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On the Adjacency Quantization in an Equation Modeling the Josephson Effect*

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Abstract. We study a two-parameter family of nonautonomous ordinary differential equations on the 2-torus. This family models the Josephson effect in superconductivity. We study its rotation number as a function of the parameters and the *Arnold tongues* (also known as the *phase locking domains*) defined as the level sets of the rotation number that have nonempty interior. The Arnold tongues of this family of equations have a number of nontypical properties: they exist only for integer values of the rotation number, and the boundaries of the tongues are given by analytic curves. (These results were obtained by Buchstaber–Karpov–Tertychnyi and Ilyashenko–Ryzhov–Filimonov.) The tongue width is zero at the points of intersection of the boundary curves, which results in adjacency points. Numerical experiments and theoretical studies carried out by Buchstaber–Karpov–Tertychnyi and Klimenko–Romaskevich show that each Arnold tongue forms an infinite chain of adjacent domains separated by adjacency points and going to infinity in an asymptotically vertical direction. Recent numerical experiments have also shown that for each Arnold tongue all of its adjacency points lie on one and the same vertical line with integer abscissa equal to the corresponding rotation number. In the present paper, we prove this fact for an open set of two-parameter families of equations in question. In the general case, we prove a weaker claim: the abscissa of each adjacency point is an integer, has the same sign as the rotation number, and does not exceed the latter in absolute value. The proof is based on the representation of the differential equations in question as projectivizations of linear differential equations on the Riemann sphere Q1
and the classical theory of linear equations with complex time.

Key words: ???

1. Introduction

1.1. Main results. We study the following family of ordinary differential equations on the torus $\mathbb{T}^2 = S^1 \times S^1$, $S^1 = \mathbb{R}/2\pi\mathbb{Z}$, with coordinates (x, t):

$$\dot{x} = \frac{dx}{dt} = \nu \sin x + a + s \sin t, \qquad a, \nu, s \in \mathbb{R},\ \nu \neq 0. \tag{1.1}$$

This family of equations, which we call *class J equations* for brevity, models the Josephson effect in superconductivity. The parameter ν is supposed to be fixed at an arbitrary nonzero value.

The period 2π flow mapping of the equation is a diffeomorphism

$$h_{a,s} \colon S^1 \to S^1$$

of the spatial circle $S^1 = S^1 \times \{0\}$. In the present paper, we study its rotation number $\rho = \rho(a, s)$ as a function of the parameters a and s. (We use the following scaling convention: the rotation number equals the rotation angle divided by 2π.) We say that the coordinate a-axis is horizontal (and refer to the a-coordinate as the *abscissa*) and the s-axis is vertical (and refer to the s-coordinate as the *ordinate*).

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Definition 1.1. The *rth Arnold tongue* is the level set $\{(a,s) \mid \rho(a,s) = r\} \subset \mathbb{R}^2$ provided that it has nonempty interior.

The rotation number of system (1.1) has the physical meaning of the mean voltage over a long time interval. The segments in which the Arnold tongues intersect horizontal lines correspond to the Shapiro steps on the voltage–current characteristic. It has been shown earlier that

• The Arnold tongues only exist for integer values of the rotation number ([4], [7], [8]).

• The boundary of each tongue $\rho = r$ consists of two analytic curves, which are the graphs of functions denoted by $a = g_r^-(s)$ and $a = g_r^+(s)$. (See [3]. Klimenko independently observed that this fact readily follows from a symmetry argument* for class J equations; see [18].)

• Each of the functions $g_r^-(s)$ and $g_r^+(s)$ has the asymptotics of the rth Bessel function at infinity,

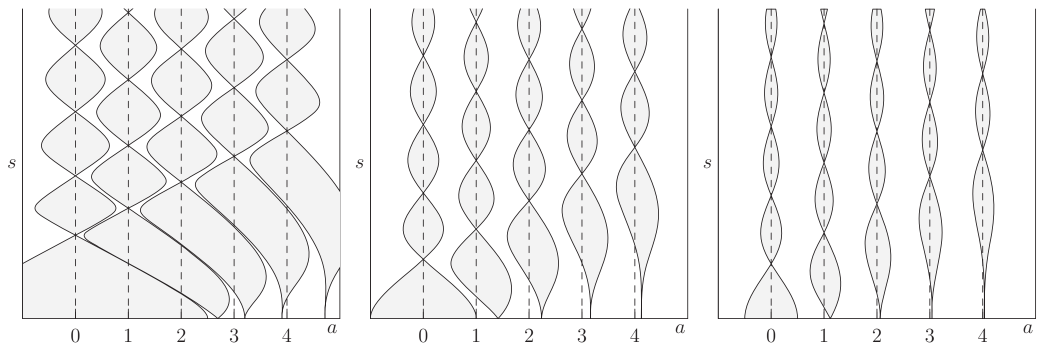
$$\begin{cases} g_r^-(s) = r - \nu J_r(-s/\nu) + o(s^{-1/2}), \\ g_r^+(s) = r + \nu J_r(-s/\nu) + o(s^{-1/2}). \end{cases} \tag{1.2}$$

This was discovered numerically and justified on the level of physical rigor in [16]; see also [10, Chap. 5], [13, Sec. 11.1], and [5]. Mathematically, this was proved in [18].

• Thus, each Arnold tongue is an infinite chain of adjacent bounded domains that go to infinity in an asymptotically vertical direction. The adjacency points of neighboring domains not lying on the horizontal axis $s = 0$ are called *adjacencies*.

Numerical experiments have revealed the following fact.

Experimental fact A. *For each $\nu \neq 0$ and every $r \in \mathbb{Z}$, all adjacencies of the rth Arnold tongue lie on the same vertical line $a = r$; see Fig.* 1.



(a) $\nu = 2{,}5$ (b) $\nu = 1$ (c) $\nu = 0{,}5$

Fig. 1. Arnold tongues 0–4. The adjacencies have integer abscissas equal to the number of the corresponding tongue. As the figure suggests, the intersection of the zeroth Arnold tongue with the line $s = 0$ is the closed interval** $[-\nu, \nu]$.

The main result of the present paper is the following theorem, which partly proves the abovementioned experimental fact.

Theorem 1.2. *Experimental fact A holds for each $\nu \neq 0$ with $|\nu| \leq 1$. For each $\nu \neq 0$, all adjacencies have integer abscissas. The abscissa of each adjacency has the same sign as the corresponding rotation number and does not exceed the latter in absolute value. The adjacencies corresponding to the zero rotation number are exactly those lying on the axis $a = 0$.*

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*A class J equation has the symmetry $(x,t) \mapsto (\pi - x, \pi - t)$. It follows that a parabolic fixed point of the period flow mapping can be only a fixed point $\pm\pi/2$ of the symmetry; see [18]. An equivalent statement was proved in [15, p. 30].

