ON INFINITE DIMENSIONAL ALGEBRAIC TRANSFORMATION GROUPS

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Abstract. We explore orbits, rational invariant functions, and quotients of the natural actions of connected, not necessarily finite dimensional subgroups of the automorphism groups of irreducible algebraic varieties. The applications of the results obtained are given.

1. Introduction. The following well-known result (see, e.g., [Bor91, Prop. I.2.2]) is one of the indispensable tools in the theory of algebraic groups:

Theorem. *Let $\varphi_i\colon T_i \to G$ $(i \in \mathcal{I})$ be a collection of morphisms from irreducible algebraic varieties T_i into an algebraic group G, and assume that the identity element of G lies in $X_i := \varphi_i(T_i)$ for each $i \in \mathcal{I}$. Then the subgroup A of G generated, as an abstract group, by the set $M := \bigcup_{i\in\mathcal{I}} X_i$ coincides with the intersection of all closed subgroups of G containing M. Moreover, A is connected and there is a finite sequence $\alpha_1, \ldots, \alpha_n$ in $\mathcal{I}$ such that $A = X_{\alpha_1}^{e_1} \cdots X_{\alpha_n}^{e_n}$, where $e_i = \pm 1$ for each i.*

Here we show that the analogous construction, applied in place of G to $\mathrm{Aut}(X)$, where X is an irreducible algebraic variety, yields a group, though not in general algebraic, but whose natural action on X surprisingly retains some basic properties of orbits and invariant fields of algebraic group actions. This leads to some applications. The main results are formulated in Section 3.

In what follows, variety means algebraic variety in the sense of Serre over an algebraically closed field k of arbitrary characteristic (so algebraic group means algebraic group over k). The standard notation and conventions of [Bo91] and [PV94] are used freely. Given a rational function $f \in k(X)$ and an element $\sigma \in \mathrm{Aut}(X)$, we denote by f^σ the rational function on X defined by $f^\sigma(\sigma(x)) = f(x)$ for every point x in the domain of definition of f.

2. Definitions and notation. Let T be an irreducible variety. Any map

$$\varphi\colon T \to \mathrm{Aut}(X), \quad t \mapsto \varphi_t,$$

determines a *family* $\{\varphi_t\}_{t\in T}$ in $\mathrm{Aut}(X)$ parameterized by T. We put

$$\varphi_T := \varphi(T)$$

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If $\mathcal{I}$ is a nonempty collection of families in $\mathrm{Aut}(X)$, then the subgroup of $\mathrm{Aut}(X)$ generated, as an abstract group, by the set $\bigcup \varphi_T$ with the union taken over all families $\{\varphi_t\}_{t\in T}$ in $\mathcal{I}$ will be called the *group generated by* $\mathcal{I}$.

We shall say that a family $\{\varphi_t\}_{t\in T}$ in $\mathrm{Aut}(X)$ is

— *injective* (see [Ram64]) if $\varphi_t \neq \varphi_s$ for all $t \neq s$;
— *unital* if $\mathrm{id}_X \in \varphi_T$;
— *algebraic* (see [Ram64]) if

$$\widetilde{\varphi}\colon T \times X \to X, \quad (t, x) \mapsto \varphi_t(x) \tag{1}$$

is a morphism.

Given a family $\{\varphi_t\}_{t\in T}$ in $\mathrm{Aut}(X)$, the family $\{\varphi_t^{-1}\}_{t\in T}$ in $\mathrm{Aut}(X)$ will be called the *inverse* of $\{\varphi_t\}_{t\in T}$. If $\{\varphi_t\}_{t\in T}, \ldots, \{\psi_s\}_{s\in S}$ is a finite sequence of families in $\mathrm{Aut}(X)$, the family

$$\{\varphi_t \circ \cdots \circ \psi_s\}_{(t,\ldots,s)\in T\times\cdots\times S} \tag{2}$$

in X will be called the *product* of $\{\varphi_t\}_{t\in T}, \ldots, \{\psi_s\}_{s\in S}$. The inverses and products of families contained in a subgroup G of $\mathrm{Aut}(X)$ are contained in G as well. The inverses and products of algebraic (resp., unital) families are algebraic, see [Ram64] (resp., unital).

Let $\mathcal{I}$ be a collection of families in $\mathrm{Aut}(X)$. We shall say that a *family* $\{\varphi_t\}_{t\in T}$ *in* $\mathrm{Aut}(X)$ *is derived from* $\mathcal{I}$ if $\{\varphi_t\}_{t\in T}$ is a product of families each of which is either a family from $\mathcal{I}$ or the inverse of such a family.

A subgroup G of $\mathrm{Aut}(X)$ is called (see [Ram64]) a *finite dimensional subgroup* if there is an integer n such that $\dim T \leqslant n$ for every injective algebraic family $\{\varphi_t\}_{t\in T}$ in this subgroup; the smallest n satisfying this property is called the *dimension of* G. If G is not finite dimensional, it is called an *infinite dimensional subgroup* of $\mathrm{Aut}(X)$.

If for every element $g \in G$ there exists a unital algebraic family $\{\varphi_t\}_{t\in T}$ in G such that $g \in \varphi_T$, then G is called (see [Ram64]) a *connected subgroup of* $\mathrm{Aut}(X)$.

If $\{\varphi_t\}_{t\in T}$ is an algebraic family such that T is a connected algebraic group and $\widetilde{\varphi}$ (given by (1)) is an action of T on X, then φ_T is a connected finite dimensional subgroup of $\mathrm{Aut}(X)$. By [Ram64, Thm.], every connected finite dimensional subgroup of $\mathrm{Aut}(X)$ is obtained in this way. Such subgroups are called *connected algebraic subgroups of* $\mathrm{Aut}(X)$.

Given a nonempty subset S of $\mathrm{Aut}(X)$, we put

$$S(x) := \{g(x) \mid g \in S\}.$$

Given a subgroup G of $\mathrm{Aut}(X)$ and a G-invariant subset Y of X, we shall say that a family $\{\varphi_t\}_{t\in T}$ in G is an *exhaustive family for the natural action of* G *on* Y if $G(y) = \varphi_T(y)$ for every point $y \in Y$.

3. Main results. In Theorems 1, 2, 3, 4 and Corollaries 1, 2 below we do *not* assume finite dimensionality of G. If G is finite dimensional, then the statement of Theorems 1 and 2 become trivial and that of Theorems 3, 4 and Corollaries 1, 2 turn into the well-known classical results of the algebraic transformation group theory (see, e.g., [PV94, Sect. 1.4, 2.3]); in particular, Theorem 4 becomes classical Rosenlicht's theorem [Ros56].

