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A Selection Principle for Functions of a Real Variable[2]

Dedicated to Professor Mikhail Vasil'evich Dolov
on the occasion of His 70th Birthday

Abstract. – *We present a new sufficient condition (which turns out to be almost necessary) in terms of the modulus of variation in order for a given sequence of real valued functions, defined on a closed interval, to contain a pointwise convergent subsequence whose limit is a bounded function with simple discontinuities. We show that many Helly type selection theorems, having to do with uniform boundedness of generalized variations, are consequences of our result.*

1. – Introduction and preliminaries

In 1912 E. Helly ([11]) proved the following celebrated selection theorem:

THEOREM A. *A uniformly bounded sequence of nondecreasing functions on a closed interval contains a pointwise convergent subsequence.*

There are several consequences and variants of this theorem for functions of bounded generalized variation. In order to present them (see Theorem B below), let us recall some of the notions of bounded generalized variation.

Let $\varphi : \mathbb{R}^+ = [0, \infty) \to \mathbb{R}^+$ be a convex function such that $\varphi(u) = 0$ if and only if $u = 0$; it follows that φ is continuous and increasing on $\mathbb{R}^+$ and admits the continuous inverse φ^{-1}.

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A function $f : [a, b] \to \mathbb{R}$ is said to be of *bounded φ-variation* on the closed interval $[a, b]$ in the sense of N. Wiener ([19]) and L. C. Young ([20]) (see also [14]) provided its *total φ-variation* defined by

$$V_\varphi(f) = \sup\left\{ \sum_{i=1}^{m} \varphi\Big(|f(x_i) - f(x_{i-1})|\Big) : m \in \mathbb{N},\ a \le x_0 < x_1 < \ldots < x_{m-1} < x_m \le b \right\}$$

is finite; in this case we write $f \in \mathrm{BV}_\varphi[a, b]$.

If in this definition $\varphi(u) = u$, the value $V_\varphi(f)$ is the *usual* C. Jordan *variation* ([13]), which will be written as $V(f)$; in this case we also write $\mathrm{BV}[a, b]$ in place of $\mathrm{BV}_\varphi[a, b]$.

It is well known ([14]) that if $\lim_{u\to+0} \varphi(u)/u = 0$, then $\mathrm{BV}[a, b]$ is a proper subset of $\mathrm{BV}_\varphi[a, b]$.

Let $\Lambda = \{\lambda_i\}_{i=1}^\infty$ be a nondecreasing sequence of positive numbers such that the series $\sum_{i=1}^\infty 1/\lambda_i$ diverges.

A function $f : [a, b] \to \mathbb{R}$ is said to be of *Λ-bounded variation* on $[a, b]$ in the sense of D. Waterman ([17]), and we write $f \in \mathrm{BV}_\Lambda[a, b]$, if the following quantity, called the *total Λ-variation* of f, is finite:

$$V_\Lambda(f) = \sup \sum_{i=1}^{m} \frac{|f(b_i) - f(a_i)|}{\lambda_i}\,,$$

where the supremum is taken over all $m \in \mathbb{N}$ and all non-ordered collections of non-overlapping intervals $[a_k, b_k] \subset [a, b]$, $k = 1, \ldots, m$.

It is well known ([18]) that if Λ is an unbounded sequence, then $\mathrm{BV}[a, b]$ is a proper subset of $\mathrm{BV}_\Lambda[a, b]$.

Now, let $\mathrm{BV}_*[a, b]$ denote one of the spaces $\mathrm{BV}[a, b]$, $\mathrm{BV}_\varphi[a, b]$ or $\mathrm{BV}_\Lambda[a, b]$, where the corresponding total variation is denoted by $V_*(\cdot)$. The following theorem is originally due to E. Helly ([11]) if $V_* = V$, and it is due to J. Musielak and W. Orlicz ([14]) if $* = \varphi$ and D. Waterman ([18]) if $* = \Lambda$:

Theorem B. *A uniformly bounded sequence $\{f_j\}$ of real valued functions on $[a, b]$, satisfying the condition $\sup_{j\in\mathbb{N}} V_*(f_j) < \infty$, contains a pointwise convergent subsequence whose pointwise limit is a function belonging to $\mathrm{BV}_*[a, b]$.*

It is the aim of this work to *remove* the condition $\sup_{j\in\mathbb{N}} V_*(f_j) < \infty$ of the uniform boundedness of variations of any kind from Theorems A and B (see Theorem 1 in Section 2). In order to do it, we need a definition.

Given $n \in \mathbb{N}$, $f : [a, b] \to \mathbb{R}$ and $\varnothing \ne E \subset [a, b]$, we set

$$\nu(n, f, E) = \sup \sum_{i=1}^{n} |f(b_i) - f(a_i)|,$$

where the supremum is taken over all points $a_1, \ldots, a_n, b_1, \ldots, b_n \in E$ such that $a_1 \le b_1 \le a_2 \le b_2 \le \ldots \le a_n \le b_n$. If $E = [a, b]$ (i.e., E is the domain of f),

$\nu(n,f,[a,b])$ will be written as $\nu(n,f)$. The sequence $\{\nu(n,f,E)\}_{n=1}^{\infty}$ is called the *modulus of variation* of f on the set E in the sense of Z. Chanturiya ([3]). Note at once (cf. Lemma 1(d), (a) in Section 3) that the modulus of variation is a nondecreasing sequence, for which the limit $\lim_{n\to\infty}\nu(n,f,E)/n\in\mathbb{R}^+$ always exists (provided the function f is bounded on the set E).

