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Modeling linear logic with implicit functions

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Abstract

Just as intuitionistic proofs can be modeled by functions, linear logic proofs, being symmetric in the inputs and outputs, can be modeled by relations (for example, cliques in coherence spaces). However generic relations do not establish any functional dependence between the arguments, and therefore it is questionable whether they can be thought as reasonable generalizations of functions. On the other hand, in some situations (typically in differential calculus) one can speak in some precise sense about an implicit functional dependence defined by a relation. It turns out that it is possible to model linear logic with implicit functions rather than general relations, an adequate language for such a semantics being (elementary) differential calculus. This results in a non-degenerate model enjoying quite strong completeness properties.

Keywords:

1. Introduction

Linear logic (**LL**), introduced by J.-Y. Girard in the late eighties [11], has become an extremely popular subject. One of the attractive features of this system consists in combining its constructive nature (a possibility of functional interpretation of proofs), typical for intuitionistic logic, with the familiar symmetries of classical logic, such as the involutivity of negation and De Morgan dualities between connectives.

From the constructive point of view, a proof should be understood as a function, or, in more modern and general terms, a morphism, that can be composed with other proofs. Typically, proofs of the implications $A \to B$ and $B \to C$ can be composed to yield a proof of $A \to C$ (the rule of syllogism). Thus one can think of a category, whose objects are formulas, and whose morphisms are equivalence classes of proofs. In other words, one assumes existence of an equivalence relation on proofs that turns the set of proofs and formulas into a well-defined category. However such a functional interpretation is non-trivial only if there exist hom-sets with *more than one element*, in other words if there exist formulas with several non-equivalent proofs.

This is the case for intuitionistic logic, whose proofs can indeed be interpreted as functions (say, λ-terms). In this sense, intuitionistic logic is constructive, in fact a prototype of a constructive logic. Whereas classical logic admits only a degenerate categorical interpretation, defined by declaring all proofs of the same formula equivalent [14], pp. 67 -116. (The corresponding category is the Boolean algebra of provably equivalent formulas.) On the other hand, classical logic enjoys a number of attractive symmetries, such as the duality between connectives, the involutive negation, the law of excluded middle — all of which are lost in the intuitionistic case.

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Linear logic combines the constructive nature of intuitionistic logic and the symmetries of classical logic. This is achieved by taking control over unlimited use of hypotheses. In **LL** each hypothesis in the proof should be used only once and exactly once. (In this paper we discuss only the so-called *multiplicative* fragment of linear logic (**MLL**). The interested reader can find an introduction to the full linear logic for example in [12].)

1.1. Denotational semantics

A *denotational model* of a constructive logic is a category where one can interpret formulas as objects and proofs as morphisms, preserving the internal categorical structure of the logic (connectives, rules, axioms etc). Finding denotational models is the problem of *denotational semantics*.

Intuitionistic logic, for example, can be interpreted simply in the category of sets and functions (although this is not a best model). Thus intuitionistic proofs, seen as λ-terms, represent "general" functions in quite a literal sense. On the other hand a functional explanation of linear logic is not completely obvious. Linear logic proofs are symmetric in the input and the output, and general functions are not. Thus linear logic proofs may correspond only to very special functions (such as linear operators) or to something more general than "general" functions. Thinking of relations as natural generalizations of functions, one often interprets linear logic proofs as relations. (This tradition goes back to Girard's work on quantitative semantics [10]. In such a semantics sets play the role of bases of vector spaces, and relations are analogous to matrices. In this paper we take somewhat more primitive view of relations, not anticipating any analogies with linear algebra.)

We note that **LL** cannot be characterized as the "general" logic of relations, the relational interpretation being very degenerate. Such an interpretation fails to capture much of the structure of **LL**, and perhaps this can be explained as follows. Linear logic proofs mix inputs and outputs indefinitely, and, thus, hide the correspondence between them. However such a correspondence is always present implicitly — for example in the form of identity links connecting dual literals in a proof-net. On the other hand a general relation does not imply any dependence between the arguments. A relation, coming from an actual **LL**-proof, always has the form of an *implicit function* — some of the arguments can be expressed as functions of the remaining ones.

This observation suggests the idea of modeling **LL** by means of implicit functions, typically in the setting of differential calculus, where a relevant theory is well developed. In differential geometry, relations, defining implicit functions, are supported at smooth submanifolds. Motivated by the above arguments, we develop a special relational interpretation of (multiplicative) linear logic, where proofs are modeled by *smooth* relations, i.e. by smooth submanifolds.

Such an interpretation however does not come for free from the usual relational semantics. One should specify the target category for the interpretation, and this is not completely trivial. An important phenomenon arising in the smooth setting is that smooth relations do not compose in general, i.e. the set-theoretic composition of smooth relations may fail to be smooth. In other words, smooth relations themselves do not form a category.

In order to get a well-defined denotational model, we interpret formulas as spaces (vector spaces or differentiable manifolds), equipped with a certain extra structure that we call the *smooth coherence space* structure, since, in some sense, it looks like a "smoothing" of the familiar coherence space structure of Girard. The extra structure (technically, two conic subsets of tangent/cotangent vectors) plays the role of a typing specification for morphisms. Morphisms between smooth coherence spaces are smooth relations satisfying the corresponding specifications.

With this definition we get a true category, i.e. our typing specifications exclude all uncomposable pairs of smooth relations. Furthermore the extra structure of "coherence" breaks the degeneracy of the usual relational interpretation, and thus we get a non-degenerate model of (multiplicative) linear logic (with the Mix-rule, to be completely pedantic). Yet more interesting, the resulting model is *complete* in a certain (unusually strong) sense. The completeness theorem says, modulo technicalities, the following. *If we fix an interpretation of formulas as "ordinary" (not coherence) spaces, i.e. vector spaces or smooth manifolds, thus getting a typical relational model, and if there exists a relation between these spaces, which remains a morphism for any lift of the model to a smooth coherence spaces model, i.e. for any consistent choice of "coherences", then the above relation is the denotation of a proof.* This can be compared with the coherence spaces semantics of Girard. We can formulate the analogous statement in the setting of usual coherence spaces, and it turns out to be *wrong*.

Thus we can argue that the tentative understanding of linear logic as the logic of implicit functions has some basis. The corresponding interpretation is not only consistent with the familiar relational interpretation, but is in fact a natural *refinement* of the latter one, which is not only sound but also complete.

