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On the superposition of the Borda and threshold preference orders for three-graded rankings

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Abstract

In the case when alternatives are ranked by several equivalent criteria on the scale of three grades (bad, average, good) we develop the axiomatics of preference functions for the superposition of the Borda and threshold preference orders and present the explicit formula for the evaluation of the enumerating preference function.

Keywords: preference order, superposition, Borda's rule, threshold rule, indifference, axiomatics, enumerating preference function

1. Introduction

In practical problems of ranking large sets (e.g., consisting of several millions of alternatives), the crucial feature is the computation of the ordinal number of an alternative in the resulting ranking. The procedure of ranking under consideration can be made more effective provided a utility function, coherent with the rankning, is found in a suitable form. Among numerous ranking procedures known in the literature we consider three methods based on Borda's and threshold preference orders and their superposition.

The simplest method of ranking alternatives over several equivalent criteria is the Borda rule of summation of individual ranks of alternatives. According to it, an alternative x is (Borda) preferred to an alternative y provided the total rank of x, which is the sum of individual ranks, is greater than the total rank of y. The deficiency of this method is that it produces massive sets with equal total ranks, and so, it is insensitive to the input information. Moreover, it is of compensatory nature: low individual ranks can be compensated by high individual ranks in the resulting ranking.

In this paper, we consider the technically simple case when alternatives are estimated by means of three grades (i.e., individual ranks): 1 meaning 'bad', 2 meaning 'average', and 3 meaning 'good'. A more subtle ranking of alternatives, as compared to the Borda ranking, can be given by applying the threshold rule [4, 5]: an alternative x is (threshold) preferred to an alternative y if and only if either the number of 'bads' for x is less than those for y, or the numbers of 'bads' are equal for both alternatives and the number of 'averages' for x is less than those for y. This

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method is extremely sensitive to the input information and is of noncompensatory nature: low individual ranks cannot be compensated by (any numbers of) high individual ranks in the resulting ranking.

The aim of this paper is to study the intermediate semicompensatory ranking procedure, which is the mixture (i.e., the superposition) of the two methods above, Borda's and threshold's. We say that an alternative x is (superpositionally) preferred to y if and only if either the sum of individual ranks for x is greater than the sum of individual ranks for y (as in Borda's rule), or the sums of individual ranks for x and y are equal and the number of 'bads' for x is less than the number of 'bads' for y (as in the threshold rule).

The paper is organized as follows. In Section 2 we define the superposition of two preference orders on a set of alternatives and in Section 3 we recall the Borda and threshold preference orders. Section 4 is devoted to the main results: the axiomatics of preference functions for the superposition of the Borda and threshold preference orders (Theorem 1) and the explicit formula for the enumerating preference function (Theorem 2). In Appendix we exhibit examples of the Borda, threshold and their superposition rankings.

2. Superposition of preference orders

Let X be a finite set (the set of alternatives) and $P \subset X \times X$ be a preference (or weak) order on X [6], i.e., P is irreflexive ($(x, x) \notin P$ for all $x \in X$), transitive ($(x, y) \in P$ and $(y, z) \in P$ imply $(x, z) \in P$) and negatively transitive ($(x, y) \notin P$ and $(y, z) \notin P$ imply $(x, z) \notin P$). The inclusion $(x, y) \in P$ is interpreted as 'x is P-preferred to y'. The indifference relation I_P on X, induced by P, is defined as the set of all pairs $(x, y) \in X \times X$ such that $(x, y) \notin P$ and $(y, x) \notin P$. Thus, $(x, y) \in I_P$ means that x and y are P-indifferent. The relation I_P is an equivalence relation on X.

A typical example of a preference order is as follows. Given a nonconstant function $F : X \to \mathbb{R}$, let $P(F)$ be the set of all pairs $(x, y) \in X \times X$ such that $F(x) > F(y)$. Then $P(F)$ is a preference order on X, and $(x, y) \in I_{P(F)}$ if and only if $F(x) = F(y)$. The function F is called a preference (or utility) function for $P(F)$ and the binary relation $P(F)$ is called F-representable.

