Synchronization of Movement for a Large-Scale Crowd

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Abstract. Real world models of large-scale crowd movement lead to computationally intractable problems implied by various classes of non-linear stochastic differential equations. Recently, cellular automata (CA) have been successfully applied to model the dynamics of vehicular traffic, ants and pedestrians' crowd movement and evacuation without taking into account mental properties. In this paper we study a large-scale crowd movement based on a CA approach and evaluated by the following three criteria: the minimization of evacuation time, maximization of instantaneous flow of pedestrians, and maximization of mentality-based synchronization of a crowd. Our computational experiments show that there exist interdependencies between the three criteria.

1 Introduction

Recently synchronization phenomena became one of the most interesting topics of investigations on nonlinear dynamics [1-7]. Analysis of the development of synchronization theory, just as of the theory of nonlinear dynamical systems, leads to a conclusion that there are two main sources of the development of such theories: the first one is an application of new mathematical methods and concepts to the systems with synchronization while the second source derives from the expanding fields of investigations with development of new mathematical objects that can serve as a background for further investigations and interpretations. Usually nonlinear dynamical systems of neutral or delay type had been the classical objects for synchronization investigations (e.g. ordinary differential equations, partial differential equations, chains of coupled maps, discrete equations with delay etc.).

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Recently some new less known mathematical objects had been found - dynamical systems with an anticipation property. It should be mentioned that a natural source of systems with anticipation are social systems which take into account mental properties of individuals (see e.g. [8, 9]). Another example of nonlinear dynamic systems with synchronization can be found in [10, 11].

In this paper we describe nonlinear models with anticipation for problems of pedestrian movements modelled by means of cellular automata (CA). This choice is enforced by the fact that CA is a well known, applicable and useful type of tools [12-15]. Some examples of cellular automata with anticipation applied to traffic and crowd movement modelling have been studied in [8, 9, 16-24].

In this chapter we study a synchronization phenomenon in pedestrian traffic, taking into account different anticipation models of pedestrians with respect to their speed, radius and shape of observed neighbourhood, including local stochastic predictions for neighbouring cells to be occupied. We consider two types of situations: evacuation of finite number of pedestrians from premises with static geometry and (endless) continuous flow of pedestrians through a corridor with simplified geometry that allows quantitative analysis on a strict background. We estimate performance of the crowd in these two situations by evacuation time (that must be minimized) and flow through the exit (that must be maximized), respectively. In fact, these two characteristics of performance are closely related, as increase in the flow at the exit leads to a faster evacuation and to decrease in the evacuation time. Further, we introduce a measure of synchronization as a measure of collaborating behaviour of (anticipating) pedestrians and claim that in a perfectly synchronized crowd the minimum value of evacuation time and the maximum value of instantaneous flow are achieved.

In section 2 we give a description of the basic CA model that was further used for introduction of the anticipation property. In section 3 we show four different implementations of anticipation property, provide the results of numerical experiments with variations of our anticipating model and describe a framework for generalization of our model for the case of arbitrary size of the extended neighbourhood and arbitrary horizon of prediction (a number of steps for which a prediction is made). In section 4 we introduce a notion of synchronization in terms of pedestrian crowd movement and discuss a relation between anticipation and synchronisation. Section 5 is devoted to conclusions and prospective of future research.

2 CA Models of Pedestrian Traffic

In this paper we study special cases of CA models with discrete space and time as follows [15-21]:

- microscopic: every pedestrian is simulated by a separate single cell;
- stochastic: local rules contain random values;
- space- and time-discrete.

The basic assumptions behind the models are:

- dynamics of pedestrian motion can be represented by a CA;
- global route is pre-determined;

- irrational behaviour is rare;
- persons are not strongly competitive, i.e. they do not hurt each other;
- individual differences can be represented by parameters determining the behaviour.

A CA has two layers (Fig. 1). The first one – data layer – embeds the information about the geometry of the scene, i.e. placement of pedestrians and obstacles. Every cell in this layer has 3 possible states: “empty”, “obstacle”, “pedestrian”.

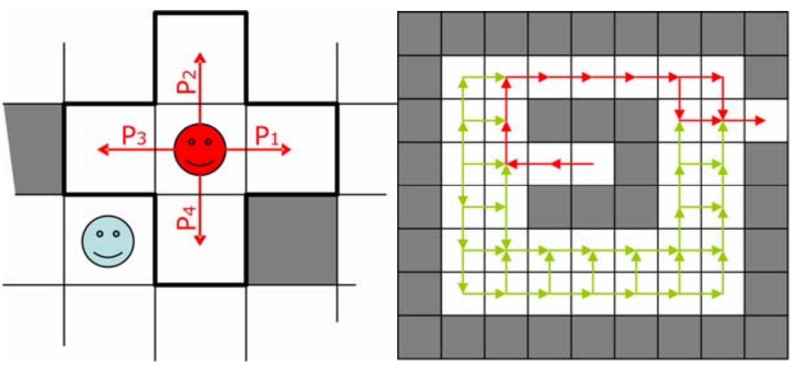
The second layer embeds a vector field of directions and stores the information about the global route. This field of directions is constructed so that to minimize evacuation time of a sole pedestrian. If there are several possibilities at a particular point, they are considered to be equally probable. Every pedestrian receives information about the global route via interaction with this so-called static floor field [21].

At every time step, for every pedestrian probabilities of shift for all the directions are being computed according to the following principles:

- if a target cell is occupied (by obstacle or other pedestrian), the corresponding probability is set to 0;
- pedestrians try to follow the optimal global route.

In fact, primary “forces” driving the pedestrians’ behaviour in our basic model are somewhat different from those used by other authors (see e.g. [21]) and include:

1. determination: striving for the global route (similarly to [21]);
2. inertia: attempt to preserve current direction of movement (relaxed in [21]);
3. randomization: tendency to random changes in direction of movement (in contrast to [21] we consider a mix of uniformly distributed movements with the properties 1 and 2).



a) b)

Fig. 1 Structure of a general CA model of pedestrian traffic: data layer (a) that embeds a description of placement of obstacles and pedestrians and auxiliary layer (b) that embeds a description of the global route in a form of vectors of direction for every cell.

The influence of the mentioned factors is determined by three parameters p_d, p_i, p_r, reflecting the primary forces 1, 2, and 3, respectively. If only the first of them is nonzero then we have a completely deterministic crowd with each member trying to follow the shortest path to an exit. However, in real life even in the most deterministic group of pedestrians there is a place for randomness.

So, what is the origin of randomness in a well organized crowd? Let us consider a situation when a pedestrian cannot follow the shortest path at a particular time step because some other pedestrian is standing at his way. Then the first one has two major options:

a) to stand still and wait while his way is free;
b) to move in any possible direction so as to pass this jam around.

In real crowds persons tend to apply a mixture of the mentioned behavioural patterns and in our model we have one more parameter ranged between 0 and 1 that expresses probability of type b) behaviour.

In order to implement probabilistic approach to resolve conflicts, arising when any two or more pedestrians attempt to move to the same target cell, at every step a sequence of pedestrians' shifts is randomly chosen.

Persons also differ in their maximum speed. These differences are implemented via division of every time step into v_{max} sub-steps $\tau_1,...,\tau_{v_{max}}$; v_{max} is a maximum possible speed over the crowd. An i-th person tries to move at a sub-step k only if $v_i<k$, where v_i – his maximum speed, and both values of v_i and k are numerical parameters independent on their informal interpretations.

2.1 Evacuation and Max Throughput: Major Behavioural Patterns and Characteristics

In the current subsection we describe major behavioural patterns of pedestrians, that can emerge within the described basic model, two types of situations were considered:

A. Evacuation from premises with true-to-life geometry;
B. Continuous flow of pedestrians through a corridor with simplified geometry that allows theoretical analysis of the flow.

The first situation is closely related to the evacuation problem of finding the minimum egress time of the crowd from premises and design of schedules or behavioural patterns that ensure minimization of that characteristic. A number of simulations were held and in Fig.2 main results can be found.

