COVERING SEMIGROUPS

VIK.S. KULIKOV AND V.M. KHARLAMOV

Abstract. We introduce and study a semigroup structure on the set of irreducible components of the Hurwitz space of marked coverings of a complex projective curve with given Galois group of the coverings and fixed ramification type. As application, we give new conditions on the ramification type that are sufficient for irreducibility of the Hurwitz spaces, suggest some bounds on the number of irreducibility components under certain more general conditions, and show that the number of irreducible components coincides with the number of topological classes of the coverings if the number of brunch points is big enough.

Introduction

Let $f : E \to F$ be a finite morphism between complex non-singular irreducible projective curves. Denote by $\mathbb{C}(E)$ and $\mathbb{C}(F)$ the fields of rational functions on E and F, respectively. The morphism f defines a finite extension $f^* : \mathbb{C}(F) \hookrightarrow \mathbb{C}(E)$ of the field $\mathbb{C}(F)$ (reciprocally, the field extension defines the covering f uniquely up to isomorphisms of coverings over a fixed base). We denote by G the Galois group of the Galois closure of this extension and call it the *Galois group of* f.

Let us fix a point $q \in F$ that is not a branch point of f and order the points of E lying over q. We call the morphism f with a fixed ordering of the points of $f^{-1}(q)$ a *marked covering*.

Consider the fundamental group $\pi_1(F \setminus B, q)$ of the complement of the branch set $B \subset F$ of a marked covering f of degree $d = \deg f$. Then, the ordering of the points of $f^{-1}(q)$ defines a homomorphism $f_* : \pi_1(F \setminus B, q) \to \mathcal{S}_d$ of $\pi_1(F \setminus B, q)$ to the symmetric group $\mathcal{S}_d$. Due to irreducibility of E, the image $\mathrm{im} f_* \subset \mathcal{S}_d$ acts transitively on $f^{-1}(q)$ and is isomorphic to G, so that we can identify $\mathrm{im} f_*$ and G and thus fix this embedding $G \hookrightarrow \mathcal{S}_d$.

The movement along a standard simple loops γ around branch points $b \in B$ the *local monodromy* $f_*(\gamma) \in G$ of f at b. The homotopy class of this standard loop, and hence the local monodromy, are defined by b uniquely only up to conjugation, in $\pi_1(F \setminus B, q)$ and G, respectively. We denote by $O \subset G$ the union of the conjugacy classes of all the local monodromies of f and call the pair (G, O) the *equipped Galois group* associated with f. The collection $\tau = (\tau_1 C_1, \dots, \tau_m C_m)$, where $C_1, \dots, C_m$ list

The first author was partially supported by grants of NSh-4713.2010.1, RFBR 11-01-00185, and by AG Laboratory HSE, RF government grant, ag. 11.G34.31.0023. The second author was partially funded by the ANR-09-BLAN-0039-01 grant of Agence Nationale de la Recherche, and is a member of FRG Collaborative Research "Mirror Symmetry & Tropical Geometry" (Award No. 0854989).

all the conjugacy classes included in O and τ_i counts the number of branch points of f with the local monodromies belonging to C_i, is called the *monodromy type* of f.

The degree d marked coverings of F with Galois group G and monodromy type τ form a so called Hurwitz space $\mathrm{HUR}_{d,G,\tau}(F)$ (for precise definitions see subsection 2.7).

In the case $F=\mathbb{P}^1$, $G=\mathcal{S}_d$ and O is the set of transpositions, the famous Clebsch – Hurwitz Theorem [3], [9] states that $\mathrm{HUR}_{d,\mathcal{S}_d,\tau}(\mathbb{P}^1)$ consists of a single irreducible component if $\tau=(nO)$ with even $n\geqslant 2(d-1)$ and it is empty otherwise. Generalizations of Clebsch – Hurwitz Theorem were obtained in [1], [18], [16], [7], and [12] – [14]. In particular, Clebsch – Hurwitz Theorem was extended to the following cases: in [1], if all but one local monodromies are transpositions; in [18], if all but two local monodromies are transpositions; in [16], if all local monodromies are either transpositions or cyclic permutations of length three; and in [12], if there are at least $3(d-1)$ transpositions among the local monodromies.

In [13], it is proved that for an equipped group $(\mathcal{S}_d,O)$ such that the first conjugacy class C_1 of O contains an odd permutation leaving fixed at least two elements, the Hurwitz space $\mathrm{HUR}_{d,\mathcal{S}_d,\tau}(\mathbb{P}^1)$ is irreducible if τ_1 is big enough. On the other hand, the example in [18] shows that $\mathrm{HUR}_{8,\mathcal{S}_8,\tau}(\mathbb{P}^1)$ consists of at least two irreducible components if $\tau=(1C_1,1C_2,1C_3)$, where C_1 is the conjugacy class of the permutation $(1,2)(3,4,5)$, C_2 is the conjugacy class of $(1,2,3)(4,5,6,7)$, and C_3 is the conjugacy class of $(1,2,3,4,5,6,7)$. Articles [7] and [14] are devoted to partial generalizations of Clebsch – Hurwitz Theorem to the case of arbitrary group G. In particular, in [14], it was proved that for a fixed equipped finite group (G,O) the number of irreducible components of $\mathrm{HUR}_{d,G,\tau}(\mathbb{P}^1)$ (if it is non-empty) does not depend on τ if all τ_i are big enough.

For higher genus, the irreducibility of $\mathrm{HUR}_{d,\mathcal{S}_d,\tau}(F)$ is proved in [8] under hypothesis that $n\geq 2d$ and all local monodromies are transpositions. After that, this result was improved, first, in [10] where the hypothesis $n\geq 2d$ was replaced by $n\geq 2d-2$, and next, in [17], where the second hypothesis was replaced by assumption that all but one local monodromies are transpositions. Let us notice that the irreducibility of the quotient of $\mathrm{HUR}_{d,\mathcal{S}_d,\tau}(F)$ by the action of the mapping class group of F (considered as a real surface) was proved in [1] under a weaker hypothesis $n>\frac{d}{2}$.

One of the aims of this article is to extend results of [12] – [14] from $F=\mathbb{P}^1$ to the case of F of arbitrary genus. The approach used there for counting the number of irreducible components of $\mathrm{HUR}_{d,G,\tau}(\mathbb{P}^1)$ is based on a systematic work with semigroups over groups; in particular, factorization semigroups $S(G,O)$ with factors belonging to O (cf., subsections 1.1 and 1.2 below) play the crucial role in this study, especially since subsets of elements of type τ of subsemigroup $S(G,O)^G_{\mathbf{1}}\subset S(G,O)$ are in a canonical bijection with the sets of irreducible components of the Hurwitz space $\mathrm{HUR}_{d,G,\tau}(\mathbb{P}^1)$.

In the present paper, to treat the coverings of projective curves (or, the same, real surfaces) of arbitrary genus we generalize the notion of factorization semigroups to that of semigroups of marked coverings. One can consider different levels of the equivalence relations of coverings and so we introduce, respectively, different species of semigroups of marked coverings. The equivalence relation of the level that is most appropriate to construction of Hurwitz spaces is based essentially on moving of branch points, while that the level most appropriate to topological classification of coverings (like in [1], for example) includes, in addition, the action on the base of coverings by the whole mapping class group. In particular, considering the coverings up to moving of branch points we introduce a semigroup $G\mathbb{S}_d(G,O)$ of marked degree d coverings with Galois group G and set of local monodromies $O\subset G$. If we consider the same coverings up to the action of the modular group, then we obtain another semigroup, which we denote by $GW\mathbb{S}_d(G,O)$. They are related by a natural epimorphism $\Phi: G\mathbb{S}_d(G,O)\to GW\mathbb{S}_d(G,O)$ of semigroups. Similar to genus 0 case, certain specific subsemigroups of these two semigroups are in a canonical bijection with the set of irreducible components of the Hurwitz space $\mathrm{HUR}_{d,G,}(F)$ and, respectively, the set of topological classes of marked degree d ramified coverings of F with Galois groups G.

By definition, the *monodromy type* of an element $s=(f:E\to F)$ belonging to one of these semigroups is the collection $\tau(s)=(\tau_1C_1,\dots,\tau_mC_m)$ of local monodromies of f. The monodromy type behaves additively and gives a homomorphism from semigroups of coverings to the semigroup $\mathbb{Z}^m_{\geqslant 0}$. Therefore, for any constant $T\in\mathbb{N}$, there appear well defined subsemigroups

$$G\mathbb{S}_{d,T}(G,O)=\{s\in G\mathbb{S}_d(G,O)\mid \tau_i(s)\geqslant T \text{ for } i=1,\dots,m\}$$

and

$$GW\mathbb{S}_{d,T}(G,O)=\{s\in GW\mathbb{S}_d(G,O)\mid \tau_i(s)\geqslant T \text{ for } i=1,\dots,m\}.$$

The main results are as follows.

Theorem 1. *For any equipped finite group (G,O) such that the elements of O generate the group G, there is a constant $T\in\mathbb{N}$ such that the restriction of Φ to $G\mathbb{S}_{d,T}(G,O)$ is an isomorphism of $G\mathbb{S}_{d,T}(G,O)$ and $GW\mathbb{S}_{d,T}(G,O)$.*

In [14], there was defined an *ambiguity index* for each equipped finite group (G,O) (see subsection 1.3).

Theorem 2. *For each equipped finite group (G,O), $O=C_1\sqcup\dots\sqcup C_m$, such that the elements of O generate the group G, there is a constant T such that for any projective irreducible non-singular curve F the number of irreducible components of each non-empty Hurwitz space $HUR_{d,G,\tau}(F)$ is equal to $a_{(G,O)}$ if $\tau_i\geqslant T$ for all $i=1,\dots,m$.*

If the elements of $O_k=C_1\sqcup\dots\sqcup C_k$ for some $k<m$ generate the group G, then there is a constant T' such that the number of irreducible components of $HUR^m_{d,G,\tau}(F)$ is less or equal to $a_{(G,O_k)}$ if $\tau_i\geqslant T'$ for all $i=1,\dots,k$.

Theorem 3. *Let C be the conjugacy class of an odd permutation $\sigma\in\mathcal{S}_d$ such that σ leaves fixed at least two elements. Then there is a constant N_C such that for any projective irreducible non-singular curve F the Hurwitz space $\mathrm{HUR}_{d,\mathcal{S}_d,\tau}(F)$ is irreducible if C enters in τ with a factor $\geq N_C$.*

The article consists of two sections. Section 1 is devoted to the algebraic part of the proof. In subsections 1.1 – 1.3 we fix notation and recall necessary definitions and results from [12] – [14]. In subsection 1.4 we introduce a notion of admissible subgroups of the automorphism groups of free groups, which is necessary for the next subsection where we define the algebraic coverings semigroups. The remaining subsections of Section 1 contain the proofs of the algebraic part of main results. Section 2 starts from two preliminary subsections where we introduce such auxiliary notions like monodromy encoding of ramified coverings and skeletons of surfaces. In 2.3 – 2.6 we introduce a series of geometric covering semigroups and prove comparison statements between algebraic and geometric covering semigroups. In the final subsections, we relate elements of the geometric coverings semigroups with irreducible components of Hurwitz spaces and complete the proofs of main theorems.

1. Semigroups over groups

1.1. **Definition of semigroups over groups.** Here, we recall basic definitions and some properties of semigroups over groups with a special emphasis to factorization semigroups (for more details, see [11] – [14]).

A collection (S, G, α, ρ), where S is a semigroup, G is a group, and $\alpha : S \to G$, $\rho : G \to \mathrm{Aut}(S)$ are homomorphisms, is called *a semigroup S over a group G* if for all $s_1, s_2 \in S$ we have

$$s_1 \cdot s_2 = \rho(\alpha(s_1))(s_2) \cdot s_1 = s_2 \cdot \lambda(\alpha(s_2))(s_1), \tag{1}$$

where $\lambda(g) = \rho(g^{-1})$.

Let $(S_1, G_1, \alpha_1, \rho_1)$ and $(S_2, G_2, \alpha_2, \rho_2)$ be two semigroups over groups G_1 and G_2, respectively. A pair (h_1, h_2) of homomorphisms $h_1 : S_1 \to S_2$ and $h_2 : G_1 \to G_2$ is called *a homomorphism of semigroups over groups* if

(*i*) $h_2 \circ \alpha_1 = \alpha_2 \circ h_1$,
(*ii*) $\rho_2(h_2(g))(h_1(s)) = h_1(\rho_1(g)(s))$ for all $s \in S_1$ and all $g \in G_1$.

In particular, if $G_1 = G_2 = G$, then a homomorphism of semigroups $\varphi : S_1 \to S_2$ is said *to be defined over G* if $\alpha_1(s) = \alpha_2(\varphi(s))$ and $\rho_2(g)(\varphi(s)) = \varphi(\rho_1(g)(s))$ for all $s \in S_1$ and $g \in G$.

1.2. **Factorization semigroups.** One of the main examples of semigroups over groups is given by, so called, factorization semigroups. To define them, consider an *equipped group* (G, O), that is, G is a group and O is a subset of G invariant under the inner automorphisms. Here and further on, we assume that:

(i) $\mathbf{1} \notin O$;
(ii) O consists of a finite number of conjugacy classes C_i of G, $O = C_1 \sqcup \dots \sqcup C_m$;
(iii) the (linear) ordering of these conjugacy classes is fixed.

By *homomorphisms of equipped groups* (G_1, O_1) and (G_2, O_2) we understand homomorphisms $f : G_1 \to G_2$ such that $f(O_1) \subset O_2$.

The factorization semigroup with factors in O is, by definition, the semigroup $S(G, O)$ generated by alphabet $\mathcal{X}_O = \{x_g \mid g \in O\}$ and subject to relations

$$x_{g_1} \cdot x_{g_2} = x_{g_2} \cdot x_{g_1^{g_2}}, \qquad g_1, g_2 \in O, \tag{2}$$

where $g_1^{g_2}$ denotes $g_2^{-1} g_1 g_2$. The homomorphism $\alpha : S(G, O) \to G$, given by $\alpha(x_g) = g$ for each $x_g \in \mathcal{X}_O$, is called the *product homomorphism. The simultaneous conjugation*

$$x_a \in \mathcal{X}_O \mapsto x_{gag^{-1}} \in \mathcal{X}_O$$

defines a homomorphism $G \to \mathrm{Aut}(S(G, O))$, which we denote by ρ. It is easy to see that under such a choice, $(S(G, O), G, \alpha, \rho)$ is a semigroup over G.

Note that there is a well defined *length homomorphism* of semigroups,

$$l : S(G, O) \to \mathbb{Z}_{\geqslant 0} = \{a \in \mathbb{Z} \mid a \geqslant 0\}$$

that is defined by $l(x_{g_1} \cdot \ldots \cdot x_{g_n}) = n$.

Put $\rho_S = \rho \circ \alpha$, $\lambda_S = \lambda \circ \alpha$, where as above $\lambda(g) = \rho(g^{-1})$.

Claim 1. ([15]) *For all* s_1, $s_2 \in S(G, O)$ *we have*

$$s_1 \cdot s_2 = s_2 \cdot \lambda_S(s_2)(s_1) = \rho_S(s_1)(s_2) \cdot s_1.$$

To each $s = x_{g_1} \cdot \ldots \cdot x_{g_n} \in S(G, O)$, we associate a subgroup G_s of G generated by the images $\alpha(x_{g_1}) = g_1, \ldots, \alpha(x_{g_n}) = g_n$ of the factors $x_{g_1}, \ldots, x_{g_n}$ and denote by G_O the subgroup of G generated by the elements of O.

Claim 2. ([12]) *The subgroup* G_s *of* G *is well defined, that is, it does not depend on a presentation of* s *as a product of generators* $x_{g_i} \in \mathcal{X}_O$.

For subgroups H and K of a group G, we put

$$S(G, O)^H = \{s \in S(G, O) \mid G_s = H\},$$
$$S(G, O)_K = \{s \in S(G, O) \mid \alpha(s) \in K\},$$

and $S(G, O)^H_K = S(G, O)_K \cap S(G, O)^H$. It is easy to see that $S(G, O)^H$ (respectively $S(G, O)^H_K$) is isomorphic to the semigroup $S(H, H \cap O)^H$ (respectively, isomorphic to $S(H, H \cap O)^H_{K \cap H}$) and the isomorphism is induced by the embedding $(H, H \cap O) \hookrightarrow (G, O)$.

Proposition 1. ([12]) *Let* (G, O) *be an equipped group and let* $s \in S(G, O)$. *We have*

1. *$\ker \rho$ coincides with the centralizer C_O of the group G_O in G;*
2. *if $\alpha(s)$ belongs to the center $Z(G_s)$ of G_s, then for each $g \in G_s$ the action $\rho(g)$ leaves fixed the element $s \in S(G, O)$;*
3. *if $\alpha(s \cdot x_g)$ belongs to the center $Z(G_{s \cdot x_g})$ of $G_{s \cdot x_g}$, then $s \cdot x_g = x_g \cdot s$,*
4. *if $\alpha(s) = \mathbf{1}$, then $s \cdot s' = s' \cdot s$ for any $s' \in S(G, O)$.*

Claim 3. ([12]) *For any equipped group* (G, O) *the semigroup* $S(G, O)_{\mathbf{1}}$ *is contained in the center of the semigroup* $S(G, O)$ *and, in particular, it is commutative.*

Note that if $g \in O$ is an element of order n, then $x_g^n \in S(G,O)_{\mathbf{1}}$.

Lemma 1. ([12]) *Let $s \in S(G,O)_{Z(G_O)}$ and $s_1 \in S(G,O)^{G_O}$, where $Z(G_O)$ is the center of G_O. Then*

$$s \cdot s_1 = \rho(g)(s) \cdot s_1 \tag{3}$$

for all $g \in G_O$.

In particular, if $s \in S(G,O)^G$, $C \subset O$ is a conjugacy class of G, and g_1^n belongs to the center $Z(G)$ of G for certain $g_1 \in C$, then for any $g_2 \in C$ we have

$$x_{g_1}^n \cdot s = x_{g_2}^n \cdot s. \tag{4}$$

Proposition 2. ([12]) *The elements of $S(G,O)_{\mathbf{1}}^G$ are fixed under the conjugation action of G.*

1.3. **Factorization semigroups over equivalent equipped groups.** Here, we introduce an additional assumption with regard to O in an equipped group (G,O): we assume that

(iv) the elements of O generate the group G.