Let P and Q be two preference orders on X. The relation $P * Q$ on X, defined by $P * Q = P \cup (I_P \cap Q)$, is called the superposition of P and Q (in this order). In other words, x is $(P * Q)$-preferred to y if and only if either x is P-preferred to y, or x and y are P-indifferent and x is Q-preferred to y. It is known [1] that $P * Q$ is a preference order on X, $I_{P*Q} = I_P \cap I_Q$ and $(P * Q) * R = P * (Q * R)$ for a preference order R on X. However, $P * Q \neq Q * P$ in general.

3. Borda's and threshold preference orders

In this paper, we assume that alternatives from X are estimated by means of the three-graded scale: 1 (bad), 2 (average), and 3 (good), i.e., X is identified with the set $\{1, 2, 3\}^n$ of all n-dimensional vectors $x = (x_1, \ldots, x_n)$ with $n \geq 3$ and components $x_i \in \{1, 2, 3\}$. Such vector-grades may be interpreted as expert grades, questionnaire data, etc.

Given $x = (x_1, \ldots, x_n) \in X = \{1, 2, 3\}^n$ and $i \in \{1, 2, 3\}$, we denote by $v_i(x)$ the multiplicity of grade i in the vector x and set $S(x) = x_1 + \cdots + x_n$. Clearly,

$$v_1(x) + v_2(x) + v_3(x) = n \quad \text{and} \quad S(x) = v_1(x) + 2v_2(x) + 3v_3(x).$$

Given $x, y \in X$, we write $(x, y) \in B$ if $S(x) > S(y)$. The relation B is a preference order on X, called the Borda preference order [9]. Note that B produces a 'coarse' ranking of X in the sense that indifference classes of alternatives $x \in X$, given by $\overline{x} = \{y \in X : S(y) = S(x)\}$, are rather massive (see Appendix for an example of B).

A more subtle ranking of X is given by the threshold preference order ([2]–[5]). Given $x, y \in X$ and $i \in \{1, 2\}$, we write $(x, y) \in V_i$ if $v_i(x) < v_i(y)$. Each relation V_i is a preference order on X such that $(x, y) \in I_{V_i}$ if and only if $v_i(x) = v_i(y)$. The superposition $V = V_1 * V_2$ is called the threshold preference order on X (note that it is a particular case of the leximin [2, 8]). More explicitly, $(x, y) \in V$ if and only if either $v_1(x) < v_1(y)$, or $v_1(x) = v_1(y)$ and $v_2(x) < v_2(y)$. The indifference relation $I_V = I_{V_1} \cap I_{V_2}$ consists of those pairs $(x, y) \in X \times X$, for which $v_1(x) = v_1(y)$ and $v_2(x) = v_2(y)$ (and so, $v_3(x) = v_3(y)$), i.e., x and y can be trasformed into each other by a permutation of their coordinates. An example of the threshold preference order V is presented in Appendix.

4. Results: axioms and enumerating preference functions

Let $P = B * V$ be the superposition of the Borda preference order B and the threshold preference order V. Then P is also a preference order on X, and we have $P = B * V_1$ and $I_P = I_V$. In other words, $(x, y) \in P$ if and only if either $S(x) > S(y)$, or $S(x) = S(y)$ and $v_1(x) < v_1(y)$. Note that $V * B = V$ (and so, $B * V \neq V * B$). An example of such superposition preference order P is given in Appendix.

Our first result addresses the representability of the relation P.