Since our interpretation is based on differential calculus, let us mention some other on-going work, which has somewhat similar flavor. Recently T. Ehrhard and L. Regnier [7], [8] introduced *differential linear logic* (and differential lambda-calculus), whereas R.Blute, R. Cockett and R.A.G. Seely [2] started developing a corresponding semantic theory of *differential categories*. These developments, however, make crucial use of the exponential fragment of **LL**, and so far we have not been able to find any convincing relation with our work.

To finish the Introduction we should say a few words about the origins of this work. In an earlier work [18, 19] we developed a similar relational model of linear logic in the setting of *symplectic* manifolds and *Lagrangian* submanifolds, while the motivating idea was to find some relations with physics, in particular with quantum mechanics. The new model, which we present in this work, is more or less a generalization of the symplectic model to the non-symplectic case.

2. Linear logic

Formulas of the multiplicative linear logic (**MLL**) are built from *positive* and *negative* literals, respectively, $p_0, \ldots, p_n, \ldots$ and $p_0^\perp, \ldots, p_n^\perp, \ldots$, by means of the binary connectives $\otimes$ (*times*, also *tensor*) and $\wp$ (*par*, also *cotensor*). Tensor and cotensor of two formulas are called their *multiplicative conjunction* and *multiplicative disjunction* respectively. *Linear negation* $A^\perp$ of the formula A is defined inductively by

$$(p^\perp)^\perp = p,\ (X \otimes Y)^\perp = X^\perp \wp Y^\perp,\ (X \wp Y)^\perp = X^\perp \otimes Y^\perp. \tag{1}$$

Thus linear negation is involutive and multiplicative connectives enjoy the De Morgan-like duality, just as in classical logic.

Linear implication (denoted by $(.) \multimap (.)$) is also defined as in classical logic:

$$A \multimap B = A^\perp \wp B \tag{2}$$

In particular the identity axiom $\vdash A \multimap A$ of linear logic is written as $\vdash A^\perp \wp A$, i.e. linear logic derives the law of excluded middle.

The most popular syntax of **MLL**-proofs is that of *proof-nets*, which can be found in [12, 5, 9]. The sequent calculus formulation of **MLL** is as follows (see [12]). The standard format of the sequent is one-sided, the two-sided formalism is obtained by rewriting $\vdash A^\perp, B$ as $A \vdash B$ etc. As usual, the one-sided sequents eventually should be understood as big disjunctions, i.e. the sequent $\vdash A_1, \ldots, A_n$ stands for the formula $A_1 \wp \ldots \wp A_n$. The rules are below.

$$\frac{}{\vdash A, A^\perp}(Identity),\quad \frac{\vdash \Gamma, A \quad \vdash A^\perp, \Delta}{\Gamma \vdash \Delta}(Cut),$$

$$\frac{\vdash \Gamma, A, B, \Delta}{\vdash \Gamma, B, A, \Delta}(Exchange),$$

$$\frac{\vdash \Gamma, A \quad \vdash B, \Delta}{\vdash \Gamma, A \otimes B, \Delta}(Times),\quad \frac{\vdash \Gamma, A, B, \Delta}{\vdash \Gamma, A \wp B, \Delta}(Par).$$

The system **MLL** + **Mix** is obtained by adding to **MLL** the following *Mix* rule:

$$\frac{\vdash \Gamma \quad \vdash \Delta}{\vdash \Gamma, \Delta}(Mix).$$

Despite sharing many attractive features of classical logic, linear logic is constructive, like the intuitionistic one, which means the possibility of functional interpretation. This means the following. Linear logic enjoys Cut-elimination with the property that the cut-free form of each proof is *unique* (up to inessential permutations of rules). This property allows us to define the equivalence relation on proofs, by declaring two proofs equivalent, if they have the same cut-free form. Then the set of linear logic formulas becomes a category whose morphisms are equivalence classes of proofs (a morphism from A to B is a proof of $A \vdash B$, i.e. of $\vdash A^\perp, B$), composition of morphisms being given by the application of the Cut rule (or of some its analogue, depending on the chosen syntax). Thinking of morphisms as generalizations of functions, this gives us a functional interpretation of proofs. The interpretation is non-trivial, because there exist hom-sets with *more than one element*. In other words there exist formulas with several non-equivalent proofs.

A *denotational model* of linear logic, is a category where one can interpret formulas as objects and proofs as morphisms, preserving the internal categorical structure of the logic. In more down-to-earth terms this means simply that the interpretation of a proof with Cut (i.e. the composition of several proofs) should coincide with the interpretation of its cut-free form.

Semantically, multiplicative linear logic can be characterized as the logic of $*$-*autonomous categories* (see [17, 4]). An example of a of $*$-autonomous category is the category **Rel** of sets and relations. The tensor $\otimes$ is defined as the Cartesian product, and the involution $(.)^\perp$ is the contravariant functor that sends each object to itself, and flips the input and the output of a morphism, i.e. a relation $\sigma \subset A \times B$ between sets A and B is sent to the relation $\sigma^\perp \subset B \times A$ between B and A, defined by the permutation of factors in $A \times B$. The category **Rel** is, in fact, an example of a *compact closed category*. These are $*$-autonomous categories with self-dual tensor, i.e. such that $(A \otimes B)^\perp \cong A^\perp \otimes B^\perp$ for all objects A, B. They yield degenerate models of **MLL**, since, by definition, the tensor and the cotensor (conjunction and disjunction) are identified. One readily checks that **Rel** yields a denotational model of **MLL** + **Mix**, and that the model is terribly degenerate. Nevertheless this semantics is very basic, and we will call it the *standard relational interpretation*.

In addition to being very degenerate, one can argue that the relational interpretation can be thought as a "functional" interpretation of logic only in a very abstract sense. Relations can be said to be generalizations of functions, but it is that level of generalization where most of the functional content is lost completely. A general relation does not establish any functional dependence between the arguments. As we argued in the Introduction, it is probably desirable to interpret proofs as something less general, typically as *implicit functions*. An adequate setting for implicit functions is differential calculus, and we propose to restrict from arbitrary relations to *smooth* relations, which always have the form of implicit functions. As we shall see below, this setting turns out to have much more structure. In fact it gives us not only a non-degenerate, but a *complete* semantics.