Theorem 1. *Let X be an irreducible variety and let G be a subgroup of* $\mathrm{Aut}(X)$*. Then the following properties are equivalent*:

(i) *G is a connected subgroup of* $\mathrm{Aut}(X)$;
(ii) *G is generated by a collection $\mathcal{I}$ of unital algebraic families in* $\mathrm{Aut}(X)$*.*

Theorem 2. *Let X be an irreducible variety and let G be a subgroup of* $\mathrm{Aut}(X)$ *generated by a collection $\mathcal{I}$ of unital algebraic families in* $\mathrm{Aut}(X)$*. Let Y be a G-invariant locally closed subvariety of X. Then there is a family derived from $\mathcal{I}$ and exhaustive for the natural action of G on Y.*

Theorem 3. *Let X be an irreducible variety and let G be a connected subgroup of* $\mathrm{Aut}(X)$*. Let Y be an irreducible G-invariant locally closed subvariety of X. Then there exists an integer $m_{G,Y}$ and a dense open subset U of Y such that* $\dim G(y) = m_{G,Y}$ *for every point $y \in U$.*

Theorem 4. *Let X be an irreducible variety and let G be a connected subgroup of* $\mathrm{Aut}(X)$*. Let Y be an irreducible G-invariant locally closed subvariety of X. Then for some G-invariant dense open subset U of Y there exists a geometric quotient, i.e., there are an irreducible variety W and a morphism $\rho\colon U \to W$ such that*

(i) *ρ is surjective, open, and the fibers of ρ are the G-orbits in U;*
(ii) *if V is an open subset of U, then*

$$\rho^*\colon k[\rho(V)] \to \{f \in k[V] \mid f \text{ is constant on the fibers of } \rho|_V\}$$

is an isomorphism of k-algebras.

Corollary 1. *Let X be an irreducible variety and let G be a connected subgroup of* $\mathrm{Aut}(X)$*. Let Y be an irreducible G-invariant locally closed subvariety of X. Then there exists a finite subset of $k(Y)^G$ that separates G-orbits of points of a dense open subset of Y.*

Corollary 2. *Let X be an irreducible variety and let G be a connected subgroup of* $\mathrm{Aut}(X)$*. Let Y be an irreducible G-invariant locally closed subvariety of X. Then the transcendence degree of the field $k(Y)^G$ over k is equal to* $\dim X - m_{G,Y}$ *(see Theorem* 3*). In particular, $k(Y)^G = k$ if and only if there is an open G-orbit in Y.*

Here are some applications of these results.

Theorem 5. *Let X be a nonunirational irreducible variety. Then there exists a nonconstant rational function on X which is G-invariant for every connected affine algebraic subgroup G of* $\mathrm{Aut}(X)$*.*

Theorem 5 shows that there is a certain rigidity for the orbits of any connected affine algebraic group G acting regularly on an irreducible nonunirational variety X: every such orbit should lie in a level variety of a certain nonconstant rational function on X not depending on G and its action on X.

We shall say that $\mathrm{Aut}(X)$ is *generically n-transitive* if there exists a dense open subset V of X such that for every point $x, y \in V^n$ lying off the union of the "diagonals", there exists an element $g \in \mathrm{Aut}(X)$ such that $g(x) = y$ for the diagonal action of $\mathrm{Aut}(X)$ on X^n.

Theorem 6. *Let X be a nonunirational irreducible variety.*

(i) *If the group* $\mathrm{Aut}(X)$ *is generically* 2*-transitive, then* $\mathrm{Aut}(X)$ *contains no nontrivial connected affine algebraic subgroups.*

(ii) *If, moreover, there is no dominant morphism $Z \to X$, where Z is an abelian variety, then* $\mathrm{Aut}(X)$ *contains no nontrivial connected algebraic subgroups.*

The other applications are discussed in Section 9.

4. Proof of Theorem 1. (i)⇒(ii): For every element $g \in G$, fix a unital algebraic family $\{\varphi_t\}_{t\in T}$ in G such that $g \in \varphi_T$; connectedness of G implies that such a family exists. Then G is generated, as an abstract group, by $\bigcup \varphi_T$ with the union taken over all the fixed families.

(ii)⇒(i): Since the inverse of any family in G is also a family in G, we may (and shall) assume that if a family belongs to $\mathcal{I}$, then its inverse belongs to $\mathcal{I}$ too. Then for every element $g \in G$, there exists a finite sequence of families $\{\varphi_t\}_{t\in T}, \ldots, \{\psi_s\}_{s\in S}$ from $\mathcal{I}$ such that $g = \varphi_{t_0} \circ \cdots \circ \psi_{s_0}$ for some $t_0 \in T, \ldots, s_0 \in S$. Hence g is contained in the product of families $\{\varphi_t\}_{t\in T}, \ldots, \{\psi_s\}_{s\in S}$ defined by (2). Therefore, G is connected. □

5. Algebraic families. This section contains several general facts utilized in the proofs of Theorems 2, 3, and 4.

Lemma 1. *Let X be an irreducible variety, let G be a connected subgroup of* $\mathrm{Aut}(X)$*, and let Y be a G-invariant locally closed subvariety of X.*

(i) *Every product of unital families in* $\mathrm{Aut}(X)$ *contains each of them.*

(ii) *If a family $\{\varphi_t\}_{t\in T}$ in G is exhaustive for the natural action of G on Y, then every family $\{\psi_s\}_{s\in S}$ in G such that $\varphi_T \subseteq \psi_S$ is also exhaustive for this action.*

(iii) *If G is generated by a collection $\mathcal{I}$ of unital algebraic families, then G is the union of all families derived from $\mathcal{I}$.*

(iv) $G|_Y := \{g|_Y \mid g \in G\}$ *is a connected subgroup of* $\mathrm{Aut}(Y)$*.*

(v) *If $\mathcal{F}$ is a finite set of algebraic families in G, then G contains a unital algebraic family $\{\varphi_t\}_{t\in T}$ such that $\varphi_T \supseteq \psi_S$ for every $\{\psi_s\}_{s\in S}$ in $\mathcal{F}$.*

Proof. (i) and (ii): This is immediate from the definitions.

(iii): The proof is similar to that of implication (ii)⇒(i) of Theorem 1.

(iv): If $\{\varphi_t\}_{t\in T}$ is a unital algebraic family in G containing an element $g \in G$, then $\{\varphi_t|_Y\}_T$ is a unital algebraic family in $G|_Y$ containing the element $g|_Y \in G|_Y$. Whence the claim.

