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Models for the Average of the Solutions to a P.D.E. with Stochastic Coefficients

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Abstract

In this study we investigate the successive approximation procedure allowed us to derive new equation, valid for the transport of dissolved matter in porous media, or in random force fields, on the macroscopic scale.

To do this the diagram technique was exploited.

Keywords: Diffusion, Porous Media, Fractional Equations, Diagram technique.

1. Introduction

In [1] we got two basic equations that describe diffusion process in two types of random porous media. Both equations contain coefficients, which are functions of some centered random process $\widetilde{\varepsilon}(x)$. Hence the solutions to initial value problems (with deterministic initial conditions) involving these equations are functional of $\widetilde{\varepsilon}(x)$. We aim at obtaining an equation for the evolution of $\langle u \rangle$, which is an average of concentration function of grains in the porous media u, and represents the concentration on the macroscopic scale.

It is desirable to have the averaging procedure universal so that it can be used for a media with arbitrarily stochastic character. For instance, the averaging method from [2] is usable only for telegraph random fields and not for fields with another characteristic. Surely, we want to have the procedure that can be applied for different equations, not only for the ones that are studied in this thesis.

We can consider two known methods, allowing approaching our objective. Both are more or less general but suffer some failures for complexity and can be more or less comfortable, according to the type of random process, $\widetilde{\varepsilon}$ belong to.

One of them is a perturbation method combined with graphic rules helping to deal with fixed-point equations whose solutions expand into series, whose successive terms in turn split into finite sums, involving more and more expressions.

Another method consists in deriving equations for the averaged concentration, but they involve new dependent variables that are functional derivatives (of different orders) of the previous ones, computed with respect to $\tilde{\varepsilon}(x)$. Then, averaging over the realizations of the random field is to be performed. The thus obtained chain of equations is infinite and should be cut according to the sense of the problem. Then we will have the self-contained system.

We can show that both methods lead to the same results when comparisons are feasible.

In this article we will concentrate on the first of mentioned approaches.

2. Perturbation Method and Diagram Technique

Upon averaging a partial differential equation with stochastic coefficients, we obtain derivatives of the average of the unknown. Besides, we obtain averages of products, containing the basic unknown and the stochastic coefficient. In Fourier space, the latter yields convolutions, and it is possible to arrive at some fixed-point equation for the Laplace-Fourier transform of the average. Iterating the fixed-point problem leads to an equation with infinite series whose successive items involve multi-point correlations of the stochastic process, in Fourier variables.

Graphics may help handling the many terms. Here we present this method, with the help of an example, obtained in [1].

2.1. A simple Small-scale Problem

Let us use according to [3, 4] the perturbation method and diagrams for equation

$$\frac{\partial u}{\partial t} - D_0 \frac{\partial^2 u}{\partial x^2} = D_0 \frac{\partial \eta}{\partial x} \cdot \frac{\partial u}{\partial x}, \tag{1}$$

that was derived in [1].

In this purpose we use instead of random function $\varepsilon(x)$ another one, which is $\eta(x) = ln\,\varepsilon(x)$. It should be mentioned that [3] considered stationary diffusion problem under random force field and so there was no time derivative term in the equation corresponding to (1) in [3]. It looks evident that because of the similarity of the above equations also the diffusion processes in random media ($\varepsilon(x)$ - random characteristics) have much in common with diffusion process under random force fields ($\eta(x)$ is something like the potential of some gradient force fields).

2.2. Laplace-Fourier Space Analysis

For (1) we will use Laplace transform over time and Fourier transform over space coordinate. We assume that both transforms are applicable.

