Automorphisms of locally conformally Kähler manifolds

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Abstract
A manifold M is locally conformally Kähler (LCK) if it admits a Kähler covering $\tilde{M}$ with monodromy acting by holomorphic homotheties. For a compact connected group G acting on an LCK manifold by holomorphic automorphisms, an averaging procedure gives a G-invariant LCK metric. Suppose that S^1 acts on an LCK manifold M by holomorphic isometries, and the lifting of this action to the Kähler cover $\tilde{M}$ is not isometric. We show that $\tilde{M}$ admits an automorphic Kähler potential, and hence (for $\dim_{\mathbb{C}} M > 2$) the manifold M can be embedded to a Hopf manifold.

Contents

1 Introduction

1.1 Locally conformally Kähler manifolds

Locally conformally Kähler (LCK) manifolds are, by definition, complex manifolds of $\dim_{\mathbb{C}} > 1$ admitting a Kähler covering with deck transforma-

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tions acting by Kähler homotheties. We shall usually denote with $\tilde{\omega}$ the Kähler form on the covering.

An equivalent definition, at the level of the manifold itself, postulates the existence of an open covering $\{U_\alpha\}$ with local Kähler metrics g_α. It requires that on overlaps $U_\alpha \cap U_\beta$, these local Kähler metrics are homothetic: $g_\alpha = c_{\alpha\beta} g_\beta$. The metrics $e^{f_\alpha} g_\alpha$ glue to a global metric whose associated two-form ω satisfies the integrability condition $d\omega = \theta \wedge \omega$, thus being locally conformal with the Kähler metrics g_α. Here $\theta\Big|_{U_\alpha} = df_\alpha$. The closed 1-form θ, which represents the cocycle $c_{\alpha\beta}$, is called **the Lee form**. Obviously, any other representative of this cocycle, $\theta' = \theta + dh$, produces another LCK metric, conformal with the initial one. This gives another definition of an LCK structure, which will be used in this paper.

Definition 1.1: Let (M, ω) be a complex Hermitian manifold, $\dim_{\mathbb{C}} M > 1$, with $d\omega = \theta \wedge \omega$, where θ is a closed 1-form. Then M is called **a locally conformally Kähler** (LCK) manifold

We refer to [DO] for an overview and to [OV3] for more recent results.

1.2 Bott-Chern cohomology and automorphic potential

Let $(\tilde{M}, \tilde{\omega})$ be a Kähler covering of an LCK manifold M, and let Γ be the deck transform group of $[\tilde{M} : M]$. Denote by $\chi : \Gamma \longrightarrow \mathbb{R}^{>0}$ the corresponding character of Γ, defined through the scale factor of $\tilde{\omega}$:

$$\gamma^* \tilde{\omega} = \chi(\gamma) \tilde{\omega}, \quad \forall \gamma \in \Gamma.$$

Definition 1.2: A differential form α on $\tilde{M}$ is called **automorphic** if $\gamma^* \alpha = \chi(\gamma)\alpha$, where $\chi : \Gamma \longrightarrow \mathbb{R}^{>0}$ is the character of Γ defined above.

A useful tool in the study of LCK geometry is the weight bundle $L \longrightarrow M$. It is a topologically trivial line bundle, associated to the representation $\mathrm{GL}(2n, \mathbb{R}) \ni A \mapsto |\det A|^{\frac{1}{n}}$, with flat connection defined as $D := \nabla_0 + \theta$, where ∇_0 is the trivial connection. It allows regarding automorphic objects on $\tilde{M}$ as objects on M with values in L.