As it can be seen in Fig. 2, "knowledge" about the shortest route (value of p_d) is critical to the behaviour of the crowd and always leads to decrease in evacuation time. With the influence of inertia the situation is not so straightforward. When the influence of determination is above some (comparatively small) threshold, inertia has rather negative effect on performance. But for an almost non-deterministic crowd ($p_d \rightarrow 0$) there exists some optimal nonzero ratio p_i/p_r that ensures the best possible performance in case of fixed other parameters (in the given example this ratio is about 10).

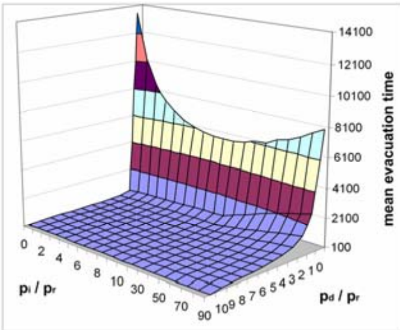
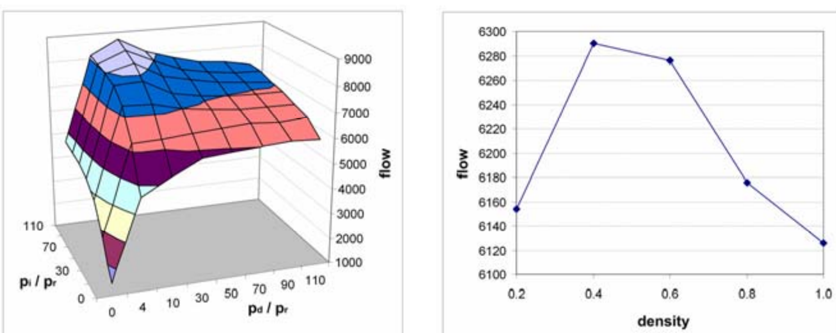


Fig. 2 Mean evacuation time for different values of parameters of the model.

Behaviour of a continuous flow of pedestrians is also of interest as it represents somewhat simpler problem and is more amenable to theoretic methods of analysis. Unlike evacuation that represents itself a transient process, continuous flow tends to a behavioural stationarity that can take different forms: the virtually constant flow, periodic oscillations. Chaotic oscillations, though being rather uncommon, are also possible but lead to difficulties in modelling and analysis. Of particular interest is a phenomenon of emergence and propagation of shock waves.

Results demonstrating the model performance in a case of continuous flow can be found in Fig.3. According to the results of simulations, in a case of continuous flow the situation is somewhat different from the case of evacuation. Now both determination and inertia have positive impact on performance; however, some peculiarities should be mentioned. First of all, for the values of p_i beyond some threshold there exists optimal finite value of p_d that maximizes the flow. Another peculiarity is that at high values of p_d impact of inertia becomes less substantial, which is illustrated by the flat right part of the surface in Fig.3a.



a) b)

Fig. 3 Average flow for different values of parameters of the model (a) and a typical density-flow diagram (b) that illustrates a phenomenon specific to crowd movement: increase in density over a certain threshold leads to a decrease in flow because pedestrians start blocking each other.

3 Anticipating Pedestrians

Starting from the basic model described in section 2, a pedestrian, capable of foreseeing the situation within his neighbourhood with the purpose to minimize his evacuation time, may be generated. Further, a pedestrian possessing this property will be referred to as an anticipating pedestrian. In the section 4 we show that this property, while being introduced at a personal level, leads to a synchronized behaviour of the whole crowd.

3.1 Basic Models of Anticipating Pedestrians

As it was mentioned at the beginning of the section 2, at every step a person determines probabilities of shift (P_k, k=1,…,4). These are these values that may be subjected to influence of an anticipation. Let us assume, that pedestrians try to avoid collisions, i.e. a person tries not to move into a particular cell of his neighbourhood if (as he predicts) it will be occupied by another person at the next time step. This may be achieved by changing the probabilities in the following manner:

$$P_k := P_k \cdot (1 - \alpha \cdot P_{k,occ}) \, , \tag{1}$$

where $\alpha \in [0,1]$ is a free parameter expressing influence of anticipation, $P_{k,occ}$ – the probability of occupation of k-th cell in a neighbourhood by one of neighbours. It is quite natural, that values P_k have to be normalized, so that their sum is equal to one (if at least one of them is nonnegative). It should be noted, that in this case all the pedestrians are assumed to have equal rights. If α is set to 1, a situation, when two pedestrians attempt let each other move and stand still, may occur. Such deadlocks can be completely excluded only by selecting the value of $\alpha < 1$. However, the number of these deadlocks can also be reduced by granting certain (e.g. fast-moving) pedestrians privileges. In this case the shift probabilities will be transformed into:

$$P_k := P_k \cdot (1 - \alpha \cdot (1 - \frac{v}{v_{max}}) \cdot P_{k,occ}) \tag{2}$$

It means that the fastest pedestrians ($v=v_{max}$) do not take care of others (their probabilities of shift remain unchanged), while slowly moving ones try to make way for those moving faster. By using in eq. 2 a somewhat greater value instead of v_{max}, the fastest pedestrians may be forced to be more "polite".

As it was described above, anticipation is closely related to an ability of foreseeing the crowd next state, so the issue of how do the pedestrians predict (in other words, how do they compute $P_{k,occ}$) arises. Two variants were considered: observation- and model-based prediction. The first variant is based upon the assumption that pedestrians preserve direction of their movement. So, $P_{k,occ}$ may be considered to be a linear function of the number of pedestrians "facing" the k-th cell (the direction of their sight is defined by the direction of their previous shift):

$$P_{k,occ} = \frac{m}{M}. \tag{3}$$

where m – number of pedestrians “facing” the k-th cell; M = n-1, in our case M = 3, and n is a number of cells in the neighbourhood.

Such an approach, though being the most simple and natural, is, at the same time, the least accurate. Thus, for the sake of comparison, the second approach was considered, according to which a target pedestrian for every cell of his neighbourhood computes P_k of its neighbours (excluding himself) and the resulting probability is defined as follows:

$$P_{k,occ} = \sum_{i=1}^{3} P_i - \sum_{i \neq j} P_i P_j + \sum_{i \neq j, j \neq k} P_i P_j P_k \tag{4}$$

It is quite evident that this approach allows a more accurate evaluation of $P_{k,occ}$, while being somewhat unnatural, as every pedestrian must know behavioural models of the others.

A number of computational experiments were held and typical performance of all the mentioned configurations of the model (corresponding to two ways of introducing anticipation given by eqs. 1 and 2, and two ways of prediction given by eqs. 3 and 4) is presented in Fig.4.

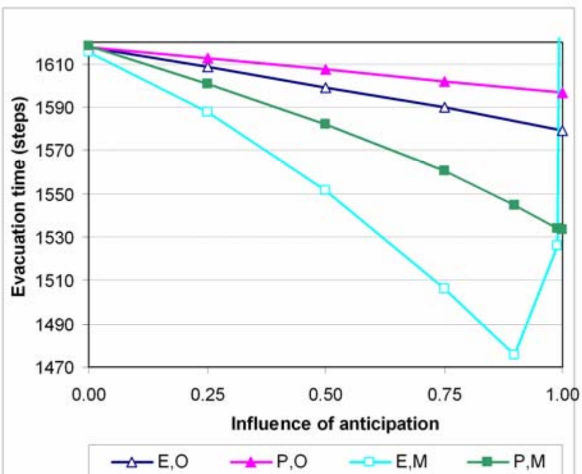


Fig. 4 Performance of CA models with different types of anticipation. (E/P – equality/priority of fast-moving pedestrians (eqs. 1 and 2, respectively); O/M – observation-/model-based prediction (eqs. 3 and 4, respectively)).

The results of simulation reveal the fact, that granting fast-moving pedestrians a priority results in greater overall evacuation time, thus making little sense. On the other hand, the more accurately $P_{k,occ}$ are computed, the better the performance. This proves the consistency of the proposed method of anticipation accounting (given by eq. 1).

