [(1-\beta)N]$ is considered in the same way. Note that

$$\max_m \sqrt{m(N-m)}/N \le 1/4 .$$

We have obtained the upper estimate for the expectation for the T_N statistics. With regard to the stochastic additive component of the statistics (like in theorem 1), the upper exponential estimates for the error probability (12) comes from the upper exponential estimates for the sums of independent, identically distributed and centered random variables satisfying the Cramer condition (see [Petrov (1987)]).

Now the simulation method verification results will be discussed.

4. Simulation Method Verification

The proposed method was tested using bidimensional vectors whose joint d.f. were characterized by (1) Clayton copula; and (2) Gumbel copula.

Clayton copula: for any $u, v \in (0,1)$ and $\kappa > 0$:

$$C_\kappa(u,v) = (u^{-\kappa} + v^{-\kappa} - 1)^{-1/\kappa}$$

Gumbel copula: for any $u, v \in (0,1)$ and $\kappa > 0$:

$$C_\kappa(u,v) = \exp[-\{(-\log u)^{1/\kappa} + (-\log v)^{1/\kappa}\}^\kappa]$$

We assume copula function does not change throughout the period, whereas its parameter κ might change at some time $m = [\theta N]$.

We begin from the analysis of the critical bounds for the method proposed for different sample sizes and copula types. Initially, we deal with a homogeneous sample, that is, without a structural shift. For each sample size N, the experiment was independently simulated 500 times. The 95th and 99th quantiles for T_N statistics maximum were estimated. 95th quantile values were then used as critical bounds for the rejection of the null hypothesis given the existence of structural shift. Simulation results are presented in tables 1-2.

Table 1

Critical Bounds, Clayton Copula, Homogeneous Set $\kappa = 0,3$

N	50	100	200	300	500	700	1000	1500	2000
95%	0,1156	0,0850	0,0615	0,0492	0,0372	0,0314	0,0278	0,0213	0,0197
99%	0,1343	0,0945	0,0674	0,0550	0,0426	0,0348	0,0323	0,0232	0,0214

Table 2

Critical Bounds, Gumbel Copula, Homogeneous Set $\kappa = 0,3$

N	50	100	200	300	500	700	1000	1500	2000
95%	0,1033	0,0749	0,0508	0,0402	0,0313	0,0243	0,0206	0,0158	0,0146
99%	0,1187	0,0836	0,0585	0,0461	0,0343	0,0292	0,0233	0,0168	0,0154

As tables 1-2 show, the critical bounds are not very sensitive to the concrete copula underlying the observations. It permits us to undertake robust parameter calibration procedure for the purpose of structural shift identification and estimation. The respective results are provided in tables 3-4 below.

Table 3

Structural Shift Identification and Estimation, Clayton Copula, Parameter Values Before and After the Structural Shift $\kappa_1 = 0,3; \kappa_2 = 1,0$; Structural Shift Parameter $\theta = 0,3$, C – Critical Bound; w_2 - IInd Type Error.

$\theta = 0,3$	$\kappa_1 = 0,3; \kappa_2 = 1,0$			
N	500	700	1000	1500
C	0,037	0,031	0,027	0,020
w_2	0,56	0,43	0,15	0,02
θ_N	0,337	0,335	0,303	0,30

Table 4

Structural Shift Identification and Estimation, Gumbel Copula, Parameter Values Before and After the Structural Shift $\kappa_1 = 0,3;\ \kappa_2 = 1,0$; Structural Shift Parameter $\theta = 0,3$, C – Critical Bound; w_2 - II[nd] Type Error.

$\theta = 0,3$	$\kappa_1 = 0,3;\ \kappa_2 = 0,7$						
N	100	200	300	500	700	1000	1500
C	0,07	0,05	0,04	0,03	0,02	0,017	0,015
w_2	0,69	0,60	0,44	0,33	0,04	0,01	0
θ_N	0,45	0,40	0,35	0,33	0,31	0,305	0,30

Based on simulation results above, we can summarize the major findings:

1) The proposed method enables us to properly identify the structural shifts in copula models and to arrive at their parameter estimates. Note, we do understand any unpredicted (rapid) change in the multivariate copula reflecting certain type of dependence in-between univariate components,
2) The critical bounds estimated do not depend either on the copula type (Clayton, Gumbel or other), or on the copula parameters under the null hypothesis. It makes them of great value when carrying out non-parametric tests for structural shift identification in copula models.