3. Some notation and terminology

In this Section we recall some basic notation and terminology from differential calculus and manifolds in order to fix the language and avoid possible confusion. For a systematic introduction into the topic see, for example, [15, 6].

For simplicity, all ambient manifolds considered in this paper will be just Euclidean vector spaces. This does not change anything in our construction and results, which are equally well applicable to a more general case. We will also assume for convenience that our vector spaces are equipped with the usual Euclidean inner product.

Given a Euclidean space M of dimension n, the tangent bundle TM of M formally is the space $M \times \mathbf{R}^n$ together with the natural projection $\pi : TM \to M$ on the first factor. It is the set of all vectors tangent to M at different points, and the projection π maps a vector to its point of tangency. The fiber $\pi^{-1}(x)$ of the projection π over the point $x \in M$ is called the *tangent space* to M at x, and is denoted T_xM. The fibers are Euclidean vector spaces, and their Euclidean vector space structures are a part of the data defining TM. However the whole bundle does not have any intrinsic vector space structure (vectors tangent at different points cannot be added). The set of zero-vectors of all tangent spaces is called the *zero section*. Although there are as many zero-vectors in TM as points in M, we will use the loose notation 0 for a zero-vector without specifying the point of tangency unless this leads to a confusion. However, it is important to stress that two vectors tangent at different points cannot be equal.

Any smooth (i.e. differentiable) map $f : M \to N$ between spaces M and N lifts to the *tangent* or *derivative* map $Tf : TM \to TN$ between the respective tangent bundles, extended to tangent vectors by $Tf : (x, v) \mapsto (f(x), Df(x)v)$, where $Df(x)$ is the Jacobian of f at the point x, also denoted by T_xf. Note that the map T_xf maps T_xM to $T_{f(x)}N$.

A smooth *submanifold* of the space $\mathbf{R}^n$ can be defined as follows.

Definition 1. *A subset $\sigma \subseteq \mathbf{R}^n$ is a smooth submanifold of dimension k, if for any $x_0 \in \sigma$ there exist a neighborhood U of x_0 and a function $F : \mathbf{R}^n \to \mathbf{R}^{n-k}$, such that the Jacobian $Df(x_0)$ of f is non-degenerate at x_0 and σ is defined by the equation $F(x) = 0$, i.e. $\sigma \cap U = F^{-1}(0) \cap U$.*

By an immediate application of the Implicit Function Theorem (see [15] (I.5) for its formulation and proof) one obtains an equivalent definition, which seems somewhat more instructive for our purposes.

Definition 2. *A subset $\sigma \subseteq \mathbf{R}^n$ is a smooth submanifold of dimension k, if for any $x_0 \in \sigma$ there exists a partition of the set $\{1, \ldots, n\}$ into two sets $\{i_1, \ldots, i_k\}$ and $\{j_1, \ldots, j_{n-k}\}$ and a smooth function $f : \mathbf{R}^k \to \mathbf{R}^{n-k}$, $(x_{i_1}, \ldots, x_{i_k}) \mapsto (x_{j_1}, \ldots, x_{j_{n-k}})$, such that in a neighborhood U of x_0, the set σ coincides with the graph of f.*

Thus a smooth submanifold locally defines an implicit function, namely the f in the above definition. Accordingly we will understand smooth submanifolds as geometric representations of implicit functions. In the sequel the word "submanifold" will always stand for "smooth submanifold".

A standard example of a submanifold is the circle $S^1 \subset \mathbf{R}^2$, defined by the equation $x^2 + y^2 = 1$. The equation has a locally unique solution at each point of S^1, for example $x = (1 - y^2)^{\frac{1}{2}}$, which defines the corresponding implicit function.

A submanifold σ of the space M has as its tangent bundle the space $T\sigma$ that consists of all vectors in TM tangent to smooth curves lying in σ. Again, the tangent bundle is equipped with the natural projection $T\sigma \to \sigma$, whose fibers (tangent spaces to σ) have intrinsic vector space structures. More precisely, the tangent space $T_x\sigma$ to σ at the point x is a vector subspace of the tangent space T_xM to the ambient space at the same point. If σ in some neighborhood is given by the equation $F(x) = 0$, then its tangent vectors v at a point x in this neighborhood are those satisfying $DF(x)v = 0$.

The vectors *normal* to σ at x are those that belong to the *annihilator* $Ann(T_x\sigma)$ of $T_x\sigma$ in T_xM (remember that our ambient spaces are assumed to be equipped with the standard Euclidean inner product). The annihilator $Ann(L)$ of a subspace L is defined as the set of vectors orthogonal to each vector in L.

An *immersed submanifold* σ of M is the manifold σ together with the immersion map $i_\sigma : \sigma \to M$ which is locally an isomorphism onto a submanifold of M. Two immersed submanifolds σ and σ' of M are identified, if there exists a smooth isomorphism $\phi : \sigma \to \sigma'$ connecting the immersion maps: $i_\sigma = i_{\sigma'} \circ \phi$. To be completely pedantic, for an immersed submanifold σ, we should accurately distinguish between points of the manifold σ and points of the image $i_\sigma(\sigma)$ of the immersion map. We do not do that, because it never leads to confusion. In the sequel we use the term "submanifold" as a shorthand for "immersed submanifold".

4. Smooth submanifolds and relations

In this Section we discuss smooth relations and how to compose them.

Recall that a relation between sets M and N is a subset of the product $M \times N$. A relation σ between sets M, N can be composed with a relation τ between N and K yielding the relation $\tau \circ \sigma$ by the formula

$$\tau \circ \sigma = \{(x, z) | x \in M, z \in K \text{ s.t. } \exists y \in N \ (x, y) \in \sigma, (y, z) \in \tau\}.$$

In this way sets and relations form a category (denoted **Rel**) with identities given by the diagonal subsets of the form $\{(x, x)\}$.

Let us consider what happens with relations in the smooth setting.

A relation σ between the spaces M and N is *smooth* if it is supported at a smooth submanifold of $M \times N$.

It turns out that smooth relations do not compose, and the standard example is as follows. We take the relations $x^2 + y^2 = 1$ and $y^2 + z^2 = 1$. Their set-theoretic composition is easily seen to be the relation $x^2 = z^2$, $|x| < 1$, $|z| < 1$. At the point $(x, z) = (0, 0)$, the defining equation has two solutions $x = z$, $x = -z$, and thus the relation does not define any single-valued function, hence it is not supported at any submanifold.