Theorem 1. *A function $F : X = \{1, 2, 3\}^n \to \mathbb{R}$ is a preference function for the superposition relation $P = B * V$ (i.e., $P = P(F)$) if and only if, given $x, y \in X$, the following four conditions (axioms) are satisfied:*

(A.1) $v_1(x) = v_1(y)$ *and* $v_3(x) = v_3(y)$ *imply* $F(x) = F(y)$;

(A.2) $v_1(x) + 1 = v_1(y)$ *and* $v_3(x) + 1 = v_3(y)$ *imply* $F(x) > F(y)$;

(A.3) $v_3(y) = 0$ *and* $v_1(x) + 1 = v_1(y) + v_3(x)$ *imply* $F(x) > F(y)$;

(A.4) $v_1(y) = 0$ *and* $v_1(x) + v_3(y) + 1 = v_3(x)$ *imply* $F(x) > F(y)$. .

A simple preference function for P is given by $F(x) = nS(x) - v_1(x)$, $x \in X$.

In order to present our second result, we recall the construction of the canonical ranking of X, generated by a preference order P on X [7]. For a subset $Y \subset X$, we denote by $\pi(Y)$ the set of most P-preferred alternatives from Y, i.e., those $y \in Y$ such that $(x, y) \notin P$ for all $x \in Y$. We set $X'_1 = \pi(X)$ and, inductively, if $k \geq 2$ and nonempty disjoint subsets $X'_1, \dots, X'_{k-1}$ of X such that $X'_1 \cup \dots \cup X'_{k-1} \neq X$ are already determined, then we set $X'_k = \pi(X \setminus (X'_1 \cup \dots \cup X'_{k-1}))$. Since X is finite, there exists a unique positive integer $s = s_P(X)$ such that $X = \bigcup_{k=1}^{s} X'_k$ (disjoint union). The value s is equal to the cardinality of the quotient set X/I_P. Setting $X_k = X'_{s-k+1}$ for $k = 1, 2, \dots, s$, we get the family $\{X_k\}_{k=1}^{s} = X/I_P$ of indifference classes (the canonical ranking of X), partitioning X, which has the following property: given $x, y \in X$, $(x, y) \in P$ if and only if there exist integers $1 \leq k_1 < k_2 \leq s$ such that $x \in X_{k_2}$ and $y \in X_{k_1}$. Also, $(x, y) \in I_P$ if and only if $x, y \in X_k$ for some integer $1 \leq k \leq s$.

Define a function $N : X \to \{1, 2, \dots, s\}$ by the following rule: given $x \in X$, there exists a unique integer $1 \leq k \leq s$ such that $x \in X_k$, and so, we set $N(x) = k$. Thus, $x \in X_k = X_{N(x)} = \{x \in X : N(x) = k\}$. The function N is called the enumerating preference function for P (it is well-defined, uniquely determined and surjective preference function for P [2]). Any other preference function $F : X \to \mathbb{R}$, representing P, can be expressed as follows: there exists an increasing function $f : \{1, 2, \dots, s\} \to \mathbb{R}$ such that $F(x) = f(N(x))$ for all $x \in X$.

In the next theorem we determine the enumerating preference function for $P = B * V$ explicitly. In this theorem $[a]$ denotes the greatest integer, which does not exceed a. Note that if $X = \{1, 2, 3\}^n$ and $P = B * V$, then $I_P = I_V$, and so, by [3, Lemma 1], $s = s_P(X) = s_V(X) = (n + 2)(n + 1)/2$.

Theorem 2. *A function N maps $X = \{1, 2, 3\}^n$ onto $\{1, 2, \dots, s\}$ and is the enumerating preference function for the superposition preference order $P = B * V$ on X if and only if it is given for $x \in X$ as follows: if $n \leq S(x) \leq 2n$, then*

$$N(x) = \left[\frac{S(x) - n}{2}\right] \cdot \left[\frac{S(x) - n + 1}{2}\right] + n + 1 - v_1(x),$$

and if $2n + 1 \leq S(x) \leq 3n$, *then*

$$N(x) = \left[\frac{S(x) - n}{2}\right] \cdot \left[\frac{S(x) - n + 1}{2}\right] + n - \frac{(S(x) - 2n + 1) \cdot (S(x) - 2n - 2)}{2} - v_1(x).$$

.

