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A POINTWISE SELECTION PRINCIPLE FOR FUNCTIONS OF A SINGLE VARIABLE WITH VALUES IN A UNIFORM SPACE

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Abstract

Given a sequence of functions, from a subset of the real line into a Hausdorff uniform space, we present a new sufficient condition for the sequence to contain a pointwise convergent subsequence. This new condition is much more weaker than the available conditions on the boundedness of generalized variations of functions, and reads in terms of some growth of moduli of variation of the functions of the sequence. Moreover, using the notion of the moduli of variation we study proper functions (i.e. those having one-sided left and right limits at each point) with respect to a dense subset and show that the Helly type selection principles involving the boundedness of generalized variations of the functions of the sequence, which are new in the context of functions with values in a uniform space, are consequences of our main result on the existence of a pointwise convergent subsequence.

Key words and phrases: moduli of variation, selection principle, pointwise convergence, proper function with respect to a dense set, uniform space, generalized variation.

1. Introduction

The classical Bolzano–Weierstrass theorem asserts that each bounded sequence of points of the real line $\mathbb{R}$ contains a convergent subsequence. The first generalization of this theorem to sequences of functions is also a classical theorem of Helly, called *Helly's selection principle*, which states that *each uniformly bounded sequence of monotone functions on a set $T \subset \mathbb{R}$ contains a pointwise convergent subsequence on T* (see [13], and also, for example, [18, Chapter VIII, Section 4, Lemma 2] if $T = [a, b]$ is a closed interval and [8, Theorem 1.3, Proof, Step 1] if T is arbitrary). Since each function of bounded

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Jordan variation is the difference of two nondecreasing bounded functions on T, Helly's selection principle is valid for a uniformly bounded sequence of functions whose Jordan variations are uniformly bounded. Further generalizations of Helly's selection principle, based on the selection principle for monotone functions, are connected with the replacement of Jordan variation by more general (or more weak) variations [1, 4, 5, 8, 12, 17, 20, 23].

In [10], whose main result is also stated in [7, 9], a new universal approach is proposed for the pointwise selection principle for functions that map $T \subset \mathbb{R}$ into a metric space X. Instead of boundedness of a generalized variation of any type for functions of a given sequence, in [10] a very weak constraint is used on the *modulus of variation* (in the sense of Chanturiya [2, 3]; also see Section 2) of these functions. It is worth noting that this constraint on the modulus of variation is not only sufficient for extracting a pointwise convergent subsequence from a given sequence of functions, but also in some cases it is necessary (e.g., in the case of the uniform convergence; also see Theorem 6 in Section 5). Moreover, the results of [10] include as particular cases many available Helly type selection principles involving the boundedness of generalized variations (see the references above).

The aim of this paper is to extend the selection principles of [7, 9, 10] to the functions with values in a uniform space X and show that the selection principles with the boundedness of generalized variations, which are new in the context under consideration, are consequences of our selection principle.

The article is organized as follows: In Section 2 we present the main definitions and formulate the central result of the paper, Theorem 1. In Section 3 we establish some properties of the moduli of variation of functions valued in a uniform space. In Section 4 we study proper functions (i.e. those having right and left one-sided limits at each point) with respect to a dense subset, which makes the idea of the moduli of variation and their role clearer. Section 5 is devoted to the proof of the central result. Finally, in Section 6 we show that many available Helly type selection principles involving the boundedness of generalized variations are consequences of our selection principle.

2. The main definitions and results

Throughout the paper we assume that $(X,\mathcal{U})$ is a Hausdorff uniform space whose uniform structure (or uniformity) $\mathcal{U}$ is defined by a complex of pseudometrics $\{d_p\}_{p\in\mathcal{P}}$ (see [15, Chapter 6]), where $\mathcal{P}$ is an index set. We describe these assumptions in slightly more detail as a reminder. Firstly, for every $p \in \mathcal{P}$, the function $d_p\colon X \times X \to \mathbb{R}^+ = [0, \infty)$ is a pseudometric on X; i.e., for all $x, y, z \in X$ this function satisfies the conditions: $d_p(x, x) = 0$, $d_p(x, y) = d_p(y, x)$, and $d_p(x, z) \leqslant d_p(x, y) + d_p(y, z)$; moreover, if $x, y \in X$ and $d_p(x, y) = 0$ for all $p \in \mathcal{P}$ imply $x = y$ then X is said to be *Hausdorff*. Secondly, the term "complex of pseudometrics $\{d_p\}_{p\in\mathcal{P}}$" means that $\{d_p\}_{p\in\mathcal{P}}$

is the family of all pseudometrics on X uniformly continuous on $X \times X$ with respect to the uniformity of the product. Recall that a pseudometric d_p is uniformly continuous on $X \times X$ with respect to the uniformity of the product (see [15, Chapter 6, Theorem 11]) if and only if $U_{p,r} \in \mathcal{U}$ for all $r > 0$, where $U_{p,r} = \big\{(x, y) \in X \times X \;\big|\; d_p(x, y) < r\big\}$. Thirdly, note that since $\{d_p\}_{p\in\mathcal{P}}$ is a complex of pseudometrics defining the uniformity $\mathcal{U}$, the family $\{U_{p,r} \mid p \in \mathcal{P},\, r > 0\}$ is a base of $\mathcal{U}$; i.e., for each $U \in \mathcal{U}$ there exist $p \in \mathcal{P}$ and $r > 0$ such that $U_{p,r} \subset U$ (also see [15, Chapter 6, Theorem 18]).

Recall that a sequence $\{x_j\}_{j=1}^{\infty} \subset X$ in a uniform space $(X,\mathcal{U})$ converges to an element $x \in X$ (as $j \to \infty$) if and only if $\lim\limits_{j\to\infty} d_p(x_j, x) = 0$ for all $p \in \mathcal{P}$. Since X is Hausdorff, this x is unique.

A subset $Y \subset X$ of the uniform space $(X,\mathcal{U})$ is called *sequentially compact* (*relatively sequentially compact*) if each sequence of points from Y contains a subsequence converging in X to an element of Y (to an element of X, respectively).

Let $\varnothing \neq T \subset \mathbb{R}$. We denote by X^T the set of all functions $f\colon T \to X$ mapping T into X. Let $\{f_j\} \equiv \{f_j\}_{j=1}^{\infty} \subset X^T$ be a sequence of functions from T into X. We say that $\{f_j\}$ *converges pointwise on* T (*uniformly on* T) to a function $f \in X^T$ provided $\lim\limits_{j\to\infty} d_p\big(f_j(t), f(t)\big) = 0$ for all $p \in \mathcal{P}$ and all $t \in T$ $\big($provided $\lim\limits_{j\to\infty} \sup_{t\in T} d_p\big(f_j(t), f(t)\big) = 0$ for all $p \in \mathcal{P}$, respectively$\big)$. A sequence $\{f_j\}$ is said to be *pointwise relatively sequentially compact* if the sequence $\{f_j(t)\}$ is relatively sequentially compact for all $t \in T$.

Given $p \in \mathcal{P}$, a positive integer $n \in \mathbb{N}$, $f \in X^T$, and $\varnothing \neq E \subset T$, we put

$$\nu_p(n, f, E) = \sup\Bigg\{\sum_{i=1}^{n} d_p\big(f(b_i), f(a_i)\big) \;\Bigg|\; \{a_i\}_{i=1}^{n},\ \{b_i\}_{i=1}^{n} \subset E \ \text{ such that } a_1 \leqslant b_1 \leqslant a_2 \leqslant b_2 \leqslant \cdots \leqslant a_{n-1} \leqslant b_{n-1} \leqslant a_n \leqslant b_n\Bigg\}.$$

The sequence $\big\{\nu_p(n, f, E)\big\}_{n=1}^{\infty} \subset [0, \infty]$ is called the *modulus of variation of* f *on* E *with respect to* d_p. The notion of the modulus of variation was first defined by Chanturiya in [2, 3] for a closed interval $E = T = [a, b]$ and $X = \mathbb{R}$ in connection with convergence problems of the theory of Fourier series and it was applied in [10] for $T \subset \mathbb{R}$ and a metric space (X, d) in order to establish a pointwise selection principle.

Note that for every $p \in \mathcal{P}$ the definition of the quantity $\nu_p(n, f, E)$ implies that it is finite for all $n \in \mathbb{N}$, so that $\nu_p(\cdot, f, E)\colon \mathbb{N} \to \mathbb{R}^+$ if and only if $\sup_{t,s\in E} d_p\big(f(t), f(s)\big) < \infty$. In what follows, all functions $f \in X^T$ under consideration are assumed *bounded* in the sense that $\sup_{t,s\in T} d_p\big(f(t), f(s)\big) < \infty$ for all $p \in \mathcal{P}$.

Following E. Landau, the condition $\lim_{n\to\infty}\mu(n)/n=0$ for a sequence $\mu:\mathbb{N}\to\mathbb{R}$ will be written briefly as $\mu(n)=o(n)$.

The main result of the paper is the following *pointwise selection principle* for functions of a single variable with values in a uniform space in terms of the moduli of variation.

Theorem 1. *Let $\varnothing\neq T\subset\mathbb{R}$ and let $(X,\mathcal{U})$ be a Hausdorff uniform space whose uniformity $\mathcal{U}$ is defined by an at most countable complex of pseudometrics $\{d_p\}_{p\in\mathcal{P}}$. Suppose that $\{f_j\}\subset X^T$ is a pointwise relatively sequentially compact sequence of functions such that*

$$\mu_p(n)\equiv\limsup_{j\to\infty}\nu_p(n,f_j,T)=o(n)\quad\text{for all}\quad p\in\mathcal{P}. \tag{1}$$

Then there exists a subsequence of $\{f_j\}$ which converges pointwise on T to a function $f\in X^T$ satisfying the condition: $\nu_p(n,f,T)\leqslant\mu_p(n)$ for all $n\in\mathbb{N}$ and $p\in\mathcal{P}$.

This theorem contains as particular cases the results of [7, 10] when X is a metric space. Some examples, illustrating the "optimality" of assumptions in Theorem 1, are given in [9, 10]. This theorem implies many available Helly type selection principles for functions of bounded and bounded generalized variations (see Section 6 below).

3. Properties of the moduli of variation

In order to prove Theorem 1 and the other results of the paper (see Section 4), we need to know some properties of the moduli of variation $\{\nu_p\}_{p\in\mathcal{P}}$. They are gathered in the following

Lemma 2. *Given $f\in X^T$, for all $\varnothing\neq E\subset T$, $n\in\mathbb{N}$, and $p\in\mathcal{P}$, we have*

(a) $\nu_p(n,f,E)\leqslant\nu_p(n+1,f,E)$;

(b) $\nu_p(n,f,E_0)\leqslant\nu_p(n,f,E)$ *for all* $\varnothing\neq E_0\subset E$;

(c) $d_p\big(f(t),f(s)\big)+\nu_p\big(n,f,(-\infty,s]\cap E\big)\leqslant\nu_p\big(n+1,f,(-\infty,t]\cap E\big)$ *for all* $t,s\in E$ *such that* $s\leqslant t$;

(d) $\nu_p(n+1,f,E)\leqslant\nu_p(n,f,E)+\dfrac{\nu_p(n+1,f,E)}{n+1}$;

(e) $\nu_p(n,f,E)\leqslant\liminf_{j\to\infty}\nu_p(n,f_j,E)$ *if the sequence of functions* $\{f_j\}\subset X^T$ *converges to* f *pointwise on* E;

(f) $\nu_p(n,g,E)\leqslant\nu_p(n,f,E)+2n\sup_{t\in E}d_p\big(f(t),g(t)\big)$ *provided that* $g\in X^T$.