In fact, smooth relations are not closed under intersections (i.e. fiber products) and projections, and this is precisely the reason why they do not compose. Indeed, for the relations $\sigma \subseteq M \times N$ and $\tau \subseteq N \times K$, their composition $\tau \circ \sigma$ is defined as the intersection of $\sigma \times \tau$ with $M \times \Delta_N \times K$, where Δ_N is the diagonal submanifold of $N \times N$, followed by the projection $\pi : M \times \Delta_N \times K \to M \times K$ along Δ_N, that is $\sigma \cap \tau = \pi(\sigma \times \tau \cap M \times \Delta_N \times K)$. If we want two smooth relations to compose, it is sufficient to ensure that the above intersection and projection preserve smoothness. This will be our strategy for building a well-defined category of smooth relations.

The two theorems below are standard results in differential geometry. See, for example [15] (II.3).

Recall that two submanifolds σ and τ of the space M are *transversal* at the point $x \in \sigma \cap \tau$ if their tangent spaces $T_x\sigma$ and $T_x\tau$ at x span the whole tangent space of M: $T_xM = T_x\sigma + T_x\tau$. Equivalently, σ and τ are transversal at x if their tangent spaces at this point have no common nonzero normal vector.

Theorem 1. *If smooth submanifolds σ and τ of the space M are transversal at each point of their intersection then their intersection $\sigma \cap \tau$ is a smooth submanifold of M.*

This is a corollary of [15] (II.3), Prop. 4.

Having found a sufficient condition for the intersection of submanifolds to be a submanifold, we need a similar condition for projections. Here it is.

Theorem 2. *Let σ be a smooth submanifold of the space M, and let f be a smooth map from M to another space N. If at any point $x \in \sigma$ no nonzero tangent vector of σ belongs to the kernel of the derivative $T_x f$ of f at x then $f(\sigma)$ is a submanifold of N.*

This is a corollary of [15] (II.3), Prop. 2.

5. Smooth coherence spaces

In this Section we construct a category of smooth relations and show that it gives a sound model of **MLL**.

In the previous Section we discussed the problems with composing smooth relations, and saw that these problems arise because the class of smooth submanifolds is not closed under intersections and projections. We also recalled the sufficient conditions for, respectively, the intersection of two submanifolds and the projection of a submanifold to be a submanifold. Observe that one of these conditions (Theorem 1) is formulated in terms of vectors normal to the submanifolds involved, and the other (Theorem 2) is formulated in terms of tangent vectors. This suggests us what kind of the extra structure on the ambient spaces is needed in order to ensure compositionality of submanifolds.

Definition 3. *A smooth coherence space A is a triple (M_A, C_A, CC_A), where M_A is a Euclidean vector space, and C_A and CC_A, respectively coherence and cocoherence, are subsets of the tangent bundle TM_A closed under scalar multiplication. A clique in the smooth coherence space A is a submanifold σ of M_A, all of whose tangent vectors belong to C_A, and all of whose normal vectors belong to CC_A.*

Remark If we were considering general (not necessarily Euclidean) vector spaces, or even general manifolds, we would define the cocoherence as a subset of the cotangent, rather than the tangent bundle. All constructions and results of this paper apply to this more general definition without any complication.

We will build a category of smooth coherence spaces with cliques as morphisms, and interpret **MLL** in it. First of all let us define the interpretation of connectives.

The *dual* $A^\perp$ of the smooth coherence space A is defined as follows. The underlying space $M_{A^\perp}$ is just M_A. The coherence $C_{A^\perp}$ is the conic subset of TM_A complementary to C_A, i.e. $C_{A^\perp} = \{v \in TM_A | v \notin C_A \text{ or } v = 0\}$. The cocoherence $CC_{A^\perp}$ is defined identically to $C_{A^\perp}$ with C_A replaced with CC_A, i.e. $CC_{A^\perp}$ is the complementary conic subset to CC_A. Note that C_A and $C_{A^\perp}$ (CC_A and $CC_{A^\perp}$) have all zero-vectors in the intersection.

Note also that, obviously, $A^\perp = A^{\perp\perp}$.

The *tensor* $A \otimes B$ of the smooth coherence spaces A and B is defined by $M_{A\otimes B} = M_A \times M_B$, $C_{A\otimes B} = C_A \times C_B$, $CC_{A\otimes B} = CC_A \times CC_B$.

All other connectives can be expressed in terms of the above, but let us spell out the definitions. The *cotensor* $A \wp B$ and the *internal hom*-space $A \multimap B$ of smooth coherence spaces A and B are defined as follows. On the level of underlying spaces: $M_{A\wp B} = M_{A\multimap B} = M_A \times M_B$. On the level of coherences: $C_{A\wp B} = \{(u, v) | 0 \neq u \in C_A \text{ or } 0 \neq v \in C_B, \text{ or } (u, v) = 0\}$, $C_{A\multimap B} = \{(u, v) | u \in C_A \text{ implies } 0 \neq v \in C_B, \text{ or } (u, v) = 0\}$. On the level of cocoherences: similar to coherences.

Now we say that a *morphism* between the smooth coherence spaces A and B is just any clique in $A \multimap B$.

Remark A reader familiar with the usual coherence spaces semantics (see [12]) can note that, if we agree that a tangent vector is in fact a pair of "infinitely close" points, then as far as coherences are concerned, all operations on smooth coherence spaces are literal translation from the language of ordinary coherence spaces. However, the structure of cocoherence is a new ingredient. In some sense it plays the role of *totality*, see [16].

Lemma 1. *Given smooth coherence spaces A, B, C and cliques σ, τ in $A \multimap B$ and $B \multimap C$ respectively, the set-theoretic composition $\tau \circ \sigma$ is a clique in $A \multimap C$.*

Proof One should check first of all that $\tau \circ \sigma$ is a smooth submanifold.

The submanifold $\delta = M_A \times \Delta_B \times M_C$ of $M_A \times M_B \times M_B \times M_C$, where Δ_B is the diagonal submanifold of $M_B \times M_B$, has normal vectors of the form $(0, v, -v, 0)$, $v \in TM_B$. One easily checks that none of the nonzero normal vectors of $\sigma \times \tau$ can be of such a form, because of the cocoherence conditions. Hence by Theorem 1, the intersection $\sigma \times \tau \cap \delta$ is a submanifold.