5. Application to Real World Data

The proposed method was applied to multivariate financial time series of interest rates to test the existence of a structural shift. The basic data set contained 21 time series for 7 maturity buckets and 3 currencies (interest rates for borrowing in a certain currency). The maturities were taken (1) overnight, (2) 1 month, (3) 3 months, (4) 6 months, (5) 1 year, (6) 3 years, (7) 5 years. For short-term maturities (less than one year) interbank rates (EURIBOR, USD LIBOR, MosPrime) were used. For long-term maturities, the interest-rate swap contract quotes for the respective interbank rates. The initial set contained daily data announced by the respective organisations (European Banking Federation for EURIBOR, British Bankers Association for USD LIBOR, National Currency Association (NVA) for MosPrime) from August 6, 2007 to May 21, 2009. Bloomberg was used as a data source. Time series graphical representations are provided in Appendix 1.

Methodology Used:

Copulas for joint distributions were estimated by a semiparametric[4] method in order to avoid marginal d.f. misspecification. Thus empirical marginals of daily log-returns were taken, and the copula was estimated parametrically. Six major copulas were regarded: Archimedean (Clayton, Gumbel), extreme value (Cochi, or Student's t with 1 degree of freedom (d.o.f.)) and elliptical (Gaussian and Student's t with 5 and 10 d.o.f.). Copula parameter estimates[5] for both methods used (IFM and ITAU) [6] can be found in Appendices 2.1 and 2.2.

Econometric Findings Interpretation.

1. The structural shift in the ruble-zone interest rate copula was estimated to be on December 3, 2008. Before the shift the joint comovement of interest rates was best characterized by a Cochi copula, afterwards by a Gaussian one. A Cochi copula has the strongest tail dependence compared to other elliptical copulas. Tail dependence indices (both upper and lower as the copula is symmetric) equaled to 85% for the IFM method and 96.2% for the rank-transformed data (ITAU method). On the contrary a Gaussian (Normal) copula has zero tail dependence, that is, the conditional probability of a simultaneous rise or fall (in the highest and lowest quantiles) of copula components is nil. Evidently, the pooled estimation provided biased results by indicating Student's t with 8 d.o.f. to fit the data best, that is, the average between the Gaussian and the Cochi copulas.

   To comment on the sources of the interest rate joint behavior it is necessary to trace the principal interest rate determinant: the refinance rate. Though in Russia it is not as linked to interbank lending rates as in case of The European Central Bank or The US Federal Reserve, it still provides a government indication of changes in the economic environment, particularly in the amount of accessible liquid funds.

   Before December 1, 2008 the Central Bank of Russian Federation (CBRF) was constantly raising the refinance rate, up to 13 % p.a. Thus the regulator was affirming that it was necessary to limit lending activity in order to limit the increase in the monetary base and to prevent the future escalation in inflation. Instead the market needed to facilitate

[4] Authors [Kim G., Silvapulle M., Silvapulle P. (2007)] argue that a semiparametric approach is preferred, enabling consistent and robust estimates compared to parametric approaches in cases when the marginal d.f. is unknown.

[5] R software was used to undertake the estimation described. The codes and data are readily available from the authors upon request.

[6] ITAU method enables researches to estimate copula parameters based not on the probability space, but on the transformed-to-ranks probability space. Would like to remark that the parameter estimate did not depend on the d.o.f. number when using ITAU method. The only thing that was influenced by the number of d.o.f. was the value of tail dependence index. Therefore we would recommend using inference-for-margins (IFM) method in R when carrying out the estimation procedure in R.

lending (starting with interbank lending) which could have been possible by decreasing the refinance rate.

