

# Solvability of linear differential systems with small exponents in the Liouvillian sense

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## Abstract

The paper is devoted to solvability of linear differential systems by quadratures, one of the classical problems of differential Galois theory. As known, solvability of a system depends entirely on properties of its differential Galois group. However, to detect the solvability or non-solvability of a given system is a difficult problem, as dependence of its differential Galois group on the coefficients of the system remains unknown. We consider systems with regular singular points as well as those with non-resonant irregular ones whose exponents (respectively, so-called formal exponents in the irregular case) are sufficiently small. It turns out that for systems satisfying such restrictions criteria of solvability can be given in terms of the coefficient matrix.

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## 1 Introduction

Consider on the Riemann sphere  $\overline{\mathbb{C}}$  a linear differential system

$$\frac{dy}{dz} = B(z)y, \quad y(z) \in \mathbb{C}^p, \quad (1)$$

of  $p$  equations with a meromorphic coefficient matrix  $B$  (whose entries are thus rational functions) and singularities at points  $a_1, \dots, a_n$ . Solvability of linear differential equations and systems in the Liouvillian sense (in other words, by quadratures) is a classical problem of differential Galois theory developed by Picard and Vessiot at the beginning of the twentieth century. In analogy to Galois theory of algebraic equations, they have connected to the system a group called the differential Galois group and shown that solvability of the system by quadratures depends entirely on properties of its differential Galois group. Later, in the middle of the century, Kolchin has completed this theory by considering other types of solvability and their dependence on properties of the differential Galois group.

However, the Picard–Vessiot–Kolchin theory reveals the cause of solvability or non-solvability of linear differential equations by quadratures rather than answers this question addressed to a fixed equation, since one does not know how the differential Galois group of an equation depends on its coefficients. In our paper we are interested in the cases when the answer to the question of solvability of the system (1) by quadratures can be given in terms of its coefficient matrix  $B$ . For example, in the case of a Fuchsian system

$$\frac{dy}{dz} = \left( \sum_{i=1}^n \frac{B_i}{z - a_i} \right) y, \quad B_i \in \text{Mat}(p, \mathbb{C}), \quad (2)$$

with sufficiently small entries of the matrices  $B_i$ , Yu. Ilyashenko and A. Khovanskii [10] (see also [15, Ch. 6, §2.3]) have obtained an explicit criterion of solvability. Namely, the following statement holds:

*There exists  $\varepsilon = \varepsilon(n, p) > 0$  such that a condition of solvability by quadratures for the Fuchsian system (2) with  $\|B_i\| < \varepsilon$  takes an explicit form: the system is solvable by quadratures, if and only if all the matrices  $B_i$  are triangular (in some basis).*

Using results of Kolchin these authors have also obtained criteria for other types of solvability of the Fuchsian system (2) with small residue matrices  $B_i$ , in terms of these matrices. Moreover, a topological version of Galois theory developed by Khovanskii at the beginning of 1970's allowed to obtain stronger results concerning non-solvability of Fuchsian systems.

Our paper may be considered as an attempt to generalize results of Ilyashenko and Khovanskii to the case of non-Fuchsian systems. It is organized as follows. In the next two sections we recall basic notions of differential Galois theory and local theory of linear differential systems which are used within the paper.

In Section 4 we describe a weak version of Ramis' theorem (due to Ilyashenko and Khovanskii) concerning the description of the local differential Galois group at a singular point of a system in terms of local meromorphic invariants of the latter. This theorem is essentially used for proving statements on the solvability of non-Fuchsian systems by quadratures. In this section we also recall a notion of the exponential torus, a subgroup of the local differential Galois group which also appears in the further proofs.

Section 5 is devoted to an approach using holomorphic vector bundles with meromorphic connections on the Riemann sphere in a context of analytic theory of linear differential equations. This approach was developed and successfully applied by A. Bolibrukh to solving inverse monodromy problems of this theory (in particular, Hilbert's 21st (the Riemann–Hilbert) problem). Here we also use it, combining with a technique from differential Galois theory, to obtain statements concerning solvability of linear differential systems by quadratures.

Section 6 concerns solvability of Fuchsian systems (more generally, systems with regular singular points) by quadratures. In particular, here we refine the Ilyashenko–Khovanskii criterion in such a way that it is sufficient the eigenvalues of the residue matrices  $B_i$  to be small (the estimate is given) rather than the matrices themselves. This refinement is naturally extended to other types of solvability as well as to the strong non-solvability of Fuchsian systems.

In Section 7 we propose a generalization of a criterion of solvability by quadratures to the case of essentially non-Fuchsian systems (systems with non-resonant irregular singular points) with small formal exponents and also discuss other types of solvability, including the local solvability by quadratures over the field of meromorphic germs at an irregular singular point.

## 2 Solvability by quadratures and the differential Galois group

In this section we recall definitions of some basic notions of differential Galois theory. Besides the solvability of a linear differential system by quadratures we consider other types of solvability and explain their dependence on properties of the differential Galois group of the system.

A *Picard–Vessiot extension* of the field  $\mathbb{C}(z)$  of rational functions corresponding to the system (1) is a differential field  $F = \mathbb{C}(z)\langle Y \rangle$  generated as a field extension of  $\mathbb{C}(z)$  by all the entries of a fundamental matrix  $Y(z)$  of the system (1). One says that the system (1) is *solvable by quadratures* if there is a fundamental matrix  $Y$  whose entries are expressed in elementary or algebraic functions and their integrals or, more formally, if the field  $F$  is contained in some

extension of  $\mathbb{C}(z)$  generated by algebraic functions, integrals and exponentials of integrals:

$$\mathbb{C}(z) = F_1 \subset \dots \subset F_m, \quad F \subseteq F_m,$$

where each  $F_{i+1} = F_i\langle x_i \rangle$  is a field extension of  $F_i$  by an element  $x_i$  which is either:

- algebraic over  $F_i$ , or
- an integral (that is, an element whose derivative belongs to  $F_i$ ), or
- an exponential of an integral (that is, an element whose logarithmic derivative belongs to  $F_i$ ).

Such an extension  $\mathbb{C}(z) \subseteq F_m$  is called *Liouvillian*, thus solvability by quadratures means that the Picard–Vessiot extension  $F$  is contained in some Liouvillian extension of the field of rational functions.

In analogy to classical Galois theory, solvability or non-solvability of a linear differential system by quadratures is related to properties of its differential Galois group. The *differential Galois group*  $\mathbf{G} = \text{Gal}(F/\mathbb{C}(z))$  of the system (1) (of the Picard–Vessiot extension  $\mathbb{C}(z) \subseteq F$ ) is the group of differential automorphisms of the field  $F$  (*i. e.*, automorphisms commuting with differentiation) that preserve the elements of the field  $\mathbb{C}(z)$ :

$$\mathbf{G} = \left\{ \sigma : F \rightarrow F \mid \sigma \circ \frac{d}{dz} = \frac{d}{dz} \circ \sigma, \sigma(f) = f \quad \forall f \in \mathbb{C}(z) \right\}.$$

As follows from the definition, the image  $\sigma(Y)$  of the fundamental matrix  $Y$  of the system (1) under any element  $\sigma$  of the Galois group is a fundamental matrix of this system again, hence,  $\sigma(Y) = Y(z)C$ ,  $C \in \text{GL}(p, \mathbb{C})$ . As every element of the differential Galois group is determined uniquely by its action on a fundamental matrix of the system, the group  $\mathbf{G}$  can be regarded as a subgroup of the matrix group  $\text{GL}(p, \mathbb{C})$  for any such given  $Y$ . Moreover this subgroup  $\mathbf{G} \subseteq \text{GL}(p, \mathbb{C})$  is algebraic, *i. e.*, closed in the *Zariski topology* of the space  $\text{GL}(p, \mathbb{C})$  (the topology whose closed sets are those determined by systems of polynomial equations), see [13, Th. 5.5]. A very good reference for the basics in differential Galois theory, especially the algebraic geometric aspects, is the book by T. Crespo and Z. Hajto [5].

The differential Galois group  $\mathbf{G}$  can be represented as a union of finite number of disjoint connected sets that are open and closed simultaneously (in the Zariski topology) and the set containing the identity matrix is called the *identity component*. The identity component  $\mathbf{G}^0 \subseteq \mathbf{G}$  is a normal subgroup of finite index [13, Lemma 4.5]. According to the Picard–Vessiot theorem, solvability of the system (1) by quadratures is equivalent to the solvability of the subgroup  $\mathbf{G}^0$  (see [13, Th. 5.12], [15, Ch. 3, Th. 5.1]). Recall that a group  $H$  is said to be *solvable*, if there exist intermediate normal subgroups  $\{e\} = H_0 \subset H_1 \subset \dots \subset H_m = H$  such that each factor group  $H_i/H_{i-1}$  is Abelian,  $i = 1, \dots, m$ .

Alongside the differential Galois group one considers the *monodromy group*  $\mathbf{M}$  of the system (1) generated by the monodromy matrices  $M_1, \dots, M_n$  corresponding to analytic continuation of the fundamental matrix  $Y$  around the singular points  $a_1, \dots, a_n$  respectively. Each  $M_i$  is defined as follows: since the operation of analytic continuation commutes with differentiation, the matrix  $Y$  considered in a neighbourhood of a non-singular point  $z_0$  goes to another fundamental matrix,  $Y(z)M_i$ , under an analytic continuation along a simple loop  $\gamma_i$  encircling a point  $a_i$ . As analytic continuation also preserves elements of the field  $\mathbb{C}(z)$  (single-valued functions), one has  $\mathbf{M} \subseteq \mathbf{G}$ . Furthermore the differential Galois group of a Fuchsian system coincides with the closure of its monodromy group in the Zariski topology (see [15, Ch. 6, Cor. 1.3]) hence a

Fuchsian system is solvable by quadratures if and only if the identity component  $\mathbf{M} \cap \mathbf{G}^0$  of its monodromy group is solvable.

Now we will discuss other types of solvability. They are defined in analogy to the solvability by quadratures and we leave this to the reader. E. Kolchin [16] has given criteria for a linear differential system to be solvable with respect to each of these types, in terms of its differential Galois group. Further we will need his results only for a system whose differential Galois group is triangular.