The kernel of the derivative of the projection $\pi : \delta \to M_A \times M_C$ along Δ_B consists of the vectors of the form $(0, v, v, 0)$, $v \in TM_B$. Again, none of the nonzero vectors tangent to $\sigma \times \tau \cap \delta$, hence to $\sigma \times \tau$, has such a form because of the coherence conditions. Hence by Theorem 2, the projection $\pi(\sigma \times \tau \cap \delta) = \tau \circ \sigma$ is a submanifold.

One should check that $\tau \cap \sigma$ satisfies the corresponding coherence and cocoherence conditions. The case of coherence being routine, let us check for cocoherence.

Let us denote $X = M_A \times M_B \times M_B \times M_C$, $\rho = \sigma \times \tau$, $\overline{X} = M_A \times M_C$. We have the submanifold $\delta \cap \rho \subset X$, the projection $\pi : \delta \to \overline{X}$, and the image $\overline{\rho} = \pi(\delta \cap \rho) = \sigma \circ \tau$. Abusing the notation we will denote the derivative of π by the same letter. Finally, let $P = CC_{A \otimes (B \wp B^{\perp}) \otimes C^{\perp}}$, $\overline{P} = CC_{A \otimes C^{\perp}}$. Pick a point x of $\delta \cap \rho$, and let $\overline{x} = \pi(x) \in \overline{\rho}$. We need to show that no non-zero vector $v \in \overline{P}$ is normal to $\overline{\rho}$ at $\overline{x}$.

Let $0 \neq v = (v_1, v_2) \in T_{\overline{x}}\overline{X}$ be in $\overline{P}$. Then the subspace

$$V = \{(tv_1, a, -a, tv_2) | t \in \mathbf{R},\ a \in T_x M_B\}$$

of $T_x X$ lies in P. Hence no non-zero vector from V is normal to ρ. In particular the annihilator $Ann(V)$ of V and $T_x\rho$ span the whole $T_x X$. Now let $\overline{u} = (u_1, u_2) \in T_{\overline{x}}\overline{X}$. Then the vector $u = (u_1, 0, 0, u_2) \in T_x X$ has a representation $u = \xi + v$, where $\xi \in Ann(V)$ and $v \in T_x\rho$. Note that ξ necessarily is of the form $\xi = (\xi_1, a, a, \xi_2)$, hence v is of the form $v = (v_1, -a, -a, v_2)$. In particular $v \in T(\rho \cap \delta)$, and $\pi(v) \in T\overline{\rho}$. On the other hand it is easy to see that $\pi(Ann(V))$ lies in (in fact, equals) the annihilator of v. Since $\overline{u} = \pi(\xi) + \pi(v)$, we have shown that $Ann(v)$ and $T_x\overline{\rho}$ span the whole $T_x\overline{X}$. But this means that v is not normal to ρ.

Thus composition of cliques is well-defined, and smooth coherence spaces form a category. (The identities being given by the diagonal submanifolds: $\delta_A = \{(x, x) | x \in M_A\} \subset M_A \times M_A$. It is straightforward to check that they are indeed cliques in $A \multimap A = A^{\perp} \wp A$.) One routinely checks that this category provides a sound model of **MLL** + **Mix** with the above defined interpretation of connectives, in other words that the category is $*$-autonomous. In fact, it is an empirical observation that a consistent refinement of the standard relational model, or, in general, of a compact closed category yields a model of **MLL**, see [13]. So we conclude with the soundness theorem.

Theorem 3. *The category of smooth coherence spaces and cliques is a denotational model of* **MLL** + **Mix**.

In the next Section we are going to discuss the more interesting completeness questions. However, one can see from the start that our model breaks the degeneracy of the standard relational model. It is easy to see that for any smooth coherence space $A \neq \{0\}$, the diagonal submanifold $\Delta \subset M_A \times M_A$ is a clique in $A^{\perp} \wp A$ (in fact, it is the denotation of the corresponding axiom), but never in $A^{\perp} \otimes A$.

6. Completeness

In this Section we discuss the completeness properties of the smooth coherence spaces model. In denotational semantics we are interested in *full completeness*, some sort of completeness on the level of proofs (to the author's knowledge, the term and the first corresponding theorem were proposed by S. Abramsky and R. Jagadeesan [1]). In the current literature, most often full completeness of a model means that all *dinatural transformations* in the category under consideration are denotations of proofs (this sort of full completeness was introduced in [3]). We are going to show, however, that the smooth coherence spaces model enjoys a somewhat stronger full completeness property.

Fix an interpretation $M : p \mapsto M_p$ of literals as Euclidean spaces and interpret the multiplicative connectives as the Cartesian products of spaces. This extends to the interpretation $A \mapsto M_A$ of all **MLL**-formulas, in fact to a relational model of **MLL**. Any choice of smooth coherence space structures (C_p, CC_p) on the spaces M_p lifts the above relational model to a smooth coherence space model. It is easy to see that the interpretation of proofs, in fact, does not depend on the lift: if the relation σ between the spaces M_A and M_B is a denotation of a proof of the formula $M_A \multimap M_B$ then σ is a clique for any consistent choice of coherences and cocoherences. The completeness theorem says that the converse is also true, provided that the spaces involved are not too degenerate.

Theorem 4 (Completeness). *Assume that all literals are interpreted as spaces of dimension greater than* 1. *Let F be an* **MLL**-*formula, and let σ be a closed, connected and nonempty submanifold of M_F. If for any choice of coherences and cocoherences for the literals the submanifold σ is a clique in the induced smooth coherence spaces model, then σ is a denotation of a proof in* **MLL+Mix**.

One can compare the stated result with the case of the ordinary coherence spaces model due to Girard (see [12]). Given a set A, the subset $\sigma = \{(x, x, x) | x \in A\}$ of A^3 is easily seen to be a clique in the coherence space $A \multimap A \otimes A$ for any choice of the coherence structure on the set A. However σ, obviously, is not a denotation of any proof. Thus the analogous theorem does not hold for the ordinary coherence spaces model. Which shows that the smooth coherence spaces model is strictly "more complete" than the ordinary one. Note also that our completeness theorem is stronger than the usual full completeness theorems saying that all dinatural transformations come from proofs, since in order to establish dinaturality one has to vary the interpretation of literals over *all* objects in the category, whereas in our case we vary only the coherence structures. The ordinary coherence spaces model, by the way, is known to be fully complete in the sense of dinatural transformations [20].