In [14], a C-graph was associated with each equipped group. By definition, the *C-graph* $\Gamma_{(G,O)}$ *of an equipped group* (G,O) is a directed labeled graph. Its vertices are labeled by elements of O, and this labeling is a bijection between O and the set of vertices, $V = \{v_g \mid g \in O\}$. Each edge of $\Gamma_{(G,O)}$ also is labeled by an element of O. Namely, two vertices v_{g_1} and v_{g_2}, $g_1, g_2 \in O$, are connected by an edge $e_{v_{g_1},v_{g_2},g}$ with label $g \in O$ if and only if $g^{-1}g_1g = g_2$. (A C-graph may contain loops, and several edges, but with distinct labels, may connect the same pair of vertices in the same direction.)

Obviously, the conjugacy classes $C_i \subset O$, $1 \leq i \leq m$, are in one-to-one correspondence with the connected components Γ_i of the C-graph $\Gamma_{(G,O)}$; more precisely, $v_g \in \Gamma_i$ if and only if $g \in C_i$.

Two equipped groups (G_1,O_1) and (G_2,O_2) are called *equivalent* if their C-graphs, $\Gamma_{(G_1,O_1)}$ and $\Gamma_{(G_2,O_2)}$, are isomorphic as C-graphs; in other words, if there is a bijection $O_1 \to O_2$ that induces an isomorphism of C-graphs between $\Gamma_{(G_1,O_1)}$ and $\Gamma_{(G_2,O_2)}$.

To each C-graph $\Gamma = \Gamma_{(G,O)}$ one associates a C-group $G_\Gamma = (\widetilde{G},\widetilde{O})$ equivalent to (G,O). Denoting by $g \mapsto \tilde{g}$ the bijection $O \to \tilde{O}$, we can describe $\tilde{G}$ as the group defined by generators $\tilde{g} \in \tilde{O}$ and the relations

$$\{\widetilde{g}_3^{-1}\widetilde{g}_1\widetilde{g}_3 = \widetilde{g}_2 \text{ if and only if there is an edge } e_{v_{g_1},v_{g_2},g_3} \in \Gamma\}$$

(equivalently, $\widetilde{g}_3^{-1}\widetilde{g}_1\widetilde{g}_3 = \widetilde{g}_2$ if and only if the relation $g_3^{-1}g_1g_3 = g_2$ holds in G). These generators are called *C-generators*. They are in one-to-one correspondence with the vertices of Γ due to the composed bijection $\tilde{g} \mapsto g \mapsto v_g$. Furhermore, two C-generators $\widetilde{g}_1$ and $\widetilde{g}_2$ belong to the same connected component of $\Gamma_{(G,O)}$ if and only if they are conjugate. The set $\widetilde{O}$ of C-generators of the C-group $\widetilde{G}$ satisfies the assumptions (i)-(iv).

The one-to-one map $\beta_{(G,O)} : \widetilde{O} \to O$ given by $\beta_{(G,O)}(\widetilde{g}) = g$ defines an epimorphism $\beta = \beta_{(G,O)} : (\widetilde{G}, \widetilde{O}) \to (G, O)$ of equipped groups and an isomorphism $\beta_* : S(\widetilde{G}, \widetilde{O}) \to S(G, O)$ of semigroups. By Claim 8 in [14], $\ker \beta$ is a subgroup of the center $Z(\widetilde{G})$ of the C-group $\widetilde{G}$.

By adding the commutativity relations one shows that the abelianization $H_1(\widetilde{G}, \mathbb{Z}) = \widetilde{G}/[\widetilde{G}, \widetilde{G}]$ of $\widetilde{G}$ is isomorphic to $\mathbb{Z}^m$. Moreover, due to a fixed ordering of the conjugacy classes $\{\widetilde{C}_1, \dots, \widetilde{C}_m\}$ it comes with a natural basis, where in terms of the abelianization homomorphism $\mathrm{ab} : \widetilde{G} \to H_1(\widetilde{G}, \mathbb{Z})$ the i-th element of the basis is given by $\mathrm{ab}(\widetilde{g})$ with $g \in C_i$, $1 \le i \le m$.

The homomorphism $\tau = \mathrm{ab} \circ \beta_*^{-1} : S(G, O) \to \mathbb{Z}^m_{\geqslant 0}$ is called the *type* homomorphism, the image $\tau(s) = (\tau_1(s), \dots, \tau_m(s)) \in \mathbb{Z}^m_{\geqslant 0}$ is called the *type* of $s \in S(G, O)$, and the ith coordinate $\tau_i(s)$ of $\tau(s)$ is called the *ith type* of s.

A C-group $\widetilde{G}$ is called *C-finite* if the number of vertices of the graph $\Gamma_{(\widetilde{G},\widetilde{O})}$ is finite. By Proposition 3 in [14], the commutant $[\widetilde{G}, \widetilde{G}]$ of a C-finite group $\widetilde{G}$ is a finite group. The order $a_{(G,O)} = |\ker \beta \cap [\widetilde{G}, \widetilde{G}]|$ of the group $\ker \beta \cap [\widetilde{G}, \widetilde{G}]$ is called the *ambiguity index* of an equipped finite group (G, O). If $O' \subset O$ are two equipments of a finite group G such that the elements of O' generate the group G, then by Corollary 2 in [14], we have $a_{(G,O)} \leqslant a_{(G,O')}$.

In [14] the following theorems are proved.

Theorem 4. *Let (G, O), $O = C_1 \sqcup \dots \sqcup C_m$, be an equipped finite group and $(\widetilde{G}, \widetilde{O}) = G_\Gamma$ with $\Gamma = \Gamma_{(G,O)}$ the C-group equivalent to (G, O). Then there is a constant $T \in \mathbb{N}$ such that for any element $s_1 \in S(G, O)^G$ with $\tau_i(s_1) \geqslant T$ for all $i = 1, \dots, m$ there exist $a_{(G,O)}$ elements $s_1, \dots, s_{a_{(G,O)}} \in S(G, O)^G$ such that*

(1) *$s_i \neq s_j$ for $1 \leqslant i < j \leqslant a_{(G,O)}$;*
(2) *$\tau(s_i) = \tau(s_1)$ for $1 \leqslant i \leqslant a_{(G,O)}$;*
(3) *$\alpha_G(s_i) = \alpha_G(s_1)$ for $1 \leqslant i \leqslant a_{(G,O)}$;*
(4) *if $s \in S(G, O)^G$, $\tau(s) = \tau(s_1)$ and $\alpha_G(s) = \alpha_G(s_1)$, then $s = s_i$ for some i, $1 \leqslant i \leqslant a_{(G,O)}$.*
(5) *if $s \in S(G, O)^G$ and $\alpha_{\widetilde{G}}(s) = \alpha_{\widetilde{G}}(s_1)$, then $s = s_1$.*

Theorem 5. *Let G be a finite group and $O' \subset O$ be two its equipments such that the elements of $O' = C_1 \sqcup \dots \sqcup C_k$ generate the group G. Then there is a constant $T = T_{O'}$ such that if for an element $s_1 \in S(G, O)^G$ the ith type $\tau_i(s_1) \geqslant T$ for all $i = 1, \dots, k$, then there are not more than $a_{(G,O')}$ elements $s_1, \dots, s_n \in S(G, O)^G$ such that*

(i) *$s_i \neq s_j$ for $1 \leqslant i < j \leqslant n$;*
(ii) *$\tau(s_i) = \tau(s_1)$ for $1 \leqslant i \leqslant n$;*
(iii) *$\alpha_G(s_i) = \alpha_G(s_1)$ for $1 \leqslant i \leqslant n$,*

where $a_{(G,O')}$ is the ambiguity index of (G, O').

Theorem 5 is exactly Theorem 7 from [14]. Theorem 4, items (1)–(4), is Theorem 6 from [14], while the item (5) is a direct consequence of ([14], Theorems 5 and 6) and the following straightforward remark.

Remark 1. The elements $s_1, \dots, s_{a_{(G,O)}} \in S(G,O)^G$, whose existence is claimed by Theorem 4, are distinguished by their valuers $\alpha_{\widetilde{G}}(s_i) \in \widetilde{G}$. Namely, for $i \neq j$ the element $\alpha_{\widetilde{G}}(s_i)\alpha_{\widetilde{G}}(s_j)^{-1}$ is a non-trivial element of $\ker \beta_{(G,O)} \cap [\widetilde{G}, \widetilde{G}]$.

1.4. **Admissible subgroups of $Aut(\mathbb{F}^{n+2p})$.** In this subsection, in order to introduce a notion of algebraic covering semigroups, we pick out some class of subgroups of the automorphism groups of free groups, called *admissible automorphism groups.*

Let $\mathbb{F}^{n+2p}$ be a free group freely generated by $n+2p$ elements. Let $\mathcal{G} = \{\gamma_1, \dots, \gamma_n\} \subset \mathbb{F}^{n+2p}$, $\mathcal{L} = \{\lambda_1, \dots, \lambda_p\} \subset \mathbb{F}^{n+2p}$, and $\mathcal{M} = \{\mu_1, \dots, \mu_p\} \subset \mathbb{F}^{n+2p}$ be three ordered sets such that the elements of $\mathcal{B} = \mathcal{G} \cup \mathcal{L} \cup \mathcal{M}$ generate the group $\mathbb{F}^{n+2p}$.

Let $\overline{n} = (n_1, \dots, n_{p+1})$ be an ordered non-negative partition of the number n, that is, an ordered $(p+1)$-tuple of non-negative integers whose sum is equal to n:

$$n = n_1 + \dots + n_{p+1}, \qquad n_i \in \mathbb{Z}_{\geqslant 0}.$$

We put $k_i = \sum_{j=1}^{i} n_j$. Each partition $\overline{n}$ defines its own ordering on $\mathcal{B}$:

$$\gamma_1, \dots, \gamma_{k_1}, \lambda_1, \mu_1, \dots, \gamma_{k_{i-1}+1}, \dots, \gamma_{k_i}, \lambda_i, \mu_i, \gamma_{k_i+1}, \dots, \gamma_{k_{i+1}}, \dots, \lambda_p, \mu_p, \gamma_{k_p+1} \dots, \gamma_n$$

(here the set $\{\gamma_{k_{i-1}+1}, \dots, \gamma_{k_i}\}$ is empty if $n_i = 0$). Denote by $\mathcal{B}_{\overline{n}}$ the set $\mathcal{B}$ with the ordering defined by partition $\overline{n}$ and call it a *frame* of $\mathbb{F}^{n+2p}$. In particular, $\mathcal{B}_{(n,0,\dots,0)} = \{\gamma_1, \dots, \gamma_n, \lambda_1, \mu_1, \dots, \lambda_p, \mu_p\}$. The element

$$\partial\mathcal{B}_{\overline{n}} = \gamma_1 \dots \gamma_{k_1}[\lambda_1, \mu_1] \dots \gamma_{k_{i-1}+1} \dots \gamma_{k_i}[\lambda_i, \mu_i]\gamma_{k_i+1} \dots \gamma_{k_{i+1}} \dots [\lambda_p, \mu_p]\gamma_{k_p+1} \dots \gamma_n$$

of $\mathbb{F}^{n+2p}$ is called *the boundary of* $\mathcal{B}_{\overline{n}}$.

Given a set $\mathcal{B}' = \mathcal{G}' \cup \mathcal{L}' \cup \mathcal{M}'$ as above and two adjacent partitions, $\overline{n}' = (\dots, n_{i-1}, n_i, n_{i+1}, n_{i+2}, \dots)$ and $\overline{n}'' = (\dots, n_{i-1}, n_i - 1, n_{i+1} + 1, n_{i+2}, \dots)$, we define *an elementary frame change* $h_{\overline{n}',\overline{n}''}$ that results both in change of the generating set and the ordering. Namely, we put

$$h_{\overline{n}',\overline{n}''}(\mathcal{B}'_{\overline{n}'}) = \mathcal{B}''_{\overline{n}''},$$

where $\mathcal{B}'' = \mathcal{G}'' \cup \mathcal{L}'' \cup \mathcal{M}''$ with $\lambda''_j = \lambda'_j$ and $\mu''_j = \mu'_j$ for all $j = 1, \dots, p$, while $\gamma''_j = \gamma'_j$ for $j \neq k_i = n_1 + \dots + n_i$ and $\gamma''_{k_i} = ([\lambda'_i, \mu'_i])^{-1}\gamma'_{k_i}[\lambda'_i, \mu'_i]$. The inverse change $h^{-1}_{\overline{n}',\overline{n}''} = h_{\overline{n}'',\overline{n}'}$ also will be called an elementary frame change. Two frames of $\mathbb{F}^{n+2p}$ are said to be *strongly equivalent* if one of them can be obtained from the other one by a finite sequence of elementary frame changes. Any composition of elementary changes transforming a frame $\mathcal{B}'_{\overline{n}'}$ into a frame $\mathcal{B}''_{\overline{n}''}$ will also be denoted by $h_{\overline{n}',\overline{n}''}$.

The proof of the following properties is straightforward.

Claim 4. *Let $\mathcal{B}'_{\overline{n}}$ and $\mathcal{B}''_{\overline{n}}$ be two frames strongly equivalent to a frame $\mathcal{B}_{(n,0,\dots,0)}$. Then $\mathcal{B}'_{\overline{n}} = \mathcal{B}''_{\overline{n}}$.*

Claim 5. *Let $\mathcal{B}'_{\overline{n}'}$ and $\mathcal{B}''_{\overline{n}''}$ be two strongly equivalent frames. Then $\partial\mathcal{B}'_{\overline{n}'} = \partial\mathcal{B}''_{\overline{n}''}$.* $\square$

The group $Aut(\mathbb{F}^{n+2p})$ naturally acts on the set of frames. This action respects the partition. Given $h \in Aut(\mathbb{F}^{n+2p})$ and a frame $\mathcal{B}_{\overline{n}}$, we put $\mathcal{B}' = h(\mathcal{B})$ and $h(\mathcal{B}_{\overline{n}}) = \mathcal{B}'_{\overline{n}}$. As usually, the orbit of $\mathcal{B}_{\overline{n}}$ under the action of a subgroup H of $Aut(\mathbb{F}^{n+2p})$ is denoted by $H\mathcal{B}_{\overline{n}}$. The following Lemma is obvious.

Lemma 2. *Let H be a subgroup of $Aut(\mathbb{F}^{n+2p})$ and $\mathcal{B}'_{\overline{n}'}$, $\mathcal{B}''_{\overline{n}''}$ two strongly equivalent frames. Then*

(i) for any $h \in H$, the frames $h(\mathcal{B}'_{\overline{n}'})$ and $h(\mathcal{B}''_{\overline{n}''})$ are strongly equivalent and $h(\mathcal{B}'_{\overline{n}'}) = h^{-1}_{\overline{n}',\overline{n}''}(h(B''_{\overline{n}''}))$,
(ii) the map $h_{\overline{n}',\overline{n}''} : H\mathcal{B}'_{\overline{n}'} \to H\mathcal{B}''_{\overline{n}''}$ is one-to-one.

Let us fix a frame $\mathcal{B}_{\mathbf{1}} = \mathcal{B}_{(n,0,\dots,0)} = \{\gamma_1, \dots, \gamma_n, \lambda_1, \mu_1, \dots, \lambda_p, \mu_p\}$ and, for each i with $2 \le i \le p+1$, put $\mathcal{B}_{\mathbf{i}} = h_{\overline{n},\overline{n}'}\mathcal{B}_{\mathbf{1}}$ where $\overline{n} = (n, 0, \dots, 0)$ and $\overline{n}' = (n-1, 0, \dots, 0, 1, 0, \dots, 0)$ with 1 on the i-th place.

We specify several auxiliary automorphisms of $\mathbb{F}^{n+2p}$.

The automorphism σ_i with $i = 1, \dots, n-1$ is defined by its action in the frame $\mathcal{B}_{\mathbf{1}}$ as follows:

$$\begin{array}{lll}
\sigma_i(\lambda_j) & = \lambda_j & \text{for } j = 1, \dots, p, \\
\sigma_i(\mu_j) & = \mu_j & \text{for } j = 1, \dots, p, \\
\sigma_i(\gamma_j) & = \gamma_j & \text{for } j \neq i, i+1, \\
\sigma_i(\gamma_i) & = \gamma_{i+1}, & \\
\sigma_i(\gamma_{i+1}) & = \gamma_i^{\gamma_{i+1}}. &
\end{array}$$

The automorphism $\xi_{i,\lambda}$ with $i = 1, \dots, p$ is defined by its action in the frame $\mathcal{B}_{\mathbf{i}}$ as follows:

$$\begin{array}{lll}
\xi_{i,\lambda}(\lambda_{j,\mathbf{i}}) & = \lambda_{j,\mathbf{i}} & \text{for } j \neq i, \\
\xi_{i,\lambda}(\mu_{j,\mathbf{i}}) & = \mu_{j,\mathbf{i}} & \text{for } j = 1, \dots, p, \\
\xi_{i,\lambda}(\gamma_{j,\mathbf{i}}) & = \gamma_{j,\mathbf{i}} & \text{for } j \neq n, \\
\xi_{i,\lambda}(\gamma_{n,\mathbf{i}}) & = \gamma_{n,\mathbf{i}}^{c_{1,i}}, & \text{where } c_{1,i} = \lambda_{i,\mathbf{i}}\mu_{i,\mathbf{i}}^{-1}\lambda_{i,\mathbf{i}}^{-1}\gamma_{n,\mathbf{i}}^{-1}, \\
\xi_{i,\lambda}(\lambda_{i,\mathbf{i}}) & = \gamma_{n,\mathbf{i}}\lambda_{i,\mathbf{i}}. &
\end{array}$$

The automorphism $\xi_{i,\mu}$ with $i = 1, \dots, p$ is defined by its action in the frame $\mathcal{B}_{\mathbf{i}}$ as follows:

$$\begin{array}{lll}
\xi_{i,\mu}(\mu_{j,\mathbf{i}}) & = \mu_{j,\mathbf{i}} & \text{for } j \neq i, \\
\xi_{i,\mu}(\lambda_{j,\mathbf{i}}) & = \lambda_{j,\mathbf{i}} & \text{for } j = 1, \dots, p, \\
\xi_{i,\mu}(\gamma_{j,\mathbf{i}}) & = \gamma_{j,\mathbf{i}} & \text{for } j \neq n, \\
\xi_{i,\mu}(\gamma_{n,\mathbf{i}}) & = \gamma_{n,\mathbf{i}}^{c_{2,i}}, & \text{where } c_{2,i} = \mu_{i,\mathbf{i}}\lambda_{i,\mathbf{i}}^{-1}\mu_{i,\mathbf{i}}^{-1}\gamma_{n,\mathbf{i}}, \\
\xi_{i,\mu}(\mu_{i,\mathbf{i}}) & = \gamma_{n,\mathbf{i}}^{-1}\mu_{i,\mathbf{i}}. &
\end{array}$$

The automorphism ζ_i with $i=1,\dots,p$ is defined by its action in the frame $\mathcal{B}_{\mathbf{i}}$ as follows:

$$\begin{array}{lll}
\zeta_i(\lambda_{j,\mathbf{i}}) & =\lambda_{j,\mathbf{i}} & \text{for } j\neq i,\\
\zeta_i(\mu_{j,\mathbf{i}}) & =\mu_{j,\mathbf{i}} & \text{for } j\neq i,\\
\zeta_i(\gamma_{j,\mathbf{i}}) & =\gamma_{j,\mathbf{i}} & \text{for } j\neq n,\\
\zeta_i(\gamma_{n,\mathbf{i}}) & =c_{3,\mathbf{i}}, & \\
\zeta_i(\lambda_{i,\mathbf{i}}) & =\lambda_{i,\mathbf{i}}^{c_{3,\mathbf{i}}}, & \\
\zeta_i(\mu_{i,\mathbf{i}}) & =\mu_{i,\mathbf{i}}^{c_{3,\mathbf{i}}}, &
\end{array}$$

where $c_{3,\mathbf{i}}=\gamma_{n,\mathbf{i}}^{[\lambda_{i,\mathbf{i}},\mu_{i,\mathbf{i}}]}$.