**Kolchin's criterion** (see also [15, Ch. 3, §8]). *Let the differential Galois group  $\mathbf{G}$  of the system (1) be triangular. Then the system is*

- 1) *solvable by integrals and algebraic functions if and only if the eigenvalues of all the matrices from  $\mathbf{G}$  are roots of the identity;*
- 2) *solvable by integrals if and only if the eigenvalues of all the matrices from  $\mathbf{G}$  are equal to the identity;*
- 3) *solvable by exponentials of integrals and algebraic functions if and only if  $\mathbf{G}$  is diagonal;*
- 4) *solvable by algebraic functions if and only if  $\mathbf{G}$  is diagonal and the eigenvalues of all its matrices are roots of the identity.*

### 3 A local form of solutions near a singular point

In this section we recall the definitions of regular and irregular singular points of a linear differential system and describe the structure of solutions near a singular point of each type.

A singular point  $z = a_i$  of the system (1) is said to be *regular* if any solution of the system has at most polynomial growth in any sector with vertex at this point, of sufficiently small radius and an opening less than  $2\pi$ . Otherwise the point  $z = a_i$  is said to be *irregular*.

A singular point  $z = a_i$  of the system (1) is said to be *Fuchsian* if the coefficient matrix  $B(z)$  has a simple pole at this point. Due to Sauvage's theorem, a Fuchsian singular point is always regular (see [9, Th. 11.1]). However, the coefficient matrix of a system at a regular singular point may in general have a pole of order greater than one. Let us write the Laurent expansion of the coefficient matrix  $B$  of the system (1) near its singular point  $z = a$  in the form

$$B(z) = \frac{B_{-r-1}}{(z-a)^{r+1}} + \dots + \frac{B_{-1}}{z-a} + B_0 + \dots, \quad B_{-r-1} \neq 0. \quad (3)$$

The number  $r$  is called the *Poincaré rank* of a system (1) at this point (or the Poincaré rank of the singular point  $z = a$ ). For example, the Poincaré rank of a Fuchsian singular point is equal to zero.

The system (1) is said to be *Fuchsian* if all its singular points are Fuchsian (then it can be written in the form (2)). The system whose singular points are all regular will be called *regular singular*.

#### 3.1 Regular singular point

According to Levelt's theorem [17], in a neighbourhood of each regular singular point  $a_i$  of the system (1) there exists a fundamental matrix of the form

$$Y_i(z) = U_i(z)(z-a_i)^{A_i}(z-a_i)^{\tilde{E}_i}, \quad (4)$$

where  $U_i(z)$  is a holomorphic matrix at the point  $a_i$ ,  $A_i = \text{diag}(\varphi_i^1, \dots, \varphi_i^p)$  is a diagonal matrix whose entries  $\varphi_i^j$  are integers organized in a non-increasing sequence,  $\tilde{E}_i = (1/2\pi\mathbf{i}) \ln \tilde{M}_i$  is an

upper-triangular matrix (the normalized logarithm of the corresponding monodromy matrix  $\widetilde{M}_i$ ) whose eigenvalues  $\rho_i^j$  satisfy the condition

$$0 \leq \operatorname{Re} \rho_i^j < 1.$$

Such a fundamental matrix is called a *Levelt matrix*, and one also says that its columns form a *Levelt basis* in the solution space of the system (in a neighborhood of the regular singular point  $a_i$ ). The complex numbers  $\beta_i^j = \varphi_i^j + \rho_i^j$  are called the (Levelt) *exponents* of the system at the regular singular point  $a_i$ .

The singular point  $a_i$  is Fuchsian if and only if the corresponding matrix  $U_i$  in the decomposition (4) is holomorphically invertible at this point, that is,  $\det U_i(a_i) \neq 0$ . It is not difficult to check that in this case the exponents of the system at the point  $a_i$  coincide with the eigenvalues of the residue matrix  $B_i$ . In the general case of a regular singularity  $a_i$  estimates for the order of the function  $\det U_i$  at this point were obtained by E. Corel [4] (see also [7]):

$$r_i \leq \operatorname{ord}_{a_i} \det U_i \leq \frac{p(p-1)}{2} r_i,$$

where  $r_i$  is the Poincaré rank of the regular singular point  $a_i$ . These estimates imply the inequalities for the sum of exponents of the regular singular system over all its singular points, which are called the *Fuchs inequalities*:

$$-\frac{p(p-1)}{2} \sum_{i=1}^n r_i \leq \sum_{i=1}^n \sum_{j=1}^p \beta_i^j \leq -\sum_{i=1}^n r_i \quad (5)$$

(the sum of exponents is an integer).

### 3.2 Irregular singular point

Let us now describe the structure of the solutions of the system (1) near one of its irregular singular points. We assume that the irregular singularity  $z = a$  of Poincaré rank  $r$  is *non-resonant*, that is, the eigenvalues  $b_1, \dots, b_p$  of the leading term  $B_{-r-1}$  of the matrix  $B(z)$  in the expansion (3) are pairwise distinct. Let us fix a matrix  $T$  reducing the leading term  $B_{-r-1}$  to its diagonal form

$$T^{-1} B_{-r-1} T = \operatorname{diag}(b_1, \dots, b_p).$$

The system possesses a uniquely determined formal fundamental matrix  $\widehat{Y}$  of the form (see [24, §§10, 11])

$$\widehat{Y}(z) = \widehat{F}(z)(z-a)^\Lambda e^{Q(z)}, \quad (6)$$

where

- a)  $\widehat{F}(z)$  is a matrix formal Taylor series in  $z-a$  and  $\widehat{F}(a) = T$ ;
- b)  $\Lambda$  is a constant diagonal matrix whose diagonal entries are called the *formal exponents* of the system (1) at the irregular singular point  $z = a$ ;
- c)  $Q(z) = \operatorname{diag}(q_1(z), \dots, q_p(z))$  is a diagonal matrix whose entries  $q_j(z)$  are polynomials in  $(z-a)^{-1}$  of degree  $r$  without a constant term,

$$q_j(z) = -\frac{b_j}{r} (z-a)^{-r} + o((z-a)^{-r}).$$

For each pair  $(b_j, b_k)$  of eigenvalues of the matrix  $B_{-r-1}$  one has  $2r$  rays starting at the point  $a$ , which are called *Stokes lines* of the system (1) at this point:

$$\left\{ z \in \mathbb{C} \mid \operatorname{Re} \frac{b_j - b_k}{(z - a)^r} = 0 \right\} = \left\{ z \in \mathbb{C} \mid \arg(z - a) = \frac{1}{r} \left( \arg(b_j - b_k) + \frac{\pi}{2} + \pi m \right), \right. \\ \left. m = 0, 1, \dots, 2r - 1 \right\}.$$

These rays are asymptotic to the corresponding curves  $\{\operatorname{Re}(q_j - q_k) = 0\}$  dividing a neighbourhood of  $a$  into domains in which the function  $e^{q_j - q_k}$  has an exponential growth or decay.

Consider a covering of a punctured neighbourhood of  $a$  by  $2r$  congruent sectors

$$S_m = \left\{ d + (m - 1) \frac{\pi}{r} - \delta < \arg(z - a) < d + m \frac{\pi}{r} + \delta \right\}, \quad m = 1, \dots, 2r,$$

with opening  $\frac{\pi}{r} + 2\delta$ . The direction  $d$ ,  $0 \leq d < 2\pi$  and small  $\delta > 0$  can be chosen in such a way that each sector contains exactly one Stokes line for each pair  $(b_j, b_k)$ . Such a covering is often referred to as the *good covering* by *good sectors*. According to Sibuya's sectorial normalization theorem (see [12, Th. 21.13, Prop. 21.17]) in each good sector  $S_m$  of a good covering there exists a *unique* actual fundamental matrix

$$Y_m(z) = F_m(z)(z - a)^\Lambda e^{Q(z)} \quad (7)$$

of the system (1) whose factor  $F_m$  has the asymptotic expansion  $\widehat{F}$  in  $S_m$ . In every intersection  $S_m \cap S_{m+1}$  the fundamental matrices  $Y_m$  and  $Y_{m+1}$  necessarily differ by a constant invertible matrix:

$$Y_{m+1}(z) = Y_m(z)C_m, \quad C_m \in \operatorname{GL}(p, \mathbb{C}),$$

and it is understood that the logarithmic term  $(z - a)^\Lambda$  is analytically continued from  $S_1$  to  $S_2$ , from  $S_2$  to  $S_3$ , ..., from  $S_{2r}$  to  $S_1$ , so that

$$Y_1(z)e^{2\pi i \Lambda} = Y_{2r}(z)C_{2r} \quad \text{in } S_{2r} \cap S_1. \quad (8)$$

The matrices  $C_1, \dots, C_{2r}$  are called (Sibuya's) *Stokes matrices* of the system (1) corresponding to the good covering  $\{S_1, \dots, S_{2r}\}$  at the non-resonant irregular singular point  $a$ . They satisfy the relation

$$e^{2\pi i \Lambda} = M C_1 \dots C_{2r}, \quad (9)$$

where  $M$  is the monodromy matrix of  $Y_1$  at  $a$ . Indeed, the fundamental matrix  $Y_1$  can be continued from  $S_1$  into  $S_2$  as  $Y_2 C_1^{-1}$ , since  $Y_1 = Y_2 C_1^{-1}$  in  $S_1 \cap S_2$ . Further it is continued from  $S_2$  into  $S_3$  as  $Y_3 (C_1 C_2)^{-1}$ , etc. Finally, in  $S_{2r}$  it becomes equal to  $Y_{2r} (C_1 \dots C_{2r-1})^{-1}$ . Then it comes back into  $S_1$  as  $Y_1 e^{2\pi i \Lambda} (C_1 \dots C_{2r})^{-1}$  according to (8), whence the relation (9) follows. It is also known that all the eigenvalues of any Stokes matrix are equal to 1, that is, the Stokes matrices are unipotent (see [12, Prop. 21.19] or [24, Th. 15.2]).

## 4 Local differential Galois groups

Alongside the differential Galois group  $\mathbf{G}$  of the system (1) one also defines *local* differential Galois groups  $\mathbf{G}_a$  corresponding to each singular point  $z = a$  of the system. Let  $\mathcal{M}_a = \mathbb{C}(\{z - a\})$  be the field of meromorphic functions at the point  $a$ . One considers a fundamental matrix  $Y_S(z)$

of the system (1) on some open sector  $S$  with vertex  $a$  (regarding its entries as elements of the field of meromorphic functions in  $S$ ) and the Picard–Vessiot extension  $\mathcal{M}_a \subseteq F_a = \mathcal{M}_a \langle Y_S \rangle$  of  $\mathcal{M}_a$  generated by all the entries of  $Y_S$ . The corresponding differential Galois group  $\mathbf{G}_a = \text{Gal}(F_a/\mathcal{M}_a)$  is the group of differential automorphisms of the field  $F_a$  that preserve the elements of the field  $\mathcal{M}_a$ . Like the global differential Galois group  $\mathbf{G}$ , this group is regarded as an algebraic subgroup of the matrix group  $\text{GL}(p, \mathbb{C})$ , where this representation depends of course on  $a$  and on  $Y_S$ .