3.2 Spatial "de-localization"

In subsection 3.1 an anticipating pedestrian was generating his prediction based on non-anticipating model of his neighbours. Thus, it is quite straightforward to make his prediction more accurate by involving an anticipating model of neighbours. For that, every pedestrian (within the neighbourhood of radius 2) is subjected to the procedure described in subsection 3.1 for the target pedestrian: calculation of P_k, calculation of $P_{k,occ}$ (eq. 4) and correction of P_k (eq. 1). It is evident, that in this case cells lying at a distance of 3 cells from the target pedestrian (centre of the neighbourhood) are involved in evaluation of $P_{k,occ}$. At the same time, pedestrians standing 2 cells apart from the centre use non-anticipating model of their neighbours (standing 3 cells apart from the centre). If they have used anticipating model instead, pedestrians standing 4 cells apart from the centre of the neighbourhood would have become involved. Thus, a neighbourhood is growing until it "covers" the entire scene (Fig. 5).

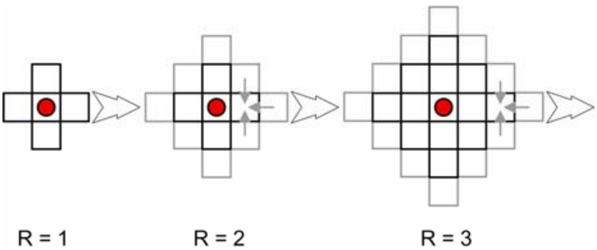


Fig. 5 Growth of the neighbourhood, inducted by the anticipating model (arrows indicate the cell for which a probability of occupation is computed, cells with grey frames get involved into the computations as they can contain possible "occupants")

It is clear that this process of neighbourhood growth must be interrupted at a certain step, because of two reasons (theoretical and computational):

- every next step destroys spatial localization of the model, thus contradicting the hypothesis of local information (a pedestrian does not know what is happening beyond his neighbourhood);
- growth of the neighbourhood makes the model more computationally intensive.

Time-cost of calculation of probabilities P_k for one pedestrian is defined by the number of cells in his (extended) neighbourhood. In our case (4-cell elementary neighbourhood) this number makes up $(R+1)^2+R^2-1 \sim R^2$, where R – radius of an extended neighbourhood. So, this radius should be limited by a certain value, through which different extent of information distribution may be simulated. From a point of view of the target pedestrian, this may be given the following interpretation: all the neighbours inside the extended neighbourhood are considered to be anticipating, unlike those standing on a border. On the other hand, pedestrians on a border may be also considered to be anticipating under an assumption that there are no pedestrians beyond the neighbourhood (in this case for these pedestrians

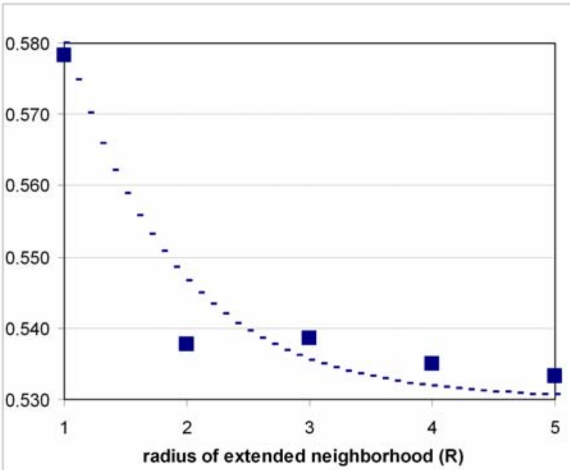


Fig. 6 Impact of neighbourhood extension on evacuation time (evacuation time is given in relative scale with 1.0 corresponding to 150 time steps).

holds $P_{k,occ} = 0$). The algorithm for implementation of the described above scheme can be found in [4].

As it becomes clear from the simulations (see Fig.6), extension of the neighbourhood has positive effect on performance of the crowd leading to shorter evacuation times. Decreasing slope of the curve in the picture is quite natural, as distant pedestrians have little influence on each other. Therefore, within the described model for practical reasons it makes sense to limit the value of R by 4 or 5.

According to the results of simulations, increasing extent of awareness of pedestrians (extending neighbourhood) leads to a decrease in evacuation time, thus certifying adequateness of the proposed approach. However, from a quantitative point of view improvements are not as high, while information about extended neighbourhood allows further optimization of behaviour via multi-step anticipation (anticipation based on multi-step prediction).

3.3 Temporal "de-localization"

The implemented models of anticipation (see subsections 3.1, 3.2) allow pedestrians to utilize additional information about their nearest environment within a neighbourhood that is somewhat broader than elementary one. Thus, comparatively to the basic model (see section 2) every pedestrian is provided with two types of additional information:

- information about placement of more distant neighbours;
- information about behavioural model of other pedestrians.

It is quite straightforward that this information is not used to its full extent, as only one-step forecasts are computed while the former suffice for multi-step prediction.

Naturally, there are several ways for constructing predictions. Let us have an extended neighbourhood of radius R with K pedestrians, one of which (the target pedestrian) is in the centre. Every pedestrian can shift in one of no more that 4 directions or stay still (5 alternatives).

The simplest way is based on an assumption that all the variants of evolution are equally probable. Then it is possible to construct a scenarios tree and to find an optimal trajectory (path from the root to one of the leaves) in that tree. Strictly speaking, the obtained graph is not a tree as branches originating from one node may intersect. This way of prediction is the worst one because of two reasons. Firstly, it has high time-space cost: in the worst case we have 5^K alternatives at the first step and $((\tau+1)^2+\tau^2)^K$ alternatives after τ steps. (Here we consider the case when maximum speed of every pedestrian is equal to 1. Thus, after τ steps every pedestrian will be within a neighbourhood of radius R=τ around his initial position, therefore he will occupy one out of no more than $(\tau+1)^2+\tau^2$ cells, and for K pedestrians there are at most $((\tau+1)^2+\tau^2)^K$ possibilities.) Secondly, there is no use of information about behavioural model of the neighbours.

At the next stage we can apply this information and try to reduce the number of alternatives (branches in the tree) at every step taking into account probabilities of shift of pedestrians. Let us have a set of pedestrians (of capacity K), a set of cells in the neighbourhood (of capacity $(R+1)^2+R^2$) and a list of probabilities for every pedestrian to shift to every cell. Now we have to find such an alternative or a set of pairs <pedestrian, cell> for which the sum of probabilities reaches the maximum. It is now clear that selection of the most probable alternative may be found as a solution to the assignment problem [25] (see Fig.7). For the sake of simplicity we are able to apply the necessary and sufficient conditions of uniqueness of an optimal assignment problem solution which can be stated as follows. The set of optimal assignment problem solutions contains a single solution if and only if all corresponding upper tolerance values are strictly positive [26]. Based on the upper tolerances we are able to adjust the number of optimal assignment problem solutions such that our computational efforts will be reasonable. The advantage of this approach is based on the fact that assignment problem has been extensively investigated and there exist efficient algorithms for solving it [27].

However, this approach needs some revision. As a matter of fact, obtained in such a way predictions are based on an assumption that probabilities for target pedestrian are known, while all these computations are held in order to adjust them. So, it makes sense to consider all possible variants for the target pedestrian. As a result, even in a case of unique solution of an assignment problem at a particular step up to 4 leaves (this value is bounded by the number of cells in an elementary neighbourhood) will be added to the tree. Thus, in order to construct a one-step prediction the following steps are necessary:

1. to construct up to 4 (number of non-zero P_k, k=1,...,4) graphs, similar to that in the Fig.7; in each such graph for the target pedestrian holds $P_i=\delta_{ik}$ (δ – Kronecker symbol);
2. to solve an assignment problem for every graph (obtained at the previous step) that will give one or several most probable variants of movement of pedestrians;

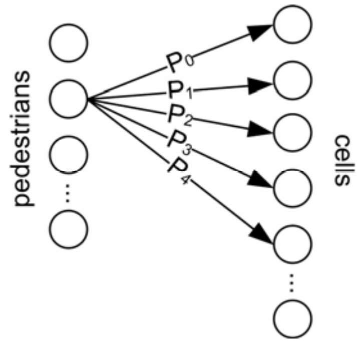


Fig. 7 One-step prediction as a solution to the assignment problem (connections are shown only for one pedestrian).