**The family of equations (1.1) is often replaced by the renormalized two-parameter family of equations $\dot{x} + \sin x = B + A\cos(\omega t)$ with fixed $\omega \in \mathbb{R}$. In the latter family, the intersection of the zeroth Arnold tongue with the line $A = 0$ is the closed interval $[-1, 1]$ for every ω.

Corollary 1.3. *For any $\nu \neq 0$ and $r \in \mathbb{Z}$, there exists an $M = M(\nu, r) > 0$ such that all adjacencies of the rth Arnold tongue with ordinates greater than M in absolute value lie on the line $a = r$.*

Corollary 1.3 follows from the integrality of the abscissas of the adjacencies (Theorem 1.2) and the asymptotic formula (1.2) for the boundary of the Arnold tongue, by which the points of the rth tongue with ordinates sufficiently large in modulus lie in the 1-neighborhood of the line $a = r$.

Remark 1.4. It is known that for each $r \in \mathbb{Z} \setminus 0$ the Arnold tongue $\{\rho(a, s) = r\}$ intersects the horizontal axis $s = 0$ exactly at one point with abscissa $\sqrt{r^2 + \nu^2}$ (see [8] and [3, Corollary 3]). This point of adjacency of neighboring components of the tongue will be called a *queer adjacency*.

1.2. Idea of the proof and outline of the paper. The proof of Theorem 1.2 is given in Section 3. It is based on the representation of the family of class J equations as a family of projectivizations of linear differential equations on the Riemann sphere (obtained by various authors; see also Section 2.2 below) and the classical theory of linear equations with complex time.

Definition 1.5. A nonsingular linear operator on a vector space is said to be *projectively identical* if its projectivization is the identity as the induced map of the projective space.

The integrality of the abscissas of adjacencies for every $\nu \neq 0$ is proved in Section 3.1. The linear equations corresponding to class J equations have two irregular nonresonant Poincaré rank 1 singular points, 0 and ∞, on the Riemann sphere (see Section 2.2). The adjacencies correspond to the parameter values for which the monodromy of the linear equation is projectively identical. It turns out that this is the case if and only if the following assertions hold:

• The germ of the linear equation in question at the irregular singular point 0 can be reduced by an analytic change of variables to its formal normal form (a direct sum of one-dimensional equations);

• The monodromy of the latter is projectively identical.

This can be derived from the classical results on the analytic classification of germs of linear equations at nonresonance irregular singular points. The residue matrix of the normal form has a unique nonzero eigenvalue, which is equal to the abscissa of the adjacency. It follows that this abscissa is an integer.

Experimental fact A for $|\nu| \leq 1$ is proved in Section 3.2. An additional elementary differential inequality (a refinement of [3, Lemma 4]) shows that for $|\nu| \leq 1$ the complement of the rth Arnold tongue to the horizontal axis $s = 0$ lies strictly between the lines $a = r \pm 1$. Thus, all its adjacencies should lie on the line $a = r$.

The general case of arbitrary ν is treated in Section 3.3. It is easily seen that the abscissa of an adjacency has the same sign as the corresponding rotation number. An additional argument concerning the corresponding Riccati equations and using the argument principle for complex solutions proves that the modulus of the abscissa of each adjacency does not exceed the modulus of the rotation number. This proves the theorem.

Some preliminary material (rotation number, linear equations, Stokes operators, and monodromy) is contained in Section 2.

2. Preliminaries

2.1. Rotation number of a flow on the torus and adjacencies. On the torus $\mathbb{T}^2 = S^1 \times S^1 = \mathbb{R}^2/2\pi\mathbb{Z}^2$ with coordinates (x, t), consider the flow given by the nonautonomous differential equation

$$\dot{x} = \frac{dx}{dt} = f(x, t) \tag{2.1}$$

with smooth right-hand side. The time t flow mapping is a diffeomorphism $h_t \colon S^1 \to S^1$ of the spatial circle. Consider the universal covering

$$\mathbb{R} \to S^1 = \mathbb{R}/2\pi\mathbb{Z}$$

over the spatial circle. The flow mappings of Eq. (2.1) can be lifted to the universal covering and induce diffeomorphisms

$$H_{q,t}\colon \mathbb{R} = \mathbb{R}\times\{q\} \to \mathbb{R} = \mathbb{R}\times\{q+t\}, \qquad H_{q,0} = \mathrm{Id}\,.$$

Recall that for any $(x, q) \in \mathbb{R}\times S^1$ there exists a limit

$$\rho = \lim_{n\to+\infty} \frac{1}{n} H_{q,2\pi n}(x) \in \mathbb{R}, \tag{2.2}$$

which depends neither on q nor on x and is called the *rotation number of the flow of Eq.* (2.1) (e.g., see [1, p. 124]).

Now consider an (arbitrary) analytic family of equations

$$v_{a,s}: \quad \dot{x} = g(x, t, s) + a, \qquad a, s \in \mathbb{R}. \tag{2.3}$$

Proposition 2.1. *The rotation number $\rho = \rho(a, s)$ of the flow of Eq.* (2.3) *is a continuous function of the parameters (a, s) and a nondecreasing function of a. If, for some parameter values (a_0, s_0), the flow mapping $h_{2\pi} = h_{a_0,s_0,2\pi}\colon S^1\times\{0\} \to S^1\times\{0\}$ of Eq.* (2.3) *has a fixed point, then the rotation number $\rho(a_0, s_0)$ is an integer. In this case, if, in addition, the flow mapping $h_{2\pi}$ is not the identity mapping, then there exists an interval I with endpoint a_0 such that*

- *For each $a \in I$, the transformation $h_{a,s_0,2\pi}$ has at least one fixed point.*
- *For each $a \in I$, the flows of all vector fields sufficiently C^1-close to v_{a,s_0} have the same rotation number $\rho(a_0, s_0)$.*