Proof We assume that the reader is familiar with the formalism of proof-nets and with the Danos-Regnier criterion. This machinery, specified for the case of **MLL+Mix**, is described in [9].

We start with two very simple algebraic observations.

Note 1. *If a vector subspace K of the vector space V lies in the union of a finite collection of vector subspaces $K_1, \ldots, K_k$ of V, then, for some $i = 1, \ldots, k$ the space K lies entirely in K_i.*

Proof Exercise.

Lemma 2. *Let V_1, V_2 be Euclidean vector spaces, $V = V_1 \times V_2$ and $\pi_i : V \to V_i$ be the natural projections, $i = 1, 2$. Let L be a subspace of V. Then, if $Ann(\pi_1(L)) = \pi_1(Ann(L))$ then $L = \pi_1(L) \times \pi_2(L)$.*

Proof Let $N = Ann(L)$, $L_i = \pi_i(L)$, $N_i = \pi_i(N)$, $i = 1, 2$. We know that $Ann(L_1) = N_1$. So $L_1 = Ann(N_1)$, and $L_1 \times \{0\} \subseteq Ann(N) = L$. The statement follows.

Now let us proceed to the proof of the Theorem.

Let $a_1, \ldots, a_n$ be an enumeration of all occurrences of literals in F. For each $i = 1, \ldots, n$ let $M_i = M_{p_i}$ and let π_i be the natural projection $M_F \to M_i$. In general, for any (occurrence of a) subformula ϕ of F let π_ϕ be the natural projection $M_F \to M_\phi$.

We will use the following notational conventions. If x is a point of σ, then $V = T_x M_F$, $L = T_x\sigma$, $N = Ann(L)$. Abusing notation, we will denote the projection $V \to V_i$ by π_i as well. We will write $x_i = \pi_i(x)$, $y_i = \pi_i(y)$ $V_i = \pi_i(V)$, $V_\phi = \pi_\phi(V)$. Similarly, for any $v \in L$, $u \in N$ we write $v_i = \pi_i(v)$, $u_i = \pi_i(u)$, $v_\phi = \pi_\phi(v)$, $u_\phi = \pi_\phi(u)$.

Note 2. *For any point $x \in \sigma$, any $i = 1, \ldots, n$ the restrictions of π_i to L and N are surjective.*

Proof Assume that there exists an i, such that $L_i = \pi_i(L)$ is not the whole V_i. Then there exists a non-zero vector $\xi \in V_i$ in the annihilator of L_i, and the vector u, defined by $u_i = \xi$, $u_j = 0$ for $j \neq i$ belongs to N. Let $p = a_i^\perp$, and choose the cocoherence $CC_p = \{t\xi | t \in \mathbf{R}\}$. By induction on F one establishes that $u \notin CC_F$, which contradicts the hypothesis of the Theorem. The argument for N is symmetric.

Now let us call any set S of unordered pairs (i, j), $i, j = 1, \ldots, n$, such that $a_i = a_j^\perp$, a *linking*. For $x \in \sigma$, let $S(x)$ be the linking consisting of all pairs (i, j) such that $a_i = a_j^\perp$, and $x_i = x_j$. Let us call a point $x \in \sigma$ *generic* if for all y in a neighborhood of x, we have $S(x) \subseteq S(y)$.

Lemma 3. *The set of generic points is dense in σ.*

Proof Let x be a point in σ, U be a neighborhood of x in σ. Let us prove that there exists a generic point $x' \in U$.

Assume that x is not generic, and let $S^0 = S(x)$. Then there exists a sequence $\{y_k\}$ converging to x in U, such that $S^0 \not\subseteq S(y_k)$ for all k. Since the set of linkings is finite, it follows that there exists a subsequence $\{x^1_k\}$ of $\{y_k\}$, such that $S(x^1_k) = S^1$ is constant. It is easy to see that $S^0 \supset S^1$: any equation from S^1 is satisfied at x by continuity, since $x = \lim_{k\to\infty} x^1_k$. If $x^1 = x^1_1$ is generic, we are done. Otherwise, by the same argument, there exists a sequence $\{x^2_k\}$ converging to x^1 in U, with $S(x^2_k) = S^2$ constant and $S^1 \supsetneq S^2$.

Repeating the argument we obtain a family of points $x, x^1, \ldots, x^i, \ldots$ in U, with the corresponding chain of linkings $S^1 \supsetneq S^2 \supsetneq \ldots \supset S^i \supsetneq \ldots$, where $S^i = S(x^i)$. Since the set of linkings is finite, the chain must stop at some i, and the corresponding point x^i is generic.

Note 3. *For any point $x \in \sigma$, any non-zero $u \in N$ there exists a linking $\{(i, j)\}$, such that $0 \neq u_i = tu_j$ (hence $x_i = x_j$) for some non-zero scalar t. In particular $(i, j) \in S(x)$.*

Proof Assume that the statement does not hold.

Then there exist non-zero $u \in N$ such that for any linking $\{(i, j)\}$ either $u_i = 0$ or $u_j = 0$ or u_i and u_j are linearly independent.

For each positive literal p define the cocoherence CC_p by

$$CC_p = \{\lambda\xi | \xi = u_j \text{ for some } j \text{ s.t. } a_j = p^\perp, \lambda \in \mathbf{R}\}. \tag{3}$$

By induction on F, one establishes $u \notin CC_F$.

Lemma 4. *At a generic point x, for any non-zero $u \in N$ there exists a pair $(i, j) \in S(x)$, such that $v_i = v_j$ for all $v \in L$ (hence $x_i = x_j$), and $0 \neq u_i = -u_j$ (where, as usual, $L = T_x\sigma$, $N = Ann(L)$).*

Proof Let $S = S(x)$. First of all, note that if $(i, j) \in S$, then automatically $v_i = v_j$ for all $v \in L$, since for all y in a neighborhood of x it holds that $y_i = y_j$.