(v): Due to (i), the proof is reduced to the case where $\mathcal{F}$ consists of a single family $\{\psi_s\}_{s\in S}$. In this case, take an element $g \in \psi_S$. Since G is connected, it contains a unital algebraic family $\{\mu_r\}_{r\in R}$ such that $g^{-1} \in \mu_R$. The product of $\{\psi_s\}_{s\in S}$ and $\{\mu_r\}_{r\in R}$ is then the sought-for family $\{\varphi_t\}_{t\in T}$. □

Lemma 2. *Let X be an irreducible variety and let G be a connected subgroup of* $\mathrm{Aut}(X)$*. Let Y be a G-invariant locally closed subvariety of X and let $Y_1, \ldots, Y_n$ be all the irreducible components of Y. Then every Y_i is G-invariant.*

Proof. Let $\{\varphi_t\}_{t\in T}$ be a unital algebraic family in G. For every $t \in T$, since $\varphi_t \in \mathrm{Aut}(X)$ and Y is φ_t-invariant, φ_t permutes $Y_1, \ldots, Y_n$. Put

$$T_{ij} := \{t \in T \mid \varphi_t(Y_i) = Y_j\}.$$

For every point $x \in Y_i$ consider the morphism

$$\widetilde{\varphi}_x \colon T \to X, \quad t \mapsto \widetilde{\varphi}(t, x) = \varphi_t(x) \tag{3}$$

(see (1)). Then, for every Y_j,

$$T_{ij} = \bigcap\nolimits_{x\in Y_i} \widetilde{\varphi}_x^{-1}(Y_j). \tag{4}$$

Since Y_j is closed, (4) implies closedness of T_{ij} in T. Unitality of φ_t implies $T_{ii} \neq \varnothing$. From $T = \bigsqcup_{j=1}^n T_{ij}$ and irreducibility of T we then infer that $T = T_{ii}$ for every i. Thus, Y_i is φ_t-invariant for every i and t. Theorem 1 then completes the proof. □

Lemma 3. *Let X be an irreducible variety and let G be a connected subgroup of* $\mathrm{Aut}(X)$. *If $\{\varphi_t\}_{t\in T}$ is an algebraic family in G, and x is a point of X, then*

(i) *$G(x)$ is an irreducible locally closed nonsingular subvariety of X;*
(ii) *$\varphi_T(x)$ is a constructible subset of $G(x)$.*

Proof. (i): This is proved in [Ram64, Lemma 2].

(ii): This follows from the definition of algebraic family and Chevalley's theorem on the image of morphism. □

Corollary 3. *Let X be an irreducible variety and let G be a connected subgroup of* $\mathrm{Aut}(X)$. *Then $k(X)^G$ is algebraically closed in $k(X)$.*

Proof. Let $f \in k(X)$ be the root of $t^n + f_1 t^{n-1} + \cdots + f_n \in k(X)^G[t]$ and let $a \in X$ be a point where f and every f_i are defined. Then $f(G(a))$ is a finite subset of k since it lies in the set of roots of $t^n + f_1(a)t^{n-1} + \cdots + f_n(a) \in k[t]$. Irreducibility of $G(a)$ then implies that this subset is a single element of k, i.e., $f|_{G(a)}$ is a constant. This means that $f \in k(X)^G$. □

Lemma 4. *Let X be an irreducible variety and let G be a connected subgroup of* $\mathrm{Aut}(X)$. *Let Y be a G-invariant locally closed subvariety of X. Let $\{\varphi_t\}_{t\in T}$ be a unital algebraic family in G such that $\varphi_T(y)$ is dense in $G(y)$ for every point $y \in Y$. Then the product of the inverse of $\{\varphi_t\}_{t\in T}$ and $\{\varphi_t\}_{t\in T}$ is the exhaustive algebraic family $\{\psi_s\}_{s\in S}$ for the natural action of G on Y.*

Proof. By the definition of $\{\psi_s\}_{s\in S}$,

$$\psi_s = \varphi_{t_1}^{-1} \circ \varphi_{t_2} \quad \text{for} \quad s = (t_1, t_2) \in S = T \times T. \tag{5}$$

Take any points $y_1, y_2 \in Y$ such that $G(y_1) = G(y_2)$. The density assumption then yields the equality $\overline{\varphi_T(y_1)} = \overline{\varphi_T(y_2)}$, where bar stands for the closure in X. By Lemma 3, this implies

$$\varphi_T(y_1) \cap \varphi_T(y_2) \neq \varnothing;$$

whence, $\varphi_{t_1}(y_2) = \varphi_{t_2}(y_1)$ for some $t_1, t_2 \in T$. Therefore, $\psi_s(y_1) = y_2$ for ψ_s defined by (5). Hence $\psi_S(y_1) = G(y_1)$ for every point $y_1 \in Y$. □

6. Proof of Theorem 2. First, we shall show that it suffices to prove the following "generic" version of Theorem 2:

Theorem 2*. *Let X, G, $\mathcal{I}$, and Y are the same as in Theorem 2 and let Y be irreducible. Then there exist a dense open G-invariant subset U in Y and a unital algebraic family $\{\varphi_t\}_{t\in T}$ in G such that*

(i) *$\{\varphi_t\}_{t\in T}$ is derived from $\mathcal{I}$;*
(ii) *$\varphi_T(y)$ is dense in $G(y)$ for every point $y \in U$.*

Indeed, assuming that Theorem 2^* is proved, we can complete the proof of Theorem 2 as follows.

The group G is connected by Theorem 1. Therefore, every irreducible component of Y is G-invariant by Lemma 2. From this and Lemma 1(i),(ii) we infer that it is sufficient to prove Theorem 2 for irreducible Y. In this case we argue by induction on $\dim Y$.

Namely, the case $\dim Y = 0$ is clear. Assume that the claim of Theorem 2 holds for irreducible G-invariant subvarieties in X of dimension $< \dim Y$ and consider the set U from Theorem 2^*. Let $Z_1, \ldots, Z_n$ be all the irreducible components of the variety $Y \setminus U$. By Lemma 2, every Z_i is G-invariant. Since $\dim Z_i < \dim Y$, the inductive assumption implies for every $i = 1, \ldots, n$ the existence of a unital algebraic family $\{\psi^{(i)}_{s_i}\}_{s_i\in S_i}$ in G such that

(a) $\{\psi^{(i)}_{s_i}\}_{s_i\in S_i}$ is derived from $\mathcal{I}$;
(b) $\{\psi^{(i)}_{s_i}\}_{s_i\in S_i}$ is exhaustive for the natural action of G on Z_i.