We have the following natural inclusion of each  $\mathbf{G}_a$  into  $\mathbf{G}$ . Consider a fundamental matrix  $Y$  of the system (1) in a neighbourhood  $U_0$  of a non-singular point  $z_0$  with respect to which the group  $\mathbf{G}$  is determined. The entries of  $Y$  are elements of the field of meromorphic functions in  $U_0$ . Being continued analytically into the sector  $S$  along some path  $\gamma$ , the matrix  $Y$  can be transformed by any element  $\sigma$  of the group  $\mathbf{G}_a$  and is continued back into  $U_0$  along  $\gamma^{-1}$ . As  $\mathbb{C}(z) \subset \mathcal{M}_a$ , the automorphism  $\sigma$  preserves rational functions and thus the transformation  $\tau_\gamma^{-1} \circ \sigma \circ \tau_\gamma$  is an element of the group  $\mathbf{G}$ , where  $\tau_\gamma$  is the operator of an analytic continuation along the path  $\gamma$  (we mean that  $\tau_\gamma$  is an operator from the solution space of (1) in  $U_0$  to that in  $S$ ). On the level of matrices, which are elements of the corresponding algebraic matrix groups  $\mathbf{G}_a$  and  $\mathbf{G}$  denoted by the same symbols, this looks as follows:

$$C_\gamma^{-1} \mathbf{G}_a C_\gamma \subset \mathbf{G},$$

where  $C_\gamma \in \text{GL}(p, \mathbb{C})$  is a matrix connecting  $Y_S$  and the analytic continuation  $\tilde{Y}_S$  of  $Y$  into  $S$  along  $\gamma$ , that is,  $\tilde{Y}_S(z) = Y_S(z)C_\gamma$ .

#### 4.1 Sibuya’s Stokes matrices as elements of a local Galois group

The first author to have described the local differential Galois group  $\mathbf{G}_a$  in terms of local matrix invariants of the system (1) at the irregular singular point  $a$  (among which the Stokes matrices are) is J.-P. Ramis. He has defined (Ramis’) Stokes matrices in a somewhat different way and shown that, in particular, they belong to the matrix group  $\mathbf{G}_a$  ([23], [19], see also [20], [22, Ch. 8]). Further Yu. Ilyashenko and A. Khovanskii [11] have shown that Sibuya’s Stokes matrices defined in the previous section also belong to  $\mathbf{G}_a$ . We will use their result and so describe it now in more details.

Consider a good covering of a neighbourhood of the singular point  $a$  of the system (1) by good sectors  $S_1, \dots, S_{2r}$  and the corresponding fundamental matrices  $Y_1, \dots, Y_{2r}$  described in the previous section. The entries of each  $Y_m$  are elements of the field of meromorphic functions in the sector  $S_m$ . Let  $\mathbf{G}_a$  be a matrix group that represents the local differential Galois group of the system (1) at the point  $a$ , with respect to the fundamental matrix  $Y_1$  in  $S_1$ . In the intersection  $S_1 \cap S_2$  one has  $Y_2(z) = Y_1(z)C_1$ , where  $C_1$  is the corresponding Stokes matrix.

**Theorem 1** (Ilyashenko–Khovanskii [11]). *The matrix  $C_1$  belongs to the group  $\mathbf{G}_a$ .*

In fact, all the Stokes matrices  $C_m$  are elements of the group  $\mathbf{G}_a$ . Indeed, let  $\mathbf{G}_{S_m}(Y_m)$  be the matrix group that represents the local differential Galois group of the system (1) at the point  $a$ , with respect to the fundamental matrix  $Y_m$  in  $S_m$  (so that  $\mathbf{G}_{S_1}(Y_1) = \mathbf{G}_a$ ). In each intersection  $S_m \cap S_{m+1}$  one has  $Y_{m+1}(z) = Y_m(z)C_m$ , and the Stokes matrix  $C_m$  belongs to  $\mathbf{G}_{S_m}(Y_m)$  in analogy to Theorem 1. But all the groups  $\mathbf{G}_{S_m}(Y_m)$  coincides with  $\mathbf{G}_a$ , as can be shown by induction with respect to  $m$ . Note that an analytic continuation of  $Y_{m+1}$  from  $S_{m+1}$  into  $S_m$  is  $Y_m C_m$ , hence

$$\mathbf{G}_{S_{m+1}}(Y_{m+1}) = \mathbf{G}_{S_m}(Y_m C_m) = C_m^{-1} \mathbf{G}_{S_m}(Y_m) C_m = \mathbf{G}_a,$$

since  $C_m \in \mathbf{G}_{S_m}(Y_m)$  and  $\mathbf{G}_{S_m}(Y_m) = \mathbf{G}_a$  by the inductive assumption.

**Remark 1.** M. Loday-Richaud in [18] gave an example of a linear differential  $(2 \times 2)$ -system that is meromorphically equivalent to a system with a non-resonant irregular singularity  $z = 0$  of Poincaré rank 1. The sector  $S = \{\operatorname{Re} z > 0\}$  does not contain any of the two Stokes lines in that example (thus,  $S$  is not "good"), that is why there are two actual fundamental matrices  $Y, \tilde{Y}$  in  $S$  having the same asymptotic. Their quotient  $C = (Y^{-1})\tilde{Y} \in \operatorname{GL}(2, \mathbb{C})$  was called also a Stokes matrix there (corresponding to the two realizations of one formal fundamental matrix in one sector), and it was shown that  $C$  does not belong to the Galois group of the system.

## 4.2 Exponential torus

As was discussed above, the local differential Galois group  $\mathbf{G}_a$  (more precisely, its matrix representation with respect to the fundamental solution  $Y_1 = Y_{S_1}$  in the good sector  $S_1$ ) contains the Stokes matrices  $C_1, \dots, C_{2r}$  as well as the formal monodromy matrix  $e^{2\pi i \Lambda}$  (see the relation (9)), but it also possesses a continuous subgroup, the *exponential torus*.

Consider the differential extension  $\mathcal{M}_a \subset \mathcal{M}_a \langle e^{q_1}, \dots, e^{q_p} \rangle$  of  $\mathcal{M}_a$  generated by the diagonal entries  $e^{q_1}, \dots, e^{q_p}$  of the exponential factor  $e^Q$  from the decomposition (6)

$$\hat{Y}(z) = \hat{F}(z)(z - a)^\Lambda e^{Q(z)}.$$

Let  $\mathcal{T}$  denote the differential Galois group of this (Picard–Vessiot) extension, that is, the group of its differential  $\mathcal{M}_a$ -automorphisms. This group may be described in more details as follows.

Let  $\{p_1, \dots, p_s\}$  be a basis of the  $\mathbb{Z}$ -module  $\mathbb{Z}q_1 + \dots + \mathbb{Z}q_p$ , so that  $\mathcal{M}_a \langle e^{q_1}, \dots, e^{q_p} \rangle = \mathcal{M}_a \langle e^{p_1}, \dots, e^{p_s} \rangle$ . As the logarithmic derivative of each  $e^{p_i}$  belongs to  $\mathcal{M}_a$ , for any element  $\sigma \in \mathcal{T}$  one has

$$\sigma(e^{p_1}) = t_1 e^{p_1}, \quad \dots, \quad \sigma(e^{p_s}) = t_s e^{p_s}, \quad (10)$$

for some  $(t_1, \dots, t_s) \in (\mathbb{C}^*)^s$ . Conversely, any element of  $(\mathbb{C}^*)^s$  defines in this way a differential  $\mathcal{M}_a$ -automorphism of  $\mathcal{M}_a \langle e^{p_1}, \dots, e^{p_s} \rangle$ , since there are no differentially-polynomial relations over  $\mathcal{M}_a$  between the functions  $e^{p_1}, \dots, e^{p_s}$ . Therefore  $\mathcal{T}$  is a torus of dimension  $s$ , that is, a linear algebraic group isomorphic to  $(\mathbb{C}^*)^s$ .

To show the inclusion  $\mathcal{T} \subset \mathbf{G}_a$ , first let us extend any  $\sigma \in \mathcal{T}$  as a differential  $\mathcal{M}_a$ -automorphism of the Picard–Vessiot extension  $\mathcal{M}_a \subset \mathcal{M}_a \langle \hat{Y} \rangle$  generated by all the entries of the matrix  $\hat{Y}$ . The latter extension may be viewed as a differential subfield of a larger differential field

$$\mathcal{L} = \widehat{\mathcal{M}}_a \langle (z - a)^{\lambda_1}, \dots, (z - a)^{\lambda_p}, e^{q_1}, \dots, e^{q_p} \rangle,$$

which is an extension of the field  $\widehat{\mathcal{M}}_a = \mathbb{C}((z - a))$  of formal Laurent series in  $z - a$  with a finite principle part generated by the diagonal entries of the matrices  $(z - a)^\Lambda$  and  $e^{Q(z)}$  regarded as formal objects so far. As

$$\widehat{\mathcal{M}}_a \langle (z - a)^{\lambda_1}, \dots, (z - a)^{\lambda_p} \rangle \cap \widehat{\mathcal{M}}_a \langle e^{q_1}, \dots, e^{q_p} \rangle = \widehat{\mathcal{M}}_a,$$

one can extend any  $\sigma \in \mathcal{T}$  given by (10) onto  $\mathcal{L}$  by letting it be the identity map on  $\widehat{\mathcal{M}}_a \langle (z - a)^{\lambda_1}, \dots, (z - a)^{\lambda_p} \rangle$ . Thus we indeed have a differential  $\widehat{\mathcal{M}}_a$ -automorphism of  $\mathcal{L}$  that preserves the Picard–Vessiot extension  $\mathcal{M}_a \subset \mathcal{M}_a \langle \hat{Y} \rangle$ . By restricting every such an automorphism to  $\mathcal{M}_a \langle \hat{Y} \rangle$  we identify  $\mathcal{T}$  with a subgroup of the differential Galois group of the Picard–Vessiot extension  $\mathcal{M}_a \subset \mathcal{M}_a \langle \hat{Y} \rangle$ . It remains to note that the latter group is isomorphic to  $\mathbf{G}_a$ , as the extensions  $\mathcal{M}_a \subset \mathcal{M}_a \langle \hat{Y} \rangle$  and  $\mathcal{M}_a \subset \mathcal{M}_a \langle Y_1 \rangle$  are isomorphic (this is provided by the mapping

of  $\widehat{Y}$  to  $Y_1$  via the operation of *r-summability* which maps here the formal factor  $\widehat{F}$  of  $\widehat{Y}$  to its *sum*  $F_1$  in the sector  $S_1$ ).