For $0\leqslant p_1\leqslant p$, denote by Br_{n,p_1} the subgroup of the group $Aut(\mathbb{F}^{n+2p})$ generated by the elements $\sigma_1,\dots,\sigma_{n-1},\xi_{1,\lambda},\dots,\xi_{p_1,\lambda},\xi_{1,\mu},\dots,\xi_{p_1,\mu},\zeta_1,\dots,\zeta_{p_1}$. Obviously, for $p_1\leqslant p$ the group Br_{n,p_1} is a subgroup of $Br_{n,p}$ and the groups $Br_{n,p_1}\subset Aut(\mathbb{F}^{n+2p_1})$ and $Br_{n,p_1}\subset Br_{n,p}\subset Aut(\mathbb{F}^{n+2p})$ are naturally isomorphic. The groups $Br_{n,p}$ will be called *algebraic braid groups.*

Claim 6. *The boundary $\partial\mathcal{B}_{\mathbf{1}}=\gamma_1\dots\gamma_n[\lambda_1,\mu_1]\dots[\lambda_p,\mu_p]\in\mathbb{F}^{n+2p}$ is fixed under the action of $Br_{n,p}$.*

Proof. Obviously, $\partial\mathcal{B}_{\mathbf{1}}$ is fixed under the actions of σ_i for $i=1,\dots,n-1$ and the actions of $\xi_{1,\lambda}$, $\xi_{1,\mu}$, ζ_1, as well as $\partial\mathcal{B}_{\mathbf{i}}$ with $i\geqslant 2$ is fixed under the actions of the automorphisms $\xi_{i,\lambda}$, $\xi_{i,\mu}$, ζ_i. Now, the statement follows from Claim 5. □

Let $(\dots,\gamma_i',\gamma_{i+1}',\dots)$ be a part of a frame $\mathcal{B}_{\overline{n}}'$ that we assume to be strongly equivalent to $\mathcal{B}_{\mathbf{1}}$. Denote by $\sigma_{i,\overline{n}}$ an automorphism of $\mathbb{F}^{n+2p}$ such that $\sigma_{i,\overline{n}}(\gamma_i')=\gamma_{i+1}'$, $\sigma_{i,\overline{n}}(\gamma_{i+1}')=(\gamma_i')^{\gamma_{i+1}'}$, and $\sigma_{i,\overline{n}}$ leaves fixed all the other elements of $\mathcal{B}_{\overline{n}}'$. Similarly, if $(\dots,\gamma_j',\lambda_i',\mu_i',\dots..)$ is a part of a frame $\mathcal{B}_{\overline{n}}'$, then denote by $\xi_{i,\overline{n},\lambda}$, $\xi_{i,\overline{n},\mu}$, and $\zeta_{\overline{n},i}$ the automorphisms of $\mathbb{F}^{n+2p}$ that leave fixed all the elements of $\mathcal{B}_{\overline{n}}'$ except γ_j', λ_i', μ_i' and act on γ_j', λ_i', μ_i' by the same formulas as $\xi_{i,\lambda}$, $\xi_{i,\mu}$, and $\zeta_{\overline{n},i}$ act on the elements γ_n, λ_i, μ_i of the frame $\mathcal{B}_{\mathbf{i}}$ (we just replace n by j).

Lemma 3. *Let $\mathcal{B}_{\overline{n}}'$ be strongly equivalent to $\mathcal{B}_{\mathbf{1}}$. Then the automorphisms $\sigma_{i,\overline{n}}$, $\xi_{i,\overline{n},\lambda}$, $\xi_{i,\overline{n},\mu}$, and $\zeta_{\overline{n},i}$ of $\mathbb{F}^{n+2p}$ belong to $Br_{n,p}$.*

Proof. Follows from Claim 5 and Lemma 2 by straightforward induction on p. □

We say that a subgroup $H_{n,p}$ of $Aut(\mathbb{F}^{n+2p})$ is *admissible* if:

1) $Br_{n,p}\subseteq H_{n,p}$;
2) for each $h\in H_{n,p}$ there is a permutation $\sigma_h\in\mathcal{S}_n$ such that $h(\gamma_i)$ is conjugate to $\gamma_{\sigma_h(i)}$;
3) for each $h\in H_{n,p}$ it holds the relation

$$h(\gamma_1\dots\gamma_n[\lambda_1,\mu_1]\dots[\lambda_p,\mu_p])=\gamma_1\dots\gamma_n[\lambda_1,\mu_1]\dots[\lambda_p,\mu_p]$$

(here $\gamma_1\dots\gamma_n=\mathbf{1}$ if $n=0$).

Let us fix a frame $\mathcal{B}_{\mathbf{1}} = \{\gamma_1, \dots, \gamma_n, \lambda_1, \mu_1, \dots, \lambda_p, \mu_p\}$ of $\mathbb{F}^{n+2p}$ and let $f : \mathbb{F}^{n+2p} \to G$ be a homomorphism to an equipped group (G, O) such that $f(\gamma_i) \in O$ (we call such an f *an equipped homomorphism* to (G, O)). Put $g_i = f(\gamma_i)$ for $1 \le i \le n$ and $a_j = f(\lambda_j)$, $b_j = f(\mu_j)$ for $1 \le j \le p$.

To each frame $\mathcal{B}_{\overline{n}}$ strongly equivalent to $\mathcal{B}_{\mathbf{1}}$, we associate a word $W_{f,\mathcal{B}_{\overline{n}}}$ in the alphabet $\mathcal{Z} = \mathcal{Z}_{(G,O)} = \mathcal{X}_O \cup \mathcal{Y}_G$, where $\mathcal{X}_O = \{x_g \mid g \in O\}$ is the alphabet we used already in subsection 1.2 and $\mathcal{Y}_G = \{y_{a,b} \mid (a, b) \in G^2\}$. We put

$$W_{f,\mathcal{B}_{\mathbf{1}}} = x_{g_1} \dots x_{g_n} y_{a_1,b_1} \dots y_{a_p,b_p}$$

and then construct the words $W_{f,\mathcal{B}_{\overline{n}}}$ iteratively by *elementary moves*: in notation used in the definition of an elementary frame change $h_{\overline{n}',\overline{n}''}(\mathcal{B}'_{\overline{n}'}) = \mathcal{B}''_{\overline{n}''}$, where $\overline{n}' = (\dots, n_{i-1}, n_i, n_{i+1}, n_{i+2}, \dots)$ and $\overline{n}'' = (\dots, n_{i-1}, n_i - 1, n_{i+1} + 1, n_{i+2}, \dots)$ are two adjacent partitions, the elementary move $W_{f,\mathcal{B}'_{\overline{n}'}} \mapsto W_{f,\mathcal{B}''_{\overline{n}''}}$ consists in the replacement of two adjacent letters $x_{g'_{k_i}} y_{a'_i,b'_i}$ in $W_{f,\mathcal{B}'_{\overline{n}'}}$ by $y_{a'_i,b'_i} x_{([a_i,b_i])^{-1} g'_{k_i} [a'_i,b'_i]}$ (as in the definition of elementary frame changes, $k_i = n_1 + \dots + n_i$).

Denote by $\overline{W}_f(G, O)$ the set of words which can be obtained from $W_{f,\mathcal{B}_{\mathbf{1}}}$ by finite sequences of elementary moves and put $\overline{W}_{n,p}(G, O) = \bigcup_f \overline{W}_f(G, O)$, where the union is taken over all equipped homomorphisms $f : \mathbb{F}^{n+2p} \to (G, O)$. We say that two words are *qf-equivalent* if they belong to the same set $\overline{W}_f(G, O)$.

Every admissible group $H_{n,p}$ acts on $\overline{W}_{n,p}(G, O)$. Namely, we put

$$h(W_{f,\mathcal{B}_{\overline{n}}}) = W_{f,\mathcal{B}'_{\overline{n}}}, B'_{\overline{n}} = h(B_{\overline{n}}).$$

In particular, if $W_{f,\mathcal{B}_{\overline{n}}}$ is obtained from $W_{f,\mathcal{B}_{\mathbf{1}}}$ by a finite sequence of elementary moves, then $h(W_{f,\mathcal{B}_{\overline{n}}})$ is obtained from $h(W_{f,\mathcal{B}_{\mathbf{1}}})$ by the same sequence of elementary moves.

Let h be an element of an admissible group $H_{n,p}$. We have $h(\gamma_i) = \gamma_{\sigma_h(i)}^{w_i}$, $h(\lambda_i) = u_i$, and $h(\mu_i) = v_i$, where w_i, u_i, v_i are some elements of $\mathbb{F}^{n+2p}$. Denote by the same letters the words $w_i = w_i(\gamma_1, \dots, \gamma_n, \lambda_1, \mu_1, \dots, \lambda_p, \mu_p)$, $u_i = u_i(\gamma_1, \dots, \gamma_n, \lambda_1, \mu_1, \dots, \lambda_p, \mu_p)$, and $v_i = v_i(\gamma_1, \dots, \gamma_n, \lambda_1, \mu_1, \dots, \lambda_p, \mu_p)$ in letters $\gamma_1, \dots, \gamma_n, \lambda_1, \mu_1, \dots, \lambda_p, \mu_p$ and their inverses representing these elements in $\mathbb{F}^{n+2p}$. Consider elements $g_1, \dots, g_n$, $a_1, b_1, \dots, a_p, b_p$ of an equipped group (G, O), where $g_1, \dots, g_n \in O$, and let us substitute g_j for γ_j, a_j for λ_j, b_j for μ_j into the words w_i, u_i, v_i and denote the corresponding elements of G by $\overline{w}_i = w_i(g_1, \dots g_n, a_1, b_1, \dots, a_p, b_p)$, $\overline{u}_i = u_i(g_1, \dots g_n, a_1, b_1, \dots, a_p, b_p)$, and $\overline{v}_i = v_i(g_1, \dots g_n, a_1, b_1, \dots, a_p, b_p)$. Denote by $\langle g_1, \dots, g_n, a_1, b_1, \dots, a_p, b_p\rangle$ a subgroup of G generated by the elements $g_1, \dots, g_n, a_1, b_1, \dots, a_p, b_p \in G$.

Claim 7. *In notations and assumptions used above, we have*

$$\langle g_1, \dots g_n, a_1, b_1, \dots, a_p, b_p\rangle = \langle g_{\sigma_h(1)}^{\overline{w}_1}, \dots, g_{\sigma_h(n)}^{\overline{w}_n}, \overline{u}_1, \overline{v}_1, \cdots, \overline{u}_p, \overline{v}_p\rangle.$$

Proof. It suffices to note that the subgroup $\langle g_1, \dots g_n, a_1, b_1, \dots, a_p, b_p\rangle$ of G is the image in G of the group $\mathbb{F}^{n+2p}$ under the homomorphism $f : \mathbb{F}^{n+2p} \to G$ given by $f(\gamma_i) = g_i$, $f(\lambda_j) = a_j$, and $f(\mu_j) = b_j$. $\square$

1.5. **Definition of covering semigroups.** Let (G,O) be an equipped group. Denote by $F\mathbb{S}(G,O)$ the free semigroup over the alphabet $\mathcal{Z}=\mathcal{Z}_{(G,O)}=\mathcal{X}_O\cup\mathcal{Y}_G$ introduced in subsection 1.4 and call $F\mathbb{S}(G,O)$ *free covering semigroup over the equipped group* (G,O).

All the covering semigroups considered below are factor semigroups of $F\mathbb{S}(G,O)$. In particular, this is the case of what we call the *quasi-free algebraic covering semigroup* $qF\mathbb{S}(G,O)$ that we define as a semigroup generated by the alphabet $\mathcal{Z}$ and subject to relations

$$x_g\cdot y_{a,b}=y_{a,b}\cdot x_{g^{[a,b]}},\quad g\in O,\ \ a,b\in G \tag{5}$$

(in other words, the elements of $qF\mathbb{S}(G,O)$ are the sets of qf-equivalent words (see subsection 1.4)).

We follow notation of Subsection 1.4. Let $\mathcal{H}=\{H_{n,p}\}_{\{n\geqslant 0,p\geqslant 0\}}$ be a collection of automorphism groups that satisfy conditions $2),3)$ from the definition of admissible automorphism groups. We associate with each $h\in H_{n,p}$ a set R_h of relations

$$x_{g_1}\cdot\dots\cdot x_{g_n}\cdot y_{a_1,b_1}\cdot\dots\cdot y_{a_p,b_p}=x_{g_{\sigma_h(1)}^{\overline{w}_1}}\cdot\dots\cdot x_{g_{\sigma_h(n)}^{\overline{w}_n}}\cdot y_{\overline{u}_1,\overline{v}_1}\cdot\dots\cdot y_{\overline{u}_p,\overline{v}_p} \tag{6}$$

taken over all $(g_1,\dots,g_n)\in O^n$ and all $(a_1,b_1,\dots,a_p,b_p)\in G^{2p}$. Denote by

$$\mathcal{R}_{\mathcal{H}}=\bigcup_{n,p}(\bigcup_{h\in H_{n,p}}R_h)$$

and consider a factor semigroup $qF\mathbb{S}(G,O)/\mathcal{R}_{\mathcal{H}}$. In particular, the semigroup

$$\mathbb{S}(G,O)=qF\mathbb{S}(G,O)/\{\mathcal{R}_{\mathcal{B}}\}$$

is called *the strong covering semigroup*, where $\mathcal{B}=\{Br_{n,p}\}_{\{n\geqslant 0,p\geqslant 0\}}$. For a collection $\mathcal{H}=\{H_{n,p}\}_{\{n\geqslant 0,p\geqslant 0\}}$ of admissible automorphism groups, we denote the semigroup $qF\mathbb{S}(G,O)/\mathcal{R}_{\mathcal{H}}$ by $\mathbb{S}_{\mathcal{H}-\mathrm{equiv}}(G,O)$ and call it an *admissible covering semigroup.*

Proposition 3. *The strong covering semigroup $\mathbb{S}(G,O)$ is isomorphic to the semigroup generated by the alphabet $\mathcal{Z}=\mathcal{Z}_{(G,O)}=\mathcal{X}_O\cup\mathcal{Y}_G$ and subject to relations*

$$x_{g_1}\cdot x_{g_2}=x_{g_2}\cdot\ x_{g_1^{g_2}} \tag{7}$$

for any $x_{g_1},x_{g_2}\in\mathcal{X}_O$, and

$$x_g\cdot y_{a,b}=y_{a,b}\cdot x_{g^{[a,b]}}, \tag{8}$$

$$x_g\cdot y_{a,b}=x_{g^{c_1}}\cdot y_{ga,b},\qquad c_1=ab^{-1}a^{-1}g^{-1}, \tag{9}$$

$$y_{a,b}\cdot x_g=y_{a,g^{-1}b}\cdot x_{g^{c_2}},\qquad c_2=ba^{-1}b^{-1}g, \tag{10}$$

$$x_g\cdot y_{a,b}=x_{g^{[a,b]}}\cdot y_{a^{g^{[a,b]}},b^{g^{[a,b]}}} \tag{11}$$

for any $x_g\in\mathcal{X}_O$ and any $y_{a,b}\in\mathcal{Y}_G$.

Proof. Follows from Claims 5, 6 and Lemmas 2, 3. □

Since every admissible automorphism group $H_{n,p}$ contains the group $Br_{n,p}$, for any collection $\mathcal{H}$ of admissible automorphism groups there is a natural epimorphism $r_{\mathcal{H}-\text{equiv}} : \mathbb{S}(G,O) \to \mathbb{S}_{\mathcal{H}-\text{equiv}}(G,O)$ of semigroups.

The semigroup $\mathbb{S}(\mathcal{S}_d, \mathcal{S}_d \setminus \{\mathbf{1}\})$ that we denote by $V\mathbb{S}_d$ will be called *a strong algebraic versal degree d covering semigroup.* Note that an embedding $i : G \hookrightarrow \mathcal{S}_d$ of a group G into $\mathcal{S}_d$ induces the semigroup embedding of $\mathbb{S}(G,O)$ into $V\mathbb{S}_d$.