3. to construct the alternatives by shifting pedestrians in the directions corresponding to solutions obtained at the previous step and to add them to the scenarios tree.

In order to construct a T-step prediction (for an arbitrary T) one has to repeat the above mentioned procedure T times. This will result in a tree, each node of which corresponds to some state of the extended neighbourhood and every simple path (sequence of nodes from the root to one of the leaves) – to one of the most probable scenarios for the nearest T steps (see Fig.8). Now we have to find the optimal one among these scenarios (strict formulation of the optimality criterion is provided in subsection 3.4). Knowledge about the optimal scenario will allow the pedestrian to make his step in a direction that will enforce this scenario and thus he will make his positive impact (in terms of minimization of evacuation time and maximization of the flow) on the overall behaviour of the crowd. In the following subsection 3.4 we describe a general framework for finding an optimal path in a scenarios tree and two approaches to solving this problem in order to demonstrate its solvability, however numerical experiments with these approaches is a matter of future research.

3.4 Finding Optimal Trajectories in a Scenarios Tree

As it was mentioned above, a scenarios tree is strictly speaking not a tree but a directed graph G(T) = (V, A) (see Fig.8b), T is a horizon of prediction (a number of steps for which a prediction is made), $V=\{v_j\}$ – a set of vertices (each of them corresponds to one of the future states of the extended neighbourhood that are present in the scenarios tree), $A \subset V \times V$ – set of arcs (each arc corresponds to a transition between two states of the extended neighbourhood). Any two vertices *i* and *j* from V are connected by an arc (*i*, *j*) if and only if there is a transition in the scenarios tree between the states of the extended neighbourhood that correspond to these two vertices. Let us denote by *s* (source, or root of the scenarios tree) the vertex of G(T), that corresponds to the current state of an extended neighbourhood, by t_i – vertices that correspond to final states of an extended neighbourhood (leaves of

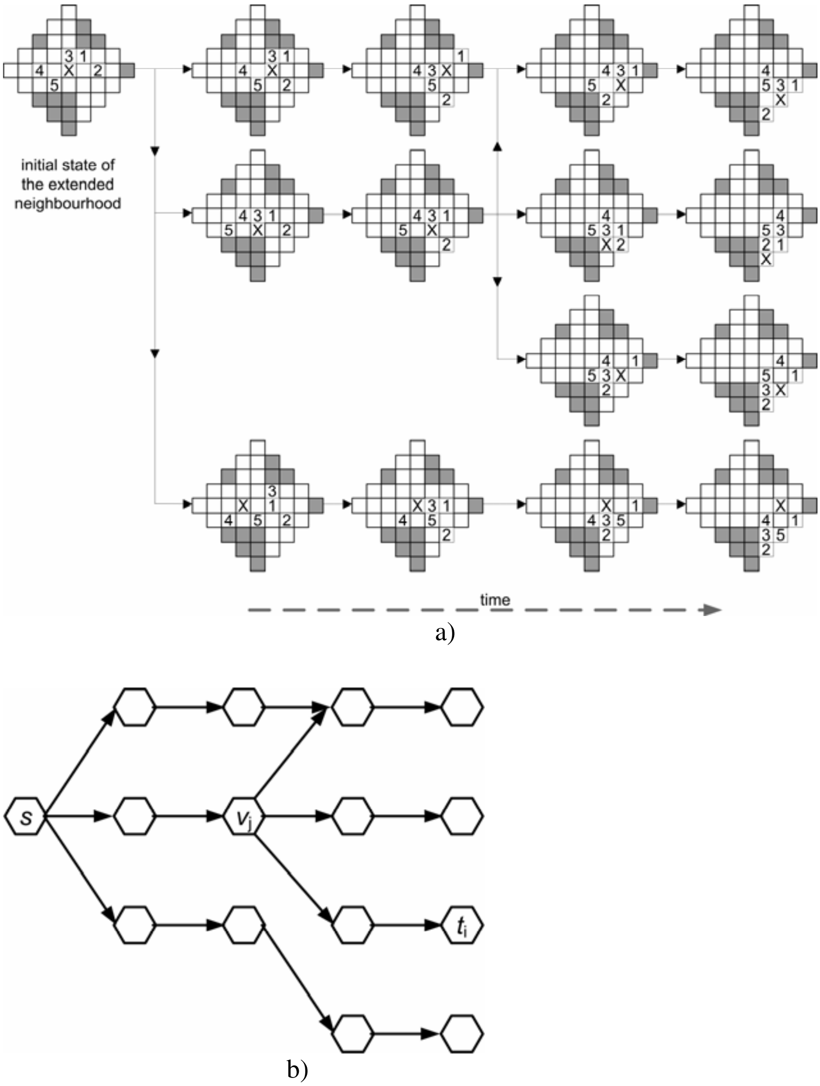


Fig. 8 An example scenarios tree (a) (X - target pedestrian, other pedestrians are numbered 1,…,5, grey cells indicate obstacles, arrows indicate one-step transitions between states of the extended neighbourhood) and a corresponding graph G(T) (b). Any chain of states of the extended neighbourhood linked by arrows reflects a possible evolution of the CA (restricted to the neighbourhood under consideration). In this case a horizon of prediction T=4, a radius of the extended neighbourhood R=4, a number of pedestrians within the neighbourhood *K*=6.

the scenarios tree), *K* is a number of pedestrians in the extended neighbourhood. Under introduced terms we can formulate the following specific properties of a graph G(T):

1. G(T) has one vertex s that has no incoming arcs (s has in-degree equal to 0);
2. G(T) has no more than $((T+1)^2+T^2)^K$ vertices t_i, that have no outgoing arcs (t_i have out-degree equal to 0);
3. G(T) has a layered structure, vertices from a layer are not connected to each other and are connected only to vertices of the next layer; there are T+1 layers;
4. every vertex (except s) has at least one and at most 5^K ingoing arcs;
5. every vertex (except t_i) has at least one and at most 5^K outgoing arcs.

Over the set of vertices (states of the extended neighbourhood) V we can define a quality function $q(v)$ ($v\in$ V), that will be used as a criterion of optimality (quality). Naturally, $q(v)$ can be defined in different ways, below is an example of how it can be done.

Let us introduce an auxiliary function C(.) (C: $V\rightarrow Z^2$) that returns coordinates of the target pedestrian. Now the optimality criterion can be defined as follows:

$$q(v) = -\min_i[\mu(C(v), X^i_{exit})],\ v\in \mathrm{V}, \tag{5}$$

where X^i_{exit} – coordinates of i-th exit, $X^i_{exit}\in Z^2$. In this case a value of the optimality criterion depends on a distance from the target pedestrian to the nearest exit, minus sign provides growth of the value of the criterion as a pedestrian approaches one of the exits. Thus, according to this criterion, scenarios that will lead the target pedestrian closer to the exit will be preferred.

Alternatively, $q(v)$ can be defined as:

$$q(v) = \mu(C(v), C(s)),\ v, s\in \mathrm{V}, \tag{6}$$

where $\mu(.)$ is a Manhattan metrics. In this case optimality of a vertex v depends on a distance between predicted position of the target pedestrian (in the state of the extended neighbourhood that corresponds to the vertex v of G(T)) and his current position (in the initial state of the extended neighbourhood that corresponds to the vertex s of G(T)). Such definition of the criterion ensures that scenarios that allow more movements of the target pedestrian will be preferred.

An advantage of the latter definition of the optimality criterion (see eq. 6) is that pedestrians do not need to know positions of exits (distances to exits). This allows to simulate situations when pedestrians either unaware of where the exits are or can hardly orientate themselves in dark or filled with smoke premises.