Proof. The first and second claims can be found in [1, pp. 130–133]. Let us prove the third claim. Assume that the flow mapping $h = h_{a_0,s_0,2\pi}$ is not the identity mapping and has a fixed point $O \in S^1$. This is an isolated fixed point by analyticity. Let $J = [P, O] \subset S^1$ be a closed interval with right endpoint O containing no other fixed points. Then $h(J) \neq J$, and either $h(J) \subset J$ or $h(J) \supset J$. Without loss of generality, we consider only the first case, $h_{a,s_0,2\pi}(J) \subset J$ for $a = a_0$. There exists an interval $I = (c, a_0) \subset \mathbb{R}$ such that the last inclusion holds for each $a \in I$ and is strict, $h_{a,s_0,2\pi}(J) \subset \mathrm{Int}(J)$, because the function $a \mapsto h_{a,s,t}(x)$ is continuous and strictly increasing for any given s, t, and x. Thus, for each $a \in I$ the mapping $h_{a,s_0,2\pi}$ takes the interval J into itself and hence has a fixed point on J and the integer rotation number equal to $\rho(a_0, s_0)$. The last inclusion survives under any C^0-small perturbations of the mapping $h_{a,s_0,2\pi}$. This, together with the continuity of the rotation number, proves Proposition 2.1. □

Now consider the family (1.1) of class J equations. For any parameter values (a, s), by

$$h_{a,s} = h_{a,s,2\pi}\colon S^1\times\{0\} \to S^1\times\{0\} \tag{2.4}$$

we denote the period 2π flow mapping of the corresponding equation. Let

$$\rho(a, s) \text{ be the corresponding rotation number.} \tag{2.5}$$

Proposition 2.2. *The rotation number $\rho(a, s)$ is a continuous function of the parameters (a, s) and a nondecreasing function of a. The adjacencies of Arnold tongues and queer adjacencies* (*see the Introduction*) *correspond exactly to the parameter values for which the corresponding flow mapping $h_{a,s}$ is the identity mapping.*

Proof. The first claim of Proposition 2.2 follows from the first claim of Proposition 2.1. Let us prove the second claim. Assume the contrary: the flow mapping corresponding to one of the adjacencies (a_0, s_0) (true or queer) is not the identity mapping. Its rotation number is an integer, and hence it has a fixed point. Consequently, there exists an interval $I \subset \mathbb{R}$ adjacent to a_0 such that for each $a \in I$ the transformation h_{a,s_0} has a fixed point and the interval $I \times \{s_0\} \subset \{s = s_0\}$ lies in the interior of the Arnold tongue $\{\rho(a, s) = \rho(a_0, s_0)\}$ (Proposition 2.1). At the same time, the line $s = s_0$ cannot meet the interior of the tongue. Indeed, otherwise a close parallel line would meet at least two distinct components of the tongue (by the definition of adjacency), which contradicts the monotonicity of the rotation number as a function of a. This contradiction proves that the flow mapping is the identity mapping. Conversely, assume that the flow mapping is the identity mapping. Then the point (a_0, s_0) belongs to an integer Arnold tongue. It cannot lie in the interior of the tongue. Indeed, for given $s = s_0$ the images of all points $x \in S^1$ under the flow mapping move in the positive (respectively, negative) direction with increasing (respectively, decreasing) parameter a. In particular, the perturbed flow mapping has no fixed points (because the unperturbed mapping is the identity mapping). Thus, the point (a_0, s_0) of the horizontal line $\mathbb{R}\times\{s_0\}$ is an isolated point of intersection of this line with the Arnold tongue $\{\rho(a,s) = \rho(a_0, s_0)\}$. Consequently, it is an adjacency (true or queer). The proof of the proposition is complete. □

2.2. Reduction of class J equations to Riccati equations. The results of this section were earlier obtained in slightly different terms in [6], [8], and [14].

Proposition 2.3. *The change of variables*

$$p = e^{ix}, \quad \tau = e^{it} \tag{2.6}$$

reduces the family of class J equations to the family of Riccati equations

$$\frac{dp}{d\tau} = \frac{1}{\tau^2}\left(\nu(1-p^2)\frac{i\tau}{2} + \left(a\tau + \frac{is(1-\tau^2)}{2}\right)p\right). \tag{2.7}$$

The latter is the projectivization of the following family of linear ordinary differential equations:

$$\dot z = \frac{A(\tau)}{\tau^2}z, \quad z = (z_1, z_2) \in \mathbb{C}^2,\ A(\tau) = \begin{pmatrix} 0 & \frac{i\nu\tau}{2} \\ \frac{i\nu\tau}{2} & \frac{is}{2}(1-\tau^2) + a\tau \end{pmatrix}, \qquad p = \frac{z_2}{z_1}. \tag{2.8}$$

Proof. By substituting the change of variables (2.6) into Eq. (1.1), we obtain

$$\sin x = \frac{1}{2i}(p - p^{-1}), \quad \sin t = \frac{1}{2i}(\tau - \tau^{-1}),$$

$$\dot p = \frac{dp}{dt} = ip\dot x = \frac{\nu p}{2}(p - p^{-1}) + iap + \frac{s}{2}(\tau - \tau^{-1})p, \quad \dot\tau = \frac{d\tau}{dt} = i\tau.$$

Consequently,

$$\frac{dp}{d\tau} = \frac{\dot p}{\dot \tau} = \frac{1}{\tau^2}\left(\frac{i\nu\tau}{2}(1-p^2) + ap\tau + \frac{is}{2}(1-\tau^2)p\right),$$

and hence we arrive at Eq. (2.7). The latter is a Riccati type equation and hence the projectivization of the linear equation

$$\dot z = \frac{dz}{d\tau} = \frac{B(\tau)}{\tau^2}z, \qquad z = (z_1, z_2) \in \mathbb{C}^2,\ p = \frac{z_2}{z_1}. \tag{2.9}$$

Equation (2.9) is determined uniquely up to normalization, because the corresponding vector function $z(\tau)$ is determined by the solution $p(\tau)$ of the Riccati equation uniquely up to multiplication by a scalar function of τ. Let us find the coefficients of Eq. (2.9). By substituting it into the formula $p = z_2/z_1$ and by differentiating the latter, we obtain

$$\frac{dp}{d\tau} = \frac{1}{\tau^2}((B_{22}(\tau) - B_{11}(\tau))p + B_{21}(\tau) - B_{12}(\tau)p^2).$$

A comparison of the last equation with Eq. (2.7) gives

$$B_{22}(\tau) - B_{11}(\tau) = a\tau + \frac{is}{2}(1-\tau^2), \qquad B_{21}(\tau) = B_{12}(\tau) = \frac{i\nu\tau}{2}.$$

Every matrix function $B = (B_{ij})(\tau)$ satisfying these relations defines a linear equation corresponding to the Riccati equation (2.7). The function $B_{11}(\tau)$ can be chosen arbitrarily, and its choice determines the matrix $B(\tau)$ uniquely. By setting $B_{11} \equiv 0$, we obtain the matrix function $A(\tau)$ in (2.8). The proof of the proposition is complete. □

2.3. Irregular singular points of linear differential equations: the Stokes phenomenon and monodromy. All results in the present section are classical and can be found in [2], [9], [12], [17], and [19].