Let us choose Fourier transformation for an arbitrary function $z(x)$ over x coordinate as follows:

$$z(x) = \int_{-\infty}^{\infty} z(k) e^{ikx} dk$$

$$z(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} z(x) e^{-ikx} dx$$

For the sake of simplicity we choose the initial conditions as $u(0,x) = \delta(x)$, where $\delta(x)$ is the Dirac delta function. In q, k variables (q - Laplace parameter, k - wave number in Fourier transform) (2.1) can be re-written as

$$qu(k_0,q) - \frac{1}{2\pi} + D_0 k_0^2 u(k_0,q) = -D_0 \int_{-\infty}^{+\infty} k\eta(k)(k_0 - k)u(k_0 - k, q)dk,$$

So that

$$u(k_0,q) = P_0(k_0,q) + P_0(k_0,q)\int_{-\infty}^{\infty}\widetilde{\eta}(k)k(k_0-k)u(k_0-k,q)dk\,, \tag{2}$$

where $P_0(k_0,q) = \dfrac{1}{2\pi\left(q + D_0k_0^2\right)}$, $\widetilde{\eta}(k) = -2\pi D_0\eta(k)$.

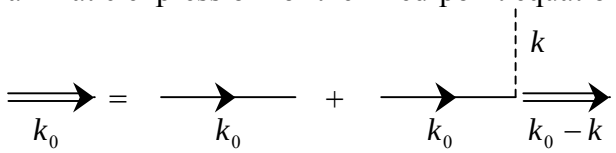
Hence $u(k_0,q)$ satisfies the fixed point equation (2). Formally solving the latter may help finding properties of $u(k_0,q)$ and of the average $<u(x,t)>$. This way, we may obtain an equation for the evolution of $<u(x,t)>$.

A method, allowing to solve (2) at least formally, may be iterating this equation, thus leading to an infinite expansion giving $u(k_0,q)$ in closed form. Describing and storing the items in the expansion can be done via formulae, containing multiple integrals.

The integrals result from an iterated process. For this reason, we can represent them with the help of drawings (or diagrams) giving the rules of the process in a less unpleasant form.

2.3. Diagram Technique

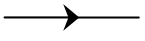
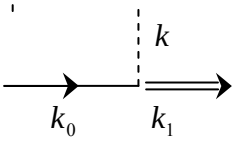
A diagrammatic expression for the fixed-point equation (2) may be:



(3)

The rules, which define the correspondence between (2) and diagram (3), are as in [4, 5].

Rules

1. each dual horizontal line  bears $u(k_0,q)$;
2. each horizontal line  bears $P_0(k_0,q)$;
3. each vertical line with wave number k bears factor $\widetilde{\eta}(k)$;
4. each triple vertex  bears factor $k\cdot k_1$;

5. for each wave number for vertical line the integration is executed.

The rules apply each time we have to consider instead of (2) an equation of the more general form

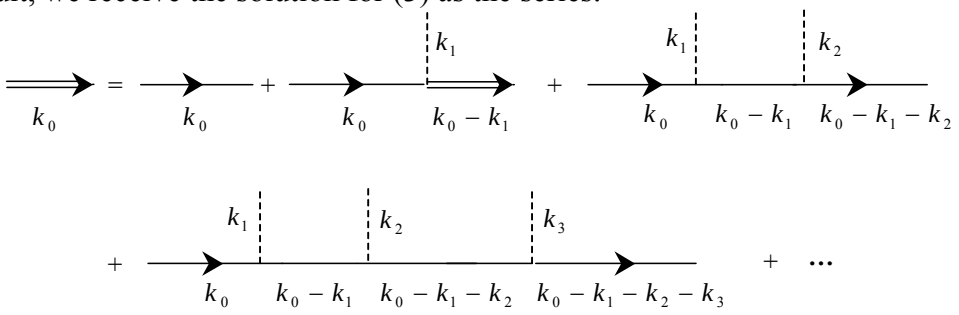
$$u(k_0,q) = P_0(k_0,q) + P_0(k_0,q)\int_{-\infty}^{\infty}\widetilde{\eta}(k)\Theta(k_0,k)u(k_0-k,q)dk \tag{2a}$$

except for rule 4, which then takes the more general form "each triple vertex bears factor $\Theta(k_0,k)$".