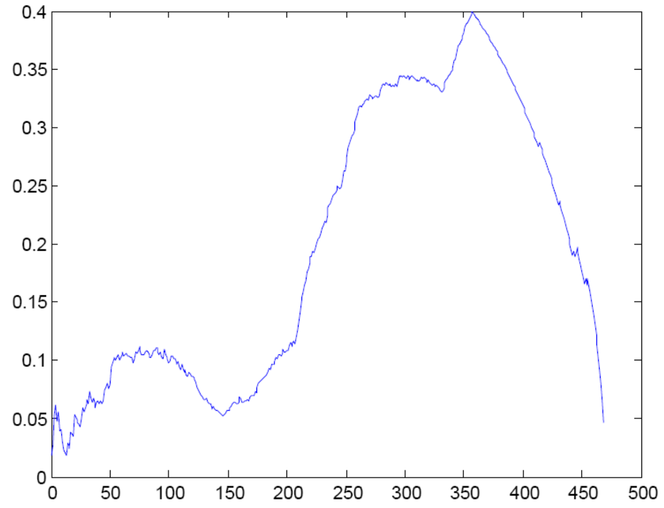

Figure 1. Key Statistics $\Psi_{l,N-l}(u)$ Dynamics Revealing the Structural Shift Point in Ruble-Zone Interest Rates Copula

The observation number can be found on horizontal axis, the statistics value – on the vertical one.

We would also like to point out the shift-even points on the figure 1 above indicating the search of structural shift moment. It can be seen that though the global maximum belongs to observation No. 348 (December 3, 2008), the local maximum exists at about November 12, 2008 (observation No. 300). That is on November 11 and 12 the CBRF initiated two consecutive up-shifts in the refinance rate to 11% and 12%, respectively (see Figure 2). If the economic environment was not characterized by a shortage of liquidity, the interest rates co-movement could probably have satisfied the normal copula assumption subsequently. Nevertheless it was that period of October-November 2008 which was called the ‘banking liquidity crisis’ when the interest rate up and down comovement was extremely strong, proving the Cochi copula to fit the data best.

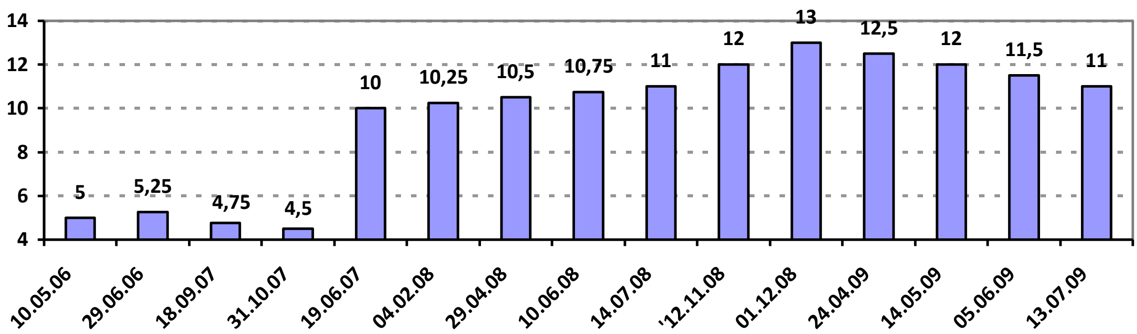

Figure 2. Key Dates of Refinance Rate Changes by the Central Bank of Russia

Horizontal axis marks the time schedule and the vertical one – the refinance rate in percentage points.

Source: http://cbr.ru/print.asp?file=/statistics/credit_statistics/refinancing_rates.htm

2. As for US Dollar-linked interest rates, the structural shift was estimated to be on July 17, 2008 (see Figure 3, Observation No. 225). For the first part of the data set (before the shift-date) the Clayton copula was found to fit best, whereas afterwards it was also a Gaussian copula. To recap a Clayton copula has a positive lower tail dependence (equal to 89.5% for the IFM method and 99% for the rank-transformed method in our case) of the components and zero for the upper tail ones.