Consider the germ of the linear ordinary differential equation

$$\dot{z} = \frac{B(\tau)}{\tau^2} z, \qquad z = (z_1, z_2) \in \mathbb{C}^2, \tag{2.10}$$

in a neighborhood of the nonresonance irregular singular point $\tau = 0$ of rank 1. By definition, this means that $B(\tau)$ is a 2×2 matrix function holomorphic at zero and the matrix $B(0)$ has distinct eigenvalues λ_1 and λ_2. Without loss of generality, we assume that $B(0)$ is diagonal,

$$B(0) = \operatorname{diag}(\lambda_1, \lambda_2), \qquad \lambda_2 - \lambda_1 \in i\mathbb{R}_+.$$

One can always ensure this by appropriate linear changes of the variables z and τ.

Definition 2.4. Two germs of equations of the form (2.10) are said to be *analytically* (respectively, *formally*) *equivalent* if there exists a change $z = H(\tau)w$ of the variable z, where $H(\tau)$ is a holomorphic invertible matrix function (respectively, a formal invertible power series in τ with matrix coefficients), taking one equation to the other.

The analytic classification of irregular nonresonance germs of linear equations (2.10) and the results given in this section were obtained in the classical papers [12], [17], and [19] and also presented in [2] and [9]. It turns out that every equation germ (2.10) is formally equivalent to a unique direct sum of one-dimensional equations of the form

$$\begin{cases} \dot{w}_1 = \dfrac{b_{10} + b_{11}\tau}{\tau^2} w_1, \\ \dot{w}_2 = \dfrac{b_{20} + b_{21}\tau}{\tau^2} w_2, \end{cases} \qquad b_{20} - b_{10} \in i\mathbb{R}_+, \tag{2.11}$$

which is called a *formal normal form*; here $b_{j0} = \lambda_j$. The corresponding normalizing series $H(\tau)$, $z = H(\tau)w$, is unique up to multiplication by a constant diagonal matrix on the right. As a rule, the normalizing series diverges. At the same time, there exists a cover of a punctured neighborhood of zero by two sectors S_0 and S_1 with vertex at zero on the complex line of the variable τ with the following property. Over each sector S_j, there exists a unique change of variables $z = H_j(\tau)w$ that transforms (2.10) into (2.11), where $H_j(\tau)$ is an analytic invertible matrix function on S_j that has a C^∞ continuation into the closure $\overline{S_j}$ and whose asymptotic Taylor series at 0 exists and coincides with the given normalizing series. The preceding assertion on the existence and uniqueness of a sectorial normalization hold in an arbitrary good sector (see the next definition); the cover consists of good sectors.

Definition 2.5. We say that a sector in $\mathbb{C}$ with vertex at 0 is *good* if it contains only one of the real semiaxes $\mathbb{R}_\pm$ and if its closure does not contain the other semiaxis (see Fig. 2).

We assume that S_0 contains the positive real semiaxis and S_1 contains the negative real semiaxis; see Fig. 2. Set $S_2 = S_0$.

The standard decomposition of the normal form (2.11) into a direct sum of one-dimensional equations defines a canonical basis in its solution space (uniquely up to multiplication of the basis functions by constants) with diagonal fundamental matrix

$$W(\tau) = \operatorname{diag}(w_1, w_2).$$

This matrix, together with the normalizing changes of variables H_j in S_j, defines canonical bases (f_{j1}, f_{j2}) in the solution spaces of Eq. (2.10) in the sectors S_j with fundamental matrices

$$Z^j(\tau) = H_j(\tau)W(\tau), \qquad j = 0, 1, 2, \tag{2.12}$$

where for each $j = 0, 1$ the $(j+1)$st branch of the fundamental matrix $W(\tau)$ on S_{j+1} is obtained from the jth branch on S_j by analytic continuation counterclockwise. The second branch of W on $S_2 = S_0$ is obtained from the zeroth branch by right multiplication by the monodromy matrix of the formal normal form (2.11). In a connected component of the intersection $S_j \cap S_{j+1}$, there are two canonical bases of solutions, one taken from S_j and one taken from S_{j+1}. As a rule, they do not coincide, and the transition between them is defined by a constant matrix C_j,

$$Z^{j+1}(\tau) = Z^j(\tau)C_j. \tag{2.13}$$

The transition operators (respectively, the matrices C_j) are called the *Stokes operators* (respectively, *matrices*). (See the above-mentioned literature.) The nontriviality of the Stokes operators is an obstruction to the analytic equivalence of Eq. (2.10) to its formal normal form (2.11) and is known as the *Stokes phenomenon*.

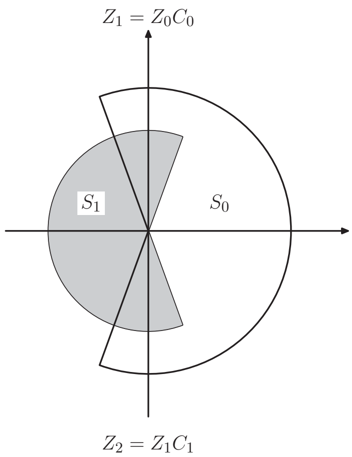

Fig. 2. The sectors S_0 and S_1 and the Stokes operators C_0 and C_1 over their intersections

Remark 2.6. The Stokes matrices (2.13) are uniquely determined up to simultaneous conjugation by the same diagonal matrix. *The Stokes matrices are unipotent.* The matrix C_0 corresponding to the upper component of the intersection of the sectors is lower triangular, and the matrix C_1 is upper triangular,

$$C_0 = \begin{pmatrix} 1 & 0 \\ c_0 & 1 \end{pmatrix}, \qquad C_1 = \begin{pmatrix} 1 & c_1 \\ 0 & 1 \end{pmatrix}. \tag{2.14}$$

Theorem 2.7 ([2], [9], [12], [17], [19])**.** *Equation* (2.10) *is analytically equivalent to its formal normal form* (2.11) *if and only if it has trivial Stokes operators. Two equations* (2.10) *are analytically equivalent if and only if they have the same formal normal forms and the same pairs of Stokes matrices* (*up to the above-mentioned simultaneous conjugation by a diagonal matrix*)*.*

Let $\gamma\colon [0,1] \to \mathbb{C} \setminus 0$ be a closed path on the punctured complex τ-line going around zero once counterclockwise. Recall that the *monodromy operator* of Eq. (2.10) is the linear operator that acts on the space of local solutions in a neighborhood of the point $\gamma(0)$ and takes each solution to the result of its analytic continuation along γ.