Finally, let us write the matrix  $T_\sigma$  representing an element  $\sigma \in \mathcal{T}$  given by (10) with respect to the fundamental solution  $Y_1$ . Let

$$\begin{aligned} q_1(z) &= m_{11}p_1(z) + \dots + m_{1s}p_s(z), \\ &\dots \quad \dots \\ q_p(z) &= m_{p1}p_1(z) + \dots + m_{ps}p_s(z), \end{aligned}$$

for some  $m_{ij} \in \mathbb{Z}$ . Then

$$\sigma(e^{q_i}) = \sigma(e^{m_{i1}p_1} \dots e^{m_{is}p_s}) = t_1^{m_{i1}} e^{m_{i1}p_1} \dots t_s^{m_{is}} e^{m_{is}p_s} = t_1^{m_{i1}} \dots t_s^{m_{is}} e^{q_i},$$

hence

$$T_\sigma = \text{diag}(\tau_1(t), \dots, \tau_p(t)),$$

where  $t = (t_1, \dots, t_s) \in (\mathbb{C}^*)^s$ ,  $\tau_i(t) = t_1^{m_{i1}} \dots t_s^{m_{is}}$ . Note that as the functions  $q_1, \dots, q_p$  are pairwise distinct, the monomials  $\tau_1(t), \dots, \tau_p(t)$  are pairwise distinct too.

## 5 Linear differential systems and meromorphic connections on holomorphic vector bundles

Let us recall some notions concerning holomorphic vector bundles and meromorphic connections in a context of linear differential equations. Here we mainly follow [3] or [12, Ch. 3] (see also [8]).

In an analytic interpretation, a holomorphic bundle  $E$  of rank  $p$  over the Riemann sphere is defined by a *cocycle*  $\{g_{\alpha\beta}(z)\}$ , that is, a collection of holomorphic matrix functions corresponding to a covering  $\{U_\alpha\}$  of the Riemann sphere:

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \text{GL}(p, \mathbb{C}), \quad U_\alpha \cap U_\beta \neq \emptyset,$$

and satisfying the conditions

$$g_{\alpha\beta} = g_{\beta\alpha}^{-1}, \quad g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = I \quad (\text{for } U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset).$$

Two holomorphically equivalent cocycles  $\{g_{\alpha\beta}(z)\}, \{g'_{\alpha\beta}(z)\}$  define the same bundle. Equivalence of cocycles means that there exists a set  $\{h_\alpha(z)\}$  of holomorphic matrix functions  $h_\alpha : U_\alpha \longrightarrow \text{GL}(p, \mathbb{C})$  such that

$$h_\alpha(z)g_{\alpha\beta}(z) = g'_{\alpha\beta}(z)h_\beta(z). \quad (11)$$

A *section*  $s$  of the bundle  $E$  is determined by a set  $\{s_\alpha(z)\}$  of vector functions  $s_\alpha : U_\alpha \longrightarrow \mathbb{C}^p$  that satisfy the conditions  $s_\alpha(z) = g_{\alpha\beta}(z)s_\beta(z)$  in intersections  $U_\alpha \cap U_\beta \neq \emptyset$ .

A *meromorphic connection*  $\nabla$  on the holomorphic vector bundle  $E$  is determined by a set  $\{\omega_\alpha\}$  of matrix meromorphic differential 1-forms that are defined in the corresponding neighbourhoods  $U_\alpha$  and satisfy gluing conditions

$$\omega_\alpha = (dg_{\alpha\beta})g_{\alpha\beta}^{-1} + g_{\alpha\beta}\omega_\beta g_{\alpha\beta}^{-1} \quad (\text{for } U_\alpha \cap U_\beta \neq \emptyset). \quad (12)$$

Under a transition to an equivalent cocycle  $\{g'_{\alpha\beta}\}$  connected with the initial one by the relations (11), the 1-forms  $\omega_\alpha$  of the connection  $\nabla$  are transformed into the corresponding 1-forms

$$\omega'_\alpha = (dh_\alpha)h_\alpha^{-1} + h_\alpha\omega_\alpha h_\alpha^{-1}. \quad (13)$$

Conversely, the existence of holomorphic matrix functions  $h_\alpha : U_\alpha \rightarrow \text{GL}(p, \mathbb{C})$  such that the matrix 1-forms  $\omega_\alpha$  and  $\omega'_\alpha$  (satisfying the conditions (12) for  $g_{\alpha\beta}$  and  $g'_{\alpha\beta}$  respectively) are connected by the relation (13) in  $U_\alpha$ , indicates the equivalence of the cocycles  $\{g_{\alpha\beta}\}$  and  $\{g'_{\alpha\beta}\}$  (one may assume that the intersections  $U_\alpha \cap U_\beta$  do not contain singular points of the connection).

Vector functions  $s_\alpha(z)$  satisfying linear differential equations  $ds_\alpha = \omega_\alpha s_\alpha$  in the corresponding  $U_\alpha$ , by virtue of the conditions (12) can be chosen in such a way that a set  $\{s_\alpha(z)\}$  determines a section of the bundle  $E$ , which is called *horizontal* with respect to the connection  $\nabla$ . Thus horizontal sections of a holomorphic vector bundle with a meromorphic connection are determined by solutions of local linear differential systems. The *monodromy of a connection* (the monodromy group) characterizes ramification of horizontal sections under their analytic continuation along loops in  $\overline{\mathbb{C}}$  not containing singular points of the connection 1-forms and is defined similarly to the monodromy group of the system (1). A connection may be called *Fuchsian* (*logarithmic*), *regular* or *irregular* depending on the type of the singular points of its 1-forms (as singular points of linear differential systems).

If a bundle is holomorphically trivial (all matrices of the cocycle can be taken as the identity matrices), then by virtue of the conditions (12) the matrix 1-forms of a connection coincide on non-empty intersections  $U_\alpha \cap U_\beta$ . Hence horizontal sections of such a bundle are solutions of a global linear differential system defined on the whole Riemann sphere. Conversely, the linear system (1) determines a meromorphic connection on the holomorphically trivial vector bundle of rank  $p$  over  $\overline{\mathbb{C}}$ . It is understood that such a bundle has the standard definition by the cocycle that consists of the identity matrices while the connection is defined by the matrix 1-form  $B(z)dz$  of coefficients of the system. But for us it will be more convenient to use the following equivalent coordinate description (a construction already appearing in [3]).

At first we consider a covering  $\{U_\alpha\}$  of the punctured Riemann sphere  $\overline{\mathbb{C}} \setminus \{a_1, \dots, a_n\}$  by simply connected neighbourhoods. Then on the corresponding non-empty intersections  $U_\alpha \cap U_\beta$  one defines the matrix functions of a cocycle,  $g'_{\alpha\beta}(z) \equiv \text{const}$ , which are expressed in terms of the monodromy matrices  $M_1, \dots, M_n$  of the system (1) *via* the operations of multiplication and taking the inverse (see [8]). In this case the matrix differential 1-forms  $\omega'_\alpha$  defining a connection are equal to zero. Further the covering  $\{U_\alpha\}$  is complemented by small neighbourhoods  $O_i$  of the singular points  $a_i$  of the system, thus we obtain the covering of the Riemann sphere  $\overline{\mathbb{C}}$ . To non-empty intersections  $O_i \cap U_\alpha$  there correspond matrix functions  $g'_{i\alpha}(z) = Y_i(z)$  of the cocycle, where  $Y_i(z)$  is a germ of a fundamental matrix of the system whose monodromy matrix at the point  $a_i$  is equal to  $M_i$  (so, for analytic continuations of the chosen germ to non-empty intersections  $O_i \cap U_\alpha \cap U_\beta$  the cocycle relations  $g_{i\alpha}g_{\alpha\beta} = g_{i\beta}$  hold). The matrix differential 1-forms  $\omega'_i$  determining the connection in the neighbourhoods  $O_i$  coincide with the 1-form  $B(z)dz$  of coefficients of the system. To prove holomorphic equivalence of the cocycle  $\{g'_{\alpha\beta}, g'_{i\alpha}\}$  to the identity cocycle it is sufficient to check the existence of holomorphic matrix functions

$$h_\alpha : U_\alpha \rightarrow \text{GL}(p, \mathbb{C}), \quad h_i : O_i \rightarrow \text{GL}(p, \mathbb{C}),$$

such that

$$\omega'_\alpha = (dh_\alpha)h_\alpha^{-1} + h_\alpha\omega_\alpha h_\alpha^{-1}, \quad \omega'_i = (dh_i)h_i^{-1} + h_i\omega_i h_i^{-1}. \quad (14)$$

Since we have  $\omega_\alpha = B(z)dz$  and  $\omega'_\alpha = 0$  for all  $\alpha$ , the first equation in (14) is rewritten as a linear system

$$d(h_\alpha^{-1}) = (B(z)dz)h_\alpha^{-1},$$

which has a holomorphic solution  $h_\alpha^{-1} : U_\alpha \rightarrow \text{GL}(p, \mathbb{C})$  since the 1-form  $B(z)dz$  is holomorphic in a simply connected neighbourhood  $U_\alpha$ . The second equation in (14) has a holomorphic solution  $h_i(z) \equiv I$ , as  $\omega_i = \omega'_i = B(z)dz$ .

One says that a bundle  $E$  has a subbundle  $E' \subset E$  of rank  $k < p$  that is *stabilized* by a connection  $\nabla$ , if the pair  $(E, \nabla)$  admits a coordinate description  $\{g_{\alpha\beta}\}, \{\omega_\alpha\}$  of the following blocked upper-triangular form:

$$g_{\alpha\beta} = \begin{pmatrix} g_{\alpha\beta}^1 & * \\ 0 & g_{\alpha\beta}^2 \end{pmatrix}, \quad \omega_\alpha = \begin{pmatrix} \omega_\alpha^1 & * \\ 0 & \omega_\alpha^2 \end{pmatrix},$$

where  $g_{\alpha\beta}^1$  and  $\omega_\alpha^1$  are blocks of size  $k \times k$  (then the cocycle  $\{g_{\alpha\beta}^1\}$  defines the subbundle  $E'$  and the 1-forms  $\omega_\alpha^1$  define the restriction  $\nabla'$  of the connection  $\nabla$  to the subbundle  $E'$ ).