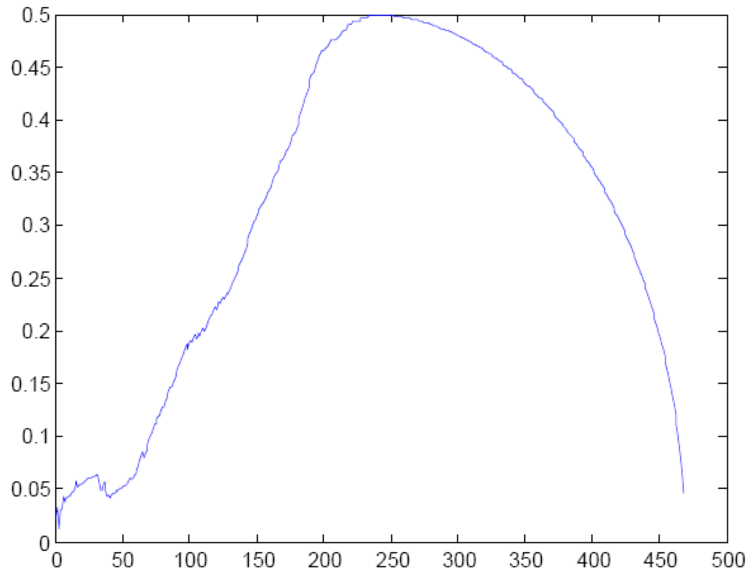

Figure 3. Key Statistics $\Psi_{l,N-l}(u)$ Dynamics Revealing the Structural Shift Point in US Dollar-Zone Interest Rates Copula

The observation number can be found on horizontal axis, the statistics value – on the vertical one.

Thus the comovement of US Dollar-zone interest rates tended to simultaneously fall, rather than to rise. It is closely related to the dynamics of the US Dollar interest rates' principal determinant – the Fed funds rate (see Figure 4).

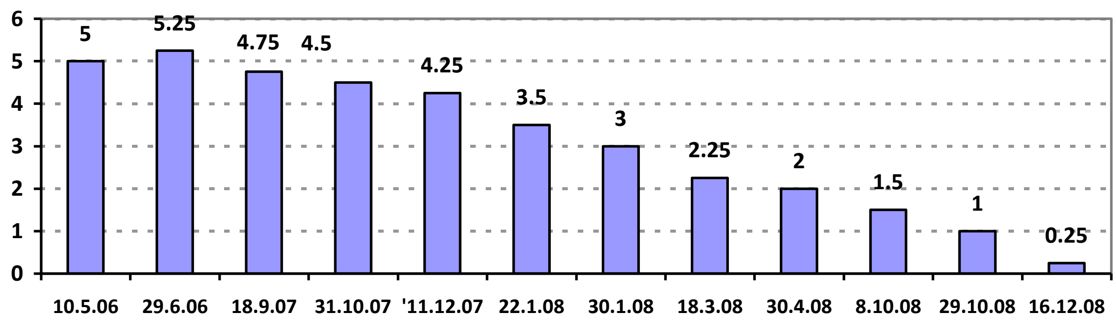

Figure 4. Key Dates of Refinance Rate Changes by the US Federal Reserve

Horizontal axis marks the time schedule and the vertical one – the refinance rate in percentage points.

Source: www.cbonds.info/index/index_detail/type_id/160/

From the very rise of credit crunch the US Federal Reserve was decreasing the refinance rate to stimulate the market participants. Nevertheless, the two periods (before and after the shift) do differ. The prior period was marked by a comparably greater decrease in refinance rate (from 4.75% to 2%, i.e. by 2.75%). The period following, by a lesser decrease (from 2% to 0.25%, i.e. by 1.75%).

Therefore we tend to interpret the results obtained as follows. During the first period before the structural shift on July 17, 2008 the market participants were expecting and were in need of a federal fund rate decrease. By contrast in the subsequent period a further decrease was not as desirable nor as vital as before. That is why a Clayton copula was identified to fit the data best for the first period and a Gaussian for the second.

3. The analysis of Euro-zone interest rate co-dynamics was not as evident, as those with the ruble-zone and USD-zone ones. Nevertheless, the period before and after the estimated structural shift date (September 24, 2008; see Figure 5, observation No. 275) can be differentiated based on the copula parameters' estimates. For the first period a Clayton copula seems to be most relevant in describing the interest rates comovement pattern (based on the maximum likelihood function value). For the second period there is no strictly preferable copula found as IFM method was unable to complete the calculation because of the infinite value of all copulas' likelihood functions (it was obtained only for the Clayton one). In addition the Clayton copula parameter estimated value decreased three times using ITAU method from 75.27 to 22.16. It can be interpreted as the decrease in the tightness of interest rate comovement. Nevertheless, the parameter significance has tripled from 1.6 to 4.81 (on the contrary, for the Student's t copula the significance of its parameter was four times lower using ITAU method). Thus we may conclude that disregarding the decrease in the measure of associativity of euro-zone interest rates, the Clayton copula has revealed itself to be more adequate in describing the joint rates dynamics.