Proof. Properties (a), (b), and (c) are immediate from the definition of the modulus of variation with respect to d_p.

The proof of (d) is essentially the same as that in [3, Lemma] for $E = [a, b]$ and $X = \mathbb{R}$; and here we recall this proof for convenience. By the definition of $\nu_p(n+1, f, E)$, for every $\varepsilon > 0$ there exist $a_i, b_i \in E$, $i = 1, \dots, n+1$ (depending on ε, p, and n, in general), such that

$$a_1 \leqslant b_1 \leqslant a_2 \leqslant b_2 \leqslant \cdots \leqslant a_n \leqslant b_n \leqslant a_{n+1} \leqslant b_{n+1}$$

and

$$\sum_{i=1}^{n+1} d_p\big(f(b_i), f(a_i)\big) \leqslant \nu_p(n+1, f, E) \leqslant \varepsilon + \sum_{i=1}^{n+1} d_p\big(f(b_i), f(a_i)\big).$$

Denoting by δ_0 the least term in the sum on the left-hand side, we find that $(n+1)\delta_0 \leqslant \nu_p(n+1, f, E)$. On the other hand, the right-hand side of the inequalities implies $\nu_p(n+1, f, E) \leqslant \varepsilon + \nu_p(n, f, E) + \delta_0$, which gives the inequality in (d) due to the arbitrariness of $\varepsilon > 0$.

(e) Let the points $a_1 \leqslant b_1 \leqslant \cdots \leqslant a_n \leqslant b_n$ be from E. By the definition of the quantity $\nu_p(n, f_j, E)$, we have

$$\sum_{i=1}^{n} d_p\big(f_j(b_i), f_j(a_i)\big) \leqslant \nu_p(n, f_j, E), \quad j \in \mathbb{N},$$

from which, for every $k \in \mathbb{N}$, we find

$$\inf_{j \geqslant k} \sum_{i=1}^{n} d_p\big(f_j(b_i), f_j(a_i)\big) \leqslant \inf_{j \geqslant k} \nu_p(n, f_j, E).$$

Passing to the limit as $k \to \infty$ and taking into account the pointwise convergence of f_j to f, we obtain

$$\sum_{i=1}^{n} d_p\big(f(b_i), f(a_i)\big) = \sum_{i=1}^{n} \lim_{j \to \infty} d_p\big(f_j(b_i), f_j(a_i)\big) \leqslant \liminf_{j \to \infty} \nu_p(n, f_j, E),$$

and it remains to take the supremum over all above-mentioned a_i and b_i, $i = 1, \dots, n$.

(f) It suffices to note only that, for all points $a_1 \leqslant b_1 \leqslant \cdots \leqslant a_n \leqslant b_n$ from E, the triangle inequality for d_p gives the following inequalities:

$$\begin{aligned}\sum_{i=1}^{n} d_p\big(g(b_i), g(a_i)\big) \leqslant & \sum_{i=1}^{n} d_p\big(g(b_i), f(b_i)\big) \\ & + \sum_{i=1}^{n} d_p\big(f(b_i), f(a_i)\big) + \sum_{i=1}^{n} d_p\big(f(a_i), g(a_i)\big) \\ & \leqslant n \sup_{t \in E} d_p\big(g(t), f(t)\big) + \nu_p(n, f, E) + n \sup_{s \in E} d_p\big(f(s), g(s)\big). \quad \square\end{aligned}$$

We observe that the inequality in Lemma 2 (d) is equivalent to

$$\frac{\nu_p(n+1,f,E)}{n+1}\leqslant\frac{\nu_p(n,f,E)}{n},\quad n\in\mathbb{N},\tag{2}$$

and so, the finite limit $\lim\limits_{n\to\infty}\nu_p(n,f,E)/n\in\mathbb{R}^+$ always exists for a bounded function $f\in X^T$.

4. Proper functions with respect to a dense subset

The aim of this section (see the last remark in Section 3) is to characterize the functions $f\colon[a,b]\to X$ on an interval $[a,b]\subset\mathbb{R}$ which satisfy the condition: $\nu_p\bigl(n,f,[a,b]\bigr)=o(n)$ for all $p\in\mathcal{P}$ (and even a bit more general condition; see Theorem 3 below).

Let $S\subset[a,b]$ be a fixed dense subset. We denote by $U_S\bigl([a,b];X\bigr)$ the set of all functions $f\colon[a,b]\to X$ such that (the Cauchy conditions with respect to S hold)

$$\lim_{S\ni t,s\to\tau-0}d_p\bigl(f(t),f(s)\bigr)=0\text{ for all }p\in\mathcal{P}\text{ at each point }\tau\in(a,b],\tag{3}$$

$$\lim_{S\ni t,s\to\tau+0}d_p\bigl(f(t),f(s)\bigr)=0\text{ for all }p\in\mathcal{P}\text{ at each point }\tau\in[a,b).\tag{4}$$

Functions f from $U_S\bigl([a,b];X\bigr)$ will be called *proper with respect to* S $\bigl($simply *proper* if $S=[a,b]$; see [21, Chapter III, Section 2; 22; 10, Lemma 3]$\bigr)$.

In the case when X is complete, a function $f\colon[a,b]\to X$ is in $U_S\bigl([a,b];X\bigr)$ if and only if at each point $\tau\in(a,b]$ the left limit $f_{|S}(\tau-)=\lim\limits_{S\ni t\to\tau-0}f(t)\in X$ of the values $f(t)$ exists with respect to S as t tends to $\tau-0$ over points of S, and at each point $\tau\in[a,b)$ the right limit $f_{|S}(\tau+)=\lim\limits_{S\ni t\to\tau+0}f(t)\in X$ with respect to S exists, where $f_{|S}$ designates the restriction of f to S.

Indeed, the condition (3) means that, given $a<\tau\leqslant b$, the pair $(f,S\ni t\to\tau-0)$ is a Cauchy directedness $\bigl($here the notation $S\ni t\to\tau-0$ in the above pair is understood to be a directed set $\bigl(S\cap[a,\tau),\succcurlyeq\bigr)$ in which the direction $\succcurlyeq$ is defined for $t,s\in S\cap[a,\tau)$ by the rule: $t\succcurlyeq s$ if and only if $t\geqslant s$, see [15, Chapter 2]$\bigr)$; and so, by the completeness of X, this pair converges in X to an element denoted by $f_{|S}(\tau-)$, so that $\lim\limits_{S\ni t\to\tau-0}d_p\bigl(f(t),f_{|S}(\tau-)\bigr)=0$ for all $p\in\mathcal{P}$. Similarly, we can establish the existence of the right limit $f_{|S}(\tau+)$ with respect to S at points $\tau\in[a,b)$ $\bigl($we should only define the direction $\succcurlyeq$ in the set $S\cap(\tau,b]$ according to the rule: $t\succcurlyeq s$ if and only if $t\leqslant s\bigr)$.

The set $U_S=U_S\bigl([a,b];\mathbb{R}\bigr)$ was first considered in [14]; and so, in the general case, $U_S\bigl([a,b];X\bigr)$ will be called a (*generalized*) *Jeffery class* as well.

The following characterization of the Jeffery class $U_S\big([a,b];X\big)$ in terms of the moduli of variation $\{\nu_p\}_{p\in\mathcal{P}}$ is given by

Theorem 3. *Let S be a dense set in $[a,b]$ and let $(X,\mathcal{U})$ be a Hausdorff uniform space whose uniformity $\mathcal{U}$ is defined by a (not necessarily countable) complex of pseudometrics $\{d_p\}_{p\in\mathcal{P}}$. Then*

$$U_S\big([a,b];X\big)=\big\{f\colon [a,b]\to X \mid \nu_p(n,f,S)=o(n)\ \ \textit{for all}\ \ p\in\mathcal{P}\big\}.$$

As particular cases, this theorem contains the results of the following papers: [2, Theorem 5], where $S=[a,b]$ and $X=\mathbb{R}$; [9, Theorem 3], where $S\subset[a,b]$ is a dense set and $X=\mathbb{R}$; and [10, Lemma 3], where $S=[a,b]$ and X is a metric space.

In order to prove Theorem 3, we need Lemma 4 (see below) of interest in its own right. Recall that a function $g\colon [a,b]\to X$ is said to be a *step function* if there exist a partition $a=c_0<c_1<\dots<c_{m-1}<c_m=b$ of the interval $[a,b]$ and elements $x_1,\dots,x_m\in X$ (depending on g) such that $g(t)=x_i$ for all $t\in(c_{i-1},c_i)$, $i=1,\dots,m$. For the moduli of variation of such a function g we have the following estimate:

$$\nu_p\big(n,g,[a,b]\big)\leqslant\sum_{i=1}^{m}\Big(d_p\big(g(c_{i-1}),x_i\big)+d_p\big(x_i,g(c_i)\big)\Big),\quad n\in\mathbb{N},\quad p\in\mathcal{P}. \tag{5}$$

Lemma 4. *Given a function $f\in U_S\big([a,b];X\big)$ and an arbitrary $p\in\mathcal{P}$, there exists a sequence of step functions $\{f_j\}\subset X^{[a,b]}$ such that*

$$\lim_{j\to\infty}\sup_{t\in S}d_p\big(f_j(t),f(t)\big)=0.$$

For $S=[a,b]$ and a Banach space X, Lemma 4 is well known [11, Chapter 7, Section 6] (where the proof is also valid for a complete metric space X). Some particular cases of Lemma 4 are also contained in [22, Theorem 1.1] $\big(S=[a,b]$ and X is a complete Hausdorff uniform space$\big)$ and [9, Theorem 3] $\big(S\subset[a,b]$ is dense and $X=\mathbb{R}\big)$.