On the other hand, Theorem 2^* and Lemma 4 imply the existence of a unital algebraic family $\{\lambda_r\}_{r\in R}$ in G such that

(c) $\{\lambda_r\}_{r\in R}$ is derived from $\mathcal{I}$;
(d) $\{\lambda_r\}_{r\in R}$ is exhaustive for the natural action of G on U.

The claim of Theorem 2 now follows from (a), (b), (c), (d) and Lemma 1(i),(ii). This completes the proof of Theorem 2 assuming that Theorem 2^* is proved. $\square$

We now turn to the proof of Theorem 2^*. Consider the map

$$\tau_Y\colon G \times Y \to Y \times Y, \quad (g, y) \mapsto (g(y), y). \tag{6}$$

Its image Γ_Y is the graph of the natural action of G on Y:

$$\Gamma_Y = \{(y_1, y_2) \in Y \times Y \mid G(y_1) = G(y_2)\}. \tag{7}$$

Claim 1. *Maintain the above notation.*

(i) *There exists a family $\{\varphi_t\}_{t\in T}$ derived from $\mathcal{I}$ such that $\tau_Y(\varphi_T \times Y)$ contains a dense open subset V of $\overline{\Gamma}_Y$, where bar stands for the closure in $Y \times Y$.*
(ii) *$\overline{\Gamma}_Y$ is irreducible.*

Proof of Claim 1. If $\{\psi_s\}_{s\in S}$ is an algebraic family in G, then the subset $\tau_Y(\psi_S \times Y)$ of Γ_Y is the image of the morphism

$$S \times Y \to Y \times Y, \quad (s, y) \mapsto (\psi_s(y), y)$$

of irreducible varieties (see (1)). Chevalley's theorem on the image of morphism then implies that $\overline{\tau_Y(\psi_S \times Y)}$ is an irreducible subvariety of $\overline{\Gamma}_Y$ and $\tau_Y(\psi_S \times Y)$ contains a dense open subset of $\overline{\tau_Y(\psi_S \times Y)}$.

From $\dim \overline{\Gamma}_Y \geqslant \dim \overline{\tau_Y(\psi_S \times Y)}$ we conclude that there exists a family $\{\varphi_t\}_{t\in T}$ derived from $\mathcal{I}$ on which the maximum of $\dim \overline{\tau_Y(\psi_S \times Y)}$ is attained when $\{\psi_s\}_{s\in S}$ runs over all families derived from $\mathcal{I}$. If $\{\psi_s\}_{s\in S}$ is a family derived from $\mathcal{I}$ such that $\varphi_T \subseteq \psi_S$, then the maximality condition and irreducibility of $\overline{\tau_Y(\psi_S \times Y)}$ imply that

$$\overline{\tau_Y(\psi_S \times Y)} = \overline{\tau_Y(\varphi_T \times Y)}. \tag{8}$$

Take an element $g \in G$. By Lemma 1(iii),(i), there is an algebraic family $\{\psi_s\}_{s\in S}$ in G such that $\varphi_T \subseteq \psi_S$ and $g \in \psi_S$. From (8) and (6) we then conclude that $\Gamma_Y \subseteq \overline{\tau_Y(\varphi_T \times Y)}$. Since $\overline{\tau_Y(\varphi_T \times Y)} \subseteq \overline{\Gamma}$, we get $\overline{\tau_Y(\varphi_T \times Y)} = \overline{\Gamma}_Y$. This completes the proof. $\square$

Endow $X \times X$ with the action of G via the second factor:

$$g \cdot (x_1, x_2) := (x_1, g(x_2)), \quad x_i \in X, g \in G. \tag{9}$$

The second projection $X \times X \to X$, $(x_1, x_2) \mapsto x_2$ is then G-equivariant and, by (7), Γ_Y and $\overline{\Gamma}_Y$ are G-invariant.

Claim 2. *$\{\varphi_t\}_{t\in T}$ and V in Claim 1 can be chosen so that V is G-invariant.*

Proof of Claim 2. Maintain the notation of Claim 1 and consider in $\overline{\Gamma}_Y$ the G-invariant dense open subset

$$V_0 := \bigcup\nolimits_{g\in G} g \cdot V. \tag{10}$$

Since V_0 is quasi-compact, its covering (10) by open subsets $g \cdot V$, $g \in G$, contains a finite subcovering:

$$V_0 = \bigcup\nolimits_i^n g_i \cdot V \quad \text{for some elements } g_1, \ldots, g_n \in G. \tag{11}$$

By Lemma 1(iii), every g_i is contained in a family derived from $\mathcal{I}$. Taking a product of $\{\varphi_t\}_{t\in T}$ with these families, we obtain a family $\{\psi_s\}_{s\in S}$ derived from $\mathcal{I}$ such that

$$\varphi_T \circ g_i^{-1} \subseteq \psi_S \text{ for every } i = 1, \ldots, n. \tag{12}$$

Since $V \subseteq \tau_Y(\varphi_T \times Y)$, from (6) and (9) we obtain

$$g_i \cdot V \subseteq \{(\varphi_t(y), g_i(y)) \mid t \in T, y \in Y\}. \tag{13}$$

This yields

$$\begin{aligned} \tau_Y(\psi_S \times Y) &= \big\{\big(\psi_s(y), y\big) \mid s \in S, y \in Y\big\} \\ &= \big\{\big(\psi_s(g_i(y)), g_i(y)\big) \mid s \in S, y \in Y\big\} \\ &\supseteq \big\{\big(\varphi_t\big(g_i^{-1}(g_i(y))\big), g_i(y)\big) \mid t \in T, y \in Y\big\} \quad \text{(by (12))} \\ &\supseteq g_i \cdot V \quad \text{(by (13))}. \end{aligned} \tag{14}$$

Thus $V_0 \subseteq \tau_Y(\psi_S \times Y)$ by (11) and (14). So, replacing $\{\varphi_t\}_{t\in T}$ and V by, resp., $\{\psi_s\}_{s\in S}$ and V_0, we may attain that V in Claim 1 is G-invariant. $\square$

To complete the proof of Theorem 2^*, consider the second projection

$$\pi_Y\colon \overline{\Gamma}_Y \to Y, \quad (y_1, y_2) \mapsto y_2; \tag{15}$$

it is a G-equivariant surjective morphism of irreducible varieties. Let $\{\varphi_t\}_{t\in T}$ and V be as in Claim 1 and let V be G-invariant by Claim 2. Since V is a dense open subset of $\overline{\Gamma}_Y$, by Chevalley's theorem on the image of morphism $\pi_Y(V)$ contains a dense open subset of Y. Let U be the union of all dense open subsets of Y lying in $\pi_Y(V)$. Since V is G-invariant and π_Y is G-equivariant, $\pi_Y(V)$ is G-invariant. Therefore, U is also G-invariant.