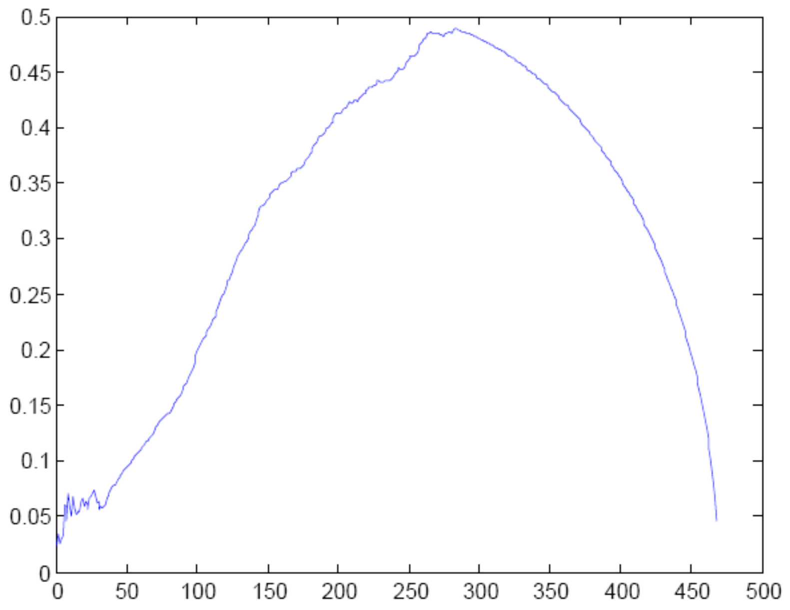

Figure. 5. Key Statistics $\Psi_{l,N-l}(u)$ Dynamics Revealing the Structural Shift Point in Euro-Zone Interest Rates Copula

The observation number can be found on horizontal axis, the statistics value – on the vertical one.

To add economic interpretation of the structural shift moment estimate, we have to trace the European Central Bank's (ECB) policy towards the refinance rate (see Figure 6).

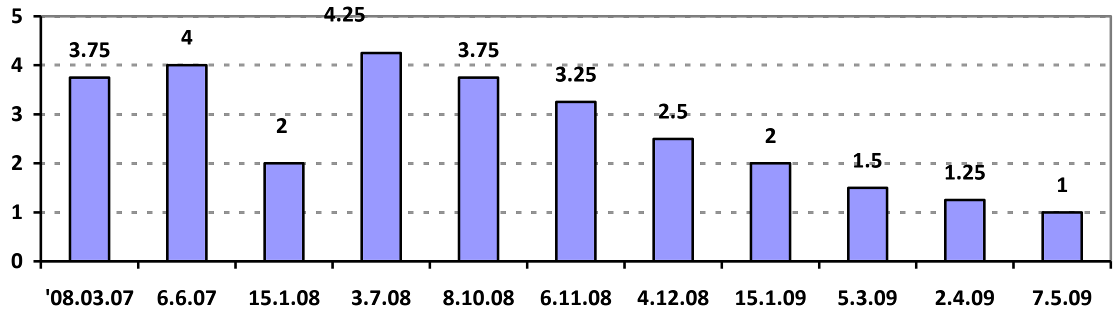

Figure 6. Key Dates of Refinance Rate Changes by the European Central Bank

The horizontal axis marks the time schedule and the vertical one – the refinance rate in percentage points.

Source: www.cbonds.info/index/index_detail/type_id/161/

During the 'before-the-shift' period (before September 24, 2008) the ECB was perpetually increasing the refinance rate in order to tighten inflationary pressures, but it was on October 8, 2008 when the ECB first cut the rate by 0.5%. Hereafter the ECB continued rate-cutting to arrive at 1% on May 5, 2009, an overall decrease by 3.25%. We argue that it was this downward movement that might be best described by a Clayton copula, rather than a Gaussian.

To conclude we would like to comment on the findings presented above and those found in the previous research. For example [Penikas, Simakova and Titova (2009)] stated that a Gumbel copula best fits the interest rate joint distribution. Current research has revealed Gaussian and Clayton copulas to be the best candidates. Aiming at understanding the differences three major issues must be accounted for.

Primarily, the earlier research did not consider the structural shift thus providing biased parameter estimates.