Definition 1.3: Let M be an LCK manifold, $\Lambda^{1,1}_{\chi,d}(\tilde{M})$ the space of closed, automorphic $(1,1)$-forms on its Kähler covering $\tilde{M}$, and let $\mathcal{C}^\infty_\chi(\tilde{M})$ be the space of automorphic functions on $\tilde{M}$. Consider the quotient

$$H^{1,1}_{BC}(M,L) := \frac{\Lambda^{1,1}_{\chi,d}(\tilde{M})}{dd^c(\mathcal{C}^\infty_\chi(\tilde{M}))},$$

where $d^c = -IdI$. This group is finite-dimensional. It is called **the Bott-Chern cohomology group of an LCK manifold** (for more details, see [OV2]). It is independent from the choice of the covering $\tilde{M}$.

Remark 1.4: The Kähler form $\tilde{\omega}$ on $\tilde{M}$ is obviously closed and automorphic. Its cohomology class $[\tilde{\omega}] \in H^{1,1}_{BC}(M,L)$ is called **the Bott-Chern class of** M. It is an important cohomology invariant of an LCK manifold, which can be considered as an LCK analogue of the Kähler class.

Definition 1.5: Let $(\tilde{M}, \tilde{\omega})$ be a Kähler covering of an LCK manifold M. We say that M is **an LCK manifold with an automorphic potential** if $\tilde{\omega} = dd^c\varphi$, for some automorphic function φ on $\tilde{M}$. Equivalently, M is an LCK manifold with an automorphic potential, if its Bott-Chern class vanishes.

Compact LCK manifolds with automorphic potential are embeddable in Hopf manifolds, see [OV2]. The existence of an automorphic potential leads to important topological restrictions on the fundamental group, see [OV3] and [KK].

The class of compact complex manifolds admitting an LCK metric with automorphic potential is stable under small complex deformation, [OV1]. This statement should be considered as an LCK analogue of Kodaira's celebrated Kähler stability theorem. The only way (known to us) to construct LCK metrics on some non-Vaisman manifolds, such as the Hopf manifolds not admitting a Vaisman structure, is by deformation, applying the stability of automorphic potential under small deformations.

1.3 Automorphisms of LCK and Vaisman manifolds

Definition 1.6: A **Vaisman manifold** is an LCK manifold (M, ω, θ) with $\nabla\theta = 0$, where θ is its Lee form, and ∇ the Levi-Civita connection.

As shown e.g. in [Ve1], a Vaisman manifold has an automorphic potential, which can be written down explicitely as $\tilde{\omega}(\pi^*\theta, \pi^*\theta)$, where $\pi^*\theta$ is the lift of the Lee form to the considered Kähler covering of M.

Compact Vaisman manifolds can be characterized in terms of their automorphisms group.

Theorem 1.7: ([KO]) Let (M, ω) be a compact LCK manifold admitting a holomorphic, conformal action of $\mathbb{C}$ which lifts to an action by non-trivial homotheties on its Kähler covering. Then (M, ω) is conformally equivalent to a Vaisman manifold. ■

Other properties of the various transformations groups of LCK manifolds were studied in [MO] and [GOP].

It was proven in [OV3] that any compact LCK manifold with automorphic potential can be obtained as a deformation of a Vaisman manifold. Many of the known examples of LCK manifolds are Vaisman (see [B] for a complete list of Vaisman compact complex surfaces), but there are also non-Vaisman ones: one of the Inoue surfaces (see [B], [Tr]), its higher-dimensional generalization in [OT], and the new examples found in [FP] on parabolic and hyperbolic Inoue surfaces. Also, a blow-up of a Vaisman manifold is still LCK (see [Tr], [Vu]), but not Vaisman, and has no automorphic potential.

In this paper, we show that LCK manifolds with automorphic potential can be characterized in terms of existence of a particular subgroup of automorphisms. In Section 2, we prove the following theorem.

Theorem 1.8: Let M be a compact complex manifold, equipped with a holomorphic S^1-action and an LCK metric (not necessarily compatible). Suppose that the weight bundle L, restricted to a general orbit of this S^1-action, is non-trivial as a 1-dimensional local system. Then M admits an LCK metric with an automorphic potential.