**Example 1.** Consider a system (1) whose monodromy matrices  $M_1, \dots, M_n$  are of the same blocked upper-triangular form, and the corresponding holomorphically trivial vector bundle  $E$  with the meromorphic connection  $\nabla$ . Suppose that in a neighbourhood of each singular point  $a_i$  of the system there exist a fundamental matrix  $Y_i(z)$  whose monodromy matrix is  $M_i$ , and a holomorphically invertible matrix  $\Gamma_i(z)$  such that  $\Gamma_i Y_i$  is a blocked upper-triangular matrix (with the same block-structure as the matrix  $M_i$ ). Let us show that to the common invariant subspace of the monodromy matrices there corresponds a vector subbundle  $E' \subset E$  that is stabilized by the connection  $\nabla$ .

We use the above coordinate description of the bundle and connection with the cocycle  $\{g'_{\alpha\beta}, g'_{i\alpha}\}$  and set  $\{\omega'_\alpha, \omega'_i\}$  of matrix 1-forms. The matrices  $g'_{\alpha\beta}$  are already blocked upper-triangular since the monodromy matrices  $M_1, \dots, M_n$  are (and  $\omega'_\alpha = 0$ ), while the matrices  $g'_{i\alpha} = Y_i$  can be transformed to such a form,  $\Gamma_i g'_{i\alpha} = \Gamma_i Y_i$ . Thus changing the matrices  $g'_{i\alpha}$  onto  $\Gamma_i g'_{i\alpha}$  and matrix 1-forms  $\omega'_i$  onto

$$\Gamma_i \omega'_i \Gamma_i^{-1} + (d\Gamma_i) \Gamma_i^{-1},$$

we pass to the holomorphically equivalent coordinate description with the cocycle matrices and connection matrix 1-forms having the same blocked upper-triangular form.

The following auxiliary lemma points to a certain block-structure of a linear differential system in the case when the corresponding holomorphically trivial vector bundle with the meromorphic connection has a *holomorphically trivial* subbundle that is stabilized by the connection.

**Lemma 1.** *If a holomorphically trivial vector bundle  $E$  of rank  $p$  over  $\overline{\mathbb{C}}$  endowed with a meromorphic connection  $\nabla$  has a holomorphically trivial subbundle  $E' \subset E$  of rank  $k$  that is stabilized by the connection, then the corresponding linear system (1) is reduced to a blocked upper-triangular form via a constant gauge transformation  $\tilde{y}(z) = Cy(z)$ ,  $C \in \text{GL}(p, \mathbb{C})$ , that is,*

$$CB(z)C^{-1} = \begin{pmatrix} B'(z) & * \\ 0 & * \end{pmatrix},$$

where  $B'(z)$  is a block of size  $k \times k$ .

**Proof.** Let  $\{s_1, \dots, s_p\}$  be a basis of global holomorphic sections of the bundle  $E$  (which are linear independent at each point  $z \in \overline{\mathbb{C}}$ ) such that the 1-form of the connection  $\nabla$  in this basis is the 1-form  $B(z)dz$  of coefficients of the linear system. Consider also a basis  $\{s'_1, \dots, s'_p\}$  of global holomorphic sections of the bundle  $E$  such that  $s'_1, \dots, s'_k$  are sections of the subbundle  $E'$ ,  $(s'_1, \dots, s'_p) = (s_1, \dots, s_p)C^{-1}$ ,  $C \in \text{GL}(p, \mathbb{C})$ .

Now choose a basis  $\{h_1, \dots, h_p\}$  of sections of the bundle  $E$  such that these are horizontal with respect to the connection  $\nabla$  and  $h_1, \dots, h_k$  are sections of the subbundle  $E'$  (this is possible

since  $E'$  is stabilized by the connection). Let  $Y(z)$  be a fundamental matrix of the system whose columns are the coordinates of the sections  $h_1, \dots, h_p$  in the basis  $\{s_1, \dots, s_p\}$ . Then

$$\tilde{Y}(z) = CY(z) = \begin{pmatrix} k \times k & * \\ 0 & * \end{pmatrix}$$

is a blocked upper-triangular matrix, since its columns are the coordinates of the sections  $h_1, \dots, h_p$  in the basis  $\{s'_1, \dots, s'_p\}$ . Consequently, the transformation  $\tilde{y}(z) = Cy(z)$  reduces the initial system to a blocked upper-triangular form.  $\square$

The degree  $\deg E$  (which is an integer) of a holomorphic vector bundle  $E$  endowed with a meromorphic connection  $\nabla$  may be defined as the sum

$$\deg E = \sum_{i=1}^n \operatorname{res}_{a_i} \operatorname{tr} \omega_i$$

of the residues of local differential 1-forms  $\operatorname{tr} \omega_i$  over all singular points of the connection (the notation "tr" is for the trace), where  $\omega_i$  is the local matrix differential 1-form of the connection  $\nabla$  in a neighbourhood of its singular point  $a_i$ . Later when calculating the degree of a bundle we will apply the Liouville formula  $\operatorname{tr} \omega_i = d \ln \det Y_i$ , where  $Y_i$  is a fundamental matrix of the local linear differential system  $dy = \omega_i y$ .

## 6 Solvability of a regular singular system with small exponents

### 6.1 Solvability by quadratures

Consider a system (1) with *regular* singular points  $a_1, \dots, a_n$  of Poincaré rank  $r_1, \dots, r_n$  respectively,

$$\frac{dy}{dz} = B(z)y, \quad y(z) \in \mathbb{C}^p.$$

Here we prove that if the real part of the exponents of (1) is sufficiently small, then generically solvability of the system (1) by quadratures implies a simple condition on the coefficient matrix  $B$ .

**Theorem 2.** *Let for some  $k \in \{1, \dots, p-1\}$  the exponents  $\beta_i^j$  of the regular singular system (1) satisfy the condition*

$$\operatorname{Re} \beta_i^j > -1/nk, \quad i = 1, \dots, n, \quad j = 1, \dots, p, \quad (15)$$

*and for each  $i = 1, \dots, n$ ,  $\beta_i^j - \beta_i^l \notin \mathbb{Q} \setminus \mathbb{Z}$ . Then solvability of the system (1) by quadratures implies existence of a constant matrix  $C \in \operatorname{GL}(p, \mathbb{C})$  such that the matrix  $CB(z)C^{-1}$  is of the following blocked form:*

$$CB(z)C^{-1} = \begin{pmatrix} B'(z) & * \\ 0 & * \end{pmatrix},$$

*where  $B'(z)$  is an upper-triangular matrix of size  $k \times k$ .*

**Remark 2.** Though the inequalities (15) restrict the real parts of the exponents from below, together with the estimates (5) they provide boundedness from above.

**Remark 3.** The sum of the Poincaré ranks of a regular singular system whose exponents satisfy the condition (15) is indeed restricted because of the Fuchs inequalities (5), namely,  $\sum_{i=1}^n r_i < p/k$ .

The proof of the theorem is based on two auxiliary lemmas.

**Lemma 2.** *Let the exponents  $\beta_i^j$  of the regular singular system (1) satisfy the condition (15). If the monodromy matrices of this system are upper-triangular, then there is a constant matrix  $C \in \text{GL}(p, \mathbb{C})$  such that the matrix  $CB(z)C^{-1}$  has the form announced in Theorem 2.*

**Proof.** We use a geometric interpretation (exposed in the previous section) according to which to the regular singular system (1) there corresponds a holomorphically trivial vector bundle  $E$  of rank  $p$  over the Riemann sphere endowed with a meromorphic connection  $\nabla$  with the regular singular points  $a_1, \dots, a_n$ .

Since the monodromy matrices  $M_1, \dots, M_n$  of the system are upper-triangular there exists, as shown in Example 1, a flag  $E^1 \subset E^2 \subset \dots \subset E^p = E$  of subbundles of rank  $1, 2, \dots, p$  respectively that are stabilized by the connection  $\nabla$ . Indeed, a fundamental matrix  $Y$  determining the monodromy matrices  $M_1, \dots, M_n$  of the system can be represented near each singular point  $a_i$  in the form

$$Y(z) = T_i(z)(z - a_i)^{E_i}, \quad E_i = (1/2\pi\mathbf{i}) \ln M_i,$$

where  $T_i$  is a meromorphic matrix at the point  $a_i$ . This matrix can be factored as  $T_i(z) = V_i(z)P_i(z)$ , with a holomorphically invertible matrix  $V_i$  at  $a_i$  and an upper-triangular matrix  $P_i$  which is a polynomial in  $(z - a_i)^{\pm 1}$  (see, for example, [7, Lemma 1]). Thus the matrix  $V_i^{-1}Y$  is upper-triangular.

Let us estimate the degree of each subbundle  $E^j$ ,  $j \leq k$ , noting that in a neighbourhood of each singular point  $a_i$  the initial system is transformed *via* a holomorphically invertible gauge transformation in a system with a fundamental matrix of the form

$$Y_i(z) = \begin{pmatrix} U_i'(z) & * \\ 0 & * \end{pmatrix} (z - a_i) \begin{pmatrix} A_i' & 0 \\ 0 & * \end{pmatrix} (z - a_i) \begin{pmatrix} E_i' & * \\ 0 & * \end{pmatrix}$$

such that the matrix  $Y_i'(z) = U_i'(z)(z - a_i)^{A_i'}(z - a_i)^{E_i'}$  is a Levelt fundamental matrix for a linear system of  $j$  equations with the regular singular point  $a_i$ . The matrix 1-form of coefficients of this system in a neighbourhood of  $a_i$  is a local 1-form of the restriction  $\nabla^j$  of the connection  $\nabla$  to the subbundle  $E^j$ , and the exponents  $\tilde{\beta}_i^1, \dots, \tilde{\beta}_i^j$  of this system (the eigenvalues of the matrix  $A_i' + E_i'$ ) form a subset of the exponents of the initial system at  $a_i$ . Therefore,

$$\text{Re } \tilde{\beta}_i^l > -1/nk, \quad l = 1, \dots, j.$$

The degree of the holomorphically trivial vector bundle  $E^p$  is equal to zero, and for  $j \leq k$  one has:

$$\begin{aligned} \deg E^j &= \sum_{i=1}^n \text{res}_{a_i} d \ln \det Y_i' = \sum_{i=1}^n \text{ord}_{a_i} \det U_i' + \sum_{i=1}^n \text{tr}(A_i' + E_i') = \\ &= \sum_{i=1}^n \text{ord}_{a_i} \det U_i' + \sum_{i=1}^n \sum_{l=1}^j \text{Re } \tilde{\beta}_i^l > -j/k \geq -1. \end{aligned}$$

As the degree of a subbundle of a holomorphically trivial vector bundle is non-positive, one has  $\deg E^j = 0$ , hence all the subbundles  $E^1 \subset \dots \subset E^k$  are holomorphically trivial (a subbundle of a holomorphically trivial vector bundle is holomorphically trivial, if its degree is equal to zero, see [12, Lemma 19.16]). Now the assertion of the lemma follows from Lemma 1.  $\square$

**Definition 1.** A matrix will be called  $N$ -resonant,  $N \in \mathbb{N}$ , if it has two eigenvalues  $\lambda_1 \neq \lambda_2$  such that  $\lambda_1^N = \lambda_2^N$ , that is,

$$|\lambda_1| = |\lambda_2|, \quad |\arg \lambda_1 - \arg \lambda_2| = \frac{2\pi}{N} j, \quad j \in \{1, 2, \dots, N-1\}.$$

Let a group  $\mathbf{M} \subset \mathrm{GL}(p, \mathbb{C})$  be generated by matrices  $M_1, \dots, M_n$ . If these matrices are sufficiently close to the identity (in the Euclidean topology) then existence of a solvable normal subgroup of finite index in  $\mathbf{M}$  implies their triangularity, see Theorem 2.7 [15, Ch. 6]. According to the remark following this theorem in [15], the requirement of proximity of the matrices  $M_i$  to the identity can be weakened as follows.