Secondly, as presented above, the refinance rate is an important determinant of interest rates comovement pattern. The previous research dealt with the period January 17, 2007 to November 17, 2008 when the CBRF was constantly increasing the rate. As stated above a Gumbel copula is characterised by a positive upper tail dependence, that is, there is a significant probability of simultanious realisations of high quantiles of all the copula components.

Thirdly, the previous research data set contained five time series of Ruble-zone interest rates, the current one contains seven for each of the three currencies chosen. In addition Archimedean copulas (including Gumbel and Clayton) have the drawback of decreasing the parameter estimate significance given the rise in the copula dimension. Another problem is that the comovement pattern is characterised by the sole parameter (elliptical copulas additionally take into account the covariance matrix). To solve the multidimensionality problem hierarchical copulas might be constructed (as proposed in, for example [Savu, Trede (2006)]).

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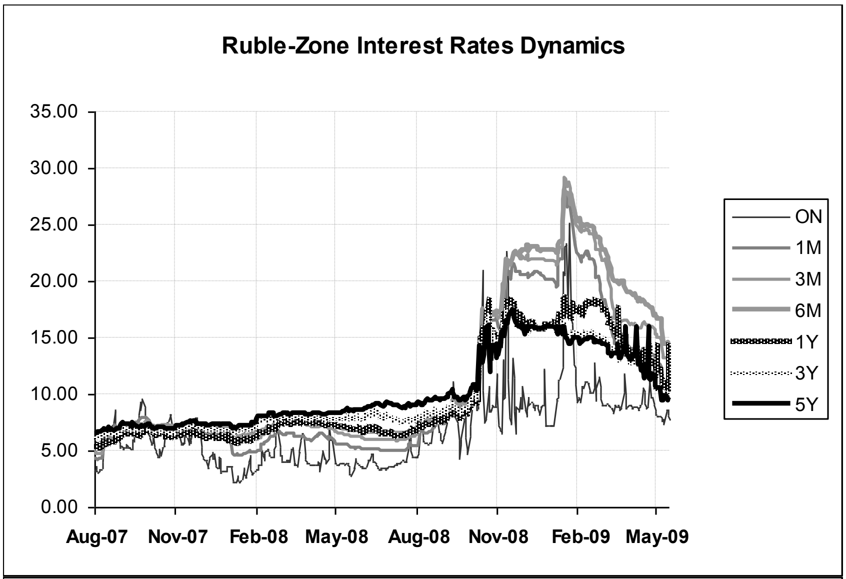
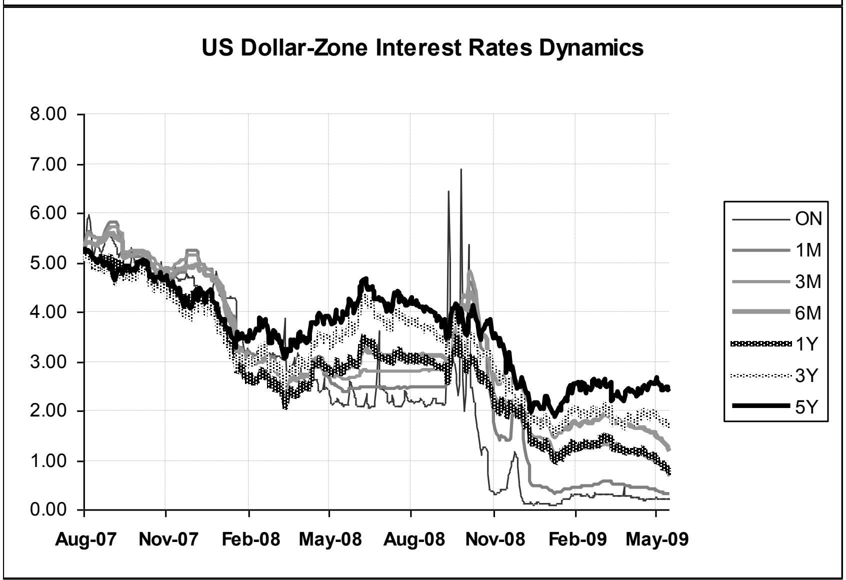
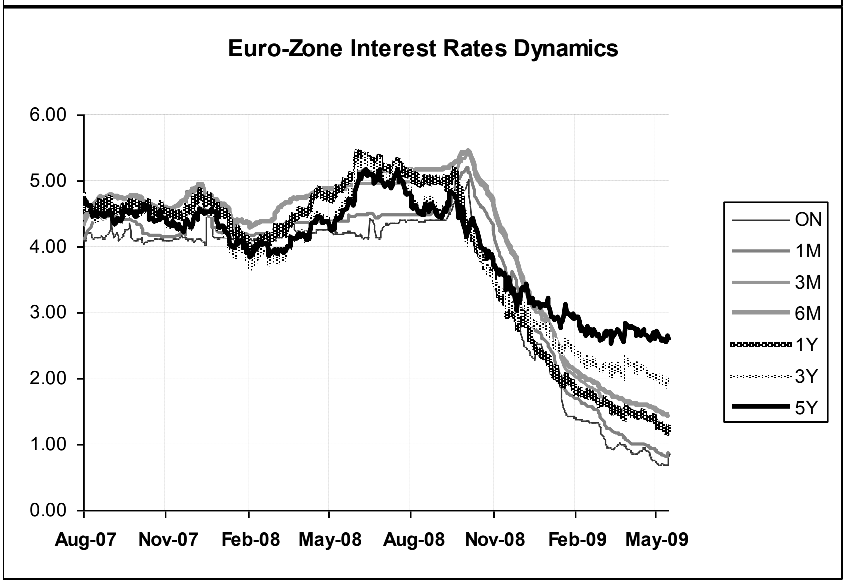
Appendix 1. Interest Rate Time Series Dynamics' Graphical Representation.