Take a point $y \in U$. Since $V \subseteq \Gamma_Y$, $\pi_Y^{-1}(y) \cap \Gamma_Y = \{(g(y), y) \mid g \in G\}$, and $V \supseteq \big\{\big(g(y), y\big) \mid g \in \varphi_T\big\}$, we have

$$\begin{aligned} \varnothing \neq V \cap \pi_Y^{-1}(y) = V \cap \Gamma_Y \cap \pi_Y^{-1}(y) &= V \cap \big\{\big(g(y), y\big) \mid g \in G\big\} && (16)\\ &\subseteq \big\{\big(g(y), y\big) \mid g \in \varphi_T\big\}. && (17)\end{aligned}$$

By Lemma 3, $\big\{\big(g(y), y\big) \mid g \in G\big\}$ is an irreducible locally closed subset of $\overline{\Gamma}_Y$. From (16) we then infer that $V \cap \big\{\big(g(y), y\big) \mid g \in G\big\}$ is a dense open subset of $\big\{\big(g(y), y\big) \mid g \in G\big\}$, and from (17) that $\varphi_T(y)$ is dense in $G(y)$. This completes the proof of Theorem 2^* and hence that of Theorem 2. ☐

7. Proof of Theorem 3. Maintain the notation of the proof of Theorem 2. There is shown that the restriction of π_Y to V is a dominant morphism of irreducible varieties $V \to Y$ whose fiber over every point y of a dense open subset U of Y is isomorphic to a dense subvariety of $G(y)$. Hence, the dimension of this fiber is $\dim G(y)$. The claim now follows from the fiber dimension theorem [Gro65, 5.6]. ☐

8. Proof of Theorem 4. By Lemma 1(iv), it suffices to give a proof for $Y = X$. We shall use the idea utilized in [Lun73, 4] for proving the existence of generic stabilizer for reductive group actions on smooth affine varieties. Below is maintained the notation used in the proof of Theorem 2.

Since any subfield of $k(X)$ containing k is finitely generated over k, replacing X by an appropriate invariant dense open subset of X we can (and shall) find an irreducible affine normal variety Z and a surjective morphism

$$\rho\colon X \to Z$$

such that $\rho^*(k(Z)) = k(X)^G$. This equality implies that ρ is a separable morphism, see, e.g., [Bor91, AG, Prop. 2.4].

The construction yields that

(q_1) $G(x) \subseteq \rho^{-1}(\rho(x))$ for every point $x \in X$.

By the fibre dimension theorem and Theorem 3, further replacing X and Z by the appropriate open sets, we can (and shall) attain the following properties:

(q_2) for every point $z \in Z$, the dimension of every irreducible component of $\rho^{-1}(z)$ is equal to $\dim X - \dim Z$;

(q_3) $\dim G(x) = \dim G(x')$ for every points $x, x' \in X$.

Lemma 3(i) and (q_3) imply that $G(x)$ is closed in X for every point $x \in X$.

By Grothendieck's generic freeness lemma [Gro65, 6.9.2], after replacing Z by a principal open subset, we can (and shall) assume that

(q_4) there exists an affine open subset X_0 of X such that $\rho(X_0) = Z$ and $k[X_0]$ is a free $\rho^*(k[Z])$-module.

Below, for any subsets $S \subseteq X$ and $R \subseteq X \times X$, we put

$$S_0 := S \cap X_0, \quad R_0 := R \cap (X_0 \times X_0).$$

Finally, replacing X by the invariant open set $\bigcup_{g \in G} g(X_0)$, we can (and shall) assume that

(q_5) the intersection of X_0 with every G-orbit in X is nonempty.

Consider now in $X \times X$ the G-invariant (with respect to action (9)) closed subset

$$X \times_Z X := \{(x_1, x_2) \in X \times X \mid \rho(x_1) = \rho(x_2)\} \tag{18}$$

and its affine open subset $(X \times_Z X)_0$.

Claim 3. *$(X \times_Z X)_0$ is dense in $X \times_Z X$.*

Proof of Claim 3. Take a point $(x_1, x_2) \in X \times_Z X$. From (18) and ($q_1$) we infer that $G(x_1) \times G(x_2) \subseteq X \times_Z X$, and from ($q_5$) and Lemma 3(i) that $(G(x_1) \times G(x_2))_0$ is a dense open subset of $G(x_1) \times G(x_2)$. Therefore, since $(x_1, x_2) \in G(x_1) \times G(x_2)$, the closure of $(G(x_1) \times G(x_2))_0$ in $X \times_Z X$ contains (x_1, x_2). Whence the claim, because $(G(x_1) \times G(x_2))_0 \subseteq (X \times_Z X)_0$. $\square$

Consider now the set

$$\Gamma := \Gamma_X \tag{19}$$

defined by (7). By (q_1), we have $\Gamma \subseteq X \times_Z X$. Since $X \times_Z X$ is closed in $X \times X$, this yields $\overline{\Gamma} \subseteq X \times_Z X$ (see Claim 1(i)).

Claim 4. *$\overline{\Gamma} = X \times_Z X$.*

First, we shall show how to deduce Theorem 4 from Claim 4.

By (19) and Claims 1(ii), 4, the variety $\overline{\Gamma} = X \times_Z X$ is irreducible. Consider its dense open subset V from Claim 2 and morphism $\pi_X \colon \overline{\Gamma} \to X$ defined by (15) for $Y = X$. If B is an irreducible component of $\overline{\Gamma} \setminus V$ such that $\pi_X(B)$ is dense in X, then, by the fiber dimension theorem, $\dim \pi_X^{-1}(x) > \dim \pi_X^{-1}(x) \cap B$ for every point $x \in X$ lying off a proper closed subset of X. This and property (q_3) imply that $V \cap \pi_X^{-1}(x)$ is dense in $\pi_X^{-1}(x)$ for every such x. On the other hand, $\pi_X^{-1}(x) = \rho^{-1}(\rho(x)) \times x$ by (18) and, as explained at the end of the proof of Theorem 2, $V \cap \pi_X^{-1}(x)$ is a dense open subset of $G(x) \times x$. Since $G(x) \subseteq \rho^{-1}(\rho(x))$, this shows that $G(x)$ is dense in $\rho^{-1}(\rho(x))$. Closedness of $G(x)$ in X then implies that $G(x) = \rho^{-1}(\rho(x))$ for every point $x \in X$ lying off a proper closed subset. This means that replacing Z by its open subset and X by the inverse image of this subset, we can (and shall) assume that ρ is an orbit map, i.e., the fibers of ρ are the G-orbits in X. Since ρ is a surjective separable morphism and Z is a normal variety, by [Bor91, Prop. II.6.6] this implies that $\rho\colon X \to Z$ is the geometric quotient. Thus the proof of Theorem 4 is completed provided that Claim 4 is proved. $\square$

So it remains to prove Claim 4.