Proof of Lemma 4 is adapted to the case under consideration from that of [11, Theorem (7.6.1)]. Fix an arbitrary $p\in\mathcal{P}$. Let $j\in\mathbb{N}$. The conditions (3) and (4) imply that for each $\tau\in[a,b]$ there exists an open interval $\big(\alpha(\tau),\beta(\tau)\big)$ with the endpoints $\alpha(\tau)$ and $\beta(\tau)$ (depending on p and j), containing the point τ, such that

$$\text{if } t,s\in\big(\alpha(\tau),\tau\big)\cap S \text{ or } t,s\in\big(\tau,\beta(\tau)\big)\cap S \text{ then } d_p\big(f(t),f(s)\big)\leqslant 1/j. \tag{6}$$

The family of intervals $\Big\{\big(\alpha(\tau),\beta(\tau)\big)\ \big|\ \tau\in[a,b]\Big\}$ forms an open cover of the interval $[a,b]$; and so, there exists a finite number of points $\tau_1,\dots,\tau_N\in$

$[a,b]$ such that $[a,b]\subset\bigcup_{k=1}^N\bigl(\alpha(\tau_k),\beta(\tau_k)\bigr)$. Among the points a, b, τ_k, $\alpha(\tau_k)$, and $\beta(\tau_k)$, $k=1,\dots,N$, we choose only those that lie on $[a,b]$ and order them in strictly ascending order. The resultant points are denoted by $c_0,c_1,\dots,c_m\in[a,b]$ so that $a=c_0<c_1<\dots<c_{m-1}<c_m=b$. By the density of S in $[a,b]$, on each interval (c_{i-1},c_i) we arbitrarily choose and fix a point $s_i\in(c_{i-1},c_i)\cap S$, $i=1,\dots,m$, and define the desired function $f_j\colon[a,b]\to X$ as follows:

$$f_j(t)=\begin{cases} f(c_i) & \text{if } t=c_i,\quad i\in\{0,1,\dots,m\},\\ f(s_i) & \text{if } t\in(c_{i-1},c_i),\quad i\in\{1,\dots,m\}.\end{cases}$$

Clearly, f_j is a step function. We show that $\sup_{t\in S}d_p\bigl(f_j(t),f(t)\bigr)$ does not exceed $1/j$. If $t\in S$ and $t=c_i$ for some $i\in\{0,1,\dots,m\}$ then $f_j(t)=f(t)$. So, our assertion will follow from the following observation: For each $i\in\{1,\dots,m\}$ there exists $k\in\{1,\dots,N\}$ such that

$$(c_{i-1},c_i)\subset\bigl(\alpha(\tau_k),\tau_k\bigr)\quad\text{or}\quad(c_{i-1},c_i)\subset\bigl(\tau_k,\beta(\tau_k)\bigr).\tag{7}$$

Indeed, if $t\in S\setminus\{c_i\}_{i=0}^m$ then $t\in(c_{i-1},c_i)\cap S$ for some $i\in\{1,\dots,m\}$; and so, by (7), both points t and s_i are in the same $\bigl(\alpha(\tau_k),\tau_k\bigr)\cap S$ or $\bigl(\tau_k,\beta(\tau_k)\bigr)\cap S$ for some $k\in\{1,\dots,N\}$. Applying the definition of f_j and (6), we find

$$d_p\bigl(f_j(t),f(t)\bigr)=d_p\bigl(f(s_i),f(t)\bigr)\leqslant 1/j.$$

It remains to prove (7). Let $i\in\{1,\dots,m\}$ and $t\in(c_{i-1},c_i)$. The definition of $c_0,c_1,\dots,c_m$ yields $t\notin\bigcup_{k=1}^N\bigl\{\alpha(\tau_k),\tau_k,\beta(\tau_k)\bigr\}$ while, according to the above-chosen finite subcover, we have

$$t\in\bigcup_{k=1}^N\Bigl[\bigl(\alpha(\tau_k),\tau_k\bigr)\cup\{\tau_k\}\cup\bigl(\tau_k,\beta(\tau_k)\bigr)\Bigr];$$

and so, $t\in\bigl(\alpha(\tau_k),\tau_k\bigr)$ or $t\in\bigl(\tau_k,\beta(\tau_k)\bigr)$ for some $k\in\{1,\dots,N\}$. If $t\in\bigl(\alpha(\tau_k),\tau_k\bigr)$ then the definition of $\{c_i\}_{i=0}^m$ and the inequalities $c_{i-1}<t<c_i$ and $\alpha(\tau_k)<t<\tau_k$ yield $\alpha(\tau_k)\leqslant c_{i-1}$ and $c_i\leqslant\tau_k$, i.e., $(c_{i-1},c_i)\subset\bigl(\alpha(\tau_k),\tau_k\bigr)$. Similarly, $t\in\bigl(\tau_k,\beta(\tau_k)\bigr)$ implies $(c_{i-1},c_i)\subset\bigl(\tau_k,\beta(\tau_k)\bigr)$, proving (7) and our Lemma, as well. □

Proof of Theorem 3.

1. *Inclusion* "$\subset$." Let $f\in U_S\bigl([a,b];X\bigr)$. Given an arbitrary $p\in\mathcal{P}$, we denote by $\{f_j\}\subset X^{[a,b]}$ the sequence of step functions (depending on p) of Lemma 4. By (5), we have $\nu_p\bigl(n,f_j,[a,b]\bigr)=o(n)$ for all $j\in\mathbb{N}$; and so, according to Lemma 2 (b), $\nu_p(n,f_j,S)=o(n)$. Lemma 2 (f) implies

$$\frac{\nu_p(n,f,S)}{n}\leqslant\frac{\nu_p(n,f_j,S)}{n}+2\sup_{t\in S}d_p\bigl(f_j(t),f(t)\bigr),\quad n,j\in\mathbb{N}.\tag{8}$$

Given $\varepsilon > 0$, choose $j = j(\varepsilon, p) \in \mathbb{N}$ such that $\sup_{t\in S} d_p\big(f_j(t), f(t)\big) \leqslant \varepsilon/3$, and then pick $n_0 = n_0(\varepsilon, j, p) \in \mathbb{N}$ such that $\nu_p(n_0, f_j, S)/n_0 \leqslant \varepsilon/3$. In view of (8) and (2), we have $\nu_p(n, f, S)/n \leqslant \varepsilon$ for all $n \geqslant n_0$. Thus, $\nu_p(n, f, S) = o(n)$ for all $p \in \mathcal{P}$.

2. *Inclusion "$\supset$."* Let $f\colon [a, b] \to X$ be such that $\nu_p(n, f, S) = o(n)$ for all $p \in \mathcal{P}$. Given $p \in \mathcal{P}$ and $n \in \mathbb{N}$, we put $\nu_{p,n}(s) = \nu_p\big(n, f, [a, s] \cap S\big)$ for all $s \in S$. By Lemma 2 (b), the function $\nu_{p,n}\colon S \to \mathbb{R}^+$ is nondecreasing; $\nu_{p,n}$ is bounded as well because there exists a number $n_0 = n_0(p) \in \mathbb{N}$ such that $\nu_p(n, f, S)/n \leqslant 1$ for all $n \geqslant n_0$. Hence, by Lemma 2 (b), (a), we conclude that $\nu_{p,n}(s) \leqslant \nu_p(n, f, S) \leqslant \max\{n_0, n\}$ for all $s \in S$. It follows that at each point $\tau \in (a, b]$ the left limit $\nu_{p,n}(\tau-) = \lim\limits_{S\ni t\to\tau-0} \nu_{p,n}(t) \in \mathbb{R}^+$ along the points of S exists. Show that at such τ the function f satisfies (3) (the conditions (4) can be verified in a similar manner). Fix an arbitrary $p \in \mathcal{P}$. If $t, s \in S$, $s \leqslant t < \tau$, by Lemma 2 (c), (d), (b), we have

$$
\begin{aligned}
d_p\big(f(t), f(s)\big) &\leqslant \nu_p\big(n+1, f, [a,t] \cap S\big) - \nu_p\big(n, f, [a, s] \cap S\big) \\
&\leqslant \nu_p\big(n, f, [a, t] \cap S\big) + \frac{\nu_p(n+1, f, [a, t] \cap S)}{n+1} - \nu_p\big(n, f, [a, s] \cap S\big) \\
&\leqslant \nu_{p,n}(t) + \frac{\nu_p(n+1, f, S)}{n+1} - \nu_{p,n}(s) \\
&\leqslant \big|\nu_{p,n}(t) - \nu_{p,n}(\tau-)\big| + \frac{\nu_p(n+1, f, S)}{n+1} + \big|\nu_{p,n}(\tau-) - \nu_{p,n}(s)\big|.
\end{aligned}
$$

Given $\varepsilon > 0$, choose a positive integer $n = n(\varepsilon, p) \in \mathbb{N}$ such that

$$\frac{\nu_p(n+1, f, S)}{n+1} \leqslant \frac{\varepsilon}{3},$$

and pick $\delta = \delta(\varepsilon, p, n) \in (0, \tau - a)$ such that

$$\big|\nu_{p,n}(t) - \nu_{p,n}(\tau-)\big| \leqslant \frac{\varepsilon}{3} \quad \text{for all} \quad t \in [\tau - \delta, \tau) \cap S.$$

By the above calculations, for all $t, s \in [\tau - \delta, \tau)$ we have $d_p\big(f(t), f(s)\big) \leqslant \varepsilon$ so that $\lim\limits_{S\ni t,s\to\tau-0} d_p\big(f(t), f(s)\big) = 0$; and it suffices to take into account the arbitrariness of $p \in \mathcal{P}$. $\square$

In view of Theorem 3, it is interesting to mention one more property of the proper functions with respect to S.

Theorem 5. *Under the conditions of Theorem 3, if $f \in U_S\big([a, b]; X\big)$ then the image $f(S)$ of S is a totally bounded subset of X. If, moreover, X is complete then $f(S)$ is relatively compact.*

Proof. We have to show that, for every $p \in \mathcal{P}$ and every $\varepsilon > 0$, the image $f(S)$ can be covered by finitely many balls $B_p(x,\varepsilon) = \{y \in X \mid d_p(y,x) < \varepsilon\}$ of d_p-radius ε centered at $x \in f(S)$. On the contrary, suppose that there are $p \in \mathcal{P}$ and $\varepsilon > 0$ such that $f(S)$ cannot be covered by finitely many above-mentioned balls. Fix $t_0 \in S$ arbitrarily and put $x_0 = f(t_0)$. Choose $x_1 \in f(S) \setminus B_p(x_0,\varepsilon)$. Then $x_1 = f(t_1)$ for some $t_1 \in S$, $t_1 \neq t_0$. By induction, if $n \in \mathbb{N}$, $n \geqslant 2$, and elements $x_0, x_1, \ldots, x_{n-1} \in f(S)$ are already chosen then, by the assumption, pick $x_n \in f(S) \setminus \bigcup_{i=0}^{n-1} B_p(x_i,\varepsilon)$ so that $x_n = f(t_n)$ for some $t_n \in S$, $t_n \neq t_i$ for all $i = 0, 1, \ldots, n-1$. In this way we obtain two sequences of pairwise distinct elements $\{t_i\}_{i=0}^{\infty} \subset S$ and $\{x_i\}_{i=0}^{\infty} \subset X$, and $d_p(x_i,x_j) \geqslant \varepsilon$ for all $i \neq j$. Without loss of generality, we may suppose that $t_{i-1} < t_i$ for all $i \in \mathbb{N}$. Then for arbitrary $n \in \mathbb{N}$ and $a_i = t_{i-1}$ and $b_i = t_i$, $i = 1, \ldots, n$, we find

$$\nu_p(n,f,S) \geqslant \sum_{i=1}^{n} d_p\big(f(b_i), f(a_i)\big) = \sum_{i=1}^{n} d_p(x_i, x_{i-1}) \geqslant n\varepsilon.$$

Consequently, $\lim_{n\to\infty} \nu_p(n,f,S)/n \geqslant \varepsilon > 0$, which contradicts the inclusion $f \in U_S\big([a,b];X\big)$ by Theorem 3.

The last assertion of the theorem follows from a well-known result (see [15, Chapter 6, Theorem 32]). □

Theorem 5 contains, as particular cases, the results of [11, Chapter 7, Section 6], where $S = [a,b]$ and X is a Banach or complete metric space; and of [22, Lemma 1.1], where $S = [a,b]$ and X is a complete Hausdorff uniform space. Theorem 5 can be extended to multifunctions with compact values along the same lines as it was done in [6, Lemma 11] for a metric space X.

5. Proof of the main result

Proof of Theorem 1. To start with, we observe that the quantity $\mu_p(n)$ in Theorem 1 is finite for all $n \in \mathbb{N}$ and $p \in \mathcal{P}$. Indeed, the condition (1) implies the existence of $n_0 = n_0(p) \in \mathbb{N}$ such that $\limsup_{j\to\infty} \nu_p(n,f_j,T) \leqslant n$ for all $n \geqslant n_0$; and so, it follows from Theorem 2 (a) that

$$\limsup_{j\to\infty} \nu_p(n,f_j,T) \leqslant \limsup_{j\to\infty} \nu_p(n_0,f_j,T) \leqslant n_0$$

if $1 \leqslant n \leqslant n_0$. We split the proof of Theorem 1 into five steps. Throughout the proof we assume with no loss of generality that $\mathcal{P} = \mathbb{N}$.