The iterative procedure for this method means that the solution of (2) in zero approximation $u_0(k_0,q) = P_0(k_0,q)$ is substituted in the right hand-side of (2) or (3). As the result, we receive the solution in first approximation

$$u_1(k_0,q) = P_0(k_0,q) + P_0(k_0,q)\int_{-\infty}^{\infty}\widetilde{\eta}(k)k(k_0-k)P_0(k_0-k,q)dk\,,$$

which is substituted into the right-hand side of (2) and leads us to the second approximation, etc. As the result, we receive the solution for (3) as the series:



(4)

2.4. Averaging

This solution for concentration $u(k_0, q)$ in (4) contains random function $\tilde{\eta}(k)$. We are interested in averaged concentration, and (4) should be averaged over realizations of random field. For the n-th diagram, which is

;

the average is

$$\left\langle P_0(k_0)\int_{\mathbf{R}^n} k_1(k_0-k_1)P_0(k_0-k_1)\tilde{\eta}(k_1)k_2(k_0-k_1-k_2)P_0(k_0-k_1-k_2)\tilde{\eta}(k_2)\ldots k_n(k_0-k_1-\ldots-k_n)P_0(k_0-k_1-\ldots-k_n)\tilde{\eta}(k_n)dk_1\ldots dk_n\right\rangle =$$

$$P_0(k_0)\int k_1(k_0-k_1)P_0(k_0-k_1)k_2(k_0-k_1-k_2)P_0(k_0-k_1-k_2)\ldots$$

$$k_n(k_0-k_1-\ldots-k_n)P_0(k_0-k_1-..-k_n)\left\langle\tilde{\eta}(k_1)\ldots\tilde{\eta}(k_n)\right\rangle dk_1\ldots dk_n$$

where the Laplace parameter q has been dropped from $P_0(k_0, q)$.

Subsequent calculations significantly depend on the stochastic properties of $\tilde{\eta}(k)$, more especially on the multi-point correlation $\left\langle\tilde{\eta}(k_1)\ldots\tilde{\eta}(k_n)\right\rangle$ (in momentum space). For translation invariant stochastic fields, $\left\langle\tilde{\eta}(k_1)\ldots\tilde{\eta}(k_n)\right\rangle$ is equal to $\delta_{k+\ldots+k_{n1}}$ times a function of $k_1,\ldots,k_{n-1}$.

Here, we also assume that $\tilde{\eta}(x) = -2\pi D_0\eta(x)$ is a Gaussian random field. This implies [6, 7] that only correlations involving an even number of wave-vectors are non zero, and that for n=2p they split according to

$$\left\langle\tilde{\eta}(k_1)\ldots\tilde{\eta}(k_n)\right\rangle = \sum\left\langle\tilde{\eta}(k_{\alpha_1})\tilde{\eta}(k_{\alpha_2})\right\rangle\ldots\left\langle\tilde{\eta}(k_{\alpha_{2p-1}})\tilde{\eta}(k_{\alpha_{2p}})\right\rangle,$$

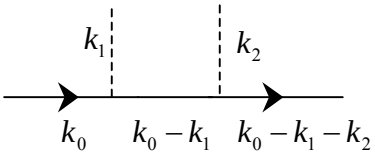
where summation is over all the possible $(2p-1)!!$ pairings of $k_1,\ldots,k_n$. This implies that the above averages in turn split into $(2p-1)!!$ sums of multiple integrals where process $\tilde{\eta}(k)$ only appears via products involving two-wave-vectors correlation functions. Hence only even-vertex averaged diagrams contribute. Each diagram with $2p$ vertices splits into a sum of $(2p-1)!!$ diagrams that we will describe below.

Indeed, in view of uniformity, we have $\left\langle\tilde{\eta}(k_1)\tilde{\eta}(k_2)\right\rangle = \psi_{\tilde{\eta}}(k_1)\delta(k_1+k_2)$, where $\psi_{\tilde{\eta}}(k)$ is the Fourier direct image of two-point correlation function $\psi_{\tilde{\eta}}(x_2-x_1) = \left\langle\tilde{\eta}(x_2)\tilde{\eta}(x_1)\right\rangle$ over (x_2-x_1). Averaged diagram with 2p vertices split into $(2p-1)!!$ integrals over R^p, corresponding to the $(2p-1)!!$ pairings. For pairing $(\alpha_1\alpha_2)..(\alpha_{2p-1}\alpha_{2p})$, we obtain