Proof of Claim 4. We divide it into three steps.

1. In view of Claim 3, it suffices to prove density of Γ_0 in $(X\times_Z X)_0$. Since $(X\times_Z X)_0$ is an affine variety, the latter is reduced to proving that if a function $f\in k[(X\times_Z X)_0]$ vanishes on Γ_0,

$$f|_{\Gamma_0}=0, \tag{20}$$

then $f=0$. To prove this, note that closedness of $(X\times_Z X)_0$ in $X_0\times X_0$ implies the existence of a function $h\in k[X_0\times X_0]$ such that

$$h|_{(X\times_Z X)_0}=f. \tag{21}$$

In turn, since $k[X_0\times X_0]=p_1^*(k[X_0])\otimes_k p_2^*(k[X_0])$, where $p_i\colon X_0\times X_0\to X_0$, $(x_1,x_2)\mapsto x_i$, there are functions $s_1,\ldots,s_m,\ t_1,\ldots,t_m\in k[X_0]$ such that

$$h=\textstyle\sum_{i=1}^m p_1^*(s_i)p_2^*(t_i), \tag{22}$$

2. By an appropriate replacement of h and $s_1,\ldots,s_m,t_1,\ldots,t_m$ we may attain that $t_1,\ldots,t_m$ are linearly independent over $\rho^*(k[Z])$. Indeed, by property (q_4), there are functions $b_1,\ldots,b_r\in k[X_0]$, linearly independent over $\rho^*(k[Z])$, such that

$$t_i=\textstyle\sum_{j=1}^r c_{ij}b_j\quad\text{for some}\ \ c_{ij}\in\rho^*(k[Z]),\ i=1,\ldots,m \tag{23}$$

In view of (22) and (23), we have

$$h=\textstyle\sum_{j=1}^r\big(\sum_{i=1}^m p_1^*(s_i)p_2^*(c_{ij})\big)p_2^*(b_j). \tag{24}$$

Take a point $x=(x_1,x_2)\in(X\times_Z X)_0$. Since $\rho(x_1)=\rho(x_2)$, we have

$$c_{ij}(x_1)=c_{ij}(x_2)\quad\text{for all } i,j. \tag{25}$$

From (24) and (25) we then obtain

$$\begin{aligned}h(x)&=\textstyle\sum_{j=1}^r\big(\sum_{i=1}^m s_i(x_1)c_{ij}(x_2)\big)b_j(x_2)\\&=\textstyle\sum_{j=1}^r\big(\sum_{i=1}^m s_i(x_1)c_{ij}(x_1)\big)b_j(x_2).\end{aligned}\tag{26}$$

Hence, if we put

$$\begin{aligned}d_j&:=\textstyle\sum_{i=1}^m s_ic_{ij}\in k[X_0],\\ \widetilde{h}&:=\textstyle\sum_{j=1}^r p_1^*(d_j)p_2^*(b_j)\in k[X_0\times X_0],\end{aligned}\tag{27}$$

then we have $h(x)=\widetilde{h}(x)$ by virtue of (26). Given (21), this yields

$$\widetilde{h}|_{(X\times_Z X)_0}=f. \tag{28}$$

From (27) and (28) we conclude that replacement of $s_1,\ldots,s_m$ and $t_1,\ldots,t_m$ by, respectively, $d_1,\ldots,d_r$ and $b_1,\ldots,b_r$ is the one we are looking for.

3. Thus, keeping the notation, we shall now assume that $t_1,\ldots,t_m$ in (22) are linearly independent over $\rho^*(k[Z])$.

Take an element $g\in G$ and let W be the domain of definition of the rational function

$$\ell=\textstyle\sum_{i=1}^m s_it_i^g\in k(X).$$

Since X is irreducible, $W \cap g(W) \cap X_0 \cap g(X_0)$ is a dense open subset of X. Let x be a point of this subset. Then the rational functions ℓ, s_i, $t_i^g \in k(X)$ are defined at x and

$$a := (x, g^{-1}(x)) \in \Gamma_0. \tag{29}$$

From this we obtain

$$\begin{aligned} \ell(x) &= \textstyle\sum_{i=1}^m s_i(x) t_i^g(x) = \sum_{i=1}^m s_i(x) t_i(g^{-1}(x)) \\ &\overset{\text{by (29)}}{=\!=} \big(\textstyle\sum_{i=1}^m p_1^*(s_i) p_2^*(t_i)\big)(a) \overset{\text{by (22)}}{=\!=} h(a) \overset{\text{by (21)}}{=\!=} f(a) \overset{\text{by (20)}}{=\!=} 0. \end{aligned}$$

So ℓ vanishes on a dense open subset of X; whence $\ell = 0$. Thus, it is proved that

$$\textstyle\sum_{i=1}^m s_i t_i^g = 0 \quad \text{for every } g \in G. \tag{30}$$

Since Z is affine and $\rho^*(k(Z)) = k(X)^G$, the field of fractions of $\rho^*(k[Z])$ is $k(X)^G$. This implies that $t_1, \ldots, t_m$ are linearly independent over $k(X)^G$. In turn, by Artin's theorem [Bou59, §1, no. 1, Thm. 1], this linear independency yields the existence of elements $g_1, \ldots, g_m \in G$ such that

$$\det\big(t_i^{g_j}\big) \neq 0. \tag{31}$$

Combining (30) and (31) we obtain $s_1 = \ldots = s_m = 0$. From this, (22), and (21), we then infer that $f = 0$, as claimed. □

9. Distinguished connected subgroups of $\mathbf{Aut}(X)$. Some collections $\mathcal{I}$ of unital algebraic families in $\mathrm{Aut}(X)$ are naturally distinguished. They generate distinguished connected subgroups $\mathrm{Aut}(X)_{\mathcal{I}}$ of $\mathrm{Aut}(X)$ that are of interest.

The first example is the collection $\mathcal{U}$ of all unital algebraic families in $\mathrm{Aut}(X)$. We shall denote $\mathrm{Aut}(X)_{\mathcal{U}}$ by $\mathrm{Aut}(X)^0$ and call the *identity component of* $\mathrm{Aut}(X)$. The group $\mathrm{Aut}(X)/\mathrm{Aut}(X)^0$ will be called *the component group of* $\mathrm{Aut}(X)$.