Optimality criterion of an arbitrary path $p=\{s,...,t_i\}$ (a sequence of vertices from the root s to one of the leaves t_i, any two consecutive ones of which are connected by an arc in G(T)) in the most general case can be defined as a linear function of optimality values of its vertices, e.g.:

$$Q(p) = \alpha_0 q(s) + \alpha_1 q(v_1) + ... + \alpha_{T-1} q(v_{T-1}) + \alpha_T q(t_i), \tag{7}$$

where $v_1,\dots,v_{T-1}\in p$, values of constants $\alpha_j\in$ IR (j=0,...,T) are up to the user's choice.

3.4.1 Network Flow Approach

Let us construct a graph $G_k(T) = (V', A')$ from $G(T) = (V, A)$ (introduced in subsection 3.4) in the following way (in this subsection we consider optimality criterion $q(.)$ defined in eq. 5):

1. $V' \supset V$;
2. $A' \supset A$;
3. every arc $(v_i,v_j)\in A'$ has the following capacity $c(v_i,v_j)=q(v_j)-q(v_i)+2$ (2 ensures positive sign of $c(v_i,v_j)$ as $|q(v_j)-q(v_i)|\leq 1$ holds)
4. arcs of type (s,v_j) have zero capacity if $j\neq k$ and infinite (large enough) capacity if $j=k$;
5. let us add one more vertex t (sink) to V' and arcs of type (t_i,t) to A' such that $c(t_i,t)=q(t_i)-q(s)+T$

In graphs $G_k(T)$ preferred transitions (arcs) have larger capacity (see Fig.9). As each of these graphs has one vertex s that can be treated as a source and one vertex t that can be treated as a sink, it is possible to solve a max-flow/min-cut problem (e.g. using the Ford-Fulkerson algorithm) [28]. It is clear that arcs that belong to optimal paths have the largest capacities and, therefore, will make up the maximum flow.

However, a graph $G_k(T)$ has one negative property: the number of arcs tends to increase from layer to layer while capacities have the same order ($c(v_i,v_j)\in\{1,2,3\}$ for all $(v_i,v_j)\in A'$). So, the closer an arc to the source s is, the faster it saturates, thus, the value of max-flow will be defined mainly by the arcs that are close to the source and capacities of distant arcs will have little influence. In order to avoid this, capacities $c(v_i,v_j)$ were adjusted by addition of the following fixed value

$$\sum_l c(t_l,t), \qquad v_i,v_j \neq t_l, \forall l:(t_l,t)\in A'. \tag{8}$$

Thus, the resulting capacities of arcs in $G_k(T)$ are as follows:

$$c(v_i,v_j)=q(v_j)-q(v_i)+2+\sum_l c(t_l,t),\ \forall l:(t_l,t)\in A' \tag{9}$$

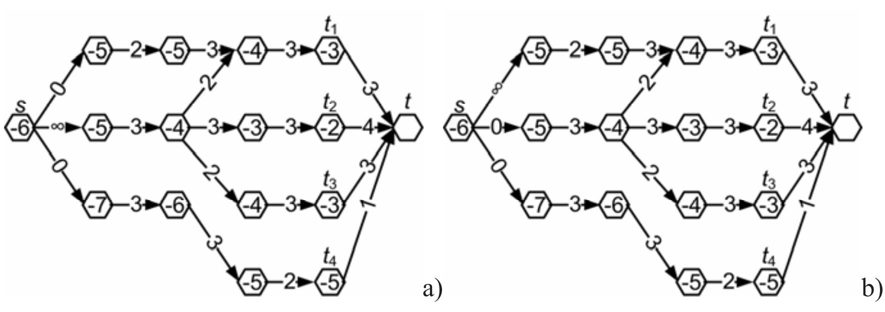


Fig. 9 Construction of $G_k(T)$: numbers in vertices correspond to $q(v_j)$, capacities of edges are determined according to eq. 9, T=4. $G_1(T)$ and $G_2(T)$ are shown.

It is quite straightforward that the value of max-flow $f_{\max}$ in this graph will satisfy

$$\min_i c(t_i, t) \le f_{\max} \le \sum_i c(t_i, t),\ \forall i : (t_i, t) \in \mathrm{A}' \tag{10}$$

- it will not be smaller than the capacity of thinnest arc incident to the sink and will not exceed total capacity of the arcs incident to the sink) and

$$\min_i (q(t_i) - q(s)) + T \le f_{\max} \le \sum_i (q(t_i) - q(s) + T),\ \forall i : (t_i, t) \in \mathrm{A}'. \tag{11}$$

Now $f_{\max}$ can be defined as $f_{\max} = \alpha_0 q(s) + \sum_i \alpha_i q(t_i)$ where $\alpha_0, \alpha_i \in \mathrm{IR}$ and sum is taken over all the leaves t_i.

Comparing the latter expression and (3.7), having $\alpha_1,...,\alpha_{T-1}$ in (3.7) equal to 0, value of max-flow can be expressed as

$$f_{\max} = \sum_i Q(p_i)\,, \tag{12}$$

where p_i – simple paths (sequences of vertices, each two consecutive ones of which are connected by an arc in $G_k(T)$) of type $(s,..,t)$ in $G_k(T)$.

Thus, the value of max-flow can serve as a consistent criterion of joint optimality of simple paths in a graph $G_k(T)$.

After having computed values of maximum flow for all $G_k(T)$ (no more that 4 – the number of arcs incident to the vertex s that is equal to the number of possible directions of shift that is bounded by the number of cells in an elementary neighbourhood) - $f_{k,\max}$, one can obtain a quantitative measure of optimality of shift in k-th direction for the target pedestrian: the higher the value of $f_{k,\max}$ is, the more opportunities he has in following the optimal trajectory (optimal for at least next T steps).

Let us introduce an auxiliary value $P_{k,f} \in [0,1)$, that will be used for adjustment of probabilities of shift P_k:

$$P_{k,f} = 1 - \frac{f_{k,\max} - \min_i(f_{i,\max}) + 1}{\max_i(f_{i,\max}) - \min_i(f_{i,\max}) + 1}\,,\ i=1,\ldots,4. \tag{13}$$

Now we can adjust the probabilities of shift P_k according to the mentioned above scheme (compare with eq. 1):

$$P_k := P_k \cdot (1 - \alpha \cdot P_{k,f})\,,\ i=1,\ldots,4. \tag{14}$$

3.4.2 Neural Network Approach

Layered structure of a scenarios tree and, therefore, of the corresponding to it graph G(T) = (V, A) (V – set of vertices that correspond to states of the extended neighbourhood in a scenarios tree, A – set of arcs that correspond to one-step transitions between states of the extended neighbourhood, that are present in a scenarios tree) allows to treat the latter as a multilayer perceptron and to apply corresponding algorithms.

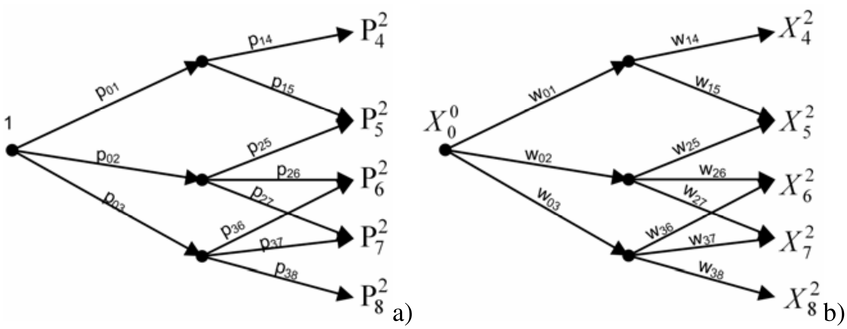


Fig. 10 An example scenarios tree (a) and a corresponding neural network (b)

Let us consider an example of a scenarios tree and a corresponding graph G(2) (see Fig.10a). Here by p_{ij} we denote a probability of transition between states i and j of the extended neighbourhood, by P_i^τ – probability of the extended neighbourhood to appear in state i after τ steps (τ =0,…,T), for the case in Fig.10 T=2, n_i – a number of descendants of i-th state of the extended neighbourhood (a number of outgoing arcs in the corresponding graph G(T)), $n_i = \text{card}\{(i,j) \mid \forall j : (i,j) \in A\}$, m_i – a number of predecessors of i-th state of the extended neighbourhood (a number of ingoing arcs in the corresponding graph G(T)), $m_i = \text{card}\{(j,i) \mid \forall j : (j,i) \in A\}$. Under the described terms, the probabilities p_{ij} of transitions are subject to:

$$\sum_j p_{ij} = 1,\ j=1,\ldots,n_i \tag{15}$$

At the same time the values P_i^τ subject to $\sum_i P_i^\tau = 1$ (i enumerates all final states of the extended neighbourhood). Moreover, the following holds:

$$P_i^\tau = \sum_j p_{ji} P_j^{\tau-1},\ \tau=1,\ldots,T,\ j=1,\ldots,m_i \tag{16}$$

$P_1^0 = 1$ – the initial state is definite and known.