1. *The first series of diagonal processes.* Applying the diagonal process, we show that there exists a subsequence of $\{f_j\}$, for which we keep the notation

of the original sequence $\{f_j\}$, and for each $p \in \mathcal{P}$ there exists a nondecreasing sequence $\gamma_p \colon \mathbb{N} \to \mathbb{R}^+$ such that

$$\lim_{j\to\infty} \nu_p(n, f_j, T) = \gamma_p(n) \leqslant \mu_p(n) \quad \text{for all } n \in \mathbb{N} \text{ and } p \in \mathcal{P}. \tag{9}$$

Let $p = 1$. Since $\limsup_{j\to\infty} \nu_1(1, f_j, T) = \mu_1(1)$, there exists a subsequence $\{f_j^{(1)}\}_{j=1}^\infty$ of $\{f_j\}$ such that $\lim_{j\to\infty} \nu_1\bigl(1, f_j^{(1)}, T\bigr) = \mu_1(1)$. We put $\gamma_1(1) = \mu_1(1)$. Inductively, if $n \in \mathbb{N}$, $n \geqslant 2$, and a subsequence $\{f_j^{(n-1)}\}_{j=1}^\infty$ of the original sequence $\{f_j\}$ with the property $\lim_{j\to\infty} \nu_1\bigl(n-1, f_j^{(n-1)}, T\bigr) = \gamma_1(n-1)$ is already chosen, we put $\gamma_1(n) = \limsup_{j\to\infty} \nu_1\bigl(n, f_j^{(n-1)}, T\bigr)$ and note that $\gamma_1(n) \leqslant \mu_1(n)$ $\bigl($and also, $\gamma_1(n-1) \leqslant \gamma_1(n)\bigr)$; and so, there exists a subsequence $\{f_j^{(n)}\}_{j=1}^\infty$ of $\{f_j^{(n-1)}\}_{j=1}^\infty$ such that $\lim_{j\to\infty} \nu_1\bigl(n, f_j^{(n)}, T\bigr) = \gamma_1(n)$. Then, for the diagonal sequence $\{f_j^{(j)}\}_{j=1}^\infty$, which we denote by $\{{}^1\!f_j\} \equiv \{{}^1\!f_j\}_{j=1}^\infty$ and say that it is *of the first stage*, we obtain

$$\lim_{j\to\infty} \nu_1\bigl(n, {}^1\!f_j, T\bigr) = \gamma_1(n) \leqslant \mu_1(n) \quad \text{for all } \ n \in \mathbb{N}. \tag{10}$$

Now (for $p = 2$), we put $\gamma_2(1) = \limsup_{j\to\infty} \nu_2\bigl(1, {}^1\!f_j, T\bigr)$ and note that $\gamma_2(1) \leqslant \limsup_{j\to\infty} \nu_2(1, f_j, T) = \mu_2(1)$. Then there is a subsequence $\{{}^1\!f_j^{(1)}\}_{j=1}^\infty$ of $\{{}^1\!f_j\}$ such that $\lim_{j\to\infty} \nu_2\bigl(1, {}^1\!f_j^{(1)}, T\bigr) = \gamma_2(1)$. We apply the induction step: if $n \geqslant 2$ and a subsequence $\{{}^1\!f_j^{(n-1)}\}_{j=1}^\infty$ of $\{{}^1\!f_j\}$ is already constructed, we put $\gamma_2(n) = \limsup_{j\to\infty} \nu_2\bigl(n, {}^1\!f_j^{(n-1)}, T\bigr)$ and, noting that $\gamma_2(n) \leqslant \mu_2(n)$, choose a subsequence $\{{}^1\!f_j^{(n)}\}_{j=1}^\infty$ of $\{{}^1\!f_j^{(n-1)}\}_{j=1}^\infty$ such that $\lim_{j\to\infty} \nu_2\bigl(n, {}^1\!f_j^{(n)}, T\bigr) = \gamma_2(n)$. Then the diagonal sequence $\{{}^1\!f_j^{(j)}\}_{j=1}^\infty$, which we denote by $\{{}^2\!f_j\} \equiv \{{}^2\!f_j\}_{j=1}^\infty$ and attach to the *second stage*, satisfies the relations

$$\lim_{j\to\infty} \nu_2\bigl(n, {}^2\!f_j, T\bigr) = \gamma_2(n) \leqslant \mu_2(n) \quad \text{for all } \ n \in \mathbb{N}. \tag{11}$$

Considering (10) and (11), induct once again: if $p \in \mathcal{P}$, $p \geqslant 3$, and a subsequence $\{{}^{p-1}\!f_j\} \equiv \{{}^{p-1}\!f_j\}_{j=1}^\infty$ of the initial sequence $\{f_j\}$ of the $(p-1)$th stage, with the property

$$\lim_{j\to\infty} \nu_{p-1}\bigl(n, {}^{p-1}\!f_j, T\bigr) = \gamma_{p-1}(n) \leqslant \mu_{p-1}(n)$$

for all $n \in \mathbb{N}$ and some nondecreasing sequence $\gamma_{p-1} : \mathbb{N} \to \mathbb{R}^+$, is already chosen, then we put $\gamma_p(1) = \limsup\limits_{j\to\infty} \nu_p\bigl(1, {}^{p-1}\!f_j, T\bigr)$ and note that

$$\gamma_p(1) \leqslant \limsup_{j\to\infty} \nu_p(1, f_j, T) = \mu_p(1).$$

It follows that there exists a subsequence $\bigl\{{}^{p-1}\!f_j^{(1)}\bigr\}_{j=1}^\infty$ of $\bigl\{{}^{p-1}\!f_j\bigr\}$ such that $\lim\limits_{j\to\infty} \nu_p\bigl(1, {}^{p-1}\!f_j^{(1)}, T\bigr) = \gamma_p(1)$. Inductively, if $n \geqslant 2$ and a subsequence $\bigl\{{}^{p-1}\!f_j^{(n-1)}\bigr\}_{j=1}^\infty$ of $\bigl\{{}^{p-1}\!f_j\bigr\}$ is already constructed, we put

$$\gamma_p(n) = \limsup_{j\to\infty} \nu_p\bigl(n, {}^{p-1}\!f_j^{(n-1)}, T\bigr),$$

employ the inequality $\gamma_p(n) \leqslant \mu_p(n)$, and choose a subsequence $\bigl\{{}^{p-1}\!f_j^{(n)}\bigr\}_{j=1}^\infty$ of $\bigl\{{}^{p-1}\!f_j^{(n-1)}\bigr\}_{j=1}^\infty$ such that $\lim\limits_{j\to\infty} \nu_p\bigl(n, {}^{p-1}\!f_j^{(n)}, T\bigr) = \gamma_p(n)$. As above, the diagonal sequence of the pth stage $\bigl\{{}^{p-1}\!f_j^{(j)}\bigr\}_{j=1}^\infty$, which we designate by $\bigl\{{}^{p}\!f_j\bigr\}_{j=1}^\infty$, satisfies the conditions

$$\lim_{j\to\infty} \nu_p\bigl(n, {}^{p}\!f_j, T\bigr) = \gamma_p(n) \leqslant \mu_p(n) \quad \text{for all} \quad n \in \mathbb{N}. \tag{12}$$

We claim that the diagonal sequence $\bigl\{{}^{j}\!f_j\bigr\}_{j=1}^\infty$, which from now on will be denoted by $\{f_j\} \equiv \{f_j\}_{j=1}^\infty$, possesses the desired properties (9). Indeed, since $\{{}^{j}\!f_j\}_{j=p}^\infty$ is a subsequence of $\{{}^{p}\!f_j\}_{j=1}^\infty$, it satisfies (12), which is what we need for (9).

2. *Application of Helly's theorem and the second series of the diagonal processes.* Prove that there exists a subsequence of the sequence $\{f_j\}$ from (9), again denoted by $\{f_j\}$, and for all $n \in \mathbb{N}$ and $p \in \mathcal{P}$ there exists a nondecreasing bounded function $\nu_{n,p}\colon T \to \mathbb{R}^+$ such that

$$\lim_{j\to\infty} \nu_p\bigl(n, f_j, (-\infty, t] \cap T\bigr) = \nu_{n,p}(t) \quad \text{for all } n \in \mathbb{N}, \ \ p \in \mathcal{P} \text{ and } t \in T. \tag{13}$$

By Lemma 2 (b), the function

$$\eta_p(n, f_j, t) = \nu_p\bigl(n, f_j, (-\infty, t] \cap T\bigr)$$

is nondecreasing in $t \in T$ for all $n \in \mathbb{N}$ and $p \in \mathcal{P} = \mathbb{N}$, and the equality in (9) implies that there exists a constant $C(n,p) \in \mathbb{R}^+$ such that $\nu_p(n, f_j, T) \leqslant C(n,p)$ for all $j \in \mathbb{N}$; and so, by Lemma 2 (b), the sequence of functions $\bigl\{\eta_p(n, f_j, \cdot)\bigr\}_{j=1}^\infty$ is uniformly bounded by $C(n,p)$ on the set T.

Apply the diagonal processes once again.

We start with the case $p=1$. The sequence $\{\eta_1(1,f_j,\cdot)\}_{j=1}^\infty$ of nondecreasing functions is uniformly bounded on T by the constant $C(1,1)$; so, by the classical Helly theorem, there exists a subsequence $\{f_j^{(1)}\}_{j=1}^\infty$ of $\{f_j\}$ and a nondecreasing bounded function $\nu_{1,1}\colon T\to\mathbb{R}^+$ such that

$$\lim_{j\to\infty}\eta_1\big(1,f_j^{(1)},t\big)=\nu_{1,1}(t)\quad\text{for all}\quad t\in T.$$

By induction, if $n\geqslant 2$ and a subsequence $\{f_j^{(n-1)}\}_{j=1}^\infty$ of the original sequence $\{f_j\}$ is already chosen then, by Helly's theorem, applied to the sequence of nondecreasing functions $\Big\{\eta_1\big(n,f_j^{(n-1)},\cdot\big)\Big\}_{j=1}^\infty$, which is uniformly bounded by $C(n,1)$ on T, we find a subsequence $\{f_j^{(n)}\}_{j=1}^\infty$ of $\{f_j^{(n-1)}\}_{j=1}^\infty$ and a nondecreasing bounded function $\nu_{n,1}\colon T\to\mathbb{R}^+$ such that

$$\lim_{j\to\infty}\eta_1\big(n,f_j^{(n)},t\big)=\nu_{n,1}(t)\quad\text{for all}\quad t\in T.$$

Then the diagonal sequence $\{f_j^{(j)}\}_{j=1}^\infty$ (*of the first stage*), which we denote by $\{{}^1\!f_j\}\equiv\{{}^1\!f_j\}_{j=1}^\infty$, satisfies the condition

$$\lim_{j\to\infty}\nu_1\big(n,{}^1\!f_j,(-\infty,t]\cap T\big)=\nu_{n,1}(t)\quad\text{for all } n\in\mathbb{N}\text{ and } t\in T. \tag{14}$$