Lemma 2.8 [9, p. 35]**.** *The monodromy matrices M of Eq.* (2.10) *and M_N of its formal normal form* (2.11) *and the Stokes matrices C_0 and C_1 introduced above satisfy the relation*

$$M = M_N C_1^{-1} C_0^{-1}. \tag{2.15}$$

Consider the projectivization of Eq. (2.10), that is, the corresponding Riccati equation for the $\overline{\mathbb{C}}$-valued function $p(\tau) = z_2(\tau)/z_1(\tau)$ obtained from Eq. (2.10) with the use of the tautological projection $\mathbb{C}^2 \setminus 0 \to \overline{\mathbb{C}} = \mathbb{CP}^1$. At the end of the paper, we use the following properties of projectivized canonical basis solutions of Eq. (2.10).

Proposition 2.9. *Assume that Eq.* (2.10) *is analytically equivalent to its formal normal form. Then the projectivizations of its canonical basis solutions are the unique solutions $p(\tau)$, holomorphic in a neighborhood of zero, of the corresponding Riccati equation. Their values at zero are zero and infinity, respectively, in the projective coordinate $p = z_2/z_1$.*

Proposition 2.9 is well known to specialists and follows from the validity of itself for the formal normal form.

Proposition 2.10. *Under the assumptions of Proposition* 2.9, *let* $\psi_1(\tau)$ *and* $\psi_2(\tau)$ *be the canonical solutions of the Riccati equation with* $\psi_1(0) = 0$ *and* $\psi_2(0) = \infty$ *mentioned there, and let*

$$z = H(\tau)w, \qquad H(\tau) = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}(\tau)$$

be a normalizing change of variables reducing the linear equation (2.10) *to normal form. Then*

$$\psi_j = \frac{h_{2j}}{h_{1j}}, \qquad j = 1, 2. \tag{2.16}$$

Proposition 2.10 follows from the fact that the canonical fundamental solution matrix of the normal form is diagonal.

3. Rotation Number and Monodromy: Proof of Theorem 1.2

3.1. Integrality of adjacencies. Consider the class J equations (1.1) and the corresponding Riccati equations (2.7) and linear equations (2.8). The flow mappings $h_{a,s}\colon S^1 \to S^1$ extend to be Möbius transformations of the Riemann sphere $\overline{\mathbb{C}} = \mathbb{CP}^1$ and coincide with the monodromy transformations of the corresponding Riccati equations around the point $0 \in \overline{\mathbb{C}}$ by Proposition 2.3. Let $M_{a,s}$ be the monodromy operator of the linear equation (2.8). By definition, the following assertion holds.

Proposition 3.1. *The extended flow mappings* $h_{a,s}$ *coincide with the projectivizations of the monodromy operators* $M_{a,s}$ *of the corresponding linear equations* (2.8).

The germ of each of the linear equations (2.8) at 0 is irregular and nonresonance; more precisely, it has the form (2.10). Its formal normal form is given by

$$\begin{cases} \dot{w}_1 = 0, \\ \dot{w}_2 = \tau^{-2}(is/2 + a\tau)w_2. \end{cases} \tag{3.1}$$

The monodromy operator $M_{N,a,s}$ of the corresponding formal normal form satisfies

$$M_{N,a,s} = \operatorname{diag}(1, e^{2\pi ia}). \tag{3.2}$$

Proposition 3.2. *A point* (a, s) *corresponds to an adjacency if and only if* $s \neq 0$ *and the corresponding monodromy operator* $M_{a,s}$ *is projectively identical, that is, is the product of the identity operator by a scalar factor.*

Proof. A point corresponds to an adjacency if and only if $s \neq 0$ and the corresponding period flow mapping of the class J equation is the identity mapping by Proposition 2.2. This, together with Proposition 3.1, proves Proposition 3.2. □

Lemma 3.3. *A point* (a, s) *corresponds to an adjacency if and only if* $a \in \mathbb{Z}$, $s \neq 0$, *and the germ at zero of the corresponding linear equation* (2.8) *is analytically equivalent to its formal normal form* (3.1).

Proof. Assume that $s \neq 0$, $a \in \mathbb{Z}$, and Eq. (2.8) is analytically equivalent to its normal form. Then the point (a, s) corresponds to an adjacency by Proposition 3.2, because the monodromy (3.2) is the identity mapping. Let us prove that the converse is true as well. Let a point (a, s), $s \neq 0$, correspond to an adjacency. Then the operator $M_{a,s}$ is projectively identical, and hence its matrix in an arbitrary basis should have trivial super- and subdiagonal entries. On the other hand, its matrix in the canonical basis of the solution space in the sector S_0 (see Fig. 2) is the product of the diagonal monodromy matrix (3.2) of the formal normal form by the inverses of the two Stokes matrices (see (2.15)). The last two matrices are unipotent, one of them is lower triangular, and the other is upper triangular. The product of these three matrices has trivial offdiagonal entries if and only if so do the last two triangular matrices, which are then the identity matrices, because they are unipotent. This precisely means that the Stokes matrices are the identity matrices, and hence the linear equation (2.8) is analytically equivalent to its formal normal form by Theorem 2.7. In particular, the monodromy of Eq. (2.8) is given by the matrix (3.2), which is projectively identical if and only if $a \in \mathbb{Z}$. Thus, we have shown that the monodromy operator of Eq. (2.8) is projectively identical if and only if the equation is analytically equivalent to its formal normal form and $a \in \mathbb{Z}$. This, together with Proposition 3.2, proves Lemma 3.3. The proof of the fact that the abscissas of the adjacencies are integers is complete. □

3.2. The case of $|\nu| \leqslant 1$: the adjacencies of the rth tongue lie on the line $a = r$. Thus, we have shown that all adjacencies lie on vertical lines with positive abscissas. Now let us show that for $|\nu| \leqslant 1$ the adjacencies of each individual Arnold tongue lie on a single integer vertical line, more precisely, that

$$\rho(a, s) = a \text{ for each adjacency } (a, s). \tag{3.3}$$

Thus, we assume that $|\nu| \leqslant 1$ in what follows.

Proposition 3.4. *If $|\nu| \leqslant 1$, then $a - 1 \leqslant \rho(a, s) \leqslant a + 1$ for any $a, s \in \mathbb{R}$. Both inequalities are strict except for the case in which $s = 0$ and $a = \pm 1$.*

The inequalities in Proposition 3.4 are stated in [3, Lemma 4]. The proof given below that the inequalities are strict reproduces that in [3] with a small addition.