Claim 8. *The map $\alpha : Z \to G$ given by $\alpha(x_g) = g$ for $x_g \in X_O$ and $\alpha(y_{a,b}) = [a,b]$ for $y_{a,b} \in Y_G$ defines a homomorphism $\mathbb{S}_{\mathcal{H}-\text{equiv}}(G,O) \to G$.*

Proof. Straightforward inspection of relations (5) and (6) shows that for each of these relations the product of the images of the left-side factors is equal in G to the product of the right-side factors. □

Further on we denote this homomorphism $\mathbb{S}_{\mathcal{H}-\text{equiv}}(G,O) \to G$ by $\alpha_{G,\mathcal{H}-\text{equiv}}$, or simply α_G, and call it *the product homomorphism.*

The action ρ of the group G on the set Z, given by

$$x_{g_1} \in X_O \mapsto \rho(g)(x_{g_1}) = x_{gg_1g^{-1}} \in X_O,$$

$$y_{a,b} \in Y_G \mapsto \rho(g)(y_{a,b}) = y_{gag^{-1},gbg^{-1}} \in Y_G,$$

defines a homomorphism $\rho_{\mathbb{S}} : G \to \text{Aut}(\mathbb{S}(G,O))$ and homomorphisms $\rho_{\mathcal{H}-\text{equiv}} : G \to \text{Aut}(\mathbb{S}_{\mathcal{H}-\text{equiv}}(G,O))$. Obviously, these actions are compatible with the homomorphism $r_{\mathcal{H}-\text{equiv}}$.

If it does not lead to a confusion, we replace the notation $\mathbb{S}_{\mathcal{H}-\text{equiv}}(G,O)$ by $\overline{\mathbb{S}}(G,O)$ and then denote the both homomorphisms $\rho_{\mathbb{S}}$ and $\rho_{\mathcal{H}-\text{equiv}}$ simply by ρ. The action $\rho(g)$ on $\overline{\mathbb{S}}(G,O)$ is called the *simultaneous conjugation* by $g \in G$. Put $\lambda(g) = \rho(g^{-1})$ and $\lambda_{\overline{\mathbb{S}}} = \lambda \circ \alpha_G$, $\rho_{\overline{\mathbb{S}}} = \rho \circ \alpha_G$.

Whatever is an admissible covering semigroup $\mathbb{S}_{\mathcal{H}-\text{equiv}}(G,O) = \overline{\mathbb{S}}(G,O)$, the collection $(\overline{\mathbb{S}}(G,O), G, \alpha_G, \rho)$ is a semigroup over the group G and the embedding $i : X_O \hookrightarrow Z$ defines an embedding $i_* : S(G,O) \hookrightarrow \overline{\mathbb{S}}(G,O)$, which is a semigroup homomorphism over G. Note also that epimorphisms $r_{\mathcal{H}-\text{equiv}} : \mathbb{S}(G,O) \to \mathbb{S}_{\mathcal{H}-\text{equiv}}(G,O)$ are also semigroup homomorphisms over G.

Using relations (7) – (11), any element $s \in \overline{\mathbb{S}}(G,O)$ can be written in a so called *reduced* form, $s = s_1 \cdot s_2$, where $s_1 \in S(G,O)$ and $s_2 = y_{a_1,b_1} \cdot \ldots \cdot y_{a_p,b_p}$ for some $a_1, b_1, \ldots, a_p, b_p \in G$. We put $\tau(s) = \tau(s_1)$ and $g(s) = p$ and call them *type of* s and *genus of* s, respectively. It is easy to see that the type and the genus of $s \in \overline{\mathbb{S}}(G,O)$ are well defined, that is, $\tau(s)$ and $g(s)$ do not depend on the reduction of s to a reduced form $s = s_1 \cdot s_2$.

Let $s_1 \cdot s_2$ with $s_1 = x_{g_1} \cdot \ldots \cdot x_{g_n}$ and $s_2 = y_{a,b} \cdot \ldots \cdot y_{a_k,b_k}$ be a reduced form of an element $s = s_1 \cdot s_2 \in \overline{\mathbb{S}}(G,O)$. As it follows from Claim 7, the subgroup of G

generated by $g_1,\dots,g_n,a_1,b_1,\dots,a_k,b_k$ does not depend on the choice of a reduced form of s. In what follows, we denote this subgroup by G_s.

As in the case of factorization semigroups, for subgroups H_1 and H_2 of a group G, we put

$$\overline{\mathbb{S}}(G,O)^{H_1}=\{s\in\overline{\mathbb{S}}(G,O)\mid G_s=H_1\},$$
$$\overline{\mathbb{S}}(G,O)_{H_2}=\{s\in\overline{\mathbb{S}}(G,O)\mid \alpha(s)\in H_2\},$$

and $\overline{\mathbb{S}}(G,O)_{H_1}^{H_2}=\overline{\mathbb{S}}(G,O)_{H_2}\cap\overline{\mathbb{S}}(G,O)^{H_1}$.

Let $G_\Gamma=(\widetilde{G},\widetilde{O})$ be the C-group equivalent to (G,O) (see subsection 1.3). For any set $\mathcal{R}_H$ of admissible relations, the epimorphism $\beta=\beta_{(G,O)}:(\widetilde{G},\widetilde{O})\to(G,O)$ of equipped groups defines an epimorphism $\beta_*=\beta_{(G,O)*}:\overline{\mathbb{S}}(\widetilde{G},\widetilde{O})=\mathbb{S}_{H-\mathrm{equiv}}(\widetilde{G},\widetilde{O})\to\overline{\mathbb{S}}(G,O)=\mathbb{S}_{H-\mathrm{equiv}}(G,O)$ over groups.

Claim 9. *The restriction of β_* to the subsemigroup $S(\widetilde{G},\widetilde{O})\subset\overline{\mathbb{S}}(\widetilde{G},\widetilde{O})$ coincides with the isomorphism of semigroups $S(\widetilde{G},\widetilde{O})$ and $S(G,O)\subset\overline{\mathbb{S}}(G,O)$ (defined in subsection 1.3).*

Proof. Obvious. □

1.6. **Solvability of some equations in strong covering semigroups.** In this subsection, we will assume that in $\mathbb{S}(G,O)$ there is the unity, $\mathbf{1}\in\mathbb{S}(G,O)$ (we add it into $\mathbb{S}(G,O)$).

Let $s_1,s_2,s_3\in\mathbb{S}(G,O)$. We say that an equation

$$s_1=s_2\cdot z\cdot s_3 \tag{12}$$

is *solvable* in $S(G,O)\subset\mathbb{S}(G,O)$ if there is an element $s\in S(G,O)$ such that $s_1=s_2\cdot s\cdot s_3$.

Note that s is a solution of equation (12) if and only if the following holds: if we write s,s_1,s_2,s_3 as products of generators of $\mathbb{S}(G,O)$, then there is a finite sequence of elementary transformations transforming the factorization of s_1 into the factorization of $s_2\cdot s\cdot s_3$, here an *elementary transformation* means a change of some pair of two neighboring factors into another one according to the one of relations (7) – (11) (reading either from the left to the right or from the right to the left).

Consider four elements $s_1,\dots,s_4$ of $\mathbb{S}(G,O)$ and let us fix their presentations as products of generators of $\mathbb{S}(G,O)$. Let S be a subset of $S(G,O)$ the elements of which have a fixed type. We say that the equations

$$s_1\cdot s\cdot s_2=s_3\cdot z\cdot s_4, \tag{13}$$

where $s\in S$, are *universally solvable* if, first, there is a solution $\overline{s}\in S$ of (13) for any s and, second, there is a finite sequence of elementary transformations which satisfy the following property: for any presentation of s as the product of generators of $S(G,O)$ there is a presentation of a solution $\overline{s}$ as the product of generators such that this sequence of elementary transformations transforms the factorization of $s_1\cdot s\cdot s_2$ into the one of $s_3 \cdot \overline{s} \cdot s_4$. The element $\overline{s}$ together with its factorization mentioned above will be called the *universal solution* (for s and its factorization) of equation (13).

Claim 10. *For any $s \in S(G,O)$ of any fixed type and any $a, b \in G$ each of the following equations*

$$s \cdot y_{a,b} = z \cdot y_{\alpha(s)a,b}, \tag{14}$$

$$y_{a,b} \cdot s = y_{a,(\alpha(s))^{-1}b} \cdot z, \tag{15}$$

$$s \cdot y_{a,b} = z \cdot y_{a,(\alpha(s)^{-1})^{[a,b]}b} \tag{16}$$

is universally solvable in $S(G,O)$.

Proof. Let $s = x_{g_1} \cdot \ldots \cdot x_{g_n}$. If $n = 1$, then a universal solvability of equation (14) follows from relation (9). Assume that equation (14) is universally solvable for any s of length $n-1$ and let us write an element s of length n in the form: $s = s_1 \cdot x_{g_n}$. We have

$$s \cdot y_{a,b} = s_1 \cdot x_{g_n} \cdot y_{a,b} = s_1 \cdot z_1 \cdot y_{g_n a,b} = \rho_{\overline{\mathbb{S}}}(s_1)(z_1) \cdot s_1 \cdot y_{g_n a,b},$$

where z_1 is an universal solution of equation $x_{g_n} \cdot y_{a,b} = z \cdot y_{g_n a,b}$. By assumption, for some z_2 we have: $s_1 \cdot y_{g_n a,b} = z_2 \cdot y_{\alpha(s_1)ga,b}$. Therefore

$$s \cdot y_{a,b} = \rho_{\overline{\mathbb{S}}}(s_1)(z_1) \cdot s_1 \cdot y_{g_n a,b} = \rho_{\overline{\mathbb{S}}}(s_1)(z_1) \cdot z_2 \cdot y_{\alpha(s_1)g_n a,b} = (\rho_{\overline{\mathbb{S}}}(s_1)(z_1) \cdot z_2) \cdot y_{\alpha(s)a,b},$$

that is, equation (14) is universally solvable always.

The proof of universal solvability of equation (15) is similar to one for equation (14). Only we must use relation (10) instead of relation (9).

To prove the universal solvability of equation (16), note that $\alpha(\lambda([a,b])(s)) = (\alpha(s))^{[a,b]}$. Therefore, by relation (8), we have

$$s \cdot y_{a,b} = y_{a,b} \cdot \lambda([a,b])(s) = y_{a,(\alpha(s)^{-1})^{[a,b]}b} \cdot z_1,$$

where z_1 is a universal solution of equation $y_{a,b} \cdot \lambda([a,b])(s) = y_{a,(\alpha(s)^{-1})^{[a,b]}b} \cdot z$. Now to prove the universal solvability of equation (16), it suffices several times to use relation (8). □

The following proposition is a generalization of Main Lemma 2.1 in [10].

Proposition 4. *Let $O \subset G$ be a finite set. Then for any*

$$h \in \langle g_1, \ldots, g_n, a_1, b_1, \ldots, a_k, b_k \rangle$$

and for any $s_{g^{-1}} \in S(G,O)$ such that $\alpha(s_{g^{-1}}) = g^{-1}$ the following equation

$$x_{g_1} \cdot \ldots \cdot x_{g_n} \cdot x_g \cdot s_{g^{-1}} \cdot y_{a_1,b_1} \cdot \ldots \cdot y_{a_k,b_k} = x_{g_1} \cdot \ldots \cdot x_{g_n} \cdot x_{g^h} \cdot z \cdot y_{a_1,b_1} \cdot \ldots \cdot y_{a_k,b_k}$$

is solvable in $S(G,O)$.

Proof. For $s_1, s_2 \in \mathbb{S}(G,O)$ denote by $G(s_1,s_2)$ the subset of G such that $h \in G(s_1,s_2)$ if and only if the equations

$$s_1 \cdot x_g \cdot s_{g^{-1}} \cdot s_2 = s_1 \cdot x_{g^h} \cdot z \cdot s_2 \tag{17}$$

are universally solvable in $S(G,O)$ for each $g \in O \subset G$ and any $s_{g^{-1}} \in S(G,O)$ of fixed type and such that $\alpha(s_{g^{-1}}) = g^{-1}$. Note that if z_1 is a solution of equation (17), then $\alpha(z_1) = (g^h)^{-1}$.

Claim 11. *Let $O \subset G$ be a finite set. Then for any $s_1, s_2 \in \mathbb{S}(G,O)$ the set $G(s_1,s_2)$ is a subgroup of G.*

Proof. Let us show that if $h \in G(s_1,s_2)$, then $h^{-1} \in G(s_1,s_2)$.

Let z_1 be an universal solution of equation (17). If we apply the inverse sequence of the sequence of elementary transformations giving the universe resolution of equation (17), then it is easy to see that $s_{g^{-1}}$ is the universal solution of the equation

$$s_1 \cdot x_{g_1} \cdot z_1 \cdot s_2 = s_1 \cdot x_{g_1^{h_1}} \cdot z \cdot s_2,$$

where $g_1 = g^h$. For any $h \in G$ the conjugation of the elements of $S(G,O)$ by h is a bijection, and for each $g \in G$ the set of different factorizations of elements $s \in S(G,O)$ of fixed type and such that $\alpha(s) = g^{-1}$ is finite. Therefore $h^{-1} \in G(s_1,s_2)$ if $h \in G(s_1,s_2)$.

Let us show that if $h_1, h_2 \in G(s_1,s_2)$, then $h_1h_2 \in G(s_1,s_2)$. It is easy to see that if z_1 is an universal solution of the equation

$$s_1 \cdot x_g \cdot s_{g^{-1}} \cdot s_2 = s_1 \cdot x_{g^{h_1}} \cdot z \cdot s_2$$

and if z_2 is an universal solution of the equation

$$s_1 \cdot x_{g^{h_1}} \cdot z_1 \cdot s_2 = s_1 \cdot x_{(g^{h_1})^{h_2}} \cdot z \cdot s_2,$$

then z_2 is an universal solution of the equation

$$s_1 \cdot x_g \cdot s_{g^{-1}} \cdot s_2 = s_1 \cdot x_{g^{h_1h_2}} \cdot z \cdot s_2. \qquad \square$$

Claim 12. *For any $s_1 = s_1' \cdot s_1''$, $s_2 = s_2' \cdot s_2'' \in \mathbb{S}(G,O)$, we have $G(s_1'', s_2') \subset G(s_1,s_2)$.*

Proof. Obvious. $\square$

Claim 13. *For any $h_1, \dots, h_n \in O$, we have $\langle h_1, \dots, h_n \rangle \subset G(x_{h_1} \cdot \dots \cdot x_{h_n}, \mathbf{1})$.*

Proof. It easily follows from Claim 11 and from the following equalities:

$$x_h \cdot x_g \cdot s_{g^{-1}} = x_{g^{(h^{-1})}} \cdot \rho(h)(s_{g^{-1}}) \cdot x_h = x_h \cdot x_{g^{(h^{-1})}} \cdot \rho(h)(s_{g^{-1}}),$$

since $x_{g^{(h^{-1})}} \cdot \rho(h)(s_{g^{-1}}) \cdot x_h = \rho_{\overline{\mathbb{S}}}(x_{g^{(h^{-1})}} \cdot \rho(h)(s_{g^{-1}}))(x_h) \cdot x_{g^{(h^{-1})}} \cdot \rho(h)(s_{g^{-1}})$ and $\alpha(x_{g^{(h^{-1})}} \cdot \rho(h)(s_{g^{-1}})) = \mathbf{1}$. $\square$

Claim 14. *For any $a, b \in G$, we have $ab^{-1}a^{-1} \in G(\mathbf{1}, y_{a,b})$.*

Proof. By Claim 10, we have

$$x_g \cdot s_{g^{-1}} \cdot y_{a,b} = x_g \cdot z_1 \cdot y_{g^{-1}a,b} = \rho(g)(z_1) \cdot x_g \cdot y_{g^{-1}a,b},$$

where z_1 is an universal solution of equation $s_{g^{-1}} \cdot y_{a,b} = z \cdot y_{g^{-1}a,b}$. By relation (9),

$$x_g \cdot y_{g^{-1}a,b} = x_{gg^{-1}aba^{-1}ggg^{-1}ab^{-1}a^{-1}} \cdot y_{a,b} = x_{g^{ab^{-1}a^{-1}}} \cdot y_{a,b}.$$

Therefore

$$x_g \cdot s_{g^{-1}} \cdot y_{a,b} = \rho(g)(z_1) \cdot x_g \cdot y_{g^{-1}a,b} = \rho(g)(z_1) \cdot x_{g^{ab^{-1}a^{-1}}} \cdot y_{a,b},$$

that is, $ab^{-1}a^{-1} \in G(\mathbf{1}, y_{a,b})$. □

Claim 15. *For any $a, b \in G$, we have $aba^{-1}b^{-1}a^{-1} \in G(\mathbf{1}, y_{a,b})$.*

Proof. Applying $ln(x_g \cdot s_{g^{-1}})$ times relation (8) and after that applying $ln(s_{g^{-1}})$ times relation (7), we have

$$x_g \cdot s_{g^{-1}} \cdot y_{a,b} = y_{a,b} \cdot x_{g^{[a,b]}} \cdot z_1 = y_{a,b} \cdot z_2 \cdot x_{g^{[a,b]}},$$

where $z_1 = \lambda([a,b])(s_{g^{-1}})$ and $z_2 = \rho(g^{[a,b]})(z_1)$. It is easy to see that $\alpha(z_2) = (g^{[a,b]})^{-1}$. By Claim 10 and relation (10),

$$y_{a,b} \cdot z_2 \cdot x_{g^{[a,b]}} = z_3 \cdot y_{a,g^{[a,b]}b} \cdot x_{g^{[a,b]}} = z_3 \cdot y_{a,b} \cdot x_{g^{[a,b]ba^{-1}b^{-1}}},$$

where z_3 is an universal solution of equation $y_{a,b} \cdot z_2 = z \cdot y_{a,g^{[a,b]}b}$.