**Lemma 3.** *There is a number  $N = N(p)$  such that if the matrices  $M_1, \dots, M_n$  are not  $N$ -resonant, then existence of a solvable normal subgroup of finite index in  $\mathbf{M}$  implies their triangularity.*

**Proof of Theorem 2.** From the theorem assumptions it follows that the identity component  $\mathbf{G}^0$  of the differential Galois group  $\mathbf{G}$  of the system (1) is solvable, hence  $\mathbf{G}^0$  is a solvable normal subgroup of  $\mathbf{G}$  of finite index. Then the monodromy group  $\mathbf{M}$  of this system also has a solvable normal subgroup of finite index, namely  $\mathbf{M} \cap \mathbf{G}^0$ .

As follows from the definition of the exponents  $\beta_i^j$  of a linear differential system at a regular singular point  $a_i$ , these are connected with the eigenvalues  $\mu_i^j$  of the monodromy matrix  $M_i$  by the relation

$$\mu_i^j = \exp(2\pi \mathbf{i} \beta_i^j).$$

Therefore,

$$\mu_i^j = \exp(2\pi \mathbf{i} (\mathrm{Re} \beta_i^j + \mathbf{i} \mathrm{Im} \beta_i^j)) = e^{-2\pi \mathrm{Im} \beta_i^j} (\cos(2\pi \mathrm{Re} \beta_i^j) + \mathbf{i} \sin(2\pi \mathrm{Re} \beta_i^j)),$$

and for any  $N \in \mathbb{N}$ , the matrices  $M_i$  are non  $N$ -resonant by the conditions on the numbers  $\beta_i^j$ . Now the assertion of the theorem follows from Lemma 2 and Lemma 3.  $\square$

As a consequence of Theorem 2 we obtain the following refinement of the Ilyashenko–Khovanskii theorem on solvability by quadratures of a Fuchsian system with small residue matrices.

**Theorem 3.** *Let the eigenvalues  $\beta_i^j$  of the residue matrices  $B_i$  of the Fuchsian system (2) satisfy the condition*

$$\mathrm{Re} \beta_i^j > -\frac{1}{n(p-1)}, \quad i = 1, \dots, n, \quad j = 1, \dots, p, \quad (16)$$

and for each  $i = 1, \dots, n$ ,  $\beta_i^j - \beta_i^l \notin \mathbb{Q} \setminus \mathbb{Z}$ . Then solvability of the Fuchsian system (2) by quadratures is equivalent to the simultaneous triangularity of the matrices  $B_i$ .

**Proof.** The necessity of simultaneous triangularity is a direct consequence of Theorem 2, since the exponents of the Fuchsian system (2) at  $a_i$  coincide with the eigenvalues of the residue matrix  $B_i$ . Sufficiency follows from a general fact that any linear differential system with an (upper-) triangular coefficient matrix is solvable by quadratures (one should begin to solve it with the last equation).  $\square$

**Remark 4.** The inequalities (16) restricting the real part of the exponents of the Fuchsian system from below also provide its boundedness from above because of the Fuchs relation  $\sum_{i=1}^n \sum_{j=1}^p \beta_i^j = 0$  (see (5)). Namely,

$$-\frac{1}{n(p-1)} < \operatorname{Re} \beta_i^j < \frac{np-1}{n(p-1)}.$$

In particular, the integer parts  $\varphi_i^j$  (so-called *Levelt's valuations*) of the numbers  $\operatorname{Re} \beta_i^j$  for such a system have to belong to the set  $\{-1, 0, 1\}$ .

**Remark 5.** If each residue matrix  $B_i$  of the Fuchsian system (2) only has one eigenvalue  $\beta_i$ , then solvability of this system by quadratures is also equivalent to the simultaneous triangularity of the matrices  $B_i$  (regardless of the values  $\operatorname{Re} \beta_i$ ). Indeed, in this case each monodromy matrix  $M_i$  only has one eigenvalue  $\mu_i = e^{2\pi i \beta_i}$ , hence is not  $N$ -resonant. Therefore solvability implies the simultaneous triangularity of the monodromy matrices and existence of a flag  $E^1 \subset E^2 \subset \dots \subset E^p = E$  of subbundles of the holomorphically trivial vector bundle  $E$  that are stabilized by the logarithmic connection  $\nabla$  (corresponding to the Fuchsian system). Since  $\deg E = \sum_{i=1}^n p\beta_i = 0$ , the degree  $\sum_{i=1}^n j\beta_i$  of each subbundle  $E^j$  is zero and all these subbundles are holomorphically trivial.

It is natural to expect that for a general Fuchsian system (with no restrictions on the exponents) solvability by quadratures not necessarily implies simultaneous triangularity of the residue matrices. This is indeed illustrated by the following example of A. Bolibrukh.

**Example 2** (A. Bolibrukh [2, Prop. 5.1.1]). There exist points  $a_1, a_2, a_3, a_4$  on the Riemann sphere and a Fuchsian system with singularities at these points, whose monodromy matrices are

$$M_1 = \begin{pmatrix} 1 & 1 & 1 & -1 & 0 & 0 & -1 \\ 0 & -1 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 2 & 2 & 2 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & -1 & 1 & 1 & -1 & 1 & -1 \\ 0 & 0 & -1 & -1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix},$$

$$M_3 = \begin{pmatrix} 1 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & -1 & -1 & 1 & -1 & 1 & 2 \\ 0 & 0 & 1 & 1 & -1 & 1 & 2 \\ 0 & 0 & 0 & -1 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \quad M_4 = \begin{pmatrix} 1 & 0 & 1 & -1 & 1 & 1 & 0 \\ 0 & -1 & 1 & -2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix},$$

whereas the coefficient matrix of this system is *not* upper-triangular. Moreover, this system cannot be transformed in an upper-triangular form neither *via* a constant gauge transformation nor even a meromorphic (rational) one preserving the singular points  $a_1, a_2, a_3, a_4$  and the pole order ( $= 1$ ) of a coefficient matrix at these points. Thus the residue matrices of this Fuchsian system are not triangular simultaneously though the system is solvable by quadratures, since its monodromy group generated by the triangular matrices is solvable.

We notice that Theorem 3 does not apply to this example as the exponents of such a system cannot satisfy the conditions (16). Indeed, for any exponent  $\beta_i^j = \varphi_i^j + \rho_i^j$  one has

$$\rho_i^j = \frac{1}{2\pi i} \ln \mu_i^j, \quad \mu_i^j \in \{-1, 1\},$$

hence  $\operatorname{Re} \rho_i^j$  is equal to 0 (for  $\mu_i^j = 1$ ) or  $1/2$  (for  $\mu_i^j = -1$ ). The inequalities (16) imply (for  $n = 4, p = 7$ )

$$\operatorname{Re} \beta_i^j > -\frac{1}{24},$$

hence  $\varphi_i^j$  is equal to 0 or 1 (see Remark 4). Therefore the sum of the exponents over all singular points is positive, which contradicts the Fuchs relation.

## 6.2 Other types of solvability

At the end of Section 2 we gave Kolchin's criterion for a linear differential system to be solvable with respect to other types of solvability formulated in terms of the differential Galois group. For a Fuchsian system with sufficiently small residue matrices, Ilyashenko and Khovanskii [10] (see also [15, Ch. 6, §2.3]) have converted these conditions on the differential Galois group to conditions on the residue matrices. Now this may be formulated in accordance with Theorem 3, while a proof is mainly still the same.

**Theorem 4.** *In the assumptions of Theorem 3, the Fuchsian system (2) is*

- 1) *solvable by integrals and algebraic functions<sup>1</sup> if and only if all the matrices  $B_i$  are (simultaneously) triangular and the eigenvalues of each  $B_i$  are rational numbers differing by an integer;*
- 2) *solvable by integrals if and only if all the matrices  $B_i$  are triangular and their eigenvalues are equal to zero;*
- 3) *solvable by exponentials of integrals and algebraic functions<sup>2</sup> if and only if all the matrices  $B_i$  are diagonal;*
- 4) *solvable by algebraic functions<sup>3</sup> if and only if all the matrices  $B_i$  are diagonal and the eigenvalues of each  $B_i$  are rational numbers differing by an integer.*

**Proof.** We note that each type of solvability implies solvability by quadratures, hence by Theorem 3 we have simultaneous triangularity of the residue matrices  $B_i$ . Therefore the differential Galois group  $\mathbf{G}$  of the system is triangular and we may apply Kolchin's criterion from Section 2. As the differential Galois group of a Fuchsian system coincides with the closure of its monodromy group, each condition of the criterion is equivalent to the same condition on the monodromy matrices (each condition determines a closed set in the Zariski topology).

So, the eigenvalues of all the matrices from  $\mathbf{G}$  are roots of the identity if and only if the eigenvalues of the monodromy matrices are, and therefore the eigenvalues of all the residue matrices  $B_i$  are rational numbers (recall that the eigenvalues of a monodromy matrix  $M_i$  are  $e^{2\pi i \beta_i^1}, \dots, e^{2\pi i \beta_i^p}$ , where  $\beta_i^1, \dots, \beta_i^p$  are the eigenvalues of a matrix  $B_i$ ). Each difference  $\beta_i^j - \beta_i^l$  should be integer according to the assumption  $\beta_i^j - \beta_i^l \notin \mathbb{Q} \setminus \mathbb{Z}$  of the theorem.