Remark 1.9: The converse statement seems to be true as well. We conjecture that given a LCK manifold M with a automorphic potential, M always admits a holomorphic S^1-action of this kind. To motivate this conjecture, consider a Hopf manifold M (Hopf manifolds are known to admit an LCK metric with an automorphic potential, see e.g. [OV1]). Suppose that M is a quotient of $\mathbb{C}^n \backslash 0$ by a group $\mathbb{Z}$ acting by linear contractions, $M = \mathbb{C}^n \backslash 0 / \langle A \rangle$, with A a linear operator with all eigenvalues α_i satisfying $|\alpha_i| < 1$.[1] Then the holomorphic diffeomorphism flow associated with the vector field $\log A$ leads to a holomorphic S^1-action on M.

[1] Such Hopf manifolds are called **linear**.

Remark 1.10: Theorem 1.7 implies that an LCK manifold M with a certain conformal action of $\mathbb{C}$ is conformally equivalent to a Vaisman manifold. By contrast, Theorem 1.8 does not postulate that the given S^1-action is compatible with the metric. Neither does Theorem 1.8 say anything about the given LCK metric on M. Instead, Theorem 1.8 says that some other LCK structure on the same complex manifold has an automorphic potential. This new metric is obtained (see Subsection 2.3) by a kind of convolution, by averaging the old one with some weight function, wich depends on the cohomological nature of the S^1-action. In particular, the original LCK metric may have no potential. In [OV2, Conjecture 6.3] it was conjectured that all LCK metrics on a Vaisman manifold have potential; this conjecture is still unsolved.

As shown in [OV2] and [OV3], Theorem 1.8 implies the following corollary.

Corollary 1.11: Let M be a compact LCK manifold of complex dimension $n \geqslant 3$. Suppose that the weight bundle L restricted to a general orbit of this S^1-action is non-trivial as a 1-dimensional local system. Then $\tilde{M}$ is diffeomorphic to a Vaisman manifold, and admits a holomorphic embedding to a Hopf manifold.

2 The proof of the main theorem

2.1 Averaging on a compact transformation group

For the sake of completeness, we recall the following procedure described in the proof of [OV2, Th. 6.1]. Let G be a compact subgroup of $\operatorname{Aut}(M)$. Averaging the Lee form θ on G, we obtain a closed 1-form θ' which is G-invariant and stays in the same cohomology class as θ: $\theta' = \theta + df$. Then $\omega' = e^{-f}\omega$ is a LCK form with Lee form θ' and conformal to ω. Hence, we may assume from the beginning that θ (corresponding to ω) is G-invariant.

Now, for any $a \in G$, $a^*\omega$ satisfies

$$d(a^*\omega) = a^*\omega \wedge a^*\theta = a^*\omega \wedge \theta. \tag{2.1}$$

Averaging ω over G and applying (2.1), we find a G-invariant Hermitian form ω' which satisfies

$$d\omega' = \omega' \wedge \theta.$$

Therefore, we may also assume that ω is G-invariant.

In conclusion, by averaging on S^1, we obtain a new LCK metric, conformal with the initial one, w.r.t. which S^1 acts by (holomorphic) isometries and whose Lee form is S^1-invariant. Hence, we may suppose from the beginning that S^1 acts by holomorphic isometries of the given LCK metric.

This implies that the lifted action of $\mathbb{R}$ acts by homotheties of the global Kähler metric with Kähler form $\tilde\omega$. Indeed, $a^*\tilde\omega = f\tilde\omega$, but $d(a^*\tilde\omega) = 0 = df\wedge\tilde\omega$, and multiplication by $\tilde\omega$ is injective on $\Lambda^1(M)$, as $\dim_{\mathbb{C}} M > 1$, hence $df\wedge\tilde\omega = 0$ implies $df = 0$.