Applying relation (8), we obtain

$$z_3 \cdot y_{a,b} \cdot x_{g^{[a,b]ba^{-1}b^{-1}}} = z_3 \cdot x_{g^{[a,b]ba^{-1}b^{-1}[b,a]}} \cdot y_{a,b} = z_3 \cdot x_{g^{aba^{-1}b^{-1}a^{-1}}} \cdot y_{a,b}$$

and to complete the proof it suffices to use the relation

$$z_3 \cdot x_{g^{aba^{-1}b^{-1}a^{-1}}} = x_{g^{aba^{-1}b^{-1}a^{-1}}} \cdot \lambda(g^{aba^{-1}b^{-1}a^{-1}})(z_3). \quad \square$$

Claim 16. *For any $a, b \in G$, we have $a, b \in G(\mathbf{1}, y_{a,b})$.*

Proof. By Claims 14 and 15, the elements $ab^{-1}a^{-1}$ and $aba^{-1}b^{-1}a^{-1}$ belong to $G(\mathbf{1}, y_{a,b})$. It follows from Claim 11 that

$$(ab^{-1}a^{-1})(aba^{-1}b^{-1}a^{-1}) = (ab)^{-1} \in G(\mathbf{1}, y_{a,b}).$$

Therefore $ab \in G(\mathbf{1}, y_{a,b})$ and hence $a = (ab^{-1}a^{-1})(ab) \in G(\mathbf{1}, y_{a,b})$. Applying one more Claim 11 to a and ab, we obtain that $b \in G(\mathbf{1}, y_{a,b})$. □

Claim 17. *For any $a_1, b_1, \dots, a_k, b_k \in G$, we have*

$$\langle a_1, b_1, \dots, a_k, b_k \rangle \subset G(\mathbf{1}, y_{a_1,b_1} \cdot \ldots \cdot y_{a_k,b_k}).$$

Proof. Let us use induction on k. In the case $k = 1$, it is Claim 16. Assume that for some $k-1$ Claim is true. Therefore, by Claim 12,

$$\langle a_1, b_1, \dots, a_{k-1}, b_{k-1}\rangle \subset G(\mathbf{1}, y_{a_1,b_1} \cdot \dots \cdot y_{a_k,b_k}).$$

Denote by $u_t = [a_1, b_1] \dots [a_t, b_t]$. We have

$$x_g \cdot s_{g^{-1}} \cdot y_{a_1,b_1} \cdot \dots \cdot y_{a_{k-1},b_{k-1}} \cdot y_{a_k,b_k} = y_{a_1,b_1} \cdot \dots \cdot y_{a_{k-1},b_{k-1}} \cdot x_{g^{u_{k-1}}} \cdot \lambda(u_{k-1})(s_{g^{-1}}) \cdot y_{a_k,b_k}.$$

Denote by $z_1 = \lambda(u_{k-1})(s_{g^{-1}})$ and let z_c (see Claim 16) be an universal solution of equation $x_{g^{u_{k-1}}} \cdot z_1 \cdot y_{a_k,b_k} = x_{g^{u_{k-1}c}} \cdot z_c \cdot y_{a_k,b_k}$, where $c = a_k$ or b_k. We have

$$\begin{gathered} x_g \cdot s_{g^{-1}} \cdot y_{a_1,b_1} \cdot \dots \cdot y_{a_k,b_k} = \\ y_{a_1,b_1} \cdot \dots \cdot y_{a_{k-1},b_{k-1}} \cdot x_{g^{u_{k-1}c}} \cdot z_c \cdot y_{a_k,b_k} = \\ x_{{g^c}^{u_{k-1}}} \cdot \lambda(u_{k-1})(z_c) \cdot y_{a_1,b_1} \cdot \dots \cdot y_{a_{k-1},b_{k-1}} \cdot y_{a_k,b_k}. \end{gathered}$$

Now, since by assumption, $u_{k-1} \in G(\mathbf{1}, y_{a_1,b_1} \cdot \dots \cdot y_{a_k,b_k})$, we obtain that the element $c \in G(\mathbf{1}, y_{a_1,b_1} \cdot \dots \cdot y_{a_k,b_k})$. □

Now the proof of Proposition 4 easily follows from Claims 11 – 17. □

Proposition 5. *Let (G, O) be an equipped finite group such that the elements of $O = C_1 \sqcup \dots \sqcup C_m$ generate the group G. Denote by n_i the number of elements of the conjugacy class C_i and by p_i the order of elements of C_i. Let $s_1 \in S(G, O)$ be such that $\tau_i(s_1) > n_i p_i$ for all i, $1 \leqslant i \leqslant m$, and $s_2 = y_{a_1,b_1} \cdot \dots \cdot y_{a_k,b_k}$ be such that $s_1 \cdot s_2 \in \mathbb{S}(G, O)^G$. Then the equation*

$$s_1 \cdot s_2 = z \cdot (y_{\mathbf{1},\mathbf{1}})^k$$

is solvable in $S(G, O)$.

Proof. Let $a_1 = g_1^{-1} \dots g_n^{-1}$, where $g_1, \dots, g_n \in O$, and let $s_1 = x_{h_1} \cdot \dots \cdot x_{h_N}$. Let $g_1 \in C_i$. Since $\tau_i(s_1) > n_i p_i$, among the factors $x_{h_1}, \dots, x_{h_N}$ there are at least $p_i + 1$ factors with the same $h_j \in C_i$. Moving p_i of these factors to the right (using relation (7)), we obtain that

$$s_1 \cdot s_2 = s_1' \cdot x_{h_j} \cdot (x_{h_j})^{p_i - 1} \cdot s_2$$

for some $s_1' \in S(G, O)$ such that $s_1' \cdot s_2 \in \overline{\mathbb{S}}(G, O)^G$.

Applying Proposition 4, we have

$$s_1' \cdot x_{h_j} \cdot (x_{h_j})^{p_i - 1} \cdot s_2 = s_1' \cdot s' \cdot x_{g_1} \cdot s_2$$

for some $s' \in S(G, O)$ such that $\tau(s') = \tau((x_{h_j})^{p_i - 1})$ and $\alpha(s') = g_1^{-1}$.

By relation (9), we have

$$s_1' \cdot s' \cdot x_{g_1} \cdot s_2 = s_1' \cdot s' \cdot x_{g_1} \cdot y_{a_1,b_1} \cdot \dots \cdot y_{a_k,b_k} = s_1' \cdot s' \cdot x_{g_1'} \cdot y_{a_1',b_1} \cdot \dots \cdot y_{a_k,b_k},$$

where g_1' is an element conjugate to g_1 and $a_1' = g_2^{-1} \dots g_n^{-1}$.

Note that $\widetilde{s}_1 = s_1' \cdot s' \cdot x_{g_1'}$ and $\widetilde{s}_2 = y_{a_1',b_1} \cdot \dots \cdot y_{a_k,b_k}$ satisfy all conditions of Proposition 5. Therefore, by induction on n, we obtain that

$$s_1 \cdot s_2 = \overline{s}_1 \cdot (y_{\mathbf{1},b_1} \cdot y_{a_2,b_2} \cdot \dots \cdot y_{a_k,b_k})$$

for some $\overline{s}_1$ which together with $\overline{s}_2 = y_{\mathbf{1},b_1} \cdot y_{a_2,b_2} \cdot \dots \cdot y_{a_k,b_k}$ satisfies all conditions of Proposition 5.

The same arguments (only instead of relation (9) we must use relation (10)) give that

$$s_1 \cdot s_2 = \widetilde{s}_1 \cdot (y_{\mathbf{1},\mathbf{1}} \cdot y_{a_2,b_2} \cdot \dots \cdot y_{a_k,b_k})$$

for some $\widetilde{s}_1$ which together with $\widetilde{s}_2 = y_{a_2,b_2} \cdot \dots \cdot y_{a_k,b_k}$ satisfies all conditions of Proposition 5.

Now to complete the proof of Proposition 5, it suffices to note that $y_{\mathbf{1},\mathbf{1}}$ commutes with any element $s \in S(G,O)$ (relation (8)) and to use induction on k. $\square$

1.7. **On the number of solutions of equation** $\alpha(z) = g$**.** Let (G,O) be an equipped group such that the elements of O generate the group G, $O = C_1 \sqcup \dots \sqcup C_m$ decomposition into a disjoint union of conjugacy classes of G. In this subsection, we investigate the following problem: *for fixed type $\overline{t} \in \mathbb{Z}^m_{\geqslant 0}$, fixed genus p and given element $h \in G$ to estimate the number of solutions in an admissible covering semigroup $\overline{\mathbb{S}}(G,O)^G$ of the equation $\alpha_G(z) = h$ under the restrictions $\tau(z) = \overline{t}$ and $g(z) = p$.*

In [14], this problem was solved in the case of $p = 0$ and $\overline{t} = (t_1, \dots, t_m)$ such that all t_i are big enough (see Theorems 4 and 5). In this subsection we generalize these results to the case of arbitrary genus, namely, we prove

Theorem 6. *Let $\overline{\mathbb{S}}(G,O) = \mathbb{S}_{H-equiv}(G,O)$ be an admissible covering semigroup over an equipped finite group (G,O), $O = C_1 \sqcup \dots \sqcup C_m$. Then there is a constant $T_1 \in \mathbb{N}$ such that if for an element $\overline{s}_1 \in \overline{\mathbb{S}}(G,O)^G$ the i-th type $\tau_i(s_1) \geqslant T_1$ for all $i = 1, \dots, m$, then there are $a_{(G,O)}$ elements $\overline{s}_1, \dots, \overline{s}_{a_{(G,O)}} \in \overline{\mathbb{S}}(G,O)^G$ such that*

1. *$\overline{s}_i \neq \overline{s}_j$ for $1 \leqslant i < j \leqslant a_{(G,O)}$;*
2. *$\tau(\overline{s}_i) = \tau(\overline{s}_1)$ for $1 \leqslant i \leqslant a_{(G,O)}$;*
3. *$g(\overline{s}_i) = g(\overline{s}_1)$ for $1 \leqslant i \leqslant a_{(G,O)}$;*
4. *$\alpha_G(\overline{s}_i) = \alpha_G(\overline{s}_1)$ for $1 \leqslant i \leqslant a_{(G,O)}$;*
5. *if $\overline{s} \in \overline{S}(G,O)^G$ is such that $\tau(\overline{s}) = \tau(\overline{s}_1)$, $g(\overline{s}) = g(\overline{s}_1)$, and $\alpha_G(\overline{s}) = \alpha_G(\overline{s}_1)$, then $\overline{s} = \overline{s}_i$ for some i, $1 \leqslant i \leqslant a_{(G,O)}$.*

Proof. Let p be the genus of $\overline{s}_1$ and T be a constant the existence of which is claimed in Theorem 4. Without loss of generality, we can assume that $T > \max_{1 \leqslant i \leqslant m} n_i p_i$, where n_i is the number of elements of the conjugacy class C_i and p_i is the order of elements of C_i. By Proposition 5 (and since $r_{\mathcal{H}-\text{equiv}}$ is an epimorphism), the element $\overline{s}_1$ can be written in the form $\overline{s}_1 = s_1 \cdot (y_{\mathbf{1},\mathbf{1}})^p$. By Theorem 4, there are exactly $a_{(G,O)}$ different elements $s_1, \dots, s_{a_{(G,O)}} \in S(G,O)^G$ satisfying conditions (1) – (4) of Theorem 6.

Consider the elements $\overline{s}_i = s_i \cdot (y_{\mathbf{1},\mathbf{1}})^p$, $1 \leqslant i \leqslant a_{(G,O)}$. By Proposition 5 and Theorem 4, they satisfy conditions (2) – (5) of Theorem 6. Let us show that they also satisfy condition (1) of Theorem 6. Assume that for some $i \neq j$ we have $\overline{s}_i = \overline{s}_j$, that is, if we write s_i and s_j as products of generators x_g, $g \in O$, then there is a finite sequens Tr of elementary transformations (see subsection 1.6) transforming the factorization of $\overline{s}_i$ into the factorization of $\overline{s}_j$. By Claim 9, the selected factorizations alow us to lift the element $\overline{s}_i$ into $\overline{\mathbb{S}}(\widetilde{G},\widetilde{O}) = \mathbb{S}_{H-\mathrm{equiv}}(\widetilde{G},\widetilde{O})$ (that is, to consider an element $\widetilde{s}_i \in \overline{\mathbb{S}}(\widetilde{G},\widetilde{O})$ with the same factorization as the one of $\overline{s}_i$), where $G_\Gamma = (\widetilde{G},\widetilde{O})$ is the C-group equivalent to (G,O). Let us apply the same sequence Tr of elementary transformations to the element $\widetilde{s}_i$. As a result we obtain an element $\widetilde{s}_j = s_j\cdot(y_{a_1,b_1}\cdot\dots\cdot y_{a_p,b_p}) = \widetilde{s}_i$ such that $a_l, b_l \in \ker\beta_{(G,O)} \subset Z(\widetilde{G})$ for $1\leqslant l\leqslant p$. But it is impossible, since in this case we have $\alpha_{\widetilde{G}}(y_{a_l,b_l}) = \mathbf{1}$ for $1\leqslant l\leqslant p$ and therefore by Remark 1, we must have $\alpha_{\widetilde{G}}(\widetilde{s}_i)\neq\alpha_{\widetilde{G}}(\widetilde{s}_j)$. □

Theorem 7. *Let G be a finite group and $O'\subset O$ be two its equipments such that the elements of $O' = C_1\sqcup\dots\sqcup C_k$ generate the group G. Then there is a constant $T = T_{O'}$ such that if for an element $\overline{s}_1 \in \overline{\mathbb{S}}(G,O)^G$ its i-th type $\tau_i(\overline{s}_1)\geqslant T$ for all $i = 1,\dots,k$, then there are not more than $a_{(G,O')}$ elements $\overline{s}_1,\dots,\overline{s}_n \in \overline{\mathbb{S}}(G,O)^G$ such that for $1\leqslant i<j\leqslant n$*

(i) $\overline{s}_i \neq \overline{s}_j$;
(ii) $\tau(\overline{s}_i) = \tau(\overline{s}_1)$;
(iii) $\alpha_G(\overline{s}_i) = \alpha_G(\overline{s}_1)$,
(iv) $g(\overline{s}_i) = g(\overline{s}_1)$,

where $a_{(G,O')}$ is the ambiguity index of (G,O').

Proof. It is similar to the proof of Theorem 6. □

Corollary 1. *Let C_1 be a conjugacy class of an odd permutation $\sigma_1\in\mathcal{S}_d$ such that σ_1 leaves fixed at least two elements. Then in the case when C_1 is contained in an equipment $O = C_1\sqcup\dots\sqcup C_m$ of $\mathcal{S}_d$, there is a constant $T = T_{C_1}$ such that for any $\sigma\in\mathcal{S}_d$, any fixed integer $p\geqslant 0$, and any $\overline{t} = (t_1,\dots,t_m)\in\mathbb{Z}^m_{\geqslant 0}$ such that $t_1\geqslant T$ the equation*

$$\alpha_{\mathcal{S}_d}(z) = \sigma \tag{18}$$

has in each covering semigroup $\overline{\mathbb{S}}(\mathcal{S}_d,O)$ at most one solution $\overline{s}$ satisfying conditions $g(\overline{s}) = p$ and $\tau(\overline{s}) = \overline{t}$. Under assumption $t_1\geqslant T$, the existence of solution of equation (18) *does not depend on p and depends only on t.*

Proof. It follows from Theorem 7 and the main result of [13]. □

Let $\overline{\mathbb{S}}(G,O)$ be an admissible covering semigroup over an equipped finite group (G,O), $O = C_1\sqcup\dots\sqcup C_m$, and let $T = (t_1,\dots,t_m)\in\mathbb{Z}_{\geqslant o}$. Denote by

$$\overline{\mathbb{S}}(G,O)_{\geqslant T} = \{\overline{s}\in\overline{\mathbb{S}}(G,O)\mid \tau_i(\overline{s})\geqslant t_i\}$$

a subsemigroup of $\overline{\mathbb{S}}(G,O)$. By Theorem 6, we have

Corollary 2. *For any equipped finite group (G,O), $O = C_1\sqcup\dots\sqcup C_m$, there is a constant $T = (t_1,\dots,t_m)\in\mathbb{Z}_{\geqslant o}$ such that the restrictions $r_{H-equiv}:\mathbb{S}(G,O)^G_{\geqslant T}\to\mathbb{S}_{H-equiv}(G,O)^G_{\geqslant T}$ of the epimorphisms $r_{H-equiv}:\mathbb{S}(G,O)\to\mathbb{S}_{H-equiv}(G,O)$ to the subsemigroup $\mathbb{S}(G,O)^G_{\geqslant T}$ are isomorphisms for any set $\mathcal{R}_H$ of admissible relations.*

2. Geometric semigroups of coverings

2.1. **Monodromy encoding of ramified coverings.** To describe ramified coverings of a given connected manifold M (in most cases, M will be a connected compact oriented surface with one hole and a base point on the boundary) we use a traditional *monodromy encoding* of non-ramified coverings and the unicity of *manifold ramified completions.*

Namely, given a non-ramified, possibly disconnected, degree d covering $\pi : \tilde{M} \to M$ over a connected manifold M with a base point $q \in M$, the lifts of a loop at q form a set of d paths in $\tilde{M}$ starting each at a different point of $\pi^{-1}(q)$ and, thus, they give rise to a permutation of the set $\pi^{-1}(q)$. This permutation depends only on the homotopy type of the loop and in this way one obtains an encoding of the covering by *a homomorphism from $\pi_1(M, q)$ to the permutation group of $\pi^{-1}(q)$*. In particular, if the covering π is equipped with a marking $\nu : I_d = [1, \dots, d] \to \pi^{-1}(q)$ it gives a well defined homomorphism $\pi_1(M, q) \to \mathcal{S}_d$.

The foliowing Proposition is well known and straightforward.

Proposition 6. *Two non-ramified marked coverings over the same based space (M, q) are isomorphic as marked coverings if and only if they define the same homomorphism $\pi_1(M, q) \to \mathcal{S}_d$. If an isomorphism exists, it is unique; equivalently, marked coverings have no non-trivial automorphisms.*

Each homomorphism $\pi_1(M, q) \to \mathcal{S}_d$ corresponds to a certain non-ramified marked degree d covering of M. The covering space is connected if and only if the action of $\pi_1(M, q)$ on $\pi^{-1}(q)$ is transitive. The orbits of the action correspond canonically to connected components of the covering space.

Manifold completions of non-ramified coverings by ramified ones are most transparent in low dimensions.

The following result is also well known and straightforward; it can be found, for example, in [5].

Proposition 7. *Let (M, q) be a based two-dimensional manifold and $B = \{P_1, \dots, P_n\}$ a finite subset of M disjoint from q and ∂M. Then each marked non-ramified covering of (U, q) with $U = M \setminus B$ has one and only one ramified completion $\tilde{M} \to M$.*

2.2. **Surfaces with a hole and their skeletons.** Our main building blocks are connected compact oriented surfaces (2-dimensional manifolds) with one hole and one marked point on the boundary. We equip them, in addition, with *a semi-skeleton*, *skeleton*, or *caudate skeleton.*

We define a semi-skeleton of a connected compact oriented 2-dimensional manifold (with one hole) to be the union of disjoint embedded bouquets of two oriented circles with a property that the complement of the union is homeomorphic to a punctured disc. Clearly, the number of connected components of a semi-skeleton is equal to the genus of the surface. We distinguish the two circles of a bouquet that represents a connected component of a semi-skeleton by means of intersection index, namely, we speak on λ- and μ-*circles* in a bouquet $C_\lambda \vee C_\mu$ by respecting the convention that $C_\lambda \cdot C_\mu = -1$ (and not 1).