Proof of Proposition 3.4. Every solution $x(t)$ of a class J equation satisfies the differential inequalities

$$a + s \sin t - 1 \leqslant \dot{x} = \nu \sin x + a + s \sin t \leqslant a + s \sin t + 1. \tag{3.4}$$

If $\sin x \not\equiv \pm 1$, then these inequalities are strict for almost all t, namely, for all t such that $\sin x(t) \neq \pm 1$. Hence the increment of the solution $x(t)$ on any interval I of length 2π lies between the integrals over the same interval of the extreme sides of (3.4). (See Chaplygin's comparison theorem [11].) These integrals are equal to $a - 1$ and $a + 1$. This implies the inequalities stated in the proposition. Now let $\sin x(t) \not\equiv 1$. The inequalities (3.4) are strict for almost all t. It follows that the increment in question lies strictly between the numbers $a \pm 1$ and is separated from these numbers uniformly over all intervals I of length 2π. Thus, the rotation number lies strictly between the numbers $a \pm 1$ by definition and in view of this uniform separation. Now assume that one of the inequalities in (3.4) is not strict. Then $\sin x(t) \equiv \pm 1$ by the preceding, and hence $x(t) \equiv \pi/2 + \pi k$, $k \in \mathbb{Z}$. A class J equation (1.1) has a constant solution $x(t)$ with $\sin x(t) \equiv \pm 1$ if and only if $s = 0$ and $a = \mp 1$; the rotation number is zero in this case. The proof of the proposition is complete. □

Consider the Arnold tongue corresponding to a given integer rotation number r. Its complement to the horizontal axis $s = 0$ lies strictly between the lines $a = r \pm 1$ by the preceding proposition. Its adjacencies do not lie on the horizontal axis (by definition) and have integer abscissas, as shown above. Hence they lie on the line $a = r$. We have proved the first assertion of Theorem 1.2.

3.3. The case of arbitrary ν: the absolute value of the abscissa of an adjacency does not exceed the absolute value of the rotation number. Here we prove the second assertion of Theorem 1.2: for each given $\nu \neq 0$, the abscissa of each adjacency has the same sign as the corresponding rotation number, and its absolute value does not exceed that of the rotation number.

First, we prove the coincidence of signs, which readily follows from the fact that the entire s-axis lies in the zeroth Arnold tongue. (See the next proposition.) Then we prove the inequality stated above by using complex Riccati equations and the argument principle for their canonical solutions.

Proposition 3.5. *For $a = 0$ and any ν and s, Eq.* (1.1) *has zero rotation number. Thus, for each $\nu \neq 0$ the entire s-axis lies in the zeroth Arnold tongue.*

Proof. For $a = 0$, Eq. (1.1) has the symmetry $(x, t) \mapsto (-x, t + \pi)$.

Hence if $x(t)$ is a solution, then so is $\widetilde{x}(t) = -x(t + \pi)$. It follows that the involution $x \mapsto -x$ conjugates the flow mappings $H_{0,2\pi}\colon \mathbb{R} \times \{0\} \to \mathbb{R} \times \{0\}$ and $H_{\pi,2\pi}\colon \mathbb{R} \times \{\pi\} \to \mathbb{R} \times \{\pi\}$ of the lifted equation (1.1). Consequently, these mappings have opposite rotation numbers defined as the limits (2.2). On the other hand, their rotation numbers are equal to the rotation number of the flow (see Section 2.1). We see that the rotation numbers in question are simultaneously equal and opposite. Hence the rotation number is zero. The proof of the proposition is complete. □

Corollary 3.6. *The abscissa of each adjacency has the same sign as the corresponding rotation number provided that the latter is not zero.*

Proof. The rotation number is zero for $a = 0$ and is a nondecreasing function of a. This proves the corollary. □

Fix some $\nu \neq 0$ and consider an arbitrary adjacency (a, s). Let $\rho = \rho(a, s)$ be the corresponding rotation number. Recall that the corresponding linear equation (2.8) is analytically equivalent to the formal normal form of itself at zero (Lemma 3.3). It follows that the $\overline{\mathbb{C}}$-valued projectivized canonical basis solutions of Eq. (2.8) are holomorphic in a full neighborhood of zero (Proposition 2.9) and hence on the entire line $\mathbb{C}$. We denote these projectivized solutions by $\psi_1(\tau)$ and $\psi_2(\tau)$ so that $\psi_1(0) = 0$ and $\psi_2(0) = \infty$ (see Proposition 2.9). Set

$$S^1_p = \{|p| = 1\} \subset \overline{\mathbb{C}},\ S^1_\tau = \{|\tau| = 1\} \subset \overline{\mathbb{C}},\ \mathbb{T}^2 = S^1_p \times S^1_\tau,\ D_1 = \{|\tau| < 1\}.$$

Remark 3.7. The torus $\mathbb{T}^2$ is invariant with respect to the Riccati equation (2.7) restricted to $\overline{\mathbb{C}} \times S^1_\tau$. The functions ψ_1 and ψ_2 are distinct solutions of this equation, and hence $\psi_1(\tau) \neq \psi_2(\tau)$ for all $\tau \in \mathbb{C}$. The restriction of each of them to the unit circle S^1_τ takes values that either all lie on one side of the unit circle S^1_p or lie on the circle itself. This follows from the invariance of the torus $\mathbb{T}^2$.

The proof of the inequality $|a| \leqslant |\rho|$ for the adjacencies in Theorem 1.2 is based on the following formula for the rotation number ρ.

Lemma 3.8. *Let* $|\psi_1|\big|_{S^1_\tau} \leqslant 1$. *Then*

$$\rho = a - 2\#(\text{poles of the function } \psi_1|_{D_1}). \tag{3.5}$$

Proof. For an arbitrary solution ψ of the Riccati equation (2.7) and any $r > 0$ such that $\psi|_{\{|\tau|=r\}} \neq \infty$, consider the variational equation along ψ. Note that the solution of the variational equation is uniquely determined up to a multiplicative constant and is holomorphic in τ in a neighborhood of the circle $\{|\tau| = r\}$. This follows from the fact that the Riccati equation, as well as the corresponding linear equation, has trivial monodromy (Proposition 3.2). Fix an arbitrary nonzero solution $v(\tau)$ of the variational equation. We equip the circle $\{|\tau| = r\}$ with counterclockwise sense and introduce the index

$$\chi_r(\psi) = \text{the increment of the argument of the solution } v(\tau) \text{ along the circle } \{|\tau| = r\}.$$

Proposition 3.9. *One has* $\chi_r(\psi_1) = a$ *for each sufficiently small* $r > 0$.