Now assume the statement does not hold, i.e. there exists $u \in N$, such that $(i, j) \in S$ implies $u_i \neq -u_j$ or $u_i = u_j = 0$. On the other hand we know that there exists $(i, j) \in S$, such that $0 \neq u_i = tu_j$ for some non-zero t. We want to pick out of S a certain maximal collection of disjoint pairs. For that purpose we partition S into disjoint clusters by the following rule: (i, j) and (i', j') belong to the same cluster, if $x_i = x_{i'}$, and $a_i = a_{i'}$ or $a_i = a^\perp_{i'}$. To each cluster C corresponds a collection $\tilde{C}$ of occurrences of literals in F - namely those which occur in pairs belonging to C, and all literals corresponding to a given cluster are either equal or dual to each other. Therefore we can associate to each cluster C an ordered pair $(p_C, p^\perp_C)$, such that every occurrence in $\tilde{C}$ is an occurrence of either p_C or $p^\perp_C$, and the number of occurrences of p_C in $\tilde{C}$ is greater or equal than the number of occurrences of $p^\perp_C$. Now, for a given C we pick a maximal collection $\{(i_1, j_1), \ldots, (i_{s_C}, j_{s_C})\} \subseteq C$ of disjoint pairs with the convention that i_k's correspond to p_C, and j_k's correspond to $p^\perp_C$ (i.e. $a_{i_k} = p_C$, and $a_{j_k} = p^\perp_C$). It is easy to see that our choice of p_C and the maximality of the collection guarantees that for any $(i, j) \in C$ if $a_j = p^\perp_C$ then $j = j_k$ for some $k = 1, \ldots, s_C$. Taking the union over all clusters, we get a maximal collection $\{(i_1, j_1), \ldots, (i_s, j_s)\}$ of disjoint pairs in S.

As we noted above, for any $v \in L$ it holds that $v_{i_k} = v_{j_k}$. It follows that the vector ξ_k defined by $(\xi_k)_{i_k} = u_{i_k}$,$(\xi_k)_{j_k} = -u_{j_k}$, $(\xi_k)_i = 0$ for all $i \neq i_k, j_k$, belongs to $N = Ann(L)$. Also, we deduce from the Note 3 that for each k there exists some $t_k \neq 0, -1$, such that $0 \neq u_{j_k} = t_k u_{i_k}$. Then the non-zero vector $\psi = u + \sum_{k=1}^{s} t_k \xi_k$ belongs to N. But ψ does not satisfy the statement of Note 3.

Indeed, let $(i, j) \in S$. Then the pair (i, j) belongs to some cluster C. There are three possibilities.

First: $(i, j) = (i_k, j_k)$ for some $k = 1, \ldots, s$. Then $\psi_i = (1 + t_k)u_i \neq 0$ (since $t_k \neq -1$), $\psi_j = 0$.

Second: $(i, j) = (j_k, i_k)$ for some $k = 1, \ldots, s$. By a similar argument, $\psi_i = 0$, $\psi_j \neq 0$.

Third: the pair (i, j) does not belong to our maximal family. But $(i, j) \in S$, hence (i, j) belongs to some cluster C. Then either $a_i = p_C$ or $a_j = p_C$. For definiteness assume $a_i = p_C$, $a_j = p^\perp_C$. It follows then that $j = j_k$ for some $k = 1, \ldots, s_C$. Hence $\psi_i = u_i \neq 0$, $u_j = 0$.

Lemma 5. *For a generic point x, the set $S(x)$ is a partition of $\{1,\dots,n\}$ into disjoint pairs $\{i_1, j_1\},\dots,\{i_k, j_k\}$ (where $k = \frac{n}{2}$).*

Proof Let $S = S(x)$. The proof is by induction.

The induction step is as follows.

For any integer l, $0 \le l \le k$ there exist l disjoint pairs $\{i_1, j_1\},\dots,\{i_l, j_l\}$ in S, such that the tangent space L is isomorphic to the product

$$L \cong L' \times L_{i_1 j_1} \times \ldots \times L_{i_l j_l}, \tag{4}$$

where L' is a subspace of $V' = \prod_{i \neq i_1, j_1,\dots,i_l, j_l} L_i$, the subspaces $L_{i_s j_s} \subset V_{i_s} \times V_{j_s}$ are defined by the respective equations $v_{i_s} = v_{j_s}$, and the above isomorphism is induced by a permutation of factors.

Note that the above statement implies that $x_{i_s} = x_{j_s}$ for $s = 1,\dots,l$, since two tangent vectors may be equal only if they are tangent at the same point.

Let us prove the statement.

For $l = 0$ there is nothing to prove.

Assume that the statement is proven for a given $l < k$, and let us prove it for $l + 1$.

For the given l, we have the factorization (4). Let $N' = Ann(L')$. Note that N' is a direct factor in N.

Now N' can be identified with a subspace of $N = Ann(L)$ by means of the natural injection $V' \to V$. Then, it follows from Lemma 4 that for each non-zero $u \in N'$, there exists a pair $(i, j) \in S(x)$, such that $v_i = v_j$ for all $v \in L$, and $u_i = -u_j$. So the subspace N' lies in the union of a finite collection of vector subspaces of V', defined by equations of the form $u_i = -u_j$, corresponding to different choices of the above (i, j). By Note 1 there exists a pair $(i, j) \in S$, such that N' lies entirely in the corresponding subspace, i.e. such that $u_i = -u_j$ for all $u \in N'$. Since N' is a direct factor in N, it follows that the projection N_{ij} of N to $V_i \times V_j$ lies in the subspace defined by the equation $u_i = -u_j$. Moreover, since the projections from N to V_i and V_j are surjective, we deduce that N_{ij} coincides with this subspace. Similarly, the projection L_{ij} of L to $V_i \times V_j$ is the subspace defined by $v_i = v_j$. We are in the situation of Lemma 2, and it follows that L_{ij} is a direct factor in L (up to an isomorphism induced by a permutation of indices).

Since for a generic x it holds that $S(x) \subset S(y)$ for all y in a neighborhood of x, it follows that the partition $\{i_1, j_1\},\dots,\{i_k, j_k\}$, constructed in the previous Lemma for a given generic x, works for all y in a neighborhood of x, i.e. for all such y's it holds that $y_{i_s} = y_{j_s}$, $s = 1,\dots,k$. Then, since generic points are dense, and σ is connected, we deduce by continuity that this partition works, in fact, for any point of σ. It follows that σ is an open subset of the submanifold σ' defined by the equations $x_{i_s} = x_{j_s}$, $s = 1,\dots,k$. But since σ is closed it must be that $\sigma = \sigma'$.