The monodromy of the weight bundle along an orbit S of the S^1-action can be computed as $\int_S \theta$, hence this monodromy is not changed by the averaging procedure. Therefore, it suffices to prove Theorem 1.8 assuming that ω is S^1-invariant.

In this case, the lift of the S^1-action on $\tilde M$ acts on the Kähler form $\tilde M$ by homotheties, and the corresponding conformal constant is equal to the monodromy of L along the orbits of S^1. Therefore, we may assume that S^1 is lifted to an $\mathbb{R}$ action on $\tilde M$ by non-trivial homotheties.

2.2 The main formula

Let now A be the vector field on $\tilde M$ generated by the $\mathbb{R}$-action. Then A is holomorphic and homothetic, i.e.

$$\operatorname{Lie}_A \tilde\omega = \lambda\tilde\omega, \quad \lambda\in\mathbb{R}^{>0}.$$

Denote:

$$A^c = IA, \quad \eta = A \lrcorner\, \tilde\omega, \quad \eta^c = I\eta.$$

Note that, by definition, $(I\alpha)(X_1,\dots,X_k) = (-1)^k\alpha(IX_1,\dots,IX_k)$.

We now prove the following formula, which is the key to the rest of our argument.

Proposition 2.1: Let A be a vector field acting on a Kähler manifold $\tilde M$ by holomorphic homotheties: $\operatorname{Lie}_A\tilde\omega = \lambda\tilde\omega$. Then

$$dd^c|A|^2 = \lambda^2\tilde\omega + \operatorname{Lie}^2_{A^c}\tilde\omega, \tag{2.2}$$

where $A^c = I(A)$.

Proof: Replacing A by $\lambda^{-1}A$, we may assume that $\lambda = 1$. By Cartan's formula,

$$\operatorname{Lie}_A\tilde\omega = d(A\lrcorner\,\tilde\omega) = d\eta,$$

and hence, as $\eta(A) = 0$,

$$\mathrm{Lie}_A \eta = d(A \lrcorner \eta) + A \lrcorner d\eta = A \lrcorner (\tilde\omega) = \eta.$$

As A is holomorphic, this implies $\mathrm{Lie}_A \eta^c = \eta^c$. But, again with Cartan's formula:

$$\mathrm{Lie}_A \eta^c = d(A \lrcorner \eta^c) + A \lrcorner d\eta^c = -d|A|^2 + A \lrcorner d\eta^c.$$

Hence:

$$d^c d|A|^2 = -d^c\eta^c + d^c(A \lrcorner d\eta^c),$$

We note that:

$$d^c\eta^c = -Id\eta = -I\tilde\omega = \tilde\omega,$$

as $\tilde\omega$ is $(1,1)$. Then, to compute $d^c(A \lrcorner d\eta^c)$, observe first that

$$\mathrm{Lie}_{A^c} \tilde\omega = d(IA \lrcorner \tilde\omega) = d\tilde\omega(IA, \cdot) = d\eta^c.$$

Thus, as $\tilde\omega$ and $\mathrm{Lie}_{A^c} \tilde\omega$ are $(1,1)$, and by Cartan's formula again:

$$\begin{aligned} d^c(A \lrcorner d\eta^c) &= -IdI(A \lrcorner \mathrm{Lie}_{A^c} \tilde\omega) = Id(A^c \lrcorner \mathrm{Lie}_{A^c} \tilde\omega) \\ &= I \,\mathrm{Lie}^2_{A^c} \tilde\omega = -\mathrm{Lie}^2_{A^c} \tilde\omega. \end{aligned}$$

This proves (2.2).