The proposition follows from the fact that it is true for the Riccati equation obtained by projectivization of the formal normal form (3.1) and that the linear equation (2.8) in question is analytically normalizable.

Proposition 3.10. *Let* $|\psi_1|\big|_{S^1_\tau} \leqslant 1$. *Then* $\chi_1(\psi_1) = \rho$.

Proof. Fix a $\tau_0 \in S^1_\tau$. The flow mappings $\overline{\mathbb{C}} \times \{\tau_0\} \to \overline{\mathbb{C}} \times \{\tau\}$ of the Riccati equation along the unit circle $|\tau| = 1$ are Möbius transformations preserving the unit disk. The index $\chi_1(\psi)$ of any solution of the Riccati equation with $\psi(\tau_0) \in \overline{D_1}$ is independent of the choice of the solution and is equal to ρ for $\psi(\tau_0) \in S^1_p$. This proves the proposition. □

Proposition 3.11. *Consider an arbitrary Riccati equation. Let* $r_2 > r_1 > 0$ *be numbers such that the Riccati equation is holomorphic in a neighborhood of the closed annulus* $A_{r_1 r_2} = \{r_1 \leqslant |\tau| \leqslant r_2\}$ *and has a meromorphic solution* $\psi(\tau)$ *in this neighborhood without poles on the boundary of the annulus. Then*

$$\chi_{r_2}(\psi) = \chi_{r_1}(\psi) - 2\#(\text{poles of the function } \psi|_{r_1<|\tau|<r_2}).$$

Proof. A nonzero solution $v(\tau)$ of the variational equation along ψ is a function ranging in the tangent bundle of the Riemann sphere, holomorphic in a neighborhood of the annulus $A_{r_1r_2}$, and vanishing nowhere. Let p be a complex coordinate on $\mathbb{C}$. It specifies the standard trivialization of the tangent bundle $T\overline{\mathbb{C}}$ over $\mathbb{C}$. The function $v(\tau)$ ranging in $T\overline{\mathbb{C}}$ and written in the standard trivialization is meromorphic, and we denote it by the same symbol $v(\tau)$. Let us show that $v(\tau)$ has the same poles with the same multiplicities as $\psi^2(\tau)$. This, together with the argument principle, will prove Proposition 3.11. Let $\tau' \in \mathrm{Int}(A_{r_1r_2})$ be a pole of ψ. Let $V \subset \overline{\mathbb{C}} \setminus 0$ be an arbitrarily small neighborhood of infinity, more precisely, the complement of a large disk centered at zero, and let $D \subset A_{r_1r_2}$ be a small closed disc centered at τ' such that $\psi(D) \subset V$. Consider the auxiliary trivialization of $T\overline{\mathbb{C}}$ over V specified by the chart with coordinate $\widetilde{p} = 1/p$. Recall that the $T\overline{\mathbb{C}}$-valued solution of the variational equation is holomorphic on the closed disk D and is represented by $\widetilde{v}(\tau) = \psi^{-2}(\tau)v(\tau)$ in the new trivialization. By construction, the function $\widetilde{v}(\tau)$ is holomorphic and vanishes nowhere in D, and $v(\tau) = \widetilde{v}(\tau)\psi^2(\tau)$. Consequently, $v(\tau)$ has the same poles with the same multiplicities as $\psi^2(\tau)$. The proof of Proposition 3.11 is complete. □

Lemma 3.8 follows from Propositions 3.9, 3.10, and 3.11. □

Lemma 3.12. *Suppose that the function $\psi_1|_{S^1_\tau}$ and $\psi_2|_{S^1_\tau}$ either take values lying on different sides of the unit circle S^1_p or $\psi_j(S^1_\tau) \subset S^1_p$ for some $j = 1, 2$. Then the rotation number of the class J equation in question is equal to the abscissa of the adjacency, $\rho = a$.*

We will derive Lemma 3.12 from Lemma 3.8 and the following proposition.

Proposition 3.13. *Under the assumptions of Lemma* 3.12, *one has*

$$|\psi_1|\big|_{\overline{D_1}} \leqslant 1, \qquad |\psi_2|\big|_{\overline{D_1}} \geqslant 1. \tag{3.6}$$

Proof. Consider the function $\phi = \psi_1/\psi_2$. It is meromorphic on $\mathbb{C}$ and has the following properties:

$$\phi \neq 1, \quad \phi(0) = 0; \quad \text{either } |\phi|\big|_{S^1_\tau} \leqslant 1 \quad \text{or } |\phi|\big|_{S^1_\tau} > 1. \tag{3.7}$$

Properties (3.7) follow from the assumptions of Lemma 3.12 and Remark 3.7. First, we analyze the case in which $|\phi|\big|_{S^1_\tau} \leqslant 1$, and then we show that the second case in which $|\phi|\big|_{S^1_\tau} > 1$ is impossible. Consider the difference $\phi(\tau) - 1$. It is meromorphic and vanishes nowhere on $\mathbb{C}$, and the increment of its argument along the circle S^1_τ is zero, since $\phi(S^1_\tau) \subset \overline{D_1}$. Consequently, the function $\phi - 1$ has no poles in the unit disk. Hence $|\phi|\big|_{D_1} \leqslant 1$ by the maximum principle, because the last inequality holds on the boundary of the disk by assumption. Thus, $|\psi_1| \leqslant |\psi_2|$ on the unit disk. On the other hand, on the unit disk we have one of the inequalities $|\psi_1| \leqslant 1 \leqslant |\psi_2|$ and $|\psi_2| \leqslant 1 \leqslant |\psi_1|$ by the assumptions of Lemma 3.12. It follows that the first inequality holds on the entire closed unit disk, and the proof of Proposition 3.13 in the first case is complete. Now assume that the second case holds, $|\phi|\big|_{S^1_\tau} > 1$; i.e., $|\phi^{-1}|\big|_{S^1_\tau} < 1$. The difference $\phi^{-1}(\tau) - 1$ is meromorphic and vanishes nowhere on $\mathbb{C}$, and the increment of its argument along S^1_τ is zero, just as before. At the same time, this difference has a pole at zero, which contradicts the argument principle. This contradiction shows that the second case is impossible and completes the proof of Proposition 3.13. □

Proof of Lemma 3.12. One has $|\psi_1|_{D_1} \leqslant 1$ (Proposition 3.13). This, together with Lemma 3.8, implies the desired assertion of Lemma 3.12. □

Lemma 3.14. *Assume that the inequalities $|\psi_1|, |\psi_2| \leqslant 1$ hold on the unit circle S^1_τ. Then $\rho \leqslant a < 0$, where a is the abscissa of the adjacency in question.*

Proof. The inequality $\rho \leqslant a$ follows from Lemma 3.8. It suffices to show that $\rho < 0$; then $a < 0$ by Corollary 3.6.