Taking into account that $P_i^\tau \in [0,1]$, (3.16) implies that:

$$0 \le \sum_j p_{ji} P_j^{\tau-1} \le 1,\ \tau=1,\ldots,T,\ j=1,\ldots,m_i \tag{17}$$

Let us now consider a multilayer perceptron that corresponds to the scenarios tree (the number of vertices and neurons, as well as a pattern of their connection, coexist, see Fig.10b). After having assigned connections in the neural network some weights w_{ij} and denoting output of i-th neuron in τ-th layer as X_i^τ, functioning of the T-layer perceptron can be expressed as:

X_1^0 - arbitrary input of the network,

$$X_i^\tau = \sigma\left(\sum_j w_{ji} X_j^{\tau-1}\right),\ \tau=1,\ldots,\mathrm{T},\ j=1,\ldots,\mathrm{m}_i \tag{18}$$

where σ(.) is a sigmoid function [29] that bounds the output of a neuron by 0 and 1.

If one defines sigmoid function σ(.) as

$$\sigma(x) = \begin{cases} x, x \in [0,1] \\ 0, x < 0 \\ 1, x > 1 \end{cases} \tag{19}$$

then it becomes straightforward that (3.16) and (3.18) are equivalent. Thus, given appropriate input, the output of the neural network can be interpreted as a vector of probabilities of final states.

On the other hand, we can consider an inverse problem: some desired final probability distribution P^T over the set of final states of extended neighbourhood is known and we have to find such a set of probabilities at the first step p_{0j} that will ensure the given final distribution. Within the framework of neural networks the following problem emerges: to find such weights w_{0j} that will transform a particular input of the network (X_1^0 = P_1^0 = 1) into a particular output ($X^\mathrm{T} = \mathrm{P}^\mathrm{T}$). In other words we have a problem of learning of a perceptron that can be resolved by a variety of algorithms, one of which is BackPropagation [29].

Remark

BackPropagation algorithm requires existence of the first derivative of σ(.), which is not true in our case (see eq. 19). There are two ways to deal with this problem:

1. to define the derivative of piecewise-linear σ(.) in 0.0 and 1.0;
2. to approximate piecewise-linear σ(.) by exponential sigmoid [29] with some accuracy level ε.

Let us now return to the problem of finding the optimal path in a scenarios tree G(T). In order to implement the described above neural network approach, arcs have to be assigned weights corresponding to probabilities of relevant transitions. The probability of transition from state *i* to state *j* is:

$$\mathrm{p}_{ij} = \prod_{l=1}^{K} \sum_{k=0}^{4} \mathrm{P}_{k,l}\delta_{k,d(l)}\ , \tag{20}$$

where K – the number of pedestrians in a neighbourhood, $\mathrm{P}_{k,l}$ – the probability of shift of l-th pedestrian in k-th direction, $d(l)$ – the direction of shift of l-th pedestrian while the neighbourhood turns from state *i* into state *j*. As the scenarios tree contains not all the transitions but the most possible ones (see subsection 3.3), in case of w_{ij}=p_{ij} eq. 15 does not hold, thus weights of arcs are to be normalized:

$$w_{ij} = \frac{\mathrm{p}_{ij}}{\sum_l \mathrm{p}_{il}}\ ,\ l=1,\ldots,\mathrm{n}_i \tag{21}$$

Input of the network is 1 (X_1^0=1). Output represents distribution of probabilities of different final states of the neighbourhood. Naturally, the more the value of optimality criterion $q(t_i)$ (see eqs. 5, 6) of some final state t_i, the higher the corresponding probability should be. Thus, the desired output of the perceptron can be defined as:

$$X_i^{\mathrm{T}} = \frac{q(t_i) - \min_j(q(t_j))}{\sum_l \left[q(t_l) - \min_j(q(t_j)) \right]}, \tag{22}$$

where j and l enumerate all neurons in the last layer of the neural network (of graph G(T)), T is a number of the output (last) layer of the neural network (of graph G(T)). After the process of learning is completed, we have desired probabilities of shift at the current step $p_{0i}=w_{0i}$ (i=1,…,n_0). Now the values of P_k can be adjusted either according to the mentioned above scheme (see eq. 1)

$$P_k := P_k \cdot (1 - \alpha \cdot (1 - p_{0k})),\ k=1,\ldots,4 \tag{23}$$

with subsequent normalization ($\sum_{k=1}^{4} P_k = 1$) or

$$P_k := (1-\alpha)P_k + \alpha \cdot p_{0k},\ k=1,\ldots,4. \tag{24}$$

3.5 Asymptotic Estimates

There is no doubt that performance of a crowd drastically depends on the behavioural characteristics of individuals (parameters of the model). It was shown above that a thorough selection of values of the parameters allows to improve the overall performance (to reduce evacuation time or to increase flow) and endowing pedestrians with such a feature as anticipation allows further improvement. But to what extent are these improvements substantial? In order to be able to answer this question we need to assess the best (theoretically) possible performance.

Let us consider the situation with continuous flow of pedestrians. In case of stationary geometry of the scene and constant boundary conditions the problem of finding the maximum flow is rather simple. According to the Ford-Fulkerson theorem,

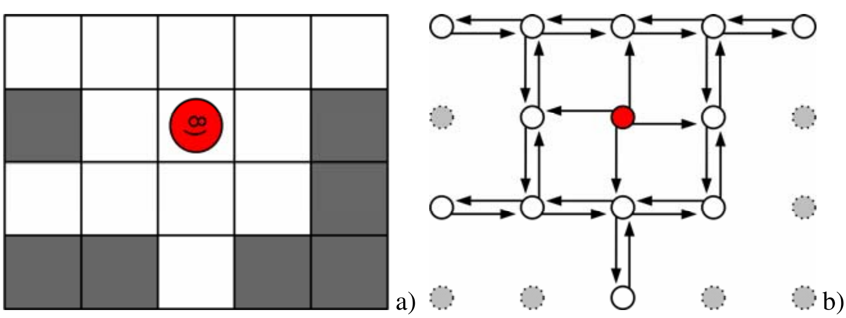


Fig. 11 An example of a scene of CA and corresponding graph (all arcs have unit capacity)

the value of maximum flow is equal to the capacity of minimum cut. Moreover, application of one of the class of so-called augmenting path algorithms will provide a bunch of paths that pedestrians have to follow in order to saturate the minimum cut (and ensure the largest possible flow).

The situation is completely different and somewhat more complex in case of evacuation. As a matter of the fact, evacuation represents a transient process that can enter stationary mode for only very short periods of time (e.g. when the whole crowd runs through a long corridor). Thus, it is not possible to apply the simple formula <time>=<number of pedestrians>/<capacity of min. cut> in this case, mainly because the width of a minimal cut varies in time (see Fig.12).