Starting from (14), we induct once again: if $p\in\mathcal{P}$, $p\geqslant 2$, and a subsequence of the $(p-1)$th stage $\{{}^{p-1}\!f_j\}_{j=1}^\infty$ of the original sequence $\{f_j\}$ is already defined, then note that the sequence of nondecreasing functions $\{\eta_p(1,{}^{p-1}\!f_j,\cdot)\}_{j=1}^\infty$ is uniformly bounded on T by $C(1,p)$; and so, by Helly's theorem, there exists a subsequence $\{{}^{p-1}\!f_j^{(1)}\}_{j=1}^\infty$ of $\{{}^{p-1}\!f_j\}_{j=1}^\infty$ and a nondecreasing bounded function $\nu_{1,p}\colon T\to\mathbb{R}^+$ such that

$$\lim_{j\to\infty}\eta_p\big(1,{}^{p-1}\!f_j^{(1)},t\big)=\nu_{1,p}(t)\quad\text{for all}\quad t\in T.$$

Inductively, if $n\geqslant 2$ and a subsequence $\{{}^{p-1}\!f_j^{(n-1)}\}_{j=1}^\infty$ of the sequence $\{{}^{p-1}\!f_j\}_{j=1}^\infty$ is already constructed then we apply Helly's theorem to the sequence of nondecreasing functions $\{\eta_p\big(n,{}^{p-1}\!f_j^{(n-1)},\cdot\big)\}_{j=1}^\infty$, uniformly bounded on T by the constant $C(n,p)$: there exists a subsequence $\{{}^{p-1}\!f_j^{(n)}\}_{j=1}^\infty$ of $\{{}^{p-1}\!f_j^{(n-1)}\}_{j=1}^\infty$ and a nondecreasing bounded function $\nu_{n,p}\colon T\to\mathbb{R}^+$ such that

$$\lim_{j\to\infty}\eta_p\big(n,{}^{p-1}\!f_j^{(n)},t\big)=\nu_{n,p}(t)\quad\text{for all}\quad t\in T.$$

Then the diagonal sequence of the pth stage $\{{}^{p-1}f_j^{(j)}\}_{j=1}^{\infty}$, for which we use the notation $\{{}^pf_j\}_{j=1}^{\infty}$, possesses the property

$$\lim_{j\to\infty} \nu_p\bigl(n, {}^pf_j, (-\infty, t]\cap T\bigr) = \nu_{n,p}(t) \quad \text{for all } n\in\mathbb{N} \text{ and } t\in T.$$

It follows that the diagonal sequence $\{{}^jf_j\}_{j=1}^{\infty}$, again denoted by $\{f_j\}$, satisfies (13) $\bigl($as well as (9)$\bigr)$.

3. *Extraction of a subsequence convergent on a countable dense subset.* Denote by Q an at most countable dense subset of T so that $Q\subset T\subset \overline{Q}$, where $\overline{Q}$ means the closure of the set Q in $\mathbb{R}$ $\bigl($the existence of Q is established as follows: if $k\in\mathbb{Z}$ is an integer and the set $T_k = T\cap[k, k+1]$ is nonempty then T_k is totally bounded and, thus, separable; and so, there exists an at most countable subset $S_k\subset T_k$ such that $T_k\subset \overline{S}_k$. It remains to put $Q=\bigcup_k S_k$ and note that $T=\bigcup_k T_k$, where the union $\bigcup_k$ is taken over all $k\in\mathbb{Z}$ such that $T_k\neq\varnothing\bigr)$. We mention that each point $t\in T$ other than a limit point for T belongs to Q. Indeed, for such a point t and some open interval (α,β), we have $T\cap(\alpha,\beta)=\{t\}$; hence, $Q\cap(\alpha,\beta)\subset T\cap(\alpha,\beta)=\{t\}$; so if we suppose that $t\notin Q$ then $Q\cap(\alpha,\beta)=\varnothing$ or $Q\subset\mathbb{R}\setminus(\alpha,\beta)$ implying $t\in T\subset\overline{Q}\subset\mathbb{R}\setminus(\alpha,\beta)$, i.e., $t\notin(\alpha,\beta)$, which contradicts the choice of (α,β).

Since, for all $n\in\mathbb{N}$ and $p\in\mathcal{P}$, the function $\nu_{n,p}$ is monotone on T, the set $Q_{n,p}\subset T$ of its discontinuity points (each of the first kind) is at most countable. Put $S = Q\cup\bigcup_{n\in\mathbb{N}}\bigcup_{p\in\mathcal{P}} Q_{n,p}$. Then S is an at most countable dense subset of T; and if $T\setminus S\neq\varnothing$ then the function

$$\nu_{n,p} \ \text{ is continuous at points } t\in T\setminus S \text{ for all } n\in\mathbb{N} \text{ and } p\in\mathcal{P}. \qquad (15)$$

Since, for every $t\in T$, the set $\bigl\{f_j(t)\bigr\}_{j=1}^{\infty}$ is relatively sequentially compact and $S\subset T$ is at most countable, we may assume without loss of generality (applying the diagonal process one more time and passing to a subsequence of $\{f_j\}$ if necessary) that, for each $s\in S$, the sequence $\{f_j(s)\}$ converges in X to an element denoted by $f(s)\in X$, so that $\lim_{j\to\infty} d_p\bigl(f_j(s), f(s)\bigr)=0$ for all $p\in\mathcal{P}$.

If $S=T$ then the proof is complete.

4. *Using the properties of the moduli of variation, and convergence everywhere on T.* Now, let $S\neq T$ and $t\in T\setminus S$. Show that $\bigl\{f_j(t)\bigr\}$ is a Cauchy sequence in X, i.e.,

$$\lim_{j,k\to\infty} d_p\bigl(f_j(t), f_k(t)\bigr) = 0$$

for all $p\in\mathcal{P}$. Fix $p\in\mathcal{P}$ arbitrarily. Let $\varepsilon>0$. By the assumption (1), $\mu_p(n)/n\to 0$ as $n\to\infty$, and so we find and fix a positive integer $n=$

$n(\varepsilon,p)\in\mathbb{N}$ such that

$$\frac{\mu_p(n+1)}{n+1}\leqslant\frac{\varepsilon}{15}. \tag{16}$$

In view of the equality in (9), there exists $J_1=J_1(\varepsilon,n,p)\in\mathbb{N}$ such that

$$\nu_p(n+1,f_j,T)\leqslant\gamma_p(n+1)+\frac{\varepsilon}{15}\quad\text{for all}\quad j\geqslant J_1. \tag{17}$$

The definition of S and the property (15) imply that t is a limit point for T and a point of continuity of $\nu_{n,p}$; so that, by the density of S in T, there exists $s=s(\varepsilon,t,n,p)\in S$ such that

$$\bigl|\nu_{n,p}(t)-\nu_{n,p}(s)\bigr|\leqslant\frac{\varepsilon}{15}. \tag{18}$$

The property (13) implies the existence of $J_2=J_2(\varepsilon,t,s,n,p)\in\mathbb{N}$ such that the following inequalities hold for all $j\geqslant J_2$:

$$\Bigl|\nu_p\bigl(n,f_j,(-\infty,t]\cap T\bigr)-\nu_{n,p}(t)\Bigr|\leqslant\frac{\varepsilon}{15}, \tag{19}$$

$$\Bigl|\nu_p\bigl(n,f_j,(-\infty,s]\cap T\bigr)-\nu_{n,p}(s)\Bigr|\leqslant\frac{\varepsilon}{15}. \tag{20}$$

Supposing (with no loss of generality) that $s<t$ and successively applying Lemma 2 (c), (d), (19), (18), (20), Lemma 2 (b), (17), the inequality in (9), and (16), for all $j\geqslant\max\{J_1,J_2\}$ we obtain

$$\begin{aligned}
d_p\bigl(f_j(t),f_j(s)\bigr)&\leqslant\nu_p\bigl(n+1,f_j,(-\infty,t]\cap T\bigr)-\nu_p\bigl(n,f_j,(-\infty,s]\cap T\bigr)\\
&\leqslant\nu_p\bigl(n+1,f_j,(-\infty,t]\cap T\bigr)-\nu_p\bigl(n,f_j,(-\infty,t]\cap T\bigr)\\
&\quad+\Bigl|\nu_p\bigl(n,f_j,(-\infty,t]\cap T\bigr)-\nu_{n,p}(t)\Bigr|+\bigl|\nu_{n,p}(t)-\nu_{n,p}(s)\bigr|\\
&\quad+\Bigl|\nu_{n,p}(s)-\nu_p\bigl(n,f_j,(-\infty,s]\cap T\bigr)\Bigr|\\
&\leqslant\frac{\nu_p\bigl(n+1,f_j,(-\infty,t]\cap T\bigr)}{n+1}+\frac{\varepsilon}{15}+\frac{\varepsilon}{15}+\frac{\varepsilon}{15}\\
&\leqslant\frac{\nu_p(n+1,f_j,T)}{n+1}+\frac{3\varepsilon}{15}\\
&\leqslant\frac{\gamma_p(n+1)}{n+1}+\frac{\varepsilon}{15(n+1)}+\frac{3\varepsilon}{15}\\
&\leqslant\frac{\mu_p(n+1)}{n+1}+\frac{4\varepsilon}{15}\leqslant\frac{\varepsilon}{3}.
\end{aligned}$$

Since the sequence $\bigl\{f_j(s)\bigr\}$ converges in the uniform space X, it follows that $\bigl\{f_j(s)\bigr\}$ is a Cauchy sequence (see [15, Chapter 6, Theorem 21]); and so there exists a positive integer $J_3 = J_3(\varepsilon, s, p) \in \mathbb{N}$ such that

$$d_p\big(f_j(s), f_k(s)\big) \leqslant \frac{\varepsilon}{3} \quad \text{for all} \quad j, k \geqslant J_3.$$

The number $J = \max\{J_1, J_2, J_3\}$ depends only on ε and p, and for all j, $k \geqslant J$ we have

$$d_p\big(f_j(t), f_k(t)\big) \leqslant d_p\big(f_j(t), f_j(s)\big) + d_p\big(f_j(s), f_k(s)\big) + d_p\big(f_k(s), f_k(t)\big) \leqslant \varepsilon.$$

Since $p \in \mathcal{P}$ is arbitrary, we conclude that $\big\{f_j(t)\big\}$ is a Cauchy sequence in X. Since this sequence is relatively sequentially compact, it admits a limit point, which we denote by $f(t) \in X$, but (see [15, Chapter 6, Theorem 21]) each Cauchy sequence in a uniform space converges to its limit point; hence we have $\lim\limits_{j\to\infty} d_p\big(f_j(t), f(t)\big) = 0$ for all $p \in \mathcal{P}$.

5. *Completion of the proof.* Since X is Hausdorff, the single-valued function $f\colon T \to X$, defined at the end of steps 3 and 4 on S and $T \setminus S$ respectively, is well defined on T and is the pointwise limit on T of the sequence $\{f_j\}$ which, by construction, is a subsequence of the original sequence. Applying Lemma 2 (e), we obtain

$$\nu_p(n, f, T) \leqslant \liminf_{j\to\infty} \nu_p(n, f_j, T) \leqslant \limsup_{j\to\infty} \nu_p(n, f_j, T) \leqslant \mu_p(n)$$

for all $n \in \mathbb{N}$ and $p \in \mathcal{P}$. The proof of Theorem 1 is complete. □

Remark 1. A *local version* of Theorem 1 holds as well. We should replace the condition (1) in this theorem by the following:

$$\limsup_{j\to\infty} \nu_p\big(n, f_j, T \cap [a, b]\big) = o(n) \quad \text{for all } a, b \in T,\ a < b, \text{ and } p \in \mathcal{P}.$$

Then a subsequence of $\{f_j\}$ converges on T to a function $f \in X^T$ such that $\nu_p\big(n, f, T \cap [a, b]\big) = o(n)$ for all $a, b \in T$, $a < b$, and $p \in \mathcal{P}$. This assertion is immediate if we apply Theorem 1 and the diagonal process over the expanding intervals.