Appendix

Here we present examples of rankings of the set $X = \{1, 2, 3\}^n$ with $n = 5$ in ascending Borda B-, threshold V- and superposition P-preference, where $P = B * V$. Each vector $x \in X$ below is written (for the sake of brevity) in the form $(x_1, x_2, \dots, x_5)_k$ with $x_1 \leq x_2 \leq \dots \leq x_5$, which represents the indifference class X_k with $k = N(x)$ and $N : X \to \{1, 2, \dots, s\}$ being the corresponding enumerating preference function.

1. For the Borda preference order B we have $s = s_B(X) = 2n + 1 = 11$ and $N(x) = S(x) - n + 1 = S(x) - 4$, and so,
   $(1,1,1,1,1)_1$, $(1,1,1,1,2)_2$, $(1,1,1,1,3)_3$, $(1,1,1,2,2)_3$, $(1,1,1,2,3)_4$, $(1,1,2,2,2)_4$,
   $(1,1,1,3,3)_5$, $(1,1,2,2,3)_5$, $(1,2,2,2,2)_5$, $(1,1,2,3,3)_6$, $(1,2,2,2,3)_6$, $(2,2,2,2,2)_6$,
   $(1,1,3,3,3)_7$, $(1,2,2,3,3)_7$, $(2,2,2,2,3)_7$, $(1,2,3,3,3)_8$, $(2,2,2,3,3)_8$,
   $(1,3,3,3,3)_9$, $(2,2,3,3,3)_9$, $(2,3,3,3,3)_{10}$, $(3,3,3,3,3)_{11}$.
2. If V is the threshold preference order, then $s = s_V(X) = (n+2)(n+1)/2 = 21$ and, by [2, Theorem 3.1],

$$N(x) = \frac{(n+2-v_1(x))\cdot(n+1-v_1(x))}{2} - v_2(x) = \frac{(7-v_1(x))\cdot(6-v_1(x))}{2} - v_2(x),$$

and so, we have

$(1,1,1,1,1)_1$, $(1,1,1,1,2)_2$, $(1,1,1,1,3)_3$, $(1,1,1,2,2)_4$, $(1,1,1,2,3)_5$, $(1,1,1,3,3)_6$, $(1,1,2,2,2)_7$,
$(1,1,2,2,3)_8$, $(1,1,2,3,3)_9$, $(1,1,3,3,3)_{10}$, $(1,2,2,2,2)_{11}$, $(1,2,2,2,3)_{12}$, $(1,2,2,3,3)_{13}$, $(1,2,3,3,3)_{14}$,
$(1,3,3,3,3)_{15}$, $(2,2,2,2,2)_{16}$, $(2,2,2,2,3)_{17}$, $(2,2,2,3,3)_{18}$, $(2,2,3,3,3)_{19}$, $(2,3,3,3,3)_{20}$, $(3,3,3,3,3)_{21}$.

3. For the superposition preference order $P = B * V$ we have $s = s_P(X) = (n+2)(n+1)/2 = 21$ and, by Theorem 2,
   $(1,1,1,1,1)_1$, $(1,1,1,1,2)_2$, $(1,1,1,1,3)_3$, $(1,1,1,2,2)_4$, $(1,1,1,2,3)_5$, $(1,1,2,2,2)_6$, $(1,1,1,3,3)_7$,
   $(1,1,2,2,3)_8$, $(1,2,2,2,2)_9$, $(1,1,2,3,3)_{10}$, $(1,2,2,2,3)_{11}$, $(2,2,2,2,2)_{12}$, $(1,1,3,3,3)_{13}$, $(1,2,2,3,3)_{14}$,
   $(2,2,2,2,3)_{15}$, $(1,2,3,3,3)_{16}$, $(2,2,2,3,3)_{17}$, $(1,3,3,3,3)_{18}$, $(2,2,3,3,3)_{19}$, $(2,3,3,3,3)_{20}$, $(3,3,3,3,3)_{21}$.

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