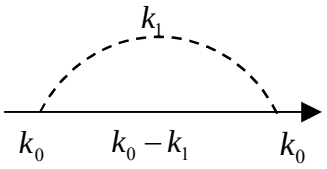
$$\int_{\mathbf{R}^{2p}}\Big[\Pi\psi(k_{\alpha_{2i-1}})k_{\alpha_{2i-1}}(k_0-k_1-\ldots-k_{\alpha_{2i-1}})k_{\alpha_{2i}}(k_0-k_1-\ldots-k_{\alpha_{2i}})\Big]\cdot$$

$$P_0(k_0-k_1-\ldots k_{\alpha_{2i-1}})P_0(k_0-k_1-\ldots k_{\alpha_{2i}})\delta(k_{\alpha_{2i-1i}}+k_{\alpha_{2i}})dk_1\ldots dk_{2p}$$

where Π stands for a product over the pairs. Hence we have an integral over R^p due to the $\delta(k_{\alpha_{2i-1i}}+k_{\alpha_{2i}})$. A graphic representation can consist in connecting the vertices, labeled by wave numbers $k_{\alpha_{2i-1}}k_{\alpha_{2i}}$ of a pair, and we will label the thus obtained arc $k_{\alpha_{2i-1}}$. Thus, diagram with two vertices

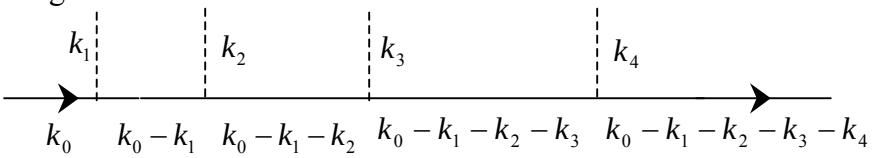


can be replaced after averaging by the diagram

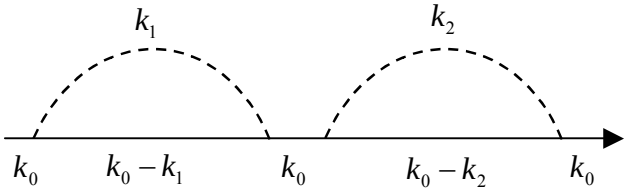


$$=-k_0P_0^2(k_0,q)\int\psi_{\tilde{\eta}}(k_1)k_1^2(k_0-k_1)dk_1\,. \tag{5}$$

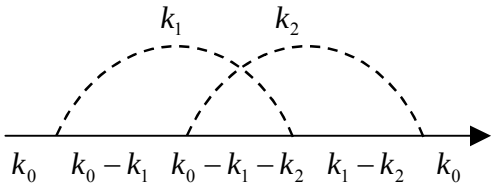
Diagram with four vertices



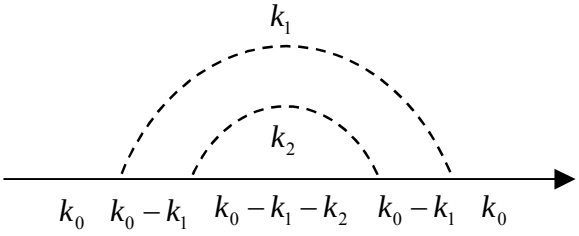
can be replaced after averaging by the sum of diagrams A, B and C defined by



for A,



for B (6)



for C.