Note that the condition (1) in Theorem 1 is *necessary* if the sequence $\{f_j\}$ converges uniformly to its limit f $\big($for another necessary condition see [10, Lemma 4 (b)]$\big)$; and so, the next assertion is, in a sense, converse to Theorem 1.

Theorem 6. *If a sequence $\{f_j\} \subset X^T$ converges uniformly on T to $f \in X^T$ such that $\nu_p(n, f, T) = o(n)$ for all $p \in \mathcal{P}$ then $\lim\limits_{j\to\infty} \nu_p(n, f_j, T) = o(n)$ for all $p \in \mathcal{P}$.*

Proof. Let $p \in \mathcal{P}$. By Lemma 2 (f), for all $n, j \in \mathbb{N}$ we have

$$\nu_p(n, f_j, T) \leqslant \nu_p(n, f, T) + 2n \sup_{t \in T} d_p\big(f(t), f_j(t)\big)$$

from which and the uniform convergence of f_j to f we infer

$$\limsup_{j \to \infty} \nu_p(n, f_j, T) \leqslant \nu_p(n, f, T) \quad \text{for all} \quad n \in \mathbb{N}.$$

From Lemma 2 (e) it follows that $\nu_p(n, f, T) \leqslant \liminf_{j \to \infty} \nu_p(n, f_j, T)$; and so, the limit $\lim_{j \to \infty} \nu_p(n, f_j, T)$ exists and equals $\nu_p(n, f, T) = o(n)$, and it remains to take into account the arbitrariness of $p \in \mathcal{P}$. $\square$

However, the condition (1) is not necessary for the pointwise convergence of $\{f_j\}$ to f (the corresponding examples are constructed in [10, Section 3; 9, Section 5] for $X = \mathbb{R}$).

Theorem 1 implies immediately that if for the sequence $\{f_j\} \subset X^T$ in Theorem 1 the condition (1) is satisfied with T replaced by $T \setminus E$, where $E \subset T$ is a set of (Lebesgue) measure zero, then a subsequence of $\{f_j\}$ converges almost everywhere (a. e.) on T to $f \in X^T$ such that $\nu_p(n, f, T \setminus E) = o(n)$ for all $p \in \mathcal{P}$.

The next more subtle result is a *selection principle for a. e. convergence* in terms of the moduli of variation for functions of a single variable with values in a uniform space.

Theorem 7. *Let T and $(X, \mathcal{U})$ satisfy the conditions of Theorem 1. Suppose that a sequence of functions $\{f_j\} \subset X^T$ is such that for almost all $t \in T$ the set $\{f_j(t)\}$ is relatively sequentially compact and for each $\varepsilon > 0$ there exists a measurable set $E_\varepsilon \subset T$ of measure at most ε such that*

$$\limsup_{j \to \infty} \nu_p(n, f_j, T \setminus E_\varepsilon) = o(n) \quad \text{for all} \quad p \in \mathcal{P}.$$

Then a subsequence of $\{f_j\}$ converges a. e. on T to $f \in X^T$ with the property: For each $\varepsilon > 0$ there exists a measurable set $E'_\varepsilon \subset T$ of measure at most ε such that $\nu_p(n, f, T \setminus E'_\varepsilon) = o(n)$ for all $p \in \mathcal{P}$.

In this theorem the a. e. convergence of a subsequence of $\{f_j\}$ to f is understood in the sense that there exists a set $E \subset T$ of measure zero such that $\lim_{j \to \infty} d_p\big(f_j(t), f(t)\big) = 0$ for all $p \in \mathcal{P}$ and $t \in T \setminus E$. By applying Theorem 1 and the diagonal process over the sets $E_1, E_{1/2}, \ldots, E_{1/k}, \ldots$ the proof of Theorem 7 follows the same lines as the proof of Theorem 6 of [10] and is therefore omitted.

6. Selection principles in some classes of functions of bounded generalized variation

Throughout this section, unless otherwise stated, we assume that $\varnothing \neq T \subset \mathbb{R}$, $(X,\mathcal{U})$ is a Hausdorff uniform space whose uniformity $\mathcal{U}$ is defined by an at most countable complex of pseudometrics $\{d_p\}_{p\in\mathcal{P}}$, and the sequence of functions $\{f_j\} \subset X^T$ is such that for each $t \in T$ the sequence $\bigl\{f_j(t)\bigr\}$ is relatively sequentially compact in X.

6.1. *Functions of bounded variation.* Given $f \in X^T$ and $p \in \mathcal{P}$, we put

$$V_p(f,T) = \sup \sum_{i=1}^{m} d_p\bigl(f(t_i), f(t_{i-1})\bigr),$$

where the supremum is taken over all partitions $\{t_i\}_{i=0}^m$ of T, i.e., $m \in \mathbb{N}$, $\{t_0, t_1, \dots, t_m\} \subset T$, and $t_{i-1} \leqslant t_i$, $i = 1, \dots, m$. If $V_p(f,T) < \infty$ for all $p \in \mathcal{P}$, we write $f \in \mathrm{BV}(T;X)$ and call f a *function of bounded* (*or finite*) *variation on* T $\bigl($for $T = [a,b]$, see [22]$\bigr)$.

As a corollary to Theorem 1 we obtain a *selection principle in* $\mathrm{BV}(T;X)$.

Theorem 8. *If, under the above conditions,* $\sup_{j\in\mathbb{N}} V_p(f_j,T) = C_p$ *is finite for all* $p \in \mathcal{P}$ *then* $\{f_j\}$ *contains a subsequence pointwise convergent on* T *whose limit* f *lies in* $\mathrm{BV}(T;X)$.

Proof. We show first that if $f \in \mathrm{BV}(T;X)$ then

$$\nu_p(n,f,T) \leqslant V_p(f,T) = \lim_{m\to\infty} \nu_p(m,f,T) \quad \text{for all } n \in \mathbb{N} \text{ and } p \in \mathcal{P}. \qquad (21)$$

Indeed, for any points $a_1 \leqslant b_1 \leqslant a_2 \leqslant b_2 \leqslant \dots \leqslant a_n \leqslant b_n$ from T we have

$$\sum_{i=1}^{n} d_p\bigl(f(b_i), f(a_i)\bigr) \leqslant V_p(f,T);$$

and so, $\nu_p(n,f,T) \leqslant V_p(f,T)$ for all $n \in \mathbb{N}$ and $p \in \mathcal{P}$. Now, if $m \in \mathbb{N}$ and points $t_0 \leqslant t_1 \leqslant \dots \leqslant t_m$ lie in T then, by Lemma 2 (a), we find

$$\sum_{i=1}^{m} d_p\bigl(f(t_i), f(t_{i-1})\bigr) \leqslant \nu_p(m,f,T) \leqslant \lim_{m\to\infty} \nu_p(m,f,T)$$

from which we infer that $V_p(f,T) \leqslant \lim_{m\to\infty} \nu_p(m,f,T) \leqslant V_p(f,T)$, and (21) follows.

By (21), for the sequence $\{f_j\}$ from Theorem 8 we conclude that

$$\sup_{j\in\mathbb{N}} \nu_p(n,f_j,T) \leqslant C_p$$

for all $n \in \mathbb{N}$ and $p \in \mathcal{P}$; and hence, $\{f_j\}$ satisfies (1). By Theorem 1, some subsequence of $\{f_j\}$, which we again denote by $\{f_j\}$, converges pointwise on T to $f \in X^T$. Since

$$V_p(f,T) \leqslant \liminf_{j\to\infty} V_p(f_j,T) \leqslant C_p \quad \text{for all} \quad p \in \mathcal{P},$$

we have $f \in \mathrm{BV}(T;X)$. Here the first inequality, expressing the sequential lower semicontinuity, can be established by analogy with Lemma 2 (e). □

Theorem 8 contains, as particular cases, the results of [4, Theorem 7.1; 5, Theorem 5.1; 1, Theorem 1] when X is a metric space.

R e m a r k 2. From the estimate (21) and Theorem 3 we find, in particular, that if $S \subset [a,b]$ is a dense set then

$$\big\{f\colon [a,b] \to X \bigm| f_{|S} \in \mathrm{BV}(S;X)\big\} \subset U_S\big([a,b];X\big).$$

This strengthens the corresponding result of [22, Theorem 1.1] established by a different method for $S = [a,b]$.

6.2. *Lipschitz continuous functions.* A function $f \in X^T$ is said to be *Lipschitz continuous on* T if

$$L_p(f,T) = \sup\Big\{d_p\big(f(t),f(s)\big)\big/\,|t-s| : t,s \in T,\ t \neq s\Big\} < \infty \quad \text{for all } p \in \mathcal{P},$$

where $L_p(f,T)$ are the *least Lipschitz constants*; in this case we write $f \in \mathrm{Lip}(T;X)$. If T is bounded then $\mathrm{Lip}(T;X)$ is embedded into $\mathrm{BV}(T;X)$; moreover, $V_p(f,T) \leqslant L_p(f,T)(\sup T - \inf T)$ for all $p \in \mathcal{P}$, $f \in \mathrm{Lip}(T;X)$.

Noting that the pointwise convergence on T of a sequence $\{f_j\} \subset X^T$ to $f \in X^T$ implies

$$L_p(f,T) \leqslant \liminf_{j\to\infty} L_p(f_j,T) \quad \text{for all} \quad p \in \mathcal{P},$$

we arrive to an analog of Theorem 8 for the space $\mathrm{Lip}(T;X)$, where the condition $\sup_{j\in\mathbb{N}} L_p(f_j,T) = C_p < \infty$ for all $p \in \mathcal{P}$ implies that the pointwise limit f of an extracted subsequence belongs to $\mathrm{Lip}(T;X)$.

6.3. *Functions of bounded* (*generalized*) φ-*variation.* Assume that the function $\varphi\colon T \times \mathbb{R}^+ \to \mathbb{R}^+$ satisfies the following conditions:

(i) for each $t \in T$ the function $\varphi(t,\cdot) = \big[u \mapsto \varphi(t,u)\big]$ is nondecreasing and continuous on $\mathbb{R}^+$ and $\lim\limits_{u\to\infty} \varphi(t,u) = \infty$;

(ii) $\varphi(t,0) = 0$ for all $t \in T$ and $\inf_{t\in T} \varphi(t,u) > 0$ for all $u > 0$.

We say that $f \in X^T$ is a *function of bounded* φ-*variation on* T (for $T = [a,b]$ and $X = \mathbb{R}$ see [12; 16, Section 10.4]) and write $f \in \mathrm{BV}_\varphi(T;X)$ if for each $p \in \mathcal{P}$ the following value is finite:

$$V_{\varphi,p}(f,T) = \sup \sum_{i=1}^{m} \varphi\Big(s_i, d_p\big(f(t_i),f(t_{i-1})\big)\Big),$$

where the supremum is taken over all $m \in \mathbb{N}$ and all collections $\{t_i\}_{i=0}^m$, $\{s_i\}_{i=1}^m \subset T$ such that $t_0 \leqslant t_1 \leqslant \cdots \leqslant t_m$ and $s_i \in [t_{i-1}, t_i] \cap T$, $i = 1, \ldots, m$.

The definition of Section 6.1 is recovered from here if $\varphi(t, u) = u$.