Proposition 3.15. *Under the assumptions of Lemma* 3.14, *the rotation number ρ is equal to the increment of the argument of the difference $\psi_2 - \psi_1$ along the unit circle.*

Proof. Recall that $\psi_1(\tau) \neq \psi_2(\tau)$ for all $\tau \in \mathbb{C}$. Consider the flow mappings $H_{1,\tau}\colon \overline{\mathbb{C}} \times \{1\} \to \overline{\mathbb{C}} \times \{\tau\}$ of the Riccati equation (2.7) for $\tau \in S^1_\tau$. The transformations $H_{1,\tau}$ are conformal automorphisms of the unit disk, and $H_{1,\tau}(\psi_j(1), 1) = (\psi_j(\tau), \tau)$ for $j = 1, 2$, because the ψ_j are solutions of Eq. (2.7). It follows that the geodesic of the Poincaré metric on D_1 joining $\psi_1(1)$ and $\psi_2(1)$ is taken to the geodesic γ_τ joining $\psi_1(\tau)$ and $\psi_2(\tau)$. Each of the ends of γ_τ runs over the unit circle and makes ρ full rotations as τ goes around the unit circle S^1_τ in the positive sense. This follows from the definition of rotation number. On the other hand, the increment of the argument of the unit direction vector of the geodesic γ_τ at the point $\psi_1(\tau)$ is equal to the increment of the argument of the straight-line vector directed from the point $\psi_1(\tau)$ to the point $\psi_2(\tau)$ on the Euclidean plane $\mathbb{R}^2 = \mathbb{C} \subset \overline{\mathbb{C}}$. This follows from the fact that the angle between these vectors is smaller than π for all τ. Thus, the increment of the argument of the function $\psi_2(\tau) - \psi_1(\tau)$ is equal to the rotation number. The proof of Proposition 3.15 is complete. □

Proposition 3.16. *Under the assumptions of Lemma* 3.14, *the increment of the argument of the difference* $\psi_2 - \psi_1$ *along the unit circle is negative.*

Proof. Consider the matrix function $H(\tau)$ of a normalizing change of variables reducing the linear equation (2.8) to the normal form (3.1), $z = H(\tau)w$. The function $H(\tau)$ is holomorphic on $\mathbb{C}$ and ranges in the set of invertible matrices. Recall that $\psi_j = h_{2j}/h_{1j}$ for $j = 1, 2$ by Proposition 2.10. Thus,

$$\psi_2 - \psi_1 = \frac{h_{11}h_{22} - h_{21}h_{12}}{h_{11}h_{12}}.$$

The numerator on the right-hand side in the last equation is the determinant of the invertible matrix function H, and hence it vanishes nowhere on $\mathbb{C}$. Hence the increment of the argument of the difference in question is minus the sum of numbers of zeros (with regard to multiplicities) of the functions h_{11} and h_{12} in the unit disk. The last sum is positive, because $h_{12}(0) = 0$. Consequently, the desired increment of the argument is negative. The proof of Proposition 3.16 is complete. □

End of proof of Lemma 3.14. The inequality $\rho < 0$ follows from Propositions 3.15 and 3.16. This, together with the argument at the beginning of the proof of Lemma 3.14, proves the lemma. □

End of proof of Theorem 1.2. We have already proved that the abscissas of the adjacencies are integers. Recall that for each adjacency each of the corresponding functions $\psi_j|_{S^1_\tau}$, $j = 1, 2$, takes values either on one side of the unit circle or on the unit circle itself. One of the following three cases holds:

(i) The functions $\psi_j|_{S^1_\tau}$, $j = 1, 2$, take values on different sides of the unit circle; here we also include the case in which $|\psi_j|\big|_{S^1_\tau} \equiv 1$ for some $j = 1, 2$.

(ii) $|\psi_j|\big|_{S^1_\tau} < 1$ for all $j = 1, 2$.

(iii) $|\psi_j|\big|_{S^1_\tau} > 1$ for all $j = 1, 2$.

In the first case, $\rho = a$ by Lemma 3.12. In the second case, $\rho \leqslant a < 0$ by Lemma 3.14. The third case can be reduced to the second by the change of variables $(x, t) \mapsto (-x, t + \pi)$ (or $(p, \tau) \mapsto (p^{-1}, -\tau)$ in the coordinates (p, τ)). The last change of variables takes a class J equation to the same equation but changes the signs of the parameter a and the rotation number. This, together with the preceding assertion, implies that the inequality $0 < a \leqslant \rho$ holds in the third case. For $a = 0$, we have $\rho = 0$ by Proposition 3.5. This proves Theorem 1.2.

3.4. Experimental fact A in the general case: state of the art. Experimental fact A claiming that $\rho = a$ for every adjacency has been proved for $|\nu| \leqslant 1$ (Theorem 1.2). It is expected to be true for every $\nu \neq 0$.

Theorem 3.17. *The difference* $\rho - a$ *is even for each adjacency. If it is nonzero*, *then either* $\rho < a < 0$ *or* $0 < a < \rho$.

Theorem 3.17 follows from Lemma 3.8 and Theorem 1.2.

Let (ν, a, s) be an adjacency. Let ψ_1 and ψ_2 be the solutions, holomorphic at zero, of the corresponding Riccati equation (2.7) in Proposition 2.10.

As is shown below, one has $\rho = a$ under the following condition:

Condition (∗) on the Riccati equation (2.7). Either $\psi_1(\tau)$ has no poles in the unit disk and $|\psi_1|\big|_{S^1_\tau} \leqslant 1$, or $\psi_2(\tau)$ has no poles in the unit disk and $|\psi_2|\big|_{S^1_\tau} \geqslant 1$.

Lemma 3.18. *Let (ν, a, s) be an adjacency $(s \neq 0)$. Then $\rho = a$ if and only if the corresponding Riccati equation satisfies condition $(*)$.*

Lemma 3.18 follows from Lemma 3.8, whose counterpart remains valid for the case in which $|\psi_2|\big|_{S^1_\tau} \geqslant 1$, with ψ_1 replaced by ψ_2 and the number of poles replaced by the number of zeros.

Conjecture C. Condition $(*)$ holds for every adjacency.

Experimental fact A is equivalent to Conjecture C by Lemma 3.18.

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Вопрос к авторам

Q1. Приведите, пожалуйста, ключевые слова и проверьте английское написание фамилии Щурова. Сохранить порядок авторов по русскому алфавиту или сделать по английскому?

Q2. Годится это издание?