Diagrams A, B, C represent

$k_0^2 P_0^3(k_0,q)\left[\int dk_1 k_1^2 (k_0-k_1)\psi_{\tilde{\eta}}(k_1)\right]^2$ for A,

$k_0 P_0^2(k_0,q)\int dk_1 dk_2 \left[k_1^2(k_0-k_1)k_2^2(k_0-k_1-k_2)k_1(k_0-k_2)\times\right.$

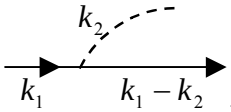
$\left.\times P_0(k_0-k_1,q)P_0(k_0-k_1-k_2,q)P_0(k_0-k_2,q)\psi_{\tilde{\eta}}(k_1)\psi_{\tilde{\eta}}(k_2)\right]$ for B, and

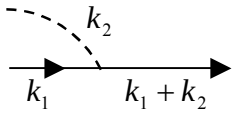
$k_0 P_0^2(k_0,q)\int dk_1 dk_2 \left[k_1^2(k_0-k_1)^2 k_2^2(k_0-k_1-k_2)P_0^2(k_0-k_1)P_0(k_0-k_1-k_2)\right]\psi_{\tilde{\eta}}(k_1)\psi_{\tilde{\eta}}(k_2)$ for C,

which is a variant of (6).

Expressions (5) and (6) were derived by direct calculations. They are equivalent to diagrams with the following rules.

Diagrammatic Rules for Averaged Equations

1. Each arc labeled k_α corresponds to wave number k_α and factor $\psi_{\tilde{\eta}}(k_\alpha)$; the arc bears integration with respect to k_α.
2. Each horizontal line ⟶ labeled k bears the factor $P_0(k,q)$.
3. Each three lines vertex with an outgoing arc 

labeled k_2 and horizontal lines labeled k_1 and (k_1-k_2) carries factor $k_2(k_1-k_2)$.
4. Each three lines vertex with an incoming arc labeled k_2 

and horizontal lines labeled k_1 and (k_1+k_2) carries factor $(-1)k_2(k_1+k_2)$.

Each time we will average an equation of the form (2a), these rules will apply, except points 3 and 4, which in general will take the form:

1. each three lines vertex with an outgoing arc labeled k_2 and horizontal lines labeled k_1 and (k_1-k_2) carries factor $\Theta(k_2,(k_1-k_2))$;
2. each three lines vertex with an incoming arc labeled k_2 and horizontal lines labeled k_1 and (k_1+k_2) carries factor $\Theta(k_2(k_1+k_2))$.

2.5. On the Macroscopic Scale

Extreme incoming and outgoing horizontal straight lines for each diagram carry the same wave number k_0 and respectively the same factor $P_0^2(k_0,q)$. It gives us the possibility not to deal with the incoming lines for the next description.

There also exists a special irreducible type of diagrams, which have no inner horizontal lines not surrounded by any arc, and reducible diagrams, with horizontal lines surrounded by arcs (and then the wave number is k_0). For instance diagram A in (6) – is reducible but B and C are irreducible.

Let us denote by Σ^i the sum of all irreducible diagrams, then the total sum Σ of all diagrams is:

$$\Sigma = \Sigma^i + \Sigma^i P_0 \Sigma^i + \sum_i \cdot P_0 \sum_i \cdot P_0 \sum_i + \cdots = \frac{\Sigma^i}{1-P_0\Sigma^i} = P_0^{-2}\left(P_0^{-1}-\Sigma^i\right)^{-1} - P_0^{-1}.$$

Hence, $\langle u(k_0,q)\rangle$ satisfies:

$$\langle u\rangle = P_0 \Sigma P_0 + P_0 = \left(P_0^{-1} - \Sigma^i\right)^{-1},$$

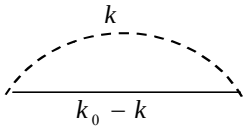
itself equivalent to

$$\left(P_0^{-1} - \Sigma^i\right)\langle u(k_0, q)\rangle = 1 \tag{7}$$

Taking into account the expression of $P_0(k_0, q)$ and turning back to physical variables x, t, we can rewrite (2.7) into the next form for $\langle u(x,t)\rangle$:

$$\frac{\partial\langle u(x,t)\rangle}{\partial t} - D_0 \frac{\partial^2\langle u(x,t)\rangle}{\partial x^2} = \frac{1}{2\pi}\widetilde{\Sigma}^i \langle u(x,t)\rangle, \tag{8}$$