The following theorem is a *selection principle in the class* $\mathrm{BV}_\varphi(T; X)$:

Theorem 9. *Let the conditions at the beginnings of Section 6 and Section 6.3 be satisfied. If* $\sup_{j\in\mathbb{N}} V_{\varphi,p}(f_j, T) = C_{\varphi,p} < \infty$ *for all* $p \in \mathcal{P}$ *then there exists a subsequence of* $\{f_j\}$ *which converges pointwise on* T *to* $f \in \mathrm{BV}_\varphi(T; X)$.

The *proof* splits into three steps: At the first two steps we justify the applicability of Theorem 1 (i.e. the condition (1)); at these steps we assume that $p \in \mathcal{P}$ is arbitrary and fixed.

1. By the definition of $V_{\varphi,p}(f_j, T)$, for fixed $t_0 \in T$ and every $j \in \mathbb{N}$, we have

$$\varphi\Big(t_0, d_p\big(f_j(t), f_j(t_0)\big)\Big) \leqslant V_{\varphi,p}(f_j, T) \leqslant C_{\varphi,p}, \quad t \in T;$$

and so, by condition (i),

$$d_p\big(f_j(t), f_j(t_0)\big) \leqslant M_{\varphi,p} \equiv \sup\big\{u \in \mathbb{R}^+ \bigm| \varphi(t_0, u) \leqslant C_{\varphi,p}\big\}, \quad t \in T.$$

It follows that, for all $t, s \in T$ and $j \in \mathbb{N}$,

$$d_p\big(f_j(t), f_j(s)\big) \leqslant d_p\big(f_j(t), f_j(t_0)\big) + d_p\big(f_j(t_0), f_j(s)\big) \leqslant 2M_{\varphi,p}\,. \tag{22}$$

In particular, (22) implies that the sequence $\{f_j\}$ is uniformly bounded, i.e., $\sup_{j\in\mathbb{N}} \sup_{t,s\in T} d_p\big(f_j(t), f_j(s)\big) \leqslant 2M_{\varphi,p} < \infty$ for all $p \in \mathcal{P}$.

2. Now, given $n, j \in \mathbb{N}$, we estimate the modulus of variation $\nu_p(n, f_j, T)$. Let $\{a_i\}_{i=1}^n$, $\{b_i\}_{i=1}^n \subset T$, $a_1 \leqslant b_1 \leqslant a_2 \leqslant b_2 \leqslant \cdots \leqslant a_n \leqslant b_n$, and $s_i \in [a_i, b_i] \cap T$, $i = 1, \ldots, n$. The definition of $V_{\varphi,p}(f_j, T)$ implies the inequalities

$$\sum_{i=1}^n \varphi\Big(s_i, d_p\big(f_j(b_i), f_j(a_i)\big)\Big) \leqslant V_{\varphi,p}(f_j, T) \leqslant C_{\varphi,p}\,, \quad j \in \mathbb{N}.$$

Taking into account (22), we obtain

$$\sum_{i=1}^n d_p\big(f_j(b_i), f_j(a_i)\big)$$
$$\leqslant \sup\left\{ \sum_{i=1}^n u_i \,\middle|\, \begin{array}{l} \{u_i\}_{i=1}^n \subset \mathbb{R}^+ \text{ such that } u_i \leqslant 2M_{\varphi,p} \\ \text{for all } i = 1, \ldots, n \text{ and } \displaystyle\sum_{i=1}^n \varphi(s_i, u_i) \leqslant C_{\varphi,p} \end{array} \right\}$$

for all $s_i \in [a_i, b_i] \cap T$, $i = 1, \dots, n$. Consequently, by the arbitrariness of $\{a_i\}_{i=1}^n$ and $\{b_i\}_{i=1}^n$, such as above, we conclude that

$$\nu_p(n, f_j, T) \leqslant \sup_{s_1,\dots,s_n} \sup \left\{ \sum_{i=1}^n u_i \;\middle|\; \{u_i\}_{i=1}^n \subset [0, 2M_{\varphi,p}] \text{ and } \sum_{i=1}^n \varphi(s_i, u_i) \leqslant C_{\varphi,p} \right\}, \tag{23}$$

where the outer supremum $\sup_{s_1,\dots,s_n}$ is taken over all collections $\{s_i\}_{i=1}^n \subset T$ such that $s_1 \leqslant s_2 \leqslant \dots \leqslant s_n$.

Denoting the right-hand side of (23) by $\xi_{\varphi,p}(n)$, we obtain the inequality

$$\sup_{j\in\mathbb{N}} \nu_p(n, f_j, T) \leqslant \xi_{\varphi,p}(n) \quad \text{for all} \quad n \in \mathbb{N}. \tag{24}$$

Show that $\xi_{\varphi,p}(n) = o(n)$.

Fix an arbitrary $\varepsilon > 0$. Let $n_0 = n_0(\varepsilon, \varphi, p) \in \mathbb{N}$ be the least positive integer such that (see condition (ii))

$$n_0 \inf_{t\in T} \varphi(t, \varepsilon) \geqslant C_{\varphi,p}\,.$$

Let $n \in \mathbb{N}$, $n \geqslant n_0$, $\{s_i\}_{i=1}^n \subset T$, $s_1 \leqslant s_2 \leqslant \dots \leqslant s_n$, and $\{u_i\}_{i=1}^n \subset \mathbb{R}^+$ be such that $u_i \leqslant 2M_{\varphi,p}$ for all $i = 1, \dots, n$ and $\sum_{i=1}^n \varphi(s_i, u_i) \leqslant C_{\varphi,p}$. Put

$$I_1(n) = \{1 \leqslant i \leqslant n \mid u_i \leqslant \varepsilon\},$$
$$I_2(n) = \{1 \leqslant i \leqslant n \mid u_i > \varepsilon\}$$

and denote by $|I_1(n)|$ and $|I_2(n)|$ the number of elements in $I_1(n)$ and $I_2(n)$, respectively. By the monotonicity of the function $u \mapsto \varphi(s, u)$ (see condition (i)), we have

$$C_{\varphi,p} \geqslant \sum_{i=1}^n \varphi(s_i, u_i) \geqslant \sum_{i\in I_2(n)} \varphi(s_i, u_i) \geqslant \sum_{i\in I_2(n)} \varphi(s_i, \varepsilon) \geqslant |I_2(n)| \inf_{t\in T} \varphi(t, \varepsilon)$$

which implies

$$|I_2(n)| \leqslant \frac{C_{\varphi,p}}{\inf_{t\in T} \varphi(t, \varepsilon)} \leqslant n_0.$$

Thus,

$$\sum_{i=1}^n u_i \leqslant \sum_{i\in I_1(n)} u_i + \sum_{i\in I_2(n)} u_i \leqslant |I_1(n)|\varepsilon + |I_2(n)| \cdot 2M_{\varphi,p} \leqslant n\varepsilon + 2n_0 M_{\varphi,p}\,.$$

Taking the supremum over all collections $\{u_i\}_{i=1}^n$ with the above properties and then the supremum over all $\{s_i\}_{i=1}^n \subset T$ such that $s_1 \leqslant s_2 \leqslant \cdots \leqslant s_n$, we obtain

$$\xi_{\varphi,p}(n) \leqslant n\varepsilon + 2n_0 M_{\varphi,p} \leqslant 2n\varepsilon \quad \text{for all} \quad n \geqslant \max\{n_0, 2n_0 M_{\varphi,p}/\varepsilon\};$$

and so, $\lim\limits_{n\to\infty} \xi_{\varphi,p}(n)/n = 0$, as required.

3. The estimate (24) and the result just proven yield the condition (1); and so, by Theorem 1, there exists a subsequence of $\{f_j\}$ (again denoted by $\{f_j\}$) which converges pointwise on T to $f \in X^T$. Show that $f \in \mathrm{BV}_\varphi(T;X)$. Given $p \in \mathcal{P}$, by the definition of $V_{\varphi,p}(f_j,T)$, for all $m \in \mathbb{N}$ and $\{t_i\}_{i=0}^m$, $\{s_i\}_{i=1}^m \subset T$ such that $t_{i-1} \leqslant t_i$ and $s_i \in [t_{i-1},t_i] \cap T$, $i = 1,\dots,m$, we have

$$\sum_{i=1}^m \varphi\Big(s_i, d_p\big(f_j(t_i), f_j(t_{i-1})\big)\Big) \leqslant V_{\varphi,p}(f_j,T), \quad j \in \mathbb{N}.$$

Passing to the limit inferior as $j \to \infty$ and taking into account the pointwise convergence of f_j to f and the continuity of functions $u \mapsto \varphi(s,u)$ (see condition (i)), we obtain

$$\sum_{i=1}^m \varphi\Big(s_i, d_p\big(f(t_i), f(t_{i-1})\big)\Big) \leqslant \liminf_{j\to\infty} V_{\varphi,p}(f_j,T).$$

Taking the supremum over all above collections $\{t_i\}_{i=0}^m$ and $\{s_i\}_{i=1}^m$, we conclude that

$$V_{\varphi,p}(f,T) \leqslant \liminf_{j\to\infty} V_{\varphi,p}(f_j,T) \leqslant C_{\varphi,p} \quad \text{for all} \quad p \in \mathcal{P},$$

which means that $f \in \mathrm{BV}_\varphi(T;X)$. $\square$

The selection principle of Theorem 9 contains, as particular cases, the results of [17, Theorem 1.3], where $\varphi(t,u) = \varphi(u)$, $T = [a,b]$, and $X = \mathbb{R}$; of [8, Theorem 1.3; 10, Section 3, Example 7], where $\varphi(t,u) = \varphi(u)$, $T \subset \mathbb{R}$, and X is a metric space; and of [12] (also see [16, Theorem 10.7 (e)]), where $\varphi\colon [a,b] \times \mathbb{R}^+ \to \mathbb{R}^+$, $T = [a,b]$, and $X = \mathbb{R}$.

R e m a r k 3. We note that the estimates (23) and (24) (if $f_j = f$ for all $j \in \mathbb{N}$) and Theorem 3 imply that if $S \subset T = [a,b]$ is a dense set then

$$\big\{f\colon [a,b] \to X \;\big|\; f_{|S} \in \mathrm{BV}_\varphi(S;X)\big\} \subset U_S\big([a,b];X\big).$$

This refines an assertion of [16, Theorem 10.9] for $S = [a,b]$ and $X = \mathbb{R}$, which was established by a different method.

R e m a r k 4. Theorem 9 can be somewhat extended. We say that $f \in X^T$ is a *function of bounded generalized* φ*-variation on* T if there exists a number $\lambda > 0$ (depending on f) such that $V_{\varphi_\lambda,p}(f,T) < \infty$ for all $p \in \mathcal{P}$, where $\varphi_\lambda(t,u) = \varphi(t,u/\lambda)$, $t \in T$, $u \in \mathbb{R}^+$. Under the conditions of Theorem 9, suppose that there exists a number $\lambda > 0$ such that $\sup_{j\in\mathbb{N}} V_{\varphi_\lambda,p}(f_j,T) = C_{\varphi,p} < \infty$ for all $p \in \mathcal{P}$. Then the pointwise limit $f \in X^T$ of an extracted subsequence of $\{f_j\}$ on T is such that $V_{\varphi_\lambda,p}(f,T) \leqslant C_{\varphi,p}$ for all $p \in \mathcal{P}$.