Theorem 7. *Let X be an irreducible variety such that* $\mathrm{Aut}(X)$ *is a finite group. Then* $\mathrm{Aut}(X)^0 = \{\mathrm{id}_X\}$.

Proof. Let $\{\varphi_t\}_{t \in T}$ be a unital algebraic family in $\mathrm{Aut}(X)$. Take a point $x \in X$. Irreducibility of T implies irreducibility of the image I_x of morphism (3). Finiteness of $\mathrm{Aut}(X)$ (resp., unitality of $\{\varphi_t\}_{t \in T}$) implies finiteness of I_x (resp., $x \in I_x$). This yields $I_x = \{x\}$, i.e., $\varphi_T = \{\mathrm{id_X}\}$; whence the claim. □

The component group of $\mathrm{Aut}(X)$, in contrast to that of an algebraic group, may be infinite.

Remark 1. If k is uncountable, then the same argument as in the proof of Theorem 7 shows that if $\mathrm{Aut}(X)$ is countable (this may happen, see Examples 1, 2 below), then $\mathrm{Aut}(X)^0 = \{\mathrm{id}_X\}$ and hence the component group of $\mathrm{Aut}(X)$ is countable.

Example 1. Let X be a surface in $\mathbf{A}^3$ defined by the equation $x_1^2 + x_2^2 + x_3^2 = x_1 x_2 x_3 + a$ where $a \in k$. By [Èl'-H74], if a is generic, then $\mathrm{Aut}(X)$ contains a subgroup of finite index which is a free product of three subgroups of order 2.

Example 2. Let $\operatorname{char} k = 0$ and let X be a smooth irreducible quartic in $\mathbf{P}^3$. Then $\operatorname{Aut}(X)^0 = \{\operatorname{id}_X\}$ by [Mat63], and, according to the classical Fano–Severi result, for a sufficiently general X there is a bijection between $\operatorname{Aut}(X)$ and the (countable) set of solutions $(a, b), a > 0$ of the Pell equation $x^2 - 7y^2 = 1$ (see [MM64, pp. 353–354]).

Example 3. Let X be the underlying variety of an algebraic torus G of dimension $n > 0$. The automorphism group $\operatorname{Aut}_{\rm gr}(G)$ of the algebraic group G is embedded in $\operatorname{Aut}(X)$ and is isomorphic to $\operatorname{GL}_n(\mathbf{Z})$. The map $G \to \operatorname{Aut}(X)$, $g \mapsto \ell_g$, where $\ell_g \colon X \to X$, $x \mapsto gx$, identifies G with a subgroup of $\operatorname{Aut}(X)$. By [Ros61, Thm. 3],

$$\operatorname{Aut}(X) = \operatorname{Aut}_{\rm gr}(G) \ltimes G. \tag{32}$$

Let $\{\varphi_t\}_{t\in T}$ be a unital algebraic family in $\operatorname{Aut}(X)$. It follows from [Ros61, Thms. 2 and 3] that there exist a morphism $\alpha \colon T \to G$ and the elements $s \in G$, $g \in \operatorname{Aut}_{\rm gr}(G)$ such that $\widetilde{\varphi}(t, x) = \ell_{\alpha(t)s}(g(x))$ for every $t \in T$, $x \in X$ (see (1)), i.e., $\varphi_t = \ell_{\alpha(t)s} \circ g$. From this, (32), and unitality of $\{\varphi_t\}_{t\in T}$ we infer that $g = \{\operatorname{id}_X\}$. Hence $\varphi_T \subseteq G$. This proves that $\operatorname{Aut}(X)^0 = G$ and the component group of $\operatorname{Aut}(X)$ is isomorphic to $\operatorname{GL}_n(\mathbf{Z})$.

Example 4. By [Ram64, Cor. 1], $\operatorname{Aut}(X)^0$ is a connected algebraic group if X is an irreducible complete variety (and, in fact, more generally, semi-complete variety).

Theorem 8. $\operatorname{Aut}(\mathbf{A}^n) = \operatorname{Aut}(\mathbf{A}^n)^0$ *for* $n \leqslant 2$.

Proof. Let $J(\mathbf{A}^n)$ and $T(\mathbf{A}^n)$ be, resp., the de Jonquières subgroup and the tame subgroup of $\operatorname{Aut}(\mathbf{A}^n)$. Recall [Ess00] that if $x_1, \ldots, x_n$ are the standard coordinate functions on $\mathbf{A}^n$, then $J(\mathbf{A}^n)$ consists of all $g \in \operatorname{Aut}(X)$ such that

$$g^*(x_i) = x_i + f_i, \quad f_i \in k[x_{i+1}, \ldots, x_n] \tag{33}$$

(it is meant that $f_n \in k$). Arbitrary polynomials f_i may occur in (33). The subgroup of $\operatorname{Aut}(X)$ generated by $J(\mathbf{A}^n)$ and $\operatorname{Aff}(\mathbf{A}^n)$ is $T(\mathbf{A}^n)$.

For every $t \in k = \mathbf{A}^1$, let g_t be the element of $J(\mathbf{A}^n)$ defined by

$$g_t^*(x_i) = x_i + tf_i, \quad f_i \in k[x_{i+1}, \ldots, x_n].$$

Then $\{g_t\}_{t\in\mathbf{A}^1}$ is the unital algebraic family in $J(\mathbf{A}^n)$ containing g. Hence $J(\mathbf{A}^n)$ is a connected subgroup of $\operatorname{Aut}(X)$. Connectedness of $\operatorname{Aff}(\mathbf{A}^n)$ then implies that $T(\mathbf{A}^n)$ is a connected subgroup of $\operatorname{Aut}(X)$ too.

The claim now follows from the equalities $\operatorname{Aut}(\mathbf{A}^1) = \operatorname{Aff}(\mathbf{A}^1)$ and $\operatorname{Aut}(\mathbf{A}^2) = T(\mathbf{A}^2)$ (the first is easy and the second follows from the so-called Automorphism theorem, cf. [Ess00, 5.1.11]). This completes the proof. □

Conjecture 1. $\operatorname{Aut}(\mathbf{A}^n) = \operatorname{Aut}(\mathbf{A}^n)^0$ for every n.

A series of examples is obtained taking $\mathcal{I}$ to be a part of the collection $\mathcal{G}$ of all algebraic families $\{\varphi_t\}_{t\in T}$ such that T is a connected algebraic group and $\widetilde{\varphi}$ defined by (1) is an action of T on X. In this case, $\operatorname{Aut}(X)_{\mathcal{I}}$ is a subgroup of $\operatorname{Aut}(X)$ generated, as an abstract group, by a collection of some connected algebraic subgroups of $\operatorname{Aut}(X)$. For $\operatorname{char} k = 0$, the subgroups $\operatorname{Aut}(X)_{\mathcal{I}}$ of this type were studied in [AFKKZ13, Sect. 1] where they are called “algebraically generated groups of automorphisms”. Propositions 1.3, 1.5 and Theorem 1.13 of [AFKKZ13] are the special cases of respectively the above Lemma 3, Theorem 2, and Theorem 4.