The eigenvalues of all the matrices from  $\mathbf{G}$  are equal to the identity if and only if the eigenvalues of the monodromy matrices are, and therefore the eigenvalues of all the residue matrices  $B_i$  are integer numbers. They cannot be negative in view of (16), but they also cannot be positive in view of the Fuchs relation  $\sum_{i=1}^n \sum_{j=1}^p \beta_i^j = 0$  (if one of them is positive, then there would be another one that is negative).

Finally,  $\mathbf{G}$  is diagonal if and only if all the monodromy matrices are. Therefore a holomorphically trivial vector bundle  $E$  over  $\overline{\mathbb{C}}$  (with a meromorphic connection  $\nabla$ ) which corresponds to the system (2) is a direct sum of line bundles stabilized by  $\nabla$ . Like in the proof of Lemma 2,

<sup>1</sup>In the assumptions of the theorem, this type of solvability is equivalent to solvability by integrals and radicals.

<sup>2</sup>In the assumptions of the theorem, this type of solvability is equivalent to solvability by exponentials of integrals.

<sup>3</sup>In the assumptions of the theorem, this type of solvability is equivalent to solvability by radicals.

it follows from (16) that each of these line bundles is holomorphically trivial. Then an evident modification of Lemma 1 proves reducibility of the Fuchsian system (2) to a diagonal form *via* a constant gauge transformation.

It is understood that for each type of solvability the sufficiency of the corresponding condition also may be proved with no use of differential Galois theory and follows from a method of solving of first order non-homogeneous linear differential equations (one begins to solve a Fuchsian system having an upper triangular coefficient matrix with the last equation and uses the method of variation of constant).  $\square$

It turns out that the class of functions Liouvillian over the field  $K = \mathbb{C}(z)$  is closed with respect to the non-algebraic operation of composition. For example, if  $f$  is Liouvillian then  $\ln f$  is also Liouvillian, as it is an integral of  $f'/f$ . For an algebraic function  $g$ , the function  $g \circ f$  is Liouvillian:  $g$  satisfies an algebraic equation  $P(z, g(z)) = 0$  over  $K$ , hence  $g \circ f$  satisfies  $P(f(z), g \circ f(z)) = 0$ , that is algebraic over  $K\langle f \rangle$ . However, if we add to the field  $K$  some non-Liouvillian function, a situation will change. Consider a field  $K\langle F \rangle$ , where  $F$  is a meromorphic non-Liouvillian function. Then the class of functions Liouvillian over this field will not be closed with respect to composition. For example,  $\ln F$  is Liouvillian over  $K\langle F \rangle$ , whereas  $F(\ln z)$  is not. Hence, non-solvability of a linear differential equation by quadratures over the field  $K\langle F \rangle$  does not guarantee that there is no, for example, a solution of the form  $F(\ln z)$ . Khovanskii has developed a topological version of Galois theory, where composition is equally involved, while a role of the Galois group is played by the monodromy group of a function (details are published in [14], [15, Ch. 5]). This theory allows to obtain stronger results on the non-solvability of linear differential systems. In particular, *if the system (1) is solvable by quadratures and meromorphic (in  $\mathbb{C}$ ) functions by using the operation of composition, then its monodromy group possesses a solvable normal subgroup of finite index* (see [15, Ch. 6, Cor. 2.13]). Systems that cannot be solved in such a way are called *strongly non-solvable*. Thus we have the following statement.

*In the assumptions of Theorem 3, the Fuchsian system (2) with non-triangular residue matrices  $B_i$  is strongly non-solvable.*

## 7 Solvability of a non-resonant irregular system with small formal exponents

### 7.1 Solvability by quadratures

Consider a system (1) with *non-resonant* irregular singular points  $a_1, \dots, a_n$  of Poincaré rank  $r_1, \dots, r_n$  respectively,

$$\frac{dy}{dz} = B(z)y, \quad y(z) \in \mathbb{C}^p.$$

If the real part of the formal exponents of this system is sufficiently small, then the following criterion of solvability by quadratures holds.

**Theorem 5.** *Let at each singular point  $a_i$  the formal exponents  $\lambda_i^j$  of the irregular system (1) satisfy the condition*

$$\operatorname{Re} \lambda_i^j > -\frac{1}{n(p-1)}.$$

*Then this system is solvable by quadratures if and only if there is a constant matrix  $C \in \operatorname{GL}(p, \mathbb{C})$  such that  $CB(z)C^{-1}$  is upper-triangular.*

**Proof.** As in Theorem 3, sufficiency does not require a special proof. Let us prove necessity.

Consider a fundamental matrix  $Y$  of the system (1) in a neighbourhood of a non-singular point  $z_0$  and the representation of the differential Galois group  $\mathbf{G}$  with respect to this matrix (the entries of  $Y$  are regarded as elements of the field of meromorphic functions at  $z_0$ ). For each singular point  $a_i$ , let  $Y_1^{a_i}, \dots, Y_{2r_i}^{a_i}$  be the fundamental matrices of the system defined respectively in the sectors  $S_1^{a_i}, \dots, S_{2r_i}^{a_i}$  of a good covering of a neighbourhood of  $a_i$  (these were defined at the end of Section 3, without the upper index  $a_i$ , which underlines here that the involved sectors and matrices correspond to a singular point  $a_i$ ).

The connection matrix between an analytic continuation  $\tilde{Y}_{S_1^{a_i}}$  of  $Y$  into a neighbourhood of each  $a_i$  (more precisely, into each first sector  $S_1^{a_i}$ ) and the corresponding  $Y_1^{a_i}$  is denoted by  $P_i$ :

$$\tilde{Y}_{S_1^{a_i}}(z) = Y_1^{a_i}(z)P_i, \quad i = 1, \dots, n.$$

As follows from the explanations of Section 4, the matrices

$$\tilde{C}_1^{a_i} = P_i^{-1}C_1^{a_i}P_i, \quad \dots, \quad \tilde{C}_{2r_i}^{a_i} = P_i^{-1}C_{2r_i}^{a_i}P_i \quad (i = 1, \dots, n)$$

belong to  $\mathbf{G}$  (here  $C_1^{a_i}, \dots, C_{2r_i}^{a_i}$  denote the Stokes matrices at the point  $a_i$  which correspond to the set of fundamental matrices  $Y_1^{a_i}, \dots, Y_{2r_i}^{a_i}$ ). Furthermore, for any set  $T_1, \dots, T_n$  of diagonal matrices from the exponential tori at the points  $a_1, \dots, a_n$  respectively, the matrices

$$\tilde{T}_i = P_i^{-1}T_iP_i, \quad i = 1, \dots, n,$$

belong to  $\mathbf{G}$ . Recall that each exponential torus consists of all matrices  $T(t) = \text{diag}(\tau_1(t), \dots, \tau_p(t))$ , where  $t = (t_1, \dots, t_s) \in (\mathbb{C}^*)^s$  is an arbitrary parameter ( $s$  is the dimension of the torus) and  $\tau_j$  are pairwise distinct monomials in  $t_1, \dots, t_s$ . Hence for any  $N \in \mathbb{N}$ , the  $N$ -th powers of  $\tau_j$  are pairwise distinct monomials too. Thus, by taking a suitable value of  $t$ , the matrices  $T_1, \dots, T_n$  may be chosen in such a way that the eigenvalues of each  $T_i$  are pairwise distinct and each  $T_i$  is non  $N$ -resonant for any  $N \in \mathbb{N}$ .

Denote by  $\widehat{\mathbf{M}}$  the group generated by the matrices  $\tilde{T}_i, \tilde{C}_1^{a_i}, \dots, \tilde{C}_{2r_i}^{a_i}$  ( $i = 1, \dots, n$ ) over all singular points. As follows from the condition of the theorem, the group  $\mathbf{G}$  possesses the solvable normal subgroup  $\mathbf{G}^0$  of finite index. Hence the subgroup  $\widehat{\mathbf{M}} \subset \mathbf{G}$  possesses the solvable normal subgroup of finite index,  $\widehat{\mathbf{M}} \cap \mathbf{G}^0$ . For any  $N \in \mathbb{N}$ , the matrices generating the group  $\widehat{\mathbf{M}}$  are non  $N$ -resonant (the matrices  $T_i$  are chosen to be non-resonant, whereas for the Stokes matrices this follows from their unipotence). Therefore, according to Lemma 3, the generators of  $\widehat{\mathbf{M}}$  are simultaneously reduced to an upper-triangular form by conjugating to some invertible matrix  $\tilde{C}$ . We may assume that they are already upper-triangular (otherwise we would consider the fundamental matrix  $Y\tilde{C}$  instead of  $Y$ ). As follows from [6, Ch. VIII, §1], the relation

$$T_i = P_i \tilde{T}_i P_i^{-1},$$

where  $\tilde{T}_i$  is an upper-triangular matrix and  $T_i$  is a diagonal matrix whose diagonal entries are pairwise distinct, implies that the matrix  $P_i$  writes  $P_i = D_i R_i$ , where  $R_i$  is an upper-triangular matrix (the conjugation  $R_i \tilde{T}_i R_i^{-1}$  annuls all the off-diagonal entries of the matrix  $\tilde{T}_i$ ) and  $D_i$  is a permutation matrix for  $T_i$  (that is, the conjugation  $D_i^{-1}T_i D_i$  permutes the diagonal entries of the matrix  $T_i$ ).