Given such a triple (F, q, S^∞), where S^∞ is a semi-skeleton of a surface F with a fixed point $q \in \partial F$, we can represent it by *an open-eyes plane diagram*, that is to draw a disc with a marked point on its boundary and $p = g(F)$ holes inside the disc, and trace the standard 4-gone identification scheme on the boundary of each hole, see the left drawing on Figure 1; the orientation of F should be induced by the counter-clock wise orientation of the disc. When it happens to be more convenient and transparent, we use also another, equivalent, presentation and draw a disc with "pince-nez", that is p pairs of holes with "bridges", see the right drawing on Figure 1 (there, the λ-circles are a and c, while b and d are the μ-circles).

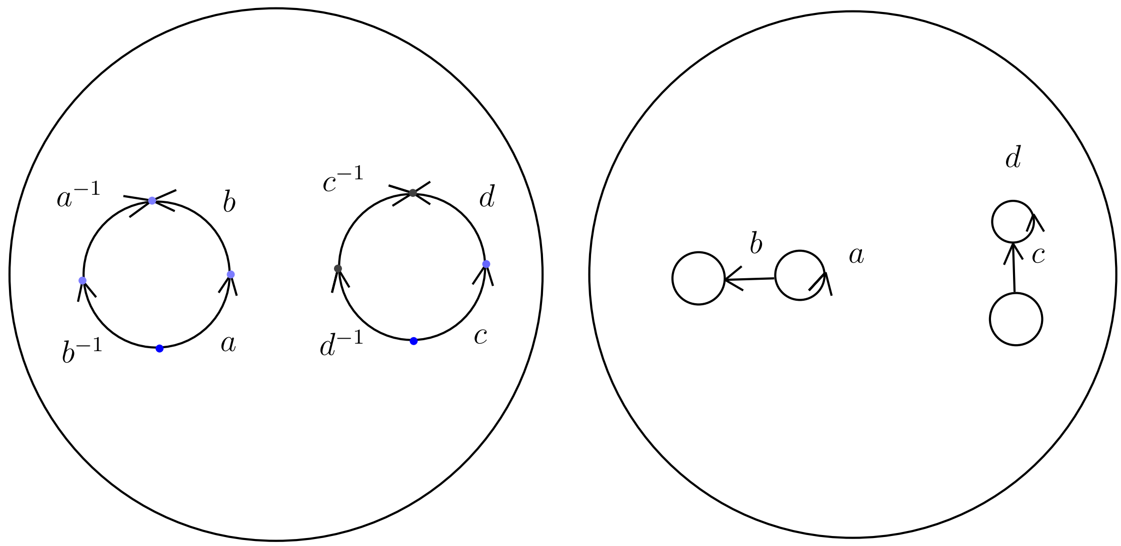
Open-eyes plane diagrams of a given triple (F, q, S^∞) are defined up to orientation preserving (stratified) homeomorphisms of the disc respecting the marked point and the orientation of the boundary identification strata. Converse statement is also true, an open-eyes plane diagram defines the triple (F, q, S^∞) up to orientation preserving homeomorphism (of triples). A similar statement is also true for diagrams with pince-nez, but as to the former one, one should take also into account the possibility for each handle to replace its λ-pince-nez presentation by the μ-pince-nez one, and vice versa.

A *skeleton* of a genus p connected compact oriented 2-dimensional manifold with one hole is, by definition, a semi-skeleton enhanced by a system of pathes that join the marked point $q \in \partial F$ with the components of the semi-skeleton, the pathes are called *strings*, they are taken disjoint and each of the p strings is chosen in such a way that in the disc model with p holes the string riches its hole at the vertex with outgoing λ- and μ-edges.

Now, let us assume that F is equipped with a finite subset $B \subset F \setminus \partial F$ (later on, such a subset B is appearing as the branch locus of a finite cover). In such a case, by a *caudate skeleton* we understand a triple (F, q, S^{cdt}) where S^{cdt} is a skeleton disjoint from B and extended by a system of *tails*, that is a collection of $n = |B|$ simple paths connecting the points of B one-by-one with q, the tails being chosen disjoint from each other and from the skeleton. In particular, S^{cdt} is homeomorphic to the wedge sum of a skeleton with n intervals.

The above notion of open-eyes plane diagrams extends to triples (F, q, S), where S is either a skeleton or caudate skeleton of F. These diagrams consist of $p = g(F)$ holes or pince-nez in a disc, a marked point on the boundary of the disc, and a system of strings, which is enhanced by a system of tails in the case of caudate skeletons, see Figure 2. Open-eyes plane diagrams of surfaces with a skeleton or, respectively, caudate skeleton are defined up to isotopies; and conversely the triple (F, q, S) is defined up to orientation preserving homeomorphisms of triples by its open-eyes plane diagram.

FIGURE 1. Plane diagrams of a genus-2 surface with its semi-skeleton.

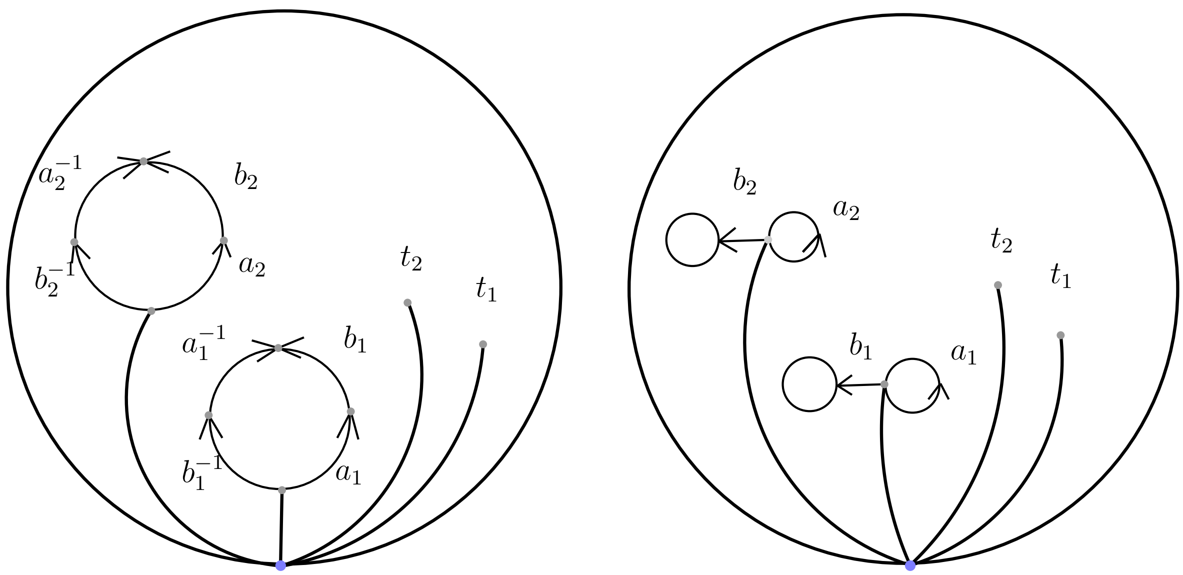


In the case of skeletons, and up to isotopy, for a given genus p there is one and only one open-eyes plane diagram. We denote this diagram by Δ_p and write (Π_p, q, Σ_p) to denote the triple that this diagram defines.

If $p = 0$, then $\Delta_0 = \Pi_0$ is just the standard disc and Σ_0 is reduced to q (the marked point on the boundary of the disc).

If $p > 0$, then the skeleton Σ_p contains a non-trivial semi-skeleton, which we denote by Σ_p^∞. A choice of a skeleton induces in a natural way an ordering on the set of the components of the semi-skeleton. Namely, we fix the ordering induced by the counter-clockwise order on the strings of the skeleton, see Figure 3, and denote the strings of Σ_p by $T_1, \ldots, T_p$ following this, counter-clockwise, order. Note that an ordering of the components of Σ_p^∞ is equivalent to a choice of the skeleton up to isotopy.

FIGURE 2. Plane diagrams of a genus-2 surface with its caudate skeleton.



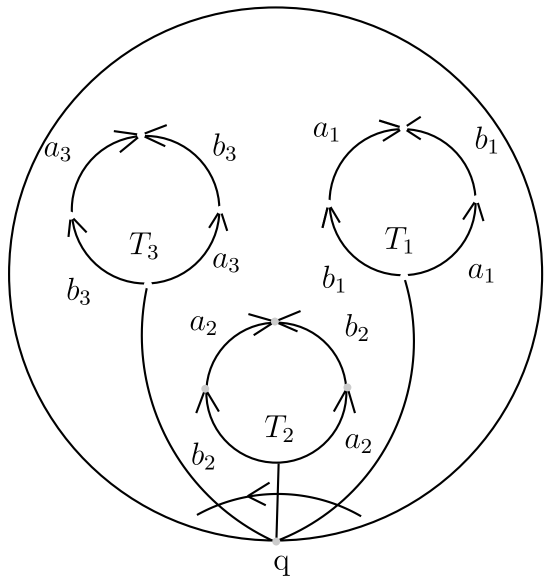
Note also that a skeleton S being given, one has a canonical choice of geometric free generators $\lambda_1, \mu_1, \dots, \lambda_p, \mu_p$ of the fundamental group $\pi_1(F, q)$, where $\lambda_i, 1 \leq i \leq p$, (respectively, μ_i) are represented by the loops $T_i \star C_{\lambda,i} \star T_i^{-1}$ (respectively, $T_i \star C_{\mu,i} \star T_i^{-1}$); the numbering respects the above ordering of the strings.

For given genus $p \geq 1$ and number $n \geq 1$ of tails, the number of isotopy classes of plane diagrams of surfaces with caudate skeletons is greater than 1 and equal to the binomial coefficient C^n_{p+n}. Indeed, similar to the case of skeletons, a choice of a caudate skeleton induces a counter-clock wise ordering on the the set of tails and strings (or, equivalently, on the set that consist of points of B and the connected components of the semi-skeleton). Conversely, the counter clock-wise ordering of the set of tails and strings determines the diagram up to isotopy.

Thus, a caudate skeleton S^{cut} being given, one gets not only a canonical choice of geometric free generators $\lambda_1, \mu_1, \dots, \lambda_p, \mu_p$ of the fundamental group $\pi_1(F, q)$, but also an extension of it to a set of geometric free generators of $\pi_1(F \setminus B, q)$ by a sequence $\gamma_1, \dots, \gamma_n$ represented by the loops $\Gamma_i \star C_i \star \Gamma_i^{-1}$, where C_i denotes a small loop around a point b_i of B and Γ_i a portion of the tail going to b_i. Note that this whole set of generators of $\pi_1(F \setminus B, q)$ is equipped with counter clock-wise ordering.

2.3. **Free semigroups of marked coverings.** We continuer to consider connected compact oriented 2-dimensional manifolds F with one hole and a marked point $q \in \partial F$ and turn to a study of their ramified finite degree coverings $f : E \to F$. Let us recall

FIGURE 3. Strings ordered counter-clock wise.



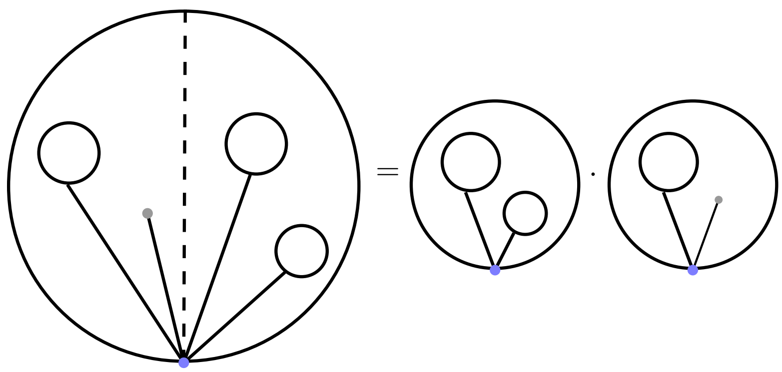
once more that *we allow disconnected covering spaces, but forbid ramifications at the boundary of* F. Our aim is to organize such coverings of a fixed degree d in a semigroup.

To achieve this goal we equip each covering with a marking, that is a numbering ν by $1, \ldots, d$ of the elements of $f^{-1}(q)$, and consider the coverings up to certain natural equivalence relations. Different choices of the equivalence relations lead to different semigroups.

We start from introducing the *free geometric degree d covering semigroup*, which we denote by $\mathrm{GF}\mathbb{S}_\mathrm{d}$. To build such a semigroup, we equip the base F of each marked covering $(f : E \to F, \nu, q)$ with a caudate skeleton S^{cdt} whose tails end at the branch points of the covering. The elements of $\mathrm{GF}\mathbb{S}_\mathrm{d}$ are the triples $(f : E \to F, \nu, S^{\mathrm{cdt}})$ considered up to homeomorphisms of coverings respecting all the ingredients; more precisely, two triples $(f_1 : E_1 \to F_1, \nu_1, S_1^{\mathrm{cdt}})$ and $(f_2 : E_2 \to F_2, \nu_2, S_2^{\mathrm{cdt}})$ are equivalent if there are homeomorphisms $\phi : E_1 \to E_2$ and $\psi : F_1 \to F_2$ such that $f_2 \circ \phi = \psi \circ f_1$, $\phi \circ \nu_1 = \nu_2$, and $\psi(S_1^{\mathrm{cdt}}) = S_2^{\mathrm{cdt}}$.

The semigroup structure on $\mathrm{GF}\mathbb{S}_\mathrm{d}$ is defined in a similar way that was used in [12] in the case of genus zero. Namely, the product $h = f \cdot g$ of two elements of $\mathrm{GF}\mathbb{S}_\mathrm{d}$ represented by marked ramified coverings $(f : E_1 \to F_1, \nu_1, S_1^{\mathrm{cdt}})$ and $(g : E_2 \to F_2, \nu_2, S_2^{\mathrm{cdt}})$ (by abuse of notation we denote by the same symbols both the elements

FIGURE 4. Plane diagram of a semigroup product.



of $\mathrm{GF}\mathbb{S}_{\mathrm{d}}$ and the underlying coverings) is given by the marked ramified covering $(h : E \to F, \nu, S_1^{cdt} \cup S_2^{cdt})$, where F and $h : E \to F$ are obtained, first downstairs, by gluing F_1 with F_2 along an arc of ∂F_2 issued from q_2 in the counter-clockwise direction and an arc of ∂F_1 issued from q_1 in the clockwise direction, (see Figure 4) and, second upstairs, by a gluing of f and g that preserves the markings over $q = q_1 = q_2$. This operation respects the equivalence relation.

We equip $\mathrm{GF}\mathbb{S}_{\mathrm{d}}$ with a map $\alpha : \mathrm{GF}\mathbb{S}_{\mathrm{d}} \to \mathcal{S}_d$ that evaluates the boundary monodromy (taken in the direction of boundary orientation). As it follows from the gluing procedure, this map is a homomorphism. Furthermore, the symmetric group $\mathcal{S}_d$ naturally acts on $\mathrm{GF}\mathbb{S}_{\mathrm{d}}$ by renumbering the points of the fibre $f^{-1}(q)$. Thus, $\mathrm{GF}\mathbb{S}_{\mathrm{d}}$ becomes in a canonical way a semigroup over $\mathcal{S}_d$.

For each $g \in \mathcal{S}_d$ denote by $X_g \in \mathrm{GF}\mathbb{S}_{\mathrm{d}}$ the element represented by a ramified covering $f : E \to \Pi_0$ with one branch point, a marked point $q \in \partial\Pi_0$, and the monodromy $\alpha(X_g)$ equal to g. Such an element is defined uniquely, as it follows, for example, from Proposition 7.

Next, consider the torus Π_1 with a hole, a marked point $q \in \partial\Pi_1$ and a skeleton Σ_1 that includes the semi-skeleton Σ_1^∞ and the string T_1. Pick a pair $a, b \in \mathcal{S}_d$ and denote by $Y_{a,b} \in \mathrm{GF}\mathbb{S}_{\mathrm{d}}$ the element represented by a non-ramified covering $f : E \to \Pi_1$ with monodromy a along the loop $T_1 \star C_\lambda \star T_1^{-1}$ and b along the loop $T_1 \star C_\mu \star T_1^{-1}$. Such an element is also defined uniquely, as it follows, for example, from Proposition 6. Note that $\alpha(Y_{a,b}) = aba^{-1}b^{-1}$.

Proposition 8. *The semigroup* $\mathrm{GF}\mathbb{S}_\mathrm{d}$ *is a free semigroup over the group* $\mathcal{S}_d$*, its set of free generators is formed by* $X_g, g \in \mathcal{S}_d \setminus \{\mathbf{1}\}$*, and* $Y_{a,b}, a, b \in \mathcal{S}_d$*.*

Proof. Follows from the following isotopy unicity: for a surface with a given caudate skeleton, its open-eyes caudate skeleton plane diagram $(\Sigma_p^{\mathrm{cdt}}, q) \subset (\Delta_p, q)$ is unique up to isotopies in (Δ_p, q). □

As a set, the semigroup $\mathrm{GF}\mathbb{S}_\mathrm{d}$ splits in a disjoint union of subsets, $(\mathrm{GF}\mathbb{S}_\mathrm{d})_{\mathrm{n,p}}$, that correspond to coverings with a given number n of branch points over surfaces of given genus p, or, saying in another way, to words with n letters X_g and p letters $Y_{a,b}$.