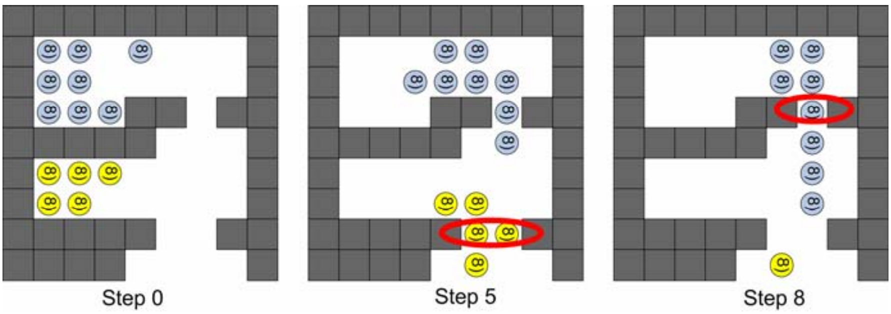


Fig. 12 Variable minimum cut emerging at different time steps of evacuation.

So, a need for some dynamic procedure occurs. Below we give an algorithm that simulates optimal evacuation for the case when all pedestrians have the same speed (equal to 1) and finds minimum egress time based on dynamically updated max-flow/min-cut problem. While solving the latter, pedestrians are treated as sources, exits - as sinks and arcs in a graph are assumed to have unit capacity.

In the algorithm the following definitions are used: by |I| we denote number of the elements in the set I, $l(p_i)$ – length of the corresponding path;

I – a set of the indices of occupied (by pedestrians) cells of an automaton;
K – subset of I that contains indices of the cells occupied by pedestrians for which paths were build at one step of the algorithm, $K \subset$ I;
n – step number.

At the output we have T – minimum evacuation time.

The Evacuation Algorithm (max flow of pedestrians and min evacuation time)

0 T:=0 //initialization
1 while |I|>0 { //while there are some pedestrians
2 build max flow => max flow F^n_{max}, saturated paths p_i, $i \in K \subset$ I, $|K|=F^n_{max}$
3 if $\max(l(p_i))=\min(l(p_i))$ // if all $l(p_i)$ are equal
4 T+=max(min(($l(p_i)$)-T , 1)
5 for all $i \in K$
6 if $l(p_i)=\min(l(p_i))$ //first pedestrians reaching the exit…

```
7        I:=I \ i                //…go out
8    }
```

The main ideas behind the given algorithm are as follows:

1. at every step a maximum set of pedestrians K that are able to reach the exit at the same time is found (number of elements in this set is bounded only by capacities of corridors);
2. as pedestrians from K can initially be at different distances from the exit (differ in values of their $l(p_i)$), only those having minimum value of $l(p_i)$ can go out at this step (lines 6,7 in the Evacuation Algorithm). At the same time, they have no influence on those behind them and, therefore, have no impact on evacuation time;
3. if at some step all $l(p_i)$ are equal (line 3) then all pedestrians from K can go out at once. This implies that they were saturating the minimum cut and being an obstacle for those behind them. This is why they have an impact on the overall evacuation time and T is adjusted in an appropriate way (line 4).

As far as we are able to assess the best possible performance we can answer two major questions:

- is there any space for improvement of performance of a crowd?
- are the approaches that we use efficient?

Let us classify the models of pedestrians according to the extent of information delocalization. Thus, by the term MP(R,T) we denote a model of pedestrian that has information about his neighbourhood of radius R for a time period of the next T steps. It should be mentioned that within our framework information about future steps is not provided to the pedestrians explicitly but is produced by them via anticipation and is not perfectly precise.

To answer the two posed questions a number of experiments were held and a summary of the results obtained can be found in the given graphs (see Fig.13).

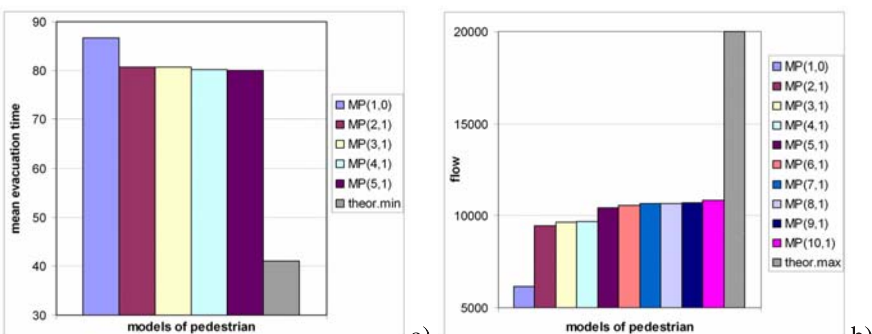


Fig. 13 Performance of the tested models of pedestrians and theoretically best performance

As it can be seen from the picture, all the tricks that we have implemented by this time have positive but rather small impact. It is a quite natural result as those models are very restricted in the amount of information that is available for pedestrians, while asymptotic estimates utilize all available information to the full extent. However, the described above framework allows to build a model MP(R,T) for arbitrary values of R and T, thus making it possible to consider an ultimate model MP(∞,∞). For practical purposes instead of infinity there must stand values that are large enough to cover the entire scene of an automaton and entire period of evacuation.

Now the most interesting question is whether the ultimate model will approach the best possible performance. And the most probable answer to this question is negative because of the following reason. The peculiarity of the model crowd (real crowds usually also have this feature) is that every pedestrian tries to improve his own performance, while, asymptotic estimates are obtained for the perfectly coordinated pedestrians.

4 Anticipation, Synchronization and Complexity

As it was shown in the previous paragraphs, anticipation in its simplest form has positive (see Fig.6) but tiny (see Fig.13) impact on the performance of the crowd. The major cause of this is that anticipating pedestrians are somewhat "egoistic", i.e. they are trying to improve their own egress time and take little care about the others. At the same time, as we know from the game theory [30], cooperating individuals can usually do better than competing ones. This fact allows coming to an idea of introduction of cooperative behaviour into the model in an explicit way. Within the described above framework this can be done by changing the criterion of optimality of predicted states of the neighbourhood $q(.)$ (see eqs. 5 and 6) in such a way that it depends not only on the position of the target pedestrian but also on the positions of his neighbours. In this case a target pedestrian will try to optimize not only his own behaviour but also that of the others (within his extended neighbourhood) and in the limit case (MP(∞,∞)) of all the crowd, thus allowing for the best possible performance. At the same time, the pedestrians in the crowd will behave in a perfectly organized and synchronized way.

Thus, under the term synchronization within the framework of pedestrian crowd movement we mean a phenomenon of cooperating behaviour of individuals that makes their movements correlated and more efficient with regard to maximization of flow or minimization of evacuation time. Thus, we can introduce a measure of synchronization as an "amount" of cooperative behaviour and claim that if for a particular crowd the value of this measure reaches its maximum, then the crowd will have the minimum possible evacuation time and produce the maximum possible flow at the exit (during its evacuation). According to this informal definition of synchronization and its measure we are able to introduce a formalized quantitative approximation to this measure. In particular, synchronization in a crowd can be measured as a percentage of conflicts (situations when two or more pedestrians try to move to the same cell) resolved in a collaborative way relatively to the total amount of conflicts. Here we can also define notions of snap

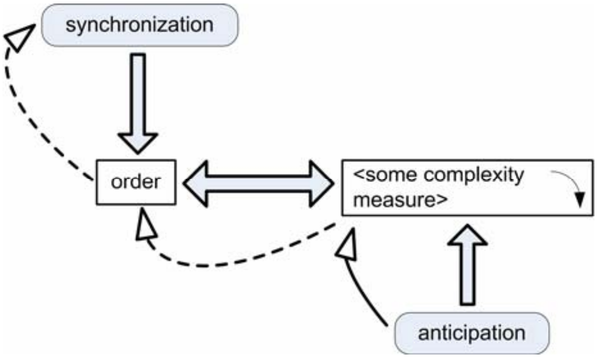


Fig. 14 Implicit relation between anticipation and synchronization via their impact on the ordered behaviour of the crowd.

synchronization and integral synchronization. According to the former we consider behaviour of pedestrians during two consecutive time steps: at first of them we can detect possible conflicts and at the second we can check how they were resolved and to calculate the value of the introduced measure of synchronization. Naturally, integral synchronization is a mean value of snap synchronization over some arbitrary long time period.