Notes: The maturities presented are as follows: ON - overnight, 1M - 1 month, 3M - 3 months, 6M - 6 months, 1Y - 1 year, 3Y - 3 years, 5Y - 5 years.

Appendix 2.1. Copula Parameter Estimates Using Inference-For-Margins (IFM) Method for The Daily Log returns of Interest Rates

Copula		Ruble-zone Rates Period Before-the-shift	Ruble-zone Rates Period After-the-shift	Ruble-zone Rates Period Pooled Data	US Dollar-zone Rates Period Before-the-shift	US Dollar-zone Rates Period After-the-shift	US Dollar-zone Rates Period Pooled Data	Euro-zone Rates Period Before-the-shift	Euro-zone Rates Period After-the-shift	Euro-zone Rates Period Pooled Data
Gumbel	Parameter	4,72	2,42	4,09	2,20	2,79	2,52	1,65		
	Z-statistics	61,09	37,53	72,01	48,84	48,01	67,96	60,34		
	ML	2 460,88	598,12	3 100,43	911,95	1 019,92	1 959,64	1 093,70	infinite	infinite
	UTDI	84,2%	66,8%	81,5%	62,9%	71,8%	68,3%	47,9%		
	LTDI	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%
Clayton	Parameter	11,23	6,08	10,12	6,26	6,46	6,42	5,21	2,71	4,31
	Z-statistics	58,12	23,26	65,23	31,60	34,09	46,47	28,77	17,03	31,66
	ML	2 524,54	584,28	3 095,12	1 093,70	991,31	2 082,37	1 420,04	763,85	2 140,48
	UTDI	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%
	LTDI	94,0%	89,2%	93,4%	89,5%	89,8%	89,8%	87,5%	77,4%	85,1%
Student's t 1 d.o.f	Parameter	0,96	0,72	0,93	0,51	0,71		0,56		
	Z-statistics	325,61	29,87	236,42	21,80	37,82		27,05		
	ML	2 880,78	538,74	3 488,68	426,46	880,44	infinite	925,83	infinite	infinite
	UTDI	85,1%	62,5%	81,2%	50,6%	62,1%		53,2%		
	LTDI	85,1%	62,5%	81,2%	50,6%	62,1%		53,2%		
Student's t 5 d.o.f	Parameter	0,98	0,88	0,97	0,78	0,88		0,75		
	Z-statistics	943,80	98,44	751,65	70,05	138,54		66,66		
	ML	2 869,21	625,04	3 527,30	829,41	1 003,76	infinite	1 244,64	infinite	infinite
	UTDI	80,6%	55,8%	76,5%	42,6%	56,5%		39,4%		
	LTDI	80,6%	55,8%	76,5%	42,6%	56,5%		39,4%		
Student's t 10 d.o.f	Parameter	0,98	0,90	0,97 **	0,82	0,90		0,78		
	Z-statistics	1 116,75	130,33	875,49 **	96,05	187,49		84,43		
	ML	2 854,39	636,73	3529,88 **	871,51	1 030,76	infinite	1 262,99	infinite	infinite
	UTDI	73,9%	45,2%	71,9%	31,3%	46,5%		26,5%		
	LTDI	73,9%	45,2%	71,9%	31,3%	46,5%		26,5%		
Guassian	Parameter	0,98	0,91	0,97	0,84 *	0,92		0,79 *		
	Z-statistics	1 374,01	187,98	1 199,99	136,30 *	279,67		112,8 *		
	ML	2 803,58	652,71	3 507,54	908,0962 *	1 074,85	infinite	1282,31 *	infinite	infinite