Some interesting parts $\mathcal{I}$ of $\mathcal{G}$ are obtained as collections of all families $\{\varphi_t\}_{t\in T}$ in $\mathcal{G}$ such that the algebraic group T has a certain property.

For instance, requiring that T is affine one obtains the collection $\mathcal{G}_{\rm aff}$. Theorems 5 and 6 give examples of dependency between the groups ${\rm Aut}(X)_{\mathcal{G}}$, ${\rm Aut}(X)_{\mathcal{G}_{\rm aff}}$ and geometric properties of X. Here is another example.

Example 5. If ${\rm Aut}(X)_{\mathcal{G}_{\rm aff}} \neq \{{\rm id}_{\rm X}\}$, then X is birationally isomorphic to the product of $\mathbf{A}^1$ and a variety of dimension $\dim X - 1$. This follows from [Mat63, Cor. 1].

Developing the idea of [Pop11, Def. 1.36], one obtains another example of interesting collection of families taking $\mathcal{I}$ to be the collection $\mathcal{G}(F)$ of all families $\{\varphi_t\}_{t\in T}$ in $\mathcal{G}$ such that T is isomorphic to a fixed connected algebraic group F.

For $F = \mathbf{G}_{\rm a}$ this yields the important subgroup ${\rm Aut}(X)_{\mathcal{G}(\mathbf{G}_{\rm a})}$ in ${\rm Aut}(X)$ introduced[1] in [Pop05, Def. 2.1] and called in this paper “∂-generated subgroup”. Its close relation to constructing a big stock of varieties with trivial Makar-Limanov invariant was demonstrated in [Pop11]. Later in [AFKKZ13] transitivity properties of ${\rm Aut}(X)_{\mathcal{G}(\mathbf{G}_{\rm a})}$ (called in this paper “the special automorphism group of X”) were studied. By [Pop11, Lemma 1.1], ${\rm Aut}(X)_{\mathcal{G}(\mathbf{G}_{\rm a})}$ coincides with the subgroup of ${\rm Aut}(X)$ generated by all connected affine subgroups of ${\rm Aut}(X)$ that have no nontrivial characters.

Another interesting case is $F = \mathbf{G}_{\rm m}$. Since the union of all maximal tori of a connected reductive group is dense in it, ${\rm Aut}(X)_{\mathcal{G}(\mathbf{G}_{\rm m})}$ coincides with the subgroup of ${\rm Aut}(X)$ generated by all connected reductive subgroups of ${\rm Aut}(X)$. This implies that

$${\rm Aut}(X)_{\mathcal{G}_{\rm aff}} = {\rm Aut}(X)_{\mathcal{G}(\mathbf{G}_{\rm a})\bigcup\mathcal{G}(\mathbf{G}_{\rm m})}.$$

Indeed, let H be a connected affine algebraic group with a maximal torus T and the unipotent radical $R_u(H)$, and let $\pi\colon H \to H/R_u(H)$ be the canonical projection. By [Bor91, Prop. 11.20], $\pi(T)$ is a maximal torus in $H/R_u(H)$. Conjugacy of maximal tori and density of their union in $H/R_u(H)$ yield $H/R_u(H) = \pi(S)$ for the subgroup S in H generated by all maximal tori. Whence the claim.

10. Proof of Theorem 5. Since G lies in ${\rm Aut}(X)_{\mathcal{G}_{\rm aff}}$, by Corollary 2 it suffices to show that neither of ${\rm Aut}(X)_{\mathcal{G}_{\rm aff}}$-orbits is open in X.

Assume the contrary and let $\mathcal{O}$ be a ${\rm Aut}(X)_{\mathcal{G}_{\rm aff}}$-orbit open in X. Take a point $x \in \mathcal{O}$. By Theorem 2, a certain family $\{\varphi_t\}_{t\in T}$ derived from $\mathcal{G}_{\rm aff}$ is exhaustive for the action of ${\rm Aut}(X)_{\mathcal{G}_{\rm aff}}$ on X. Then $\mathcal{O}$ is the image of morphism (3). Since $\mathcal{O}$ is open in X, this morphism is dominant. On the other hand, the definitions of derived family and $\mathcal{G}_{\rm aff}$ imply that T is a product of underlying varieties of connected affine algebraic groups. But such underlying varieties are rational (see [Pop13, Lemma 2] for a four-lines proof; we

[1]At the irrelevant assumption $X = \mathbf{A}^n$.

failed to find an earlier reference for a proof valid in arbitrary characteristic). Hence T is a rational variety. This and dominance of morphism (3) then imply that X is unirational — a contradiction. □

10. Proof of Theorem 6. (i): Assume the contrary and let C be a nontrivial connected affine algebraic subgroup of $\mathrm{Aut}(X)$. Then there exists a point $x \in V$ such that $V \cap C(x)$ is an irreducible locally closed set of positive dimension. Hence there exists a point $y \in V \cap C(x)$, $y \neq x$. By the condition of 2-transitivity, for every point $z \in V$, $z \neq x$, there exists an element $g \in \mathrm{Aut}(X)$ such that $g(x) = x$, $g(z) = y$. This implies that for the subgroup $H := g^{-1} \circ C \circ g$ we have $z \in H(x)$. Therefore, for the connected subgroup G of $\mathrm{Aut}(X)$ generated by all conjugates of C in $\mathrm{Aut}(X)$ we have $V \subseteq G(x)$; whence $G(x)$ is open in X. From this, arguing as in the proof of Theorem 5, we deduce that X is unirational — a contradiction.

(ii): Assume the contrary and let A be a connected algebraic subgroup of $\mathrm{Aut}(X)$. Since, by (i), A contains no nontrivial connected affine algebraic subgroups, the structure theorem on algebraic groups [Bar55], [Ros56] implies that A is an abelian variety. The same argument as in the proof of (i) then shows that one of the orbits $\mathcal{O}$ of the connected subgroup G of $\mathrm{Aut}(X)$ generated by all conjugates of A in $\mathrm{Aut}(X)$ is open in X and there exists a surjective morphism $Z \to \mathcal{O}$, where Z is a product of several copies of the underlying variety of A. Since Z is then the underlying variety of an abelian variety too, this contradicts the assumption on X. □

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