However, when collaboration is not introduced explicitly, is there a place for synchronization? In order to answer this question, let us check the relation between one-step anticipation and synchronized behaviour of pedestrians. In Fig.14 it is shown that both these phenomena are implicitly biased via an effect that they have on behaviour and some quantitative characteristic of the crowd – a measure of complexity or disorder. But how can we measure complexity of behaviour of the crowd. The most straightforward measure of complexity is dispersion of some characteristic of the crowd, e.g. evacuation time (that varies from one simulation to other) or flow (that varies from one time step to another). However, such statistical approach demands a substantial number of simulations to be held in order to obtain consistent estimates of dispersion. That is why we proposed another, entropy based, measure that allows estimating the complexity given only a snapshot of the scene.

From the common intuition we assume that behavioural complexity of the crowd is determined by behavioural complexities of its members. This fact makes it possible to define the complexity of the crowd as a sum of behavioural complexities of pedestrians (in fact it is better to use a normalized sum, i.e. divided by the number of pedestrians). As it was mentioned at the very beginning of the chapter (section 2), the behaviour of every pedestrian is driven by the probabilities of shift P_k (k=1,…,4) that depend on parameters of the model and situation in the neighbourhood of that pedestrian. Thus, we can define a snap complexity of the behaviour as Shannon's entropy: $H=-\Sigma P_k \ln(P_k)$ (k=1,…,4).

In order to define relation between this measure and anticipation a number of simulations were held and results are reflected in Fig.15. Curves in the figure correspond to different snapshots of the scene, that were obtained in the following

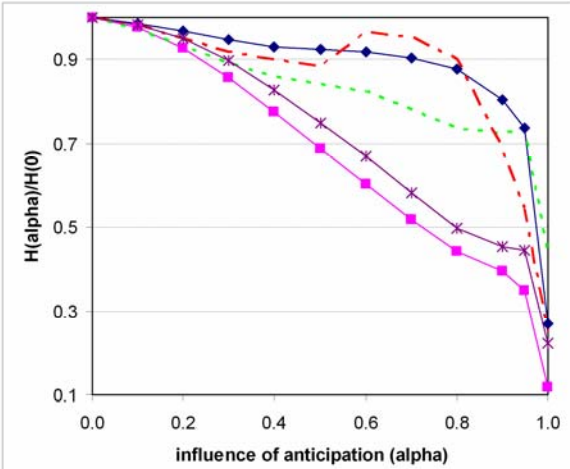


Fig. 15 Decrease of behavioural complexity with increasing influence of anticipation. Curves correspond to different snapshots of the scene (configuration of obstacles and number of pedestrians are the same in all case, only their relative positions are different).

way. The initial conditions (geometry of obstacles, initial placement of pedestrians and their individual parameters) were the same for all trials. Then the model was run for 20 time steps and for each obtained in this way snapshot we varied the value of the parameter α that expresses influence of an anticipation and measured behavioural complexity. As the models that we consider in this paper are stochastic, the obtained snapshots differ in relative positions of pedestrians and depending on how they blocked each other different curves were produced. According to the obtained numerical data, anticipation leads to a decrease in complexity, however this decline is not always monotonous and allows local peaks that do not exceed complexity of non-anticipating crowd.

As far as we have an implicit estimate of the relation between anticipation and synchronization, it is natural to perform some explicit procedures in order to check consistency of the proposed approach. In further simulations we were trying to estimate an extent of coherent behaviour in the crowd. Under the coherent behaviour we mean that pedestrians that started from one point should have same trajectories, thus leading to a more organized and synchronized crowd. In order to obtain numerical data on the issue, we have considered a dense (density ~ 75%) continuous flow of pedestrians through a straight corridor with one L-shaped bend. As a measure of coherency we used a correlation coefficient between vectors of direction of pedestrians' movement at specially chosen points that are critical to the overall performance (in this particular case we considered points at the edge of the bend). The results of simulations are given in the picture below (see Fig.16).

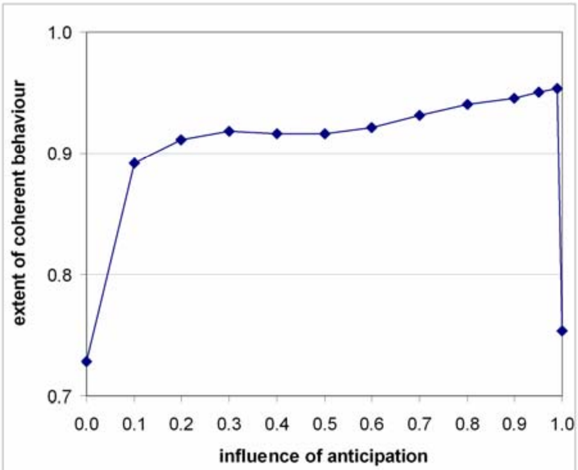


Fig. 16 Coherency of behaviour vs. influence of anticipation.

5 Summary and Future Research Directions

Up-to-date literature suggests to study a synchronization phenomenon by application of mathematical models for nonlinear stochastic dynamical systems of either neutral or delay type, based for example on stochastic differential equations, partial differential equations, chains of coupled maps, discrete equations with delay, etc. (see [1-7]). Each of the above mentioned models leads to intractable computational difficulties in finding the solutions within them. One of the inherent obstacles in finding such solutions is the so called "mentality" property induced by multi-valuedness of the corresponding solutions as functions of complex time. In some simple cases these solutions can be expressed as the inversion of a single hyper-elliptic integral. The associated Riemann surfaces are known to be infinitely sheeted coverings of the complex time plane, ramified at an infinite set of points whose projection in the complex time plane is dense [31]. Note that another way for considering such systems is provided by the theory of attractors [32]. We resolve such difficulties by using cellular automata tools with incorporated adjusted classical max-flow-min-cut models [25]. They allow studying global properties of pedestrian flows in terms similar to asymptotic behaviour of solutions, periodic orbits and their stability or sensitive dependence on initial conditions and on input parameters.

Our experiments show that by increasing the probability (see eq. 1), characterizing a personal anticipation, the maximization of flow and minimization of evacuation time of a crowd are achieved. If this probability is close to 1, then the indicated performance becomes essentially worse. The synchronization of a crowd measured by the number of egoistic pedestrians' behaviour will follow the same dependency as that probability. Note that a crowd is well synchronized even if it is not moving at all, but the above mentioned criteria will be infinitely dropped. In other words our measure of synchronization is applicable only for a crowd with at least a single movement. The following management rule might be extracted from our experiments for the minimization of crowd's evacuation time: all fast moving pedestrians should provide a preference for occupation of nearest free cells to slowly moving ones.

In this chapter we have considered a rather novel class of nonlinear models with anticipation, provided results of simulations of these models as well as interpretations of the obtained solutions. One of the most important results is the emergence of models that combine intrinsic multi-valuedness (which makes the models naturally suitable for studying the processes related to decision-making) with tractability (of their parameters, structure and results of simulations) and computational feasibility. Moreover, we proposed a framework for modelling optimal decision-making of the anticipating pedestrian based on a scenarios tree, and developed two approaches (based on the max-flow/min-cut problem and the problem of multilayer perceptron learning, correspondingly), that can be implemented within this framework. However, computational experiments with both of them remain a matter of future research. Though the proposed models in general case are not amenable to standard mathematical methods of analysis, we managed to estimate their performance in extreme cases.

We have also considered such at first sight different phenomena as anticipation, synchronization and complexity of behaviour and proposed a scheme of their implicit relations. So, another major result is an attempt to bridge the gap between the three: more anticipating crowd has lesser behavioural complexity and is more synchronized.

The described investigations open new possibilities for research on nonlinear dynamics and synchronization. May be the most interesting and important is the transition of concepts from classical theory: synchronization, self-organization, bifurcation and chaos, attractors, etc. to the case of multi-valued dynamical systems. Such transition will require the transformations of the notion and definitions of different types of synchronization, changes in self-organization theory, new aspects of bifurcation theory, definition of chaos etc. Moreover, interpretations of new solutions and properties of these novel mathematical models will lead to advances in the field of different practically important systems, for example in supply chain management [33].

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