Here operator $\widetilde{\Sigma}^i$ corresponds in physical variables to Σ^i. In general, it is integro-differential. Equations (7) and (8) give only the formal solution to our problem, since an approximation for Σ^i should be calculated. To do this, we will assume that fluctuations of the random field are small, so that it is enough for Σ^i to take into account only the diagram with the first degree of $\psi_{\tilde{\eta}}(k)$, so that we have

$\Sigma^i =$ 

$$-\frac{k_0}{2\pi}\int_{-\infty}^{\infty} \frac{k^2(k_0 - k)}{q + D_0(k_0 - k)^2}\hat{\psi}_{\tilde{\eta}}(k)dk \tag{9}$$

With a_0 being the amplitude of fluctuations, assumed to be small $(a_0 << 1)$, we denote by l the correlation length. Because $\tilde{\eta} = -2\pi D_0 \eta$, we have $\psi_{\tilde{\eta}}(k) = 4\pi^2 D_0^2 \hat{\psi}_{\eta}(k)$ the above equation is equivalent to

$$\Sigma^i = -2\pi D_0^2 a_0^2 k_0 \int_{-\infty}^{\infty} \frac{k^2(k_0 - k)}{\left[q + D_0(k_0 - k)^2\right]}\hat{\psi}_{\eta}(k)dk.$$

Specializing the correlation function $\hat{\psi}_{\tilde{\eta}}(k)$ allows to compute the leading orders in the right-hand side of (2.9), but is not necessary if we focus on the macroscopic scale. We are looking for a large scale equation for the evolution of. This can be achieved if we measure x with a scale, large with respect to the one, we mean to characterize complexity or disorder in the medium. Such a scale may be the correlation length l of Gaussian process η. Non Gaussian processes, or higher dimensions, may need more than one length-scale. Here we will take the limit "l tending to zero" with fixed k and q, and expand Σ^i with respect to parameter l.

We can prove that under not very restrictive assumptions for the correlation function, the leading orders in this expansion are

$$\Sigma^i = 2\pi D_0 a_0^2 k_0^2\left(1 + \nu_2 \tau q - \nu_1 \tau^{1/2} q^{1/2}\right),\ \tau = l^2/D_0 \cdot \tag{10}$$

with ν_1 and ν_2 being positively valued constants, only depending on the shape of function $\hat{\psi}_{\tilde{\eta}}(k)$.

In x, t variables, this corresponds to operator:

$$\widetilde{\Sigma}^i = -2\pi D_0 a_0^2 \frac{\partial^2}{\partial x^2}\left(1 + \nu_2 \tau D_t^1 - \nu_1 \tau^{1/2} D_t^{1/2}\right).$$

Here, according to [1] we use the fractional differential operator over time $D_t^{1/2}$ of order $1/2$ and D_t^1 defined by $D_t^1 v = \frac{\partial v}{\partial t} + v(0^+)\delta(t)$. By substituting $\widetilde{\Sigma}^i$ into (8), we receive finally

$$\frac{\partial\langle u(x,t)\rangle}{\partial t} - D_0\left(1 - a_0^2\right)\frac{\partial^2\langle u(x,t)\rangle}{\partial x^2} = D_0 a_0^2\left(\nu_1 \tau^{1/2} D_t^{1/2} - \nu_2\right)\frac{\partial^2\langle u(x,t)\rangle}{\partial x^2} \tag{11}$$

which is an integro-differential equation for the evolution of the averaged concentration in the presence of the random force. Let us note that in accordance with [3], we have the renormalization of diffusion coefficient, which is smaller than in the small scale equation. This is very common for the similar tasks.

On the right-hand side of Equation (11) we can see terms mixing derivatives with respect to x and derivatives with respect to t, with small coefficients $\nu_1 \tau^{1/2}$ and $\nu_2 \tau$. Parameter ν_2 may be set equal to zero or not, depending on we only keep leading order or not in the expansion of Σ^i with respect to small parameter l. The effect of $D_0 a_0^2 \nu_2 \tau D_t^1 \frac{\partial^2 \langle u(x,t) \rangle}{\partial x^2}$, whose coefficient is even smaller

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