In a neighbourhood of each  $a_i$  we pass from the set of fundamental matrices  $Y_1^{a_i}, \dots, Y_{2r_i}^{a_i}$ , which correspond to the sectors  $S_1^{a_i}, \dots, S_{2r_i}^{a_i}$ , to the fundamental matrices

$$\tilde{Y}_m^{a_i}(z) = Y_m^{a_i}(z)P_i \quad (\text{in particular, } \tilde{Y}_1^{a_i} = Y)$$

connected to each other in the intersections  $S_m^{a_i} \cap S_{m+1}^{a_i}$  by the relations

$$\tilde{Y}_{m+1}^{a_i}(z) = \tilde{Y}_m^{a_i}(z) \tilde{C}_m^{a_i}.$$

From the decomposition of  $P_i$  above and from (7) it follows that the matrices  $\tilde{Y}_m^{a_i}$  can be written

$$\tilde{Y}_m^{a_i}(z) = F_m^{a_i}(z)(z - a_i)^{\Lambda_i} e^{Q_i(z)} D_i R_i = F_m^{a_i}(z) D_i (z - a_i)^{\Lambda'_i} e^{Q'_i(z)} R_i,$$

where

$$\Lambda'_i = D_i^{-1} \Lambda_i D_i, \quad Q'_i(z) = D_i^{-1} Q_i(z) D_i$$

are diagonal matrices obtained from the corresponding matrices  $\Lambda_i, Q_i(z)$  by a permutation of the diagonal entries. Therefore, in the intersections  $S_m^{a_i} \cap S_{m+1}^{a_i}$  we have the relations

$$F_{m+1}^{a_i}(z) D_i (z - a_i)^{\Lambda'_i} e^{Q'_i(z)} R_i = F_m^{a_i}(z) D_i (z - a_i)^{\Lambda'_i} e^{Q'_i(z)} R_i \tilde{C}_m^{a_i}.$$

Thus in the sectors  $S_1^{a_i}, \dots, S_{2r_i}^{a_i}$ , which form a covering of a punctured neighbourhood of  $a_i$ , there are holomorphically invertible matrices  $F_1^{a_i}(z) D_i, \dots, F_{2r_i}^{a_i}(z) D_i$  respectively such that in the intersections  $S_m^{a_i} \cap S_{m+1}^{a_i}$  their quotients

$$(F_m^{a_i}(z) D_i)^{-1} F_{m+1}^{a_i}(z) D_i = (z - a_i)^{\Lambda'_i} e^{Q'_i(z)} R_i \tilde{C}_m^{a_i} R_i^{-1} e^{-Q'_i(z)} (z - a_i)^{-\Lambda'_i}$$

are upper-triangular matrices. As for each  $m = 1, \dots, 2r_i$ , the matrix  $F_m^{a_i}(z) D_i$  has the same asymptotic expansion  $\hat{F}_i(z) D_i$  in the corresponding  $S_m^{a_i}$ , there exists a matrix  $\Gamma_i(z)$  holomorphically invertible at  $a_i$  such that all the matrices

$$\tilde{F}_m^{a_i}(z) = \Gamma_i(z) F_m^{a_i}(z) D_i, \quad m = 1, \dots, 2r_i,$$

are upper-triangular (according to [1, Prop. 3]). In particular,

$$\Gamma_i(z) Y(z) = \Gamma_i(z) \tilde{Y}_1^{a_i}(z) = \Gamma_i(z) F_1^{a_i}(z) D_i (z - a_i)^{\Lambda'_i} e^{Q'_i(z)} R_i = \tilde{F}_1^{a_i}(z) (z - a_i)^{\Lambda'_i} e^{Q'_i(z)} R_i$$

is an upper-triangular matrix. Hence for the pair  $(E, \nabla)$  consisting of a holomorphically trivial vector bundle  $E$  of rank  $p$  over  $\bar{\mathbb{C}}$  and meromorphic connection  $\nabla$  on it which corresponds to the system (1), one has a flag  $E^1 \subset E^2 \subset \dots \subset E^p = E$  of subbundles of rank  $1, 2, \dots, p$  respectively that are stabilized by  $\nabla$  (see Example 1).

The final part of the proof, estimates for the degree of each subbundle  $E^j, j \leq p-1$ , proceeds as for Lemma 2. As  $E^j$  is stabilized by  $\nabla$ , in a neighbourhood of each singular point  $a_i$  the initial system is transformed *via* a holomorphically invertible gauge transformation in a system with a formal fundamental matrix of the form (6),

$$\hat{Y}_i(z) = \begin{pmatrix} \hat{F}_i^1(z) & * \\ 0 & * \end{pmatrix} (z - a_i) \begin{pmatrix} \Lambda_i^1 & 0 \\ 0 & \Lambda_i^2 \end{pmatrix} e \begin{pmatrix} Q_i^1(z) & 0 \\ 0 & Q_i^2(z) \end{pmatrix},$$

such that  $\hat{Y}_i^1(z) = \hat{F}_i^1(z) (z - a_i)^{\Lambda_i^1} e^{Q_i^1(z)}$  is a formal fundamental matrix (6) of a local linear differential system of  $j$  equations that corresponds to the restriction  $\nabla^j$  (near  $a_i$ ) of the connection  $\nabla$  to the subbundle  $E^j$  (existence of the blocked formal fundamental matrix  $\hat{Y}_i$  is proved, for example, in [18, Lemma III.3.8]). Therefore, the formal exponents  $\tilde{\lambda}_i^1, \dots, \tilde{\lambda}_i^j$  of this system (the eigenvalues of the diagonal matrix  $\Lambda_i^1$ ) form a subset of the formal exponents of the initial system at  $a_i$ , hence

$$\operatorname{Re} \tilde{\lambda}_i^l > -\frac{1}{n(p-1)}, \quad l = 1, \dots, j.$$

Thus for the degree  $\deg E^j$  one has

$$\deg E^j = \sum_{i=1}^n \operatorname{res}_{a_i} d \ln \det \widehat{Y}_i^1 = \sum_{i=1}^n \operatorname{ord}_{a_i} \det \widehat{F}_i^1 + \sum_{i=1}^n \operatorname{tr} \Lambda_i^1 + \sum_{i=1}^n \operatorname{res}_{a_i} d \operatorname{tr} Q_i^1.$$

Recall that  $\det \widehat{F}_i^1(a_i) \neq 0$  and  $\operatorname{tr} Q_i^1$  is a polynomial in  $1/(z - a_i)$ , hence

$$\deg E^j = \sum_{i=1}^n \operatorname{tr} \Lambda_i^1 = \sum_{i=1}^n \sum_{l=1}^j \operatorname{Re} \tilde{\lambda}_i^l > -\frac{j}{p-1} \geq -1.$$

This implies (as at the end of the proof of Lemma 2) all  $\deg E^j = 0$  and all the subbundles  $E^1 \subset \dots \subset E^{p-1} \subset E$  are holomorphically trivial, whence the assertion of the theorem follows.  $\square$

## 7.2 Other types of solvability

According to the behavior of solutions of a linear differential system near its irregular singular point, the system (1) having at least one irregular singular point is not solvable by integrals and algebraic functions. For the solvability by exponentials of integrals and algebraic functions the following criterion holds.

**Theorem 6.** *In the assumptions of Theorem 5, the system (1) is solvable by exponentials of integrals and algebraic functions<sup>4</sup> if and only if there exists a constant matrix  $C \in \operatorname{GL}(p, \mathbb{C})$  such that the matrix  $CB(z)C^{-1}$  is diagonal.*

**Proof.** Sufficiency follows immediately from the integration of a system with the diagonal coefficient matrix. To prove necessity, we note that solvability by exponentials of integrals and algebraic functions implies solvability by quadratures, hence by Theorem 5 the coefficient matrix  $B$  is reduced to a triangular form *via* a constant gauge transformation. Therefore the differential Galois group  $\mathbf{G}$  of the system (1) is triangular and we may apply Kolchin's criterion from Section 2. According to this criterion  $\mathbf{G}$  is diagonal. Further we proceed as in the proof of Theorem 5 having furthermore significant simplifications. Namely, as  $\widehat{\mathbf{M}} \subset \mathbf{G}$  is diagonal, the Stokes matrices are trivial, hence the fundamental matrix  $Y(z)$  near each singular point  $a_i$  has the form

$$Y(z) = F_i(z)(z - a_i)^{\Lambda_i} e^{Q_i'(z)},$$

where the matrix  $F_i(z)$  is holomorphically invertible at  $a_i$ . The monodromy matrices of  $Y$  are diagonal, the matrix  $F_i^{-1}(z)Y(z)$  is also diagonal, therefore a holomorphically trivial vector bundle  $E$  of rank  $p$  over  $\overline{\mathbb{C}}$  (with a meromorphic connection  $\nabla$  on it) which corresponds to the system (1) is a direct sum of line bundles stabilized by  $\nabla$  (see Example 1). From the bounds on the formal exponents it follows again that each of these line bundles is holomorphically trivial. Now a modification of Lemma 1, as we said in the proof of Theorem 4, proves reducibility of the system (1) to a diagonal form *via* a constant gauge transformation.  $\square$

As was mentioned before, a topological version of Galois theory proves that a Fuchsian system non-solvable by quadratures is also strongly non-solvable. For linear differential equations with irregular singular points this property fails. For example, the equation

$$y'' + zy = 0$$

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<sup>4</sup>In the assumptions of the theorem, this type of solvability is equivalent to solvability by exponentials of integrals.

with one (irregular) singular point  $z = \infty$  is not solvable by quadratures (see [13, Ch. V]) but its monodromy is trivial and all the solutions are meromorphic in  $\mathbb{C}$ . Thus, the next natural question is to obtain some sufficient condition for the strong non-solvability of a linear differential system with irregular singular points (formulated in terms of the coefficient matrix rather than monodromy group).

We conclude this subsection by discussing the local solvability of a linear differential system by quadratures near its singular point  $z = a$ . To define this type of solvability, one naturally changes the base field  $\mathbb{C}(z)$  of rational functions to the field  $\mathcal{M}_a$  of meromorphic germs at  $a$  and considers Liouvillian extensions of  $\mathcal{M}_a$ . If a Picard–Vessiot extension  $\mathcal{M}_a \subset F_a$  corresponding to the system is contained in some Liouvillian extension of  $\mathcal{M}_a$ , then the system is said to be *locally solvable by quadratures* near the singular point  $z = a$ . In analogy to the global case, local solvability by quadratures is equivalent to the solvability of the identity component  $\mathbf{G}_a^0$  of the local differential Galois group  $\mathbf{G}_a$  of the system.

If the singular point  $z = a$  is regular, then the system is always locally solvable by quadratures near this point. This follows both from the form (4) of a fundamental matrix,

$$Y(z) = U(z)(z - a)^A(z - a)^{\tilde{E}},$$

where  $U(z)$  is a holomorphic matrix at  $a$ , or from the fact that the local differential Galois group in this case is the Zariski closure of the local monodromy group, which is cyclic hence solvable. On the other hand, in the case of non-resonant irregular singular point the following criterion of the local solvability by quadratures is obtained along the proof of Theorem 5.

**Theorem 7.** *A linear differential system is locally solvable by quadratures near its non-resonant irregular singular point if and only if the coefficient matrix is locally reduced to an upper-triangular form via a holomorphically invertible gauge transformation at this point.*

If a linear differential system only has one singular point (for example, the entries of the coefficient matrix are polynomials) then its local differential Galois group at this point coincides with the global one (see [21, Prop. 1.3]). Hence solvability of such a system by quadratures is equivalent to the local solvability. Thus we obtain the following consequence of Theorem 7.

**Corollary 1.** *Consider a linear differential system with a polynomial coefficient matrix whose irregular singular point  $z = \infty$  is non-resonant. This system is solvable by quadratures if and only if its coefficient matrix is locally reduced to an upper-triangular form via a holomorphically invertible gauge transformation at infinity.*

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