The above partition establishes a bijection between the sets of positive and negative occurrences of literals in F, and it is clear how to construct a proof-structure from it — connect each a_{i_s} and a_{j_s} by an axiom link. Let ρ be the constructed proof-structure.

Lemma 6. *The proof-structure ρ is a proof-net.*

Proof Assume this is not the case. Recall that a *switching* of the proof-structure ρ is any graph obtained from ρ by deleting for every $\wp$-link of ρ one of the two edges forming the link. By the Danos-Regnier criterion there exists a cyclic switching α of ρ. So there exists a closed path z in α, which can be represented as a closed path, which traverses each of its vertices except the starting one exactly once. Let s be the number of the axiom links traversed by z. It is easy to see that, in terms of the formulas labeling its consecutive vertices, z must necessarily have a presentation of the form $z = (a_1, b_1 \dots, a_i, b_i, \dots, a_s, b_s, \dots, a_1)$, where all a_i's and b_i's are literals, $a_i = b_i^\perp$, and a_i and b_i are connected by an axiom link, $i = 1,\dots,s$, and dots stand for compound formulas.

Let us say that the above a_i's and b_i's are, respectively, *false* and *true* occurrences of literals in F. Let us say that all other occurrences of literals in F are *inessential*. Note that, if $\bar{F}$ is the tree of subformulas of F considered as a subgraph of ρ, then the connected components of $z \cap \bar{F}$ are paths starting at true occurrences and ending at false occurrences.

Now pick a point $x \in \sigma$, and let $V = T_x\sigma$. It follows from the factorization (4) that there exists a vector $v \in V$, such that for any occurrence a of a literal in F it holds that $v_a = 0$ iff the occurrence a is inessential. Moreover, since the dimensions of all spaces involved are greater than 1, it follows that v can be chosen such that for any two distinct occurrences a_1 and a_2 of the same literal a, the projections v_{a_1} and v_{a_2} are linearly independent in V_a.

For each positive literal p let us define the coherence $C_p \subseteq V_p$ by

$$C_p = \{tv_a | a \text{ is a true occurrence of } p \text{ in } F, t \in \mathbf{R}\} \cup \{0\}, \tag{5}$$

and extend the definition to compound formulas inductively. We claim that $v \notin C_F$.

This follows from the following observation.

Lemma 7. *Let A be (an occurrence of) a subformula in F. Let $\bar{A}$ be the tree of subformulas of A, considered as a subgraph of ρ. Then the following statements hold.*

If $v_A = 0$ then z does not meet $\bar{A}$.

If $0 \neq v_A \in C_A$ then there exists a true occurrence a in A, such that the connected component of $z \cap \bar{A}$ starting at a has the endpoints a and A.

Proof The first statement of the Lemma is straightforward. The second is proven by induction on the formula.

Let A be a positive literal. If the occurrence is inessential, then $v_A = 0$. If the occurrence is false, then v_A can be C_A only if there exists a true occurrence A' of the same literal, such that $v_A = tv_{A'}$ for some t. But by our choice of v the projections v_A and $v_{A'}$ are linearly independent. So A must be a true occurrence.

Let A be a negative literal, $A = p^\perp$. Again, the occurrence A cannot be inessential. If the occurrence A is false then the dual occurrence B of p, connected to A by an axiom link is true. Therefore $v_A = v_B$ is in C_p and not in $C_{p^\perp}$, since $v_A \neq 0$. So A must be a true occurrence again.

Let $A = A_1 \wp A_2$. If $0 \neq v_A \in C_A$ then at least for some $i = 1, 2$ it holds that $0 \neq v_{A_i} \in C_{A_i}$. Assume for definiteness that the i in question equals 1. By the induction hypothesis there exists a true occurrence a in A_1 such that the connected component z_1 of $z \cap \bar{A_1}$ starting at a has the endpoints a and A_1. That means that the connected component z_A of $z \cap \bar{A}$ leaves the tree $\bar{A_1}$ after traversing A_1, and the only way for z_A to leave $\bar{A_1}$ is along the edge (A_1, A). It follows that in the switching α it is the edge (A_2, A) and not (A_1, A), which was erased. But if the edge (A_2, A) was erased, the path z_A has no way to continue after getting to A, hence A is its endpoint.

Let $A = A_1 \otimes A_2$. If $0 \neq v_A \in C_A$, then $0 \neq v_{A_i} \in C_{A_i}$, $i = 1, 2$, and at least for some $i = 1, 2$ it holds that $v_{A_i} \neq 0$. Assume for definiteness that the i in question equals 1. By the induction hypothesis there exists a true occurrence a in A_1 such that the connected component z_1 of $z \cap \bar{A_1}$ starting at a has the endpoints a and A_1. If $v_{A_2} = 0$, then z does not meet $\bar{A_2}$, hence the connected component z_A of $z \cap \bar{A}$ has no way to continue after getting to A, hence A is its endpoint. Assume that $v_{A_2} \neq 0$. Again, by the induction hypothesis there exists a true occurrence a' in A_2 such that the connected component z_2 of $z \cap \bar{A_2}$ starting at a' has the endpoints a and A_2. Then the path z_A after traversing A goes up along the edge (A, A_2) and continues to traverse z_2 in the reverse order, ending at a'. Then the path z_A is a connected component of $z \cap \bar{F}$, with the endpoints a and a'. But then one of these occurrences must be false, which is a contradiction. Thus $v_{A_2} = 0$, and the statement holds.

The previous Lemma implies that if $v \in C_F$ then there exists a connected component of $z \cap \bar{F}$, one of whose endpoints is F. But this is impossible, since all connected components of $z \cap \bar{F}$ start and end at literals. This finishes the proof of the Theorem.

Note that the restriction that the interpretations of literals must be of dimension greater than one is essential. If we set $M_p = \mathbf{R}$ then the submanifold $\sigma = \{(x, y, x, y) | x, y \in M_p\}$ of M_p^4 is a clique in the smooth coherence space $F = (p \otimes p) \wp (p^\perp \otimes p^\perp)$ for any consistent choice of coherences and cocoherences, but the formula F is not derivable.

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