Notes: ML - the value of the maximum likelihood function; UTDI and LTDI stand for the vaule of the upper and lower tail dependence indexes; * the estimate for the Student's t copula with 100 d.o.f. was taken as the first proxy for the Gaussian copula estimate; ** the maximum ML value presented belongs to the 8 d.o.f. Student's t copula case; yellow marks the best copula chosen based on the maximum ML value.

Appendix 2.2. Copula Parameter Estimates Using ITAU Method for The Rank-Transformed Probabilities of Daily Log returns of Interest Rates.

Copula		Ruble-zone Rates Period			US Dollar-zone Rates Period			Euro-zone Rates Period		
		Before-the-shift	After-the-shift	Pooled Data	Before-the-shift	After-the-shift	Pooled Data	Before-the-shift	After-the-shift	Pooled Data
Gumbel	Parameter	49,92	14,16	52,93	36,07	11,03	42,53	38,63	12,08	45,09
	Z-statistics	3,32	5,68	5,65	1,83	4,98	2,05	1,64	5,25	1,83
	UTDI	98,6%	95,0%	98,7%	98,1%	93,5%	98,4%	98,2%	94,1%	98,5%
	LTDI	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%
Clayton	Parameter	97,85	26,33	103,86	70,14	20,07	83,07	75,27	22,16	88,19
	Z-statistics	3,25	5,28	5,55	1,78	4,53	2,00	1,60	4,81	1,78
	UTDI	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%
	LTDI	99,3%	97,4%	99,3%	99,0%	96,6%	99,2%	99,1%	96,9%	99,2%
Student's t 1 d.o.f	Parameter	0,9971	0,9901	0,9982	0,9989	0,9832	0,9992	0,9988	0,9830	0,9993
	Z-statistics	1 389	362	3 148	884	188	1 403	832	216	1 352
	UTDI	96,2%	93,0%	97,0%	97,6%	90,8%	98,0%	97,6%	90,8%	98,1%
	LTDI	96,2%	93,0%	97,0%	97,6%	90,8%	98,0%	97,6%	90,8%	98,1%
Student's t 5 d.o.f	Parameter	0,9971	0,9901	0,9982	0,9989	0,9832	0,9992	0,9988	0,9830	0,9993
	Z-statistics	1 389	362	3 148	884	188	1 403	832	216	1 352
	UTDI	92,9%	86,9%	94,3%	95,6%	82,9%	96,3%	95,4%	82,8%	96,4%
	LTDI	92,9%	86,9%	94,3%	95,6%	82,9%	96,3%	95,4%	82,8%	96,4%
Student's t 10 d.o.f	Parameter	0,9971	0,9901	0,9982	0,9989	0,9832	0,9992	0,9988	0,9830	0,9993
	Z-statistics	1 389	362	3 148	884	188	1 403	832	216	1 352
	UTDI	90,2%	82,0%	92,2%	93,9%	76,6%	94,9%	93,7%	76,4%	95,0%
	LTDI	90,2%	82,0%	92,2%	93,9%	76,6%	94,9%	93,7%	76,4%	95,0%
Guassian	Parameter	0,9971	0,9901	0,9982	0,9989	0,9832	0,9992	0,9988	0,9830	0,9993
	Z-statistics	1 389	362	3 148	884	188	1 403	832	216	1 352

Notes: UTDI and LTDI stand for the value of the upper and lower tail dependence indexes;

Henry I. Penikas
Senior Lecturer at Mathematical Economics and Econometrics Department,
Junior Research Fellow at International Laboratory of Decision Choice and Analysis,
National Research University Higher School of Economics.
E-mail: penikas@gmail.com