2.4. **Very strong semigroup of marked coverings.** As a next step, we replace caudate skeletons by skeletons, that is forget the tails going to the branch points, and thus construct another semigroup over $\mathcal{S}_d$ replacing everywhere in the above construction of $\mathrm{GF}\mathbb{S}_\mathrm{d}$ the caudate skeletons by skeletons. We call this new semigroup the *very strong semigroup of degree d marked coverings* and denote it by $\mathrm{GV}\mathbb{S}_\mathrm{d}$. The forgetful map $\mathrm{GF}\mathbb{S}_\mathrm{d} \to \mathrm{GV}\mathbb{S}_\mathrm{d}$ consisting in replacing a caudate skeleton by the skeleton is a well defined homomorphism of semigroups over $\mathcal{S}_d$. Let us denote by Kh: us
the same symbols $X_g, Y_{a,b}$ the images in $\mathrm{GV}\mathbb{S}_\mathrm{d}$ of the above free generators $X_g, Y_{a,b} \in \mathrm{GF}\mathbb{S}_\mathrm{d}$.

Proposition 9. *The elements* $X_g, g \in \mathcal{S}_d \setminus \{\mathbf{1}\}$*, and* $Y_{a,b}, a, b \in \mathcal{S}_d$*, form a set of generators of the semigroup* $\mathrm{GV}\mathbb{S}_\mathrm{d}$*. They satisfy the relations*

$$X_g \cdot Y_{a,b} = Y_{a,b} \cdot X_{g^{[a,b]}} \tag{19}$$

and

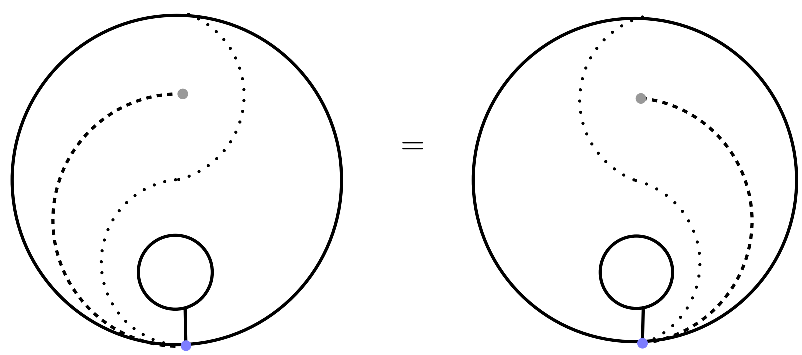
$$X_{g_1} \cdot X_{g_2} = X_{g_2} \cdot X_{g_1^{g_2}} \tag{20}$$

for any $g_1, g_2, a, b \in \mathcal{S}_d$*. These are the defining relations of* $\mathrm{GV}\mathbb{S}_\mathrm{d}$*.*

Proof. For a surface with a given skeleton and branch locus, an enhancing of its open-eyes skeleton plane diagram $(\Sigma_p, q) \subset (\Delta_p, q)$ to an open-eyes caudate skeleton diagram $(\Sigma_p^{\mathrm{cdt}}, q) \subset (\Delta_p, q)$ with a given ordering of strings and tails is unique up to isotopies of $(\Sigma_p \cup B, q)$ in (Δ_p, q). The relation (19) reflects the elementary change of ordering (see Figure 5), and (20) the standard Artin-Hurwitz half-twist of two points of B in $\Delta_p \setminus \Sigma_p$ (cf., [14]). □

Similar to $\mathrm{GF}\mathbb{S}_\mathrm{d}$, the semigroup $\mathrm{GV}\mathbb{S}_\mathrm{d}$ splits as a set in a disjoint union of subsets, $(\mathrm{GV}\mathbb{S}_\mathrm{d})_{\mathrm{n,p}}$, that correspond to coverings with a given number n of branch points over surfaces of given genus p.

FIGURE 5. Plane diagram of an elementary change of ordering.



2.5. **Strong versal geometric covering semigroups.** These semigroups are most close to the classical theory of Hurwitz spaces.

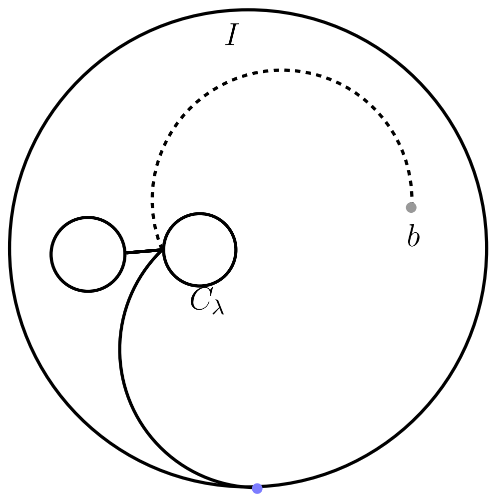
We equip the base of each covering with a skeleton and consider the triples $(f_1 : E_1 \to F_1, \nu_1, S_1)$ and $(f_2 : E_2 \to F_2, \nu_2, S_2)$, where S_1 and S_2 are skeletons of F_1 and F_2, respectively, up to the equivalence generated by two binary relations: first, up to isotopy of coverings with fixed base, marking, and skeleton, but moving branch points (which can move, in particular, through the skeleton); second, up to homeomorphisms respecting all the ingredients, that is up to homeomorphisms $\phi : E_1 \to E_2$ and $\psi : F_1 \to F_2$ such that $f_2 \circ \phi = \psi \circ f_1$, $\phi \circ \nu_1 = \nu_2$, and $\psi(S_1^\infty) = S_2^\infty$. The only, but major, difference with respect to the previous very strong covering semigroups is that we authorize the branch points to cross the skeleton.

Taking into account this additional equivalence relation, we obtain another semigroup over $\mathcal{S}_d$, which we denote by $\mathrm{G}\mathbb{S}_\mathrm{d}$ and call the *strong versal geometric degree d covering semigroup*. The quotient map $\mathrm{GV}\mathbb{S}_\mathrm{d} \to \mathrm{G}\mathbb{S}_\mathrm{d}$ is a homomorphism of semigroups over $\mathcal{S}_d$ and, set theoretically, it splits in quotient maps $(\mathrm{GV}\mathbb{S}_\mathrm{d})_{\mathrm{n,p}} \to (\mathrm{G}\mathbb{S}_\mathrm{d})_{\mathrm{n,p}}$.

For each n and p, let us fix the surface F (of genus p) and its skeleton S, and place the branch locus B (of cardinality n) to be disjoint from S. Then the braid group on n strands, that is the group $Br_n(F, \partial F)$ of isotopy classes of orientation preserving identical on the boundary self-homeomorphisms of (F, B), becomes to act naturally on $(\mathrm{GV}\mathbb{S}_\mathrm{d})_{\mathrm{n,p}}$, and, as it follows also directly from the definitions, the fibers of the quotient map $(\mathrm{GV}\mathbb{S}_\mathrm{d})_{\mathrm{n,p}} \to (\mathrm{G}\mathbb{S}_\mathrm{d})_{\mathrm{n,p}}$ are the orbits of this action.

Let us recall that, on the other hand, the braid group $Br_n(F, \partial F)$ can be canonically identified with the fundamental group $\pi_1(F^{(n)} \setminus \Delta)$, where $F^{(n)}$ is the symmetric product of n copies of F and Δ is the discriminant locus, that is, the set of those n-tuples that contain fewer than n distinct points. More precisely, we start from fixing a set $B = \{P_1, \dots, P_n\} \subset F \setminus \partial F$ consisting of n distinct points and treat

FIGURE 6. Simple path joining a branch point with a bouquet vertex.



$Br_n(F, \partial F)$ as the group of homotopy classes of geometric braids, where as is usual: by a *geometric braid* on F based at B we understand an n-tuple $\Psi = (\psi_1, \ldots, \psi_n)$ of paths $\psi_i : [0,1] \to F \setminus \partial F$ such that

(1) $\psi_i(0) = P_i$ and $\psi_i(1) \in B$ for each $i = 1, \ldots, n$;
(2) $\psi_1(t), \ldots, \psi_n(t)$ are distinct points of $F \setminus \partial F$ for each $t \in [0,1]$;

and multiplication is given by concatenation of paths.

By a λ- (respectively, μ-) move we understand a geometric braid whose all but one strands are constant and the remaining one follows a path $I \star C_\lambda \star I^{-1}$ (respectively, $I \star C_\mu \star I^{-1}$) where I is a simple path in the complement of $S \cup (B \setminus \{b\})$ joining a point $b \in B$ with the vertex of a bouquet $C_\lambda \vee C_\mu \subset S$, see Figure 6. The standard Artin-Hurwitz (half-twist) geometric braids exchanging two points of B will be called H-moves.

The following proposition is well known. In fact, it follows easily, for example, from the exact sequences

$$1 \to \pi_1(F \setminus B') \to PBr_n(F, \partial F) \to PBr_{n-1}(F, \partial F) \to 1$$

and

$$1 \to PBr_n(F, \partial F) \to Br_n(F, \partial F) \to \mathcal{S}_n \to 1,$$

where the second sequence is the definition of the pure braid groups, PBr_n, and B' denotes $B \setminus \{P_1\}$.

Proposition 10. ([4, 6]) *The braid group $Br_n(F, \partial F)$ is generated by H-, λ-, and μ-moves.* □

The following claim is an immediate consequence of the above Proposition.

Corollary 3. *The relations imposed by the partial quotient maps* $(\mathrm{GV}\mathbb{S}_\mathrm{d})_{2,0} \to (\mathrm{G}\mathbb{S}_\mathrm{d})_{2,0}$ *and* $(\mathrm{GV}\mathbb{S}_\mathrm{d})_{1,1} \to (\mathrm{G}\mathbb{S}_\mathrm{d})_{1,1}$ *imply all the other relations imposed by the quotient map* $\mathrm{GV}\mathbb{S}_\mathrm{d} \to \mathrm{G}\mathbb{S}_\mathrm{d}$. □

To describe finally the semigroup $G\mathbb{S}_d$ in terms of generators and relations, let us denote by the same symbols $X_g, Y_{a,b}$ the images in $\mathrm{G}\mathbb{S}_\mathrm{d}$ of $X_g, Y_{a,b} \in \mathrm{GV}\mathbb{S}_\mathrm{d}$.

Proposition 11. *The elements* $X_g, g \in \mathcal{S}_d \setminus \{\mathbf{1}\}$, *and* $Y_{a,b}, a, b \in \mathcal{S}_d$, *form a set of generators of the semigroup* $\mathrm{G}\mathbb{S}_\mathrm{d}$. *They satisfy the relations:*

$$X_{g_1} \cdot X_{g_2} = X_{g_2} \cdot X_{g_1^{g_2}} \tag{21}$$

for any $g_1, g_2 \in \mathcal{S}_d$; *and*

$$X_g \cdot Y_{a,b} = Y_{a,b} \cdot X_{g^{[a,b]}}, \tag{22}$$

$$X_g \cdot Y_{a,b} = X_{g^{c_1}} \cdot Y_{ga,b}, \quad c_1 = ab^{-1}a^{-1}g^{-1}, \tag{23}$$

$$Y_{a,b} \cdot X_g = Y_{a,g^{-1}b} \cdot X_{g^{c_3}}, \quad c_2 = ba^{-1}b^{-1}g, \tag{24}$$

for any $g, a, b \in \mathcal{S}_d$. *These are the defining relations of* $\mathrm{G}\mathbb{S}_\mathrm{d}$.

Proof. Due to Propositions 9, 10 and Corollary 3, the only new, with respect to Proposition 9, relations are given by the λ- and μ-moves in $(\mathrm{GV}\mathbb{S}_\mathrm{d})_{1,1}$.

Under pulling the branch point through λ- or μ-cuts as it is shown in Figures 7 and 8, we obtain the the relations (23) and (24). In particular, the elements g' appearing in the relations $X_g \cdot Y_{a,b} = X_{g'} \cdot Y_{ga,b}$ and, respectively, $Y_{a,b} \cdot X_g = Y_{a,g^{-1}b} \cdot X_{g'}$ (see Figures 7 and 8) can be found, for example, from the identities expressing the unchanged monodromy along the hole: $gaba^{-1}b^{-1} = g'a'b(a')^{-1}b^{-1}, a' = ga$ under the move across the λ-cut, as is shown in Figure 7, and $gaba^{-1}b^{-1} = g'ab'a^{-1}(b')^{-1}, b' = g^{-1}b$ under the μ-cut, as is shown in Figure 8. □

Corollary 4. *The elements* $X_g, g \in \mathcal{S}_d \setminus \{\mathbf{1}\}$, *and* $Y_{a,b}, a, b \in \mathcal{S}_d$, *satisfy the relation*

$$X_g \cdot Y_{a,b} = X_{g^{[a,b]}} \cdot Y_{a^{g^{[a,b]}}, b^{g^{[a,b]}}} \tag{25}$$

Proof. This relation follows from the relations (22), (23), and (24). It expresses the braiding of a branch point around the handle (that is around the hole on the open-eyes plane diagram) . □

Due to Proposition 11 and Corollary 4, there arises a canonical morphism

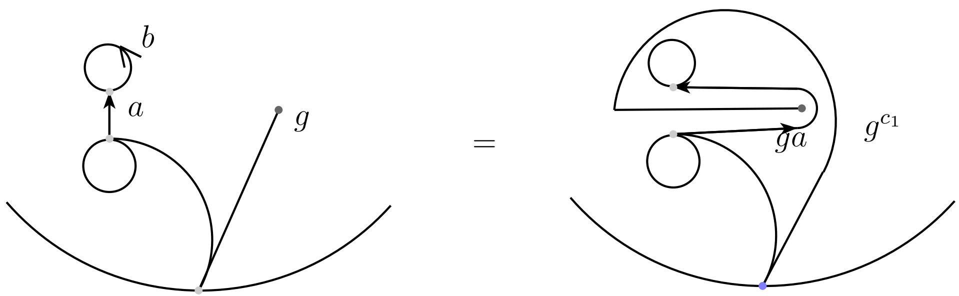
$$\varrho_d : \mathrm{G}\mathbb{S}_\mathrm{d} \to \mathbb{S}_\mathrm{d}$$

that maps X_g to x_g and $Y_{a,b}$ to $y_{a,b}$, see subsection 1.5.

Proposition 12. *The morphism* $\varrho_d : \mathrm{G}\mathbb{S}_\mathrm{d} \to \mathbb{S}_\mathrm{d}$ *is an isomorphism of semigroups over* $\mathcal{S}_d$.

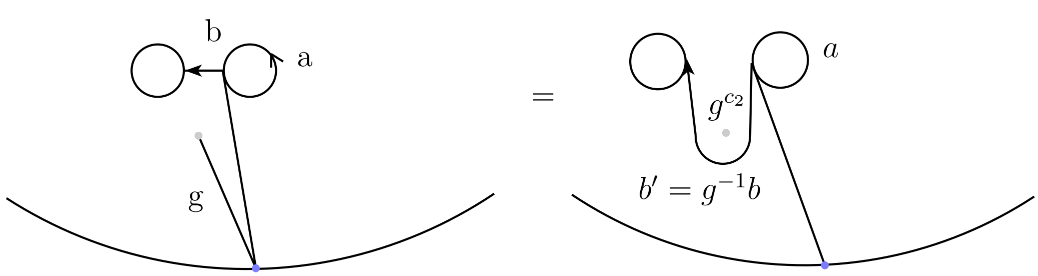
Proof. The morphism ϱ_d literally translates the list of generators and defining relations in $\mathrm{G}\mathbb{S}_\mathrm{d}$ given in Proposition 11 and extended by (25) into the list of generators and defining relations in $\mathbb{S}_d$ given in Proposition 3. □

FIGURE 7. Moving a branch point through an λ-cut in an XY-product



$$X_g Y_{a,b} = X_{g^{c_1}} Y_{ga,b}$$
$$c_1 = ab^{-1}a^{-1}g^{-1}$$

FIGURE 8. Moving a branch point through a μ-cut in a YX-product



$$Y_{a,b} X_g = Y_{a,g^{-1}b} X_{g^{c_2}}$$
$$c_2 = ba^{-1}b^{-1}g$$

2.6. **Admissible geometric covering semigroups.** Again, to define new geometric covering semigroups, we start from the very strong semigroup of degree d marked coverings, $\mathrm{GV}\mathbb{S}_{\mathrm{d}}$, and add supplementary relations between the triples $(f : E \to F, \nu, S^{cdt})$ representing the elements of $\mathrm{GV}\mathbb{S}_{\mathrm{d}}$. For that purpose, it is convenient to introduce some auxiliary category $\mathcal{F}$.

The objects of $\mathcal{F}$ are the triples (F, q, S^{cdt}) where F is a connected compact oriented surface with one hole, q is a fixed point on its boundary, and S^{cut} is a caudate skeleton of F; we denote the set of ends of the tails of S^{cdt} by B, its cardinality by n, and the genus of F by p. The morphisms of $\mathcal{F}$ are the preserving orientation homeomorphisms of triples (F, q, B).[1]

We denote by $Homeo$ the whole set of morphisms of $\mathcal{F}$; by $Homeo_{n,p}$ the subset of $Homeo$ consisting of the morphisms between the triples (F, q, S^{cdt}) with given $|B| = n$ and $g(F) = p$; and by $H(F, q, S^{cdt})$ the group consisting of the self-homeomorphisms $(F, q, B) \to (F, q, B)$.[2] We say that a collection of subsets $\widetilde{H}_{n,p} \subset Homeo_{n,p}$, $n \geqslant 0$, $p \geqslant 0$, is *geometrically admissible*, if it contains the isotopies of B in $F \setminus \partial F$ and for each two triples (F, q, S^{cdt}) and (F, q, S^{cdt}) with the same n and p there is a morphism $(F, q, S^{cdt}) \to (F, q, S^{cdt})$ belonging to $\widetilde{H}_{n,p}$.

For each n and p, we fix the triple (F, q, S^{cdt}). The caudate skeleton S^{cdt} defines a frame in the free group $\pi_1(F \setminus B, q)$, while each element $\psi \in H(F, q, S^{cdt})$ defines an automorphism of this free group $\mathbb{F}^{n+2p} = \pi_1(F \setminus B, q)$. Let us denote by $H_{n,p}$ the subgroups of $Aut(\mathbb{F}^{n+2p})$, $\mathbb{F}^{n+2p} = \pi_1(F \setminus B, q)$, representing $\widetilde{H}_{n,p} \cap H(F, q, S^{cut})$.