2.3 The second averaging argument

Clearly, the action of the Lie derivative on $\Omega^\bullet(M)$ can be extended to the Bott-Chern cohomology groups by $\mathrm{Lie}_X[\alpha] = [\mathrm{Lie}_X \alpha]$. Then (2.2) tells us that

$$\mathrm{Lie}^2_{A^c}[\tilde\omega] = -\lambda^2[\tilde\omega],$$

where $[\tilde\omega]$ is the class of $\tilde\omega$ in the Bott-Chern cohomology group $H^2_{BC}(M, L) = H^2_{BC}(\tilde M)$. This implies that

$$V := \mathrm{span}\{[\tilde\omega], \mathrm{Lie}_{A^c}[\tilde\omega]\} \subset H^2_{BC}(M, L)$$

is 2-dimensional. Then, obviously, Lie_{A^c} acts on V with two 1-dimensional eigenspaces, corresponding to $\sqrt{-1}\,\lambda$ and $-\sqrt{-1}\,\lambda$. As Lie_{A^c} acts on V essentially as a rotation with $\lambda\pi/2$, the flow of A^c, e^{tA^c}, will satisfy:

$$e^{tA^c}[\tilde\omega] = [\tilde\omega], \text{ for } t = 2n\pi\lambda^{-1},\ n \in \mathbb{Z}.$$

We also note that

$$\int_0^{2\pi\lambda^{-1}} e^{tA^c}[\tilde\omega]dt = 0. \qquad (2.3)$$

Let now

$$\tilde{\omega}_W := \int_0^{2\pi\lambda^{-1}} e^{tA^c}\tilde{\omega}dt.$$

This new form is obtained as a sum of Kähler forms with the same automorphy, hence it is also an automorphic Kähler form. Its Bott-Chern class is equal to $\int_0^{2\pi} e^{tA^c}[\tilde{\omega}]dt$, and thus it vanishes by (2.3).

In conclusion, $\tilde{\omega}_W$ is a Kähler form with trivial Bott-Chern class, and hence it admits a global automorphic potential. We proved Theorem 1.8.

Remark 2.2: Another way to arrive at a Kähler form with potential is by averaging using a kind of convolution. Let

$$\psi = \begin{cases} \cos t + 1, & \text{for } t \in [-\pi, \pi] \\ 0, & \text{for } t \notin [-\pi, \pi]. \end{cases}$$

Define

$$\tilde{\omega}_\psi = \int_{\mathbb{R}} e^{t\lambda^{-1}A^c}\tilde{\omega}\psi(t)dt.$$

One can see that $\operatorname{Lie}_{\lambda^{-1}A^c}\tilde{\omega}_\psi = \tilde{\omega}_{\psi'}$ and $\operatorname{Lie}^2_{\lambda^{-1}A^c}\tilde{\omega}_\psi = \tilde{\omega}_{\psi''}$. Then, (2.2) becomes

$$\begin{aligned} dd^c|A|^2_\psi &= \lambda^2\tilde{\omega}_\psi + \operatorname{Lie}^2_{A^c}\tilde{\omega}_\psi \\ &= \lambda^2(\tilde{\omega}_\psi + \tilde{\omega}_{\psi''}) = \lambda^2 \int_{\mathbb{R}} e^{t\lambda^{-1}A^c}\tilde{\omega}(\psi + \psi'')(t)dt. \end{aligned}$$

where $|A|^2_\psi$ means square length of A taken with respect to the metric ω_ψ. As $\psi'' + \psi = 1$ on $[-\pi, \pi]$, we see that $dd^c|A|^2_\psi > 0$ and hence $|A|^2_\psi$ is a Kähler potential for $\tilde{\omega}_\psi$. On the other hand, one can verify that

$$dd^c(|A|^2_\psi) = \operatorname{Lie}^2_{A^c}\tilde{\omega}_\psi + \lambda^2\tilde{\omega}_{\psi''} = \tilde{\omega}_W,$$

where $\tilde{\omega}_W = \int_{-\pi}^{\pi} e^{tA^c}\tilde{\omega}dt$.

Therefore, this averaging construction with "weight" ψ gives the same form $\tilde{\omega}_W = \int_{-\pi}^{\pi} e^{tA^c}\tilde{\omega}dt$ which we have obtained by the means of averaging with the circle.

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