6.4. *Functions of* (*generalized*) Φ*-bounded variation.* Let $\Phi = \{\varphi_k\}_{k=1}^\infty$ be a sequence of φ-functions, i.e., each function $\varphi_k\colon \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, nondecreasing, unbounded, and such that $\varphi_k(u) = 0$ only at $u = 0$. The sequence Φ is said to be a Φ*-sequence* [20] if Φ satisfies the following two conditions:

$$\varphi_{k+1}(u) \leqslant \varphi_k(u) \quad \text{for all} \quad k \in \mathbb{N} \quad \text{and} \quad u \in \mathbb{R}^+, \tag{25}$$

$$\sum_{k=1}^\infty \varphi_k(u) = \infty \quad \text{for all} \quad u > 0. \tag{26}$$

These two conditions on Φ are assumed throughout Section 6.4.

We say that $f \in X^T$ is a *function of* Φ*-bounded variation on* T (see [20, 19] if $T = [a,b]$ and $X = \mathbb{R}$) if for each $p \in \mathcal{P}$ the following quantity is finite:

$$V_{\Phi,p}(f,T) = \sup \sum_{k=1}^m \varphi_k\Big(d_p\big(f(b_{\sigma(k)}), f(a_{\sigma(k)})\big)\Big),$$

where the supremum is taken over all $m \in \mathbb{N}$, all $\{a_k\}_{k=1}^m$, $\{b_k\}_{k=1}^m \subset T$ such that $a_1 \leqslant b_1 \leqslant a_2 \leqslant b_2 \leqslant \cdots \leqslant a_m \leqslant b_m$, and all permutations $\sigma\colon \{1,\dots,m\} \to \{1,\dots,m\}$ (the notation $V_{\Phi,p}(f,T)$ looks like the corresponding notation in Section 6.3, but this will not lead to ambiguities in the sequel).

The definition in Section 6.1 is recovered from here if $\varphi_k(u) = u$ for all $k \in \mathbb{N}$.

The next result is a *selection principle* in the class of functions of Φ-bounded variation which map T into a uniform space X.

Theorem 10. *Suppose that the conditions at the beginnings of Sections* 6 *and* 6.4 *are satisfied. If* $\sup_{j\in\mathbb{N}} V_{\Phi,p}(f_j,T) = C_p < \infty$ *for all* $p \in \mathcal{P}$ *then* $\{f_j\}$ *contains a subsequence which converges pointwise on* T *to a function* $f \in X^T$ *such that* $V_{\Phi,p}(f,T) \leqslant C_p$ *for all* $p \in \mathcal{P}$.

Proof. 1. At the first step we show that the sequence $\{f_j\}$ of Theorem 10 satisfies the condition (1) of Theorem 1. To this end, we fix $p \in \mathcal{P}$ arbitrarily.

Given $j \in \mathbb{N}$, by the definition of $V_{\Phi,p}(f_j,T)$, we find that for all $t, s \in T$ the following inequality holds:

$$\varphi_1\Big(d_p\big(f_j(t), f_j(s)\big)\Big) \leqslant V_{\Phi,p}(f_j,T) \leqslant C_p$$

from which we obtain

$$\sup_{t,s\in T} d_p\bigl(f_j(t), f_j(s)\bigr) \leqslant M_p \equiv \sup\bigl\{u \in \mathbb{R}^+ \bigm| \varphi_1(u) \leqslant C_p\bigr\}, \quad j \in \mathbb{N}. \tag{27}$$

Let $n \in \mathbb{N}$ and let $\{a_i\}_{i=1}^n, \{b_i\}_{i=1}^n \subset T$ be arbitrary collections of numbers such that $a_1 \leqslant b_1 \leqslant a_2 \leqslant b_2 \leqslant \cdots \leqslant a_n \leqslant b_n$. Given $j \in \mathbb{N}$, by the definition of $V_{\Phi,p}(f_j,T)$, for every permutation $\sigma\colon \{1,\dots,n\} \to \{1,\dots,n\}$ we have

$$\sum_{k=1}^n \varphi_k\Bigl(d_p\bigl(f_j(b_{\sigma(k)}), f_j(a_{\sigma(k)})\bigr)\Bigr) \leqslant V_{\Phi,p}(f_j,T) \leqslant C_p\,;$$

and so, the definition of the modulus of variation $\nu_p(n,f_j,T)$ implies

$$\sup_{j\in\mathbb{N}} \nu_p(n,f_j,T) \leqslant \sup \sum_{i=1}^n u_i, \tag{28}$$

where the supremum on the right-hand side of this inequality is taken over all collections of n numbers $\{u_i\}_{i=1}^n \subset \mathbb{R}^+$ such that $\bigl($see (27)$\bigr)$ $\max_{1\leqslant i\leqslant n} u_i \leqslant M_p$ and

$$\sum_{k=1}^n \varphi_k(u_{\sigma(k)}) \leqslant C_p \quad \text{for all permutations} \quad \sigma\colon \{1,\dots,n\} \to \{1,\dots,n\}. \tag{29}$$

Denote by $\xi_p(n)$ the right-hand side of (28) and show that $\xi_p(n) = o(n)$. By (26), the following positive integer is well defined for every $\varepsilon > 0$:

$$n_0 = n_0(\varepsilon,p) \equiv \min\Biggl\{ n \in \mathbb{N} \Biggm| \sum_{k=1}^n \varphi_k(\varepsilon) > C_p \Biggr\}.$$

Now, let $n \in \mathbb{N}$, $n \geqslant n_0$, and $\{u_i\}_{i=1}^n \subset [0, M_p]$ be an arbitrary collection of n numbers satisfying (29). We put

$$I_1(n) = \{1 \leqslant k \leqslant n \mid u_k \leqslant \varepsilon\},$$
$$I_2(n) = \{1 \leqslant k \leqslant n \mid u_k > \varepsilon\},$$

and denote by $\bigl|I_1(n)\bigr|$ and $\bigl|I_2(n)\bigr|$ the number of elements in $I_1(n)$ and $I_2(n)$. We show that $\bigl|I_2(n)\bigr| \leqslant n_0$. Indeed, if $\bigl|I_2(n)\bigr| > n_0$ then $I_2(n) = \{k_1,\dots,k_{i_0}\}$ for some $i_0 \in \{n_0,\dots,n\}$ and $k_i \in \{1,\dots,n\}$, $i = 1,\dots,i_0$. Define the permutation $\sigma\colon \{1,\dots,n\} \to \{1,\dots,n\}$ by the rule:

$$\sigma(i) = \begin{cases} k_i & \text{if } i \in \{1,\dots,i_0\}; \\ \text{arbitrary} & \text{if } i \in \{i_0+1,\dots,n\}. \end{cases}$$

Then, since each function φ_k is nondecreasing, by the definition of the number n_0 and the permutation σ, we obtain

$$\sum_{k=1}^{n}\varphi_k\bigl(u_{\sigma(k)}\bigr) \geqslant \sum_{i\in I_2(n)}\varphi_i\bigl(u_{\sigma(i)}\bigr) \geqslant \sum_{i=1}^{n_0}\varphi_i(u_{k_i}) \geqslant \sum_{i=1}^{n_0}\varphi_i(\varepsilon) > C_p\,,$$

which contradicts (29). Therefore, the sum under the supremum sign on the right-hand side of (28) can be estimated as follows:

$$\sum_{i=1}^{n}u_i = \sum_{k\in I_1(n)}u_k + \sum_{k\in I_2(n)}u_k \leqslant \bigl|I_1(n)\bigr|\varepsilon + \bigl|I_2(n)\bigr|M_p \leqslant n\varepsilon + n_0 M_p \leqslant 2n\varepsilon$$

for all $n \geqslant N_0 \equiv \max\{n_0, n_0M_p/\varepsilon\}$. By the arbitrariness of collection $\{u_i\}_{i=1}^n$ satisfying (29), we infer from here that $\xi_p(n)/n \leqslant 2\varepsilon$ for all $n \geqslant N_0$; hence $\lim\limits_{n\to\infty}\xi_p(n)/n = 0$.

2. The just-proven result and (28) imply that the sequence $\{f_j\}$ satisfies (1); and so, by Theorem 1, $\{f_j\}$ contains a pointwise convergent subsequence on T, which we again denote by $\{f_j\}$ and its limit, by $f \in X^T$. Establish that $V_{\Phi,p}(f,T) < \infty$ for all $p \in \mathcal{P}$. Let $p \in \mathcal{P}$, $m \in \mathbb{N}$, $\{a_i\}_{i=1}^m$, $\{b_i\}_{i=1}^m \subset T$, $a_1 \leqslant b_1 \leqslant \cdots \leqslant a_m \leqslant b_m$, and $\sigma\colon \{1,\dots,m\} \to \{1,\dots,m\}$ be a permutation. Then

$$\sum_{k=1}^{m}\varphi_k\Bigl(d_p\bigl(f_j(b_{\sigma(k)}), f_j(a_{\sigma(k)})\bigr)\Bigr) \leqslant V_{\Phi,p}(f_j,T), \quad j \in \mathbb{N}.$$

Passing to the limit inferior as $j \to \infty$ and considering the pointwise convergence of f_j to f and the continuity of φ_k, we find

$$\sum_{k=1}^{m}\varphi_k\Bigl(d_p\bigl(f(b_{\sigma(k)}), f(a_{\sigma(k)})\bigr)\Bigr) \leqslant \liminf_{j\to\infty} V_{\Phi,p}(f_j,T).$$

Therefore,

$$V_{\Phi,p}(f,T) \leqslant \liminf_{j\to\infty} V_{\Phi,p}(f_j,T) \leqslant C_p\,,$$

and it remains to take into account the arbitrariness of $p \in \mathcal{P}$. □

Theorem 10 contains, as particular cases, the results of [23, Theorem 5], when $T = [a,b]$, $X = \mathbb{R}$, and $\varphi_k(u) = u/\lambda_k$, where $0 < \lambda_k \leqslant \lambda_{k+1}$ for $k \in \mathbb{N}$ and $\sum_{k=1}^\infty 1/\lambda_k = \infty$; and of [20, Theorem 2.8; 19, Theorem 2.6; 9, Section 6.1], where $T = [a,b]$, $X = \mathbb{R}$, and Φ is an arbitrary Φ-sequence.

Remark 5. If $S \subset [a,b] = T$ is a dense set then, by the estimates (28) and (29) and Theorem 3, we have the inclusion

$$\bigl\{f\colon [a,b] \to X \bigm| V_{\Phi,p}(f,S) < \infty \ \text{ for all } \ p \in \mathcal{P}\bigr\} \subset U_S\bigl([a,b];X\bigr).$$

R e m a r k 6. Theorem 10 can be somewhat strengthened. We say that $f \in X^T$ is a *function of generalized* Φ*-bounded variation on* T if there exists a number $\lambda > 0$ (depending on f) such that $V_{\Phi_\lambda,p}(f,T) < \infty$ for all $p \in \mathcal{P}$, where $\Phi_\lambda = \{\varphi_{k,\lambda}\}_{k=1}^\infty$ and $\varphi_{k,\lambda}(u) = \varphi_k(u/\lambda)$, $k \in \mathbb{N}$, $u \in \mathbb{R}^+$. In the framework of Theorem 10, suppose that there exists a number $\lambda > 0$ such that

$$\sup_{j\in\mathbb{N}} V_{\Phi_\lambda,p}(f_j,T) = C_p < \infty \quad \text{for all } p \in \mathcal{P}.$$

Then a subsequence of $\{f_j\}$ converges pointwise on T to $f \in X^T$ such that $V_{\Phi_\lambda,p}(f,T) < \infty$ for all $p \in \mathcal{P}$.

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