As follows directly from the definitions, $H_{n,p}$ are admissible subgroups of $Aut(\mathbb{F}^{n+2p})$, if the collection $\widetilde{H}_{n,p} \subset Homeo_{n,p}$ is geometrically admissible. For example, one get geometrically admissible collections by considering the homeomorphisms preserving λ-circles up to isotopy (respectively, μ-circles), or taking the whole sets, $\widetilde{H}_{n,p} = Homeo_{n,p}$. In fact, one can easily go other way round and, starting from a collection of admissible subgroups $H_{n,p}$ of $Aut(\mathbb{F}^{n+2p})$, build a geometrically admissible collection by attributing to $\tilde{H}_{n,p} \cap H(F, q, S^{cdt})$ all the elements in $H(F, q, S^{cdt})$ that act in $\mathbb{F}^{n+2p} = \pi_1(F \setminus B, q)$ as elements of $H_{n,p}$ (with respect to the frame defined by S^{cdt}).

Let $\tilde{\mathcal{H}} = \{\widetilde{H}_{n,p}\}_{n \geqslant 0, p \geqslant 0}$ be a geometrically admissible collection. Assume in addition that this collection of homeomorphisms is closed under the boundary connected sum of the triples (F, q, B) (see definition of the product in the semigroup $\mathrm{GF}\mathbb{S}_{\mathrm{d}}$). We say that two triples $(f_1 : E_1 \to F_1, \nu_1, S_1^{cdt})$ and $(f_2 : E_2 \to F_2, \nu_2, S_2^{cdt})$ are *$\tilde{\mathcal{H}}$-equivalent* if there are homeomorphisms $\phi : E_1 \to E_2$ and $\psi : (F_1, q_1, B_1) \to (F_2, q_2, B_2)$ such that $f_2 \circ \phi = \psi \circ f_1$, $\phi \circ \nu_1 = \nu_2$, and $\psi \in \widetilde{H}_{n,p}$. By means of such an additional equivalence relation, we obtain a semigroup over the group $\mathcal{S}_d$ taking the quotient

[1] Note that morphisms are not supposed to respect skeletons.

[2] Note that $H(F, q, S^{cut})$ is in a canonical bijection with the set of self-morphisms $(F, q, S^{cdt}) \to (F, q, S^{cdt})$.

$\mathrm{G}\mathbb{S}_{\mathrm{d},\tilde{\mathcal{H}}-\mathrm{equiv}} = \mathrm{GV}\mathbb{S}_{\mathrm{d}}/\{\mathrm{f}_1 \overset{\tilde{\mathcal{H}}}{\sim} \mathrm{f}_2\}$; we call the semigroups thus obtained *admissible versal geometric degree d covering semigroups*. In particular, if $\widetilde{H}_{n,p} = Homeo_{n,p}$ for each n and p, the semigroup $\mathrm{G}\mathbb{S}_{\mathrm{d},\tilde{\mathcal{H}}-\mathrm{equiv}}$ is called the *weak versal geometric degree d covering semigroup* and denoted by $GW\mathbb{S}_d$.

Let G be a subgroup of $\mathcal{S}_d$ and $O \subset G$ be some its equipment. A subsemigroup $\mathrm{G}\mathbb{S}_{\mathrm{d},\tilde{\mathcal{H}}-\mathrm{equiv}}(G,O)$ of the semigroup $\mathrm{G}\mathbb{S}_{\mathrm{d},\tilde{\mathcal{H}}-\mathrm{equiv}}$ generated by X_g, $g \in O$, and $Y_{a,b}$, $a, b \in G$, is a semigroup over G. Its elements are $\mathcal{H}$-equivalence classes of degree d coverings with local monodromies in O and Galois groups contained in G, and we call this semigroup an *admissible geometric degree d covering semigroup with local monodromies in O and Galois group in G*.

The following statement is straightforward.

Claim 18. *Any admissible geometric degree d covering semigroup* $\mathrm{G}\mathbb{S}_{\mathrm{d},\tilde{\mathcal{H}}-\mathrm{equiv}}(G,O)$ *is isomorphic over G to the admissible algebraic covering semigroup* $\mathbb{S}_{\mathcal{H}-\mathrm{equiv}}(G,O)$*, where* $\mathcal{H} = \{H_{n,p}\}_{n\geqslant 0,p\geqslant 0}$ *is the collection of subgroups representing* $\tilde{\mathcal{H}} = \{\widetilde{H}_{n,p}\}_{n\geqslant 0,p\geqslant 0}$ *in* $Aut(\mathbb{F}^{n+2p})$*.*

2.7. **Construction of Hurwitz spaces of marked coverings.** Here, we adapt Fulton's construction of Hurwitz spaces, see [2], to the case of marked coverings.

Let $D \subset \overline{F}$ be an open disc in a projective irreducible non-singular algebraic curve $\overline{F}$. We put $F = \overline{F} \setminus D$, choose a point $q \in \partial F$, and fix an n-point set $B \subset F \setminus \partial F$.

Let us recall that for any surface F its braid group on n strands, $Br_n(F, \partial F)$, can be seen as fundamental group $\pi_1(F^{(n)} \setminus \Delta)$, where $F^{(n)}$ is the symmetric product of n copies of F and Δ is the discriminant locus.

Due to our assumptions, the fundamental group $\pi_1(F \setminus B, q)$ is isomorphic to the free group $\mathbb{F}^{n+2p}$ where p is the genus of F, and in such a way the braid group $Br_n(F, \partial F)$, which acts naturally (the right action) on $\pi_1(F \setminus B, q) \simeq \mathbb{F}^{n+2p}$, becomes anti-isomorphic to the algebraic braid group $Br_{n,p}$ introduced in Subsection 1.4 (this is usually called *Artin presentation theorem* and follows from a comparison of the actions of the generators of these groups on $\mathbb{F}^{n+2p} \simeq \pi_1(F \setminus B, q)$).

To detail these identifications, let us pick a caudate skeleton S^{cdt} of F the set of ends of whose tails is B. In notation of subsection 2.2, the choice of S^{cdt} defines the set $\{\lambda_1, \mu_1, \dots, \lambda_p, \mu_p\}$ of free generators of the group $\pi_1(F, q)$ and loops γ_i around the points of B, so that $\gamma_1, \dots, \gamma_n, \lambda_1, \mu_1, \dots, \lambda_p, \mu_p$ are free generators of $\pi_1(F \setminus B, q) \simeq \mathbb{F}^{n+2p}$ and $\gamma_1 \dots \gamma_n [\lambda_1, \mu_1] \dots [\lambda_p, \mu_p] = \partial F$ in $\pi_1(F \setminus B, q)$ (as usual ∂F is taken counter-clockwise).

The set $\{\gamma_1, \dots, \gamma_n, \lambda_1, \mu_1, \dots, \lambda_p, \mu_p\}$ is a frame of the free group $\pi_1(F \setminus B, q) \simeq \mathbb{F}^{n+2p}$, which in accordance with notation of Subsection 1.4 we denote by $\boldsymbol{\mathcal{B}_1}$. As to the standard generators $\sigma_1, \dots, \sigma_{n-1}, \xi_{1,\lambda}, \dots, \xi_{p,\lambda}, \xi_{1,\mu}, \dots, \xi_{p,\mu}, \zeta_1, \dots, \zeta_p$ of the algebraic braid group $Br_{n,p}$, they turn in terms of the geometric braid group $Br_n(F, \partial F)$ into H-moves, λ- and μ-moves, and braiding of a point around a handle (see Subsection 2.5).

This anti-isomorphism $Br_n(F,\partial F)\to Br_{n,p}$ defines a right action of $Br_n(F,\partial F)$ on the set of words $W_{\mathcal{B}_1}(G,O)=\bigcup_f W_{f,\mathcal{B}_1}$, where the union is taken over all equipped homomorphisms $f:\mathbb{F}^{n+2p}\to(G,O)$ to an equipped group (G,O). Obviously, the subset $W^G_{\mathcal{B}_1,\mathbf{1}}(G,O)$ of words in $W_{\mathcal{B}_1}(G,O)$ representing the elements of the semigroup $\mathbb{S}(G,O)^G_{\mathbf{1}}$ is invariant under the action of $Br_n(F,\partial F)$. Therefore this action defines homomorphisms $\omega=\omega_{n,p,(G,O)}:\pi_1(F^{(n)}\setminus\Delta)=Br_n(F,\partial F)\to\mathcal{S}_{|W_{\mathcal{B}_1}(G,O)|}$ and $\omega^G_{\mathbf{1}}:\pi_1(F^{(n)}\setminus\Delta)=Br_n(F,\partial F)\to\mathcal{S}_{|W^G_{\mathcal{B}_1,\mathbf{1}}(G,O)|}$ to the symmetric groups $\mathcal{S}_{|W_{\mathcal{B}_1}(G,O)|}$ and $\mathcal{S}_{|W^G_{\mathcal{B}_1,\mathbf{1}}(G,O)|}$ acting, respectively, on the sets $W_{\mathcal{B}_1}(G,O)$ and $W^G_{\mathcal{B}_1,\mathbf{1}}(G,O)$.

Put $F_0=F\setminus\partial F$. As it follows from Proposition 6, the homomorphisms ω and $\omega^G_{\mathbf{1}}$ define a $|W_{\mathcal{B}_1}(G,O)|$-sheeted unramified covering

$$\theta_n=\theta_n(G,O):\widetilde{\mathrm{HUR}}_{(G,O),n}(F)\to F_0^{(n)}\setminus\Delta$$

and a $|W^G_{\mathcal{B}_1,\mathbf{1}}(G,O)|$-sheeted unramified covering

$$\theta^G_n=\theta_n(G,O)^G_{\mathbf{1}}:\mathrm{HUR}_{(G,O),n}(F)\to F_0^{(n)}\setminus\Delta,$$

respectively. Furthermore, there is a canonical embedding $j:\mathrm{HUR}_{(G,O),n}(F)\hookrightarrow\widetilde{\mathrm{HUR}}_{(G,O),n}(F)$ such that $\theta^G_n=\theta_n\circ j$. Moreover, the both coverings are marked (by words of $W_{\mathcal{B}_1}(G,O)$ and $W^G_{\mathcal{B}_1,\mathbf{1}}(G,O)$, respectively) over the point in $F_0^{(n)}\setminus\Delta$ represented by B, and j respects the markings.

According to the usual construction of covering spaces by means of the groupoid of homotopy classes of paths, the covering space $\widetilde{\mathrm{HUR}}_{(G,O),n}(F)$ as a set is the set of pairs (B',f'), where $B'\in F_0^{(n)}\setminus\Delta$ and $f':\pi_1(F\setminus B',q)\to G$ are epimorphisms such that the conjugacy classes of their values at the loops around the points of B' belong to O and $f'(\partial F)=\mathbf{1}$. We call $\widetilde{\mathrm{HUR}}_{(G,O),n}(F)$ the *Hurwitz space of marked n-branched coverings of F with equipped Galois group* (G,O). This construction being functorial, a choice of an embedding $i:G\hookrightarrow\mathcal{S}_d$ provides an embedding of $\widetilde{\mathrm{HUR}}_{(G,O),n}(F)$ into $\widetilde{\mathrm{HUR}}_{d,n}=\widetilde{\mathrm{HUR}}_{(\mathcal{S}_d,\mathcal{S}_d\setminus\{\mathbf{1}\}),n}(F)$, which we call the *Hurwitz space of marked n-branched degree d coverings of F.*

The advantage of considering marked coverings is that the Hurwitz spaces $\widetilde{\mathrm{HUR}}_{d,n}$ come then with a universal family of coverings, $\mathcal{F}_{d,n}\to\widetilde{\mathrm{HUR}}_{d,n}$. Such a family can be obtained as manifold completion (see Subsection 2.1) of the unramified covering of $U=\{(p,B'):p\notin B'\}\subset F\times(F_0^{(n)}\setminus\Delta)$, that is the covering defined by the homomorphism $\pi_1(U,(q,B))\to\mathcal{S}_{I_d\times W_{\mathcal{B}_1}(G,O)}$, $G=\mathcal{S}_d$, $O=\mathcal{S}_d\setminus\{1\}$ that sends the images of elements $\varsigma\in\pi_1(F\setminus B,q)$ to permutations $(x,w)\mapsto(\alpha(\varsigma)x,w)$ and the images of elements $\chi\in\pi_1(F^{(n)}\setminus\Delta)$ to permutations $(k,w)\mapsto(k,\omega(\chi)w)$.

Claim 19. *The connected components of the spaces $\widetilde{HUR}_{(G,O),n}(F)$ and $HUR_{(G,O),n}(F)$ are in one-to-one correspondence with the elements of the strong covering semigroup $\mathbb{S}(G,O)$ and its subsemigroup $\mathbb{S}(G,O)^G_{\mathbf{1}}$, respectively.*

Proof. By Proposition 6 and due to existence of universal families, the connected components of $\widetilde{\mathrm{HUR}}_{(G,O),n}(F)$ (respectively, $\mathrm{HUR}_{(G,O),n}(F)$) are in one-to-one correspondence with the orbits of the action of $Br_n(F)$ on the set $W_{\mathcal{B}_\mathbf{1}}(G,O)$ (respectively, $W^G_{\mathcal{B}_\mathbf{1},\mathbf{1}}(G,O)$). Via the anti-isomorphism $Br_n(F)\to Br_{n,p}$ and due to the definition of the strong covering semigroups, these orbits coincide with the elements of $\mathbb{S}(G,O)$. $\square$

Let $F=\overline{F}\setminus D$ and $F'=\overline{F}\setminus D'$, where $D'\subset D$ are open discs in a closed genus p oriented surface $\overline{F}$ without boundary such that the marked point $q\in\partial D\cap\partial D'$. Then there is a natural embedding $j_{F,F'}:\mathrm{HUR}_{(G,O),n}(F)\hookrightarrow\mathrm{HUR}_{(G,O),n}(F')$ which is compatible with the covering maps and the embedding $j_{F,F'}:F_0^{(n)}\setminus\Delta\hookrightarrow F_0'^{(n)}\setminus\Delta$. Let $i:G\hookrightarrow\mathcal{S}_d$ be an embedding of a group G such that its image acts transitively on I_d. If a word w_{f_*} represents an element of $\mathbb{S}(G,O)^G_\mathbf{1}$, then, first, the covering space E of the d sheeted marked covering $f:E\to F$ is connected, and, second, the covering f can be extended uniquely to a d sheeted marked (at $q\in\overline{F}$) covering $f:\overline{E}\to\overline{F}$ unbranched at the points of D. The embeddings $\mathrm{HUR}_{(G,O),n}(\overline{F}\setminus D_i)\hookrightarrow\mathrm{HUR}_{(G,O),n}(\overline{F}\setminus D_{i+1})$ corresponding to an infinite sequence of open discs

$$\dots\subset D_{i+1}\subset D_i\subset\dots\subset D_1$$

such that $\cap_{i=1}^{\infty}D_i=\emptyset$ and $q\in\partial D_i$ for all i define an unramified covering $\theta_n(G,O)^G_\mathbf{1}:\mathrm{HUR}_{(G,O),n}(\overline{F},q)\to(\overline{F}\setminus q)^{(n)}\setminus\Delta$ the covering space of which is called the *Hurwitz space* of marked (at a point $q\in\overline{F}$) coverings of a projective algebraic curve $\overline{F}$ with equipped Galois group (G,O) and branched at n points.

2.8. **Proof of Theorems 1, 2, and 3.** Due to Proposition 12: Theorem 1 follows from Corollary 2; Theorem 2 follows from Claim 19, Theorems 6, and 7; and Theorem 3 follows from Claim 19 and Corollary 1.

References

[1] I. Berstein and A.L. Edmonds: *On the classification of generic branched coverings of surfaces.* Illinois J. Math. Volume 28, Issue 1 (1984), 64 – 82.

[2] W. Fulton: *Hurwitz schemes and irreducibility of moduli of algebraic curves.* Ann. of Math., **90:3** (1969), 542 – 575.

[3] A. Clebsch: *Zür Theorie der Riemann'schen Fläche.* Math. Ann., 6 (1872), 216 – 230.

[4] E. Fadell , L. P. Neuwirth:*Configuration spaces,* Math. Scand. 10 (1962), 111 – 118.

[5] R. H. Fox: *Covering Spaces with Singularities.* In Algebraic Geometry and Topology. A Symposium in honor of S. Lefschetz. Editors: Fox et al. Princeton Univ. Press (1957) pp.243 – 257.

[6] R. H. Fox, L. P. Neuwirth:*The braid groups,* Math. Scand. 10 (1962), 119 – 126.

[7] M.D. Fried and H. Völklein: *The inverse Galois problem and rational points on moduli space.* Math. Ann., 290, (1991), 771 – 800.

[8] T. Graber, J. Harris, J. Starr: *A note on Hurwitz schemes of covers of a positive genus curve.* arXiv:math/0205056

[9] A. Hurwitz: *Ueber Riemann'she Flächen mit gegebenen Verweigugspunkten.* Math. Ann., 39, (1981), 1 – 61.

[10] V. Kanev: *Irreducibility of Hurwitz spaces.* arXiv: math/0509154v1 [math.AG] 7 Sep 2005.
[11] Vik.S. Kulikov: *Hurwitz curves.* UMN **62:6** (2007), 3 – 86.
[12] Vik.S. Kulikov: *Factorization semigroups and irreducible components of Hurwitz space,* Izv. Math. **75:4** (2011), 711 – 748.
[13] Vik.S. Kulikov: *Factorization semigroups and irreducible components of Hurwitz space. II,* (accepted in Izv. Math.; primiry version is in arXiv:1011.3619).
[14] Vik.S. Kulikov: *Factorizations in finite groups,* arXiv:1105.1939 (submitted in Sb. Math.).
[15] V. Kharlamov and Vik.S. Kulikov: *On braid monodromy factorizations.* Izv. Math. 67:3 (2003), 499 – 534.
[16] A. N. Protopopov: *Topological classification of branched coverings of the two-dimensional sphere.* Journal of Mathematical Sciences. 52:1 (2003), 2832–2846.
[17] F. Vetro: *Irreducibility of Hurwitz spaces for coverings with one special fibre.* Indag. Math. (N.S.), vol. 17 (2006), no. 1, 115 – 127.
[18] B. Wajnryb: *Orbits of Hurwitz action for coverings of a sphere with two special fibres.* Indag. Math. (N.S.), vol. 7 (1996), no. 4, 549 – 558.

Steklov Mathematical Institute
E-mail address: kulikov@mi.ras.ru

UFR de Mathématiques et IRMA, Université de Strasbourg et CNRS
E-mail address: kharlam@math.unistra.fr