In order to get a better feeling of the notion of modulus of variation, let us take a look at how it behaves on some well known classes of functions: the following relations are valid for all $n\in\mathbb{N}$ (for their proofs and appropriate references see the Appendix in Section 7):

$$\text{if } f:[a,b]\to\mathbb{R} \text{ is monotone, then } \nu(n,f)=|f(b)-f(a)|; \tag{1}$$

$$\text{if } f\in\mathrm{BV}[a,b], \text{ then } \nu(n,f)\le V(f)=\sup_{n\in\mathbb{N}}\nu(n,f); \tag{2}$$

$$\text{if } f\in\mathrm{BV}_\varphi[a,b], \text{ then } \nu(n,f)\le n\varphi^{-1}\Big(\frac{1}{n}\Big)\max\{1,V_\varphi(f)\}; \tag{3}$$

$$\text{if } f\in\mathrm{BV}_\Lambda[a,b], \text{ then } \nu(n,f)\le n\bigg(\sum_{i=1}^{n}\frac{1}{\lambda_i}\bigg)^{-1}V_\Lambda(f); \tag{4}$$

$$\text{if } f:[a,b]\to\mathbb{R} \text{ is continuous, then } \nu(n,f)\le C\,n\,\omega\Big(\frac{b-a}{n},f\Big), \tag{5}$$

where $C>0$ is an absolute constant (independent of f) and $\omega(u,f)$ is the modulus of continuity of f on $[a,b]$, i.e., if $0<u\le b-a$, then

$$\omega(u,f)\equiv\omega(u,f,[a,b])=\sup\big\{|f(x)-f(y)|:x,\,y\in[a,b],\,|x-y|\le u\big\},$$

and $\omega(0,f)=\lim_{u\to+0}\omega(u,f)=0$.

From the right hand sides of (1)–(5) one can clearly see that the condition $\lim_{n\to\infty}\nu(n,f)/n=0$ enters into the picture. As an illustration, we note that this condition characterizes functions with simple discontinuities (i.e., those that have finite limits from the left and from the right at each point of $[a,b]$) as the following Chanturiya's result shows ([3, Theorem 5]):

THEOREM C. *A function $f:[a,b]\to\mathbb{R}$ has left and right finite limits at all points of $[a,b]$ if and only if $\lim_{n\to\infty}\nu(n,f)/n=0$.*

As a consequence of (1)–(4) and Theorem C, we get the well known property that functions from classes (1)–(4) have only simple discontinuities ([3, 5, 6, 13, 14, 18, 19, 20]).

Our paper is organized as follows. In Section 2 we present our main results: Theorems 1 and 2. The properties of the modulus of variation needed for our purposes are considered in Section 3. In Section 4 a generalization of Theorem C is given and in Section 5 the main results of the paper are proved. Finally, in Section 6 we show that Helly type selection theorems due to M. Schramm ([16]) and P. C. Bhakta ([2]) are consequences of our Theorem 1.

2. – Main results

Our first main result is the following *pointwise selection principle*:

Theorem 1. *Let $\{f_j\}$ be a uniformly bounded sequence of real valued functions on $[a,b]$ such that*

$$\lim_{n\to\infty}\Big(\frac{1}{n}\limsup_{j\to\infty}\nu(n,f_j)\Big)=0. \tag{6}$$

Then it contains a subsequence which converges pointwise on $[a,b]$ to a bounded function $f:[a,b]\to\mathbb{R}$ satisfying $\lim_{n\to\infty}\nu(n,f)/n=0$.

This theorem will be proved in Section 5. Now two remarks are in order.

Remark 1. In Theorem 1 conditions of Theorem C may not be satisfied for all functions f_j, but they always hold for the pointwise limit function f (see Remark 5(c) in Section 5).

Remark 2. Selection theorems A and B (and some others — see Section 6) follow from our Theorem 1: one should take into account (1)–(5) and the following well known property of generalized variations $V_*(\cdot)$ considered above: if a sequence $f_j:[a,b]\to\mathbb{R}$ converges pointwise to f, then $V_*(f)\le\liminf_{j\to\infty}V_*(f_j)$.

Condition (6) is the best possible for the extraction of a regular pointwise convergent subsequence (that is, convergent to a function f satisfying conditions of Theorem C) in the following sense: although it is *not necessary* for the pointwise convergence (see Remark 5 in Section 5), it is *necessary* for the uniform convergence and it is "*almost*" *necessary* for the pointwise convergence. This can be seen from the following Theorem 2, our second main result, which will be proved in Section 5:

Theorem 2. (a) *If a sequence $\{f_j\}$ of real valued functions on $[a,b]$ converges uniformly to a function f such that $\lim_{n\to\infty}\nu(n,f)/n=0$, then*

$$\lim_{n\to\infty}\Big(\frac{1}{n}\lim_{j\to\infty}\nu(n,f_j)\Big)=0.$$

(b) *If $\{f_j\}$ is a sequence of measurable real valued functions on $[a,b]$ which converges pointwise (or almost everywhere) to a function $f:[a,b]\to\mathbb{R}$ satisfying the condition $\lim_{n\to\infty}\nu(n,f)/n=0$, then for each $\varepsilon>0$ there exists a measurable set $E=E(\varepsilon)\subset[a,b]$ with (Lebesgue) measure $\le\varepsilon$ such that*

$$\lim_{n\to\infty}\Big(\frac{1}{n}\lim_{j\to\infty}\nu(n,f_j,[a,b]\setminus E)\Big)=0.$$

3. – Auxiliary lemma

In order to prove Theorems 1 and 2 and Theorem 3 from Section 4, we need a lemma expressing the main properties of the modulus of variation.

LEMMA 1. *Given* $f:[a,b]\to\mathbb{R}$, $\varnothing\ne E\subset[a,b]$ *and* $n\in\mathbb{N}$, *we have*:

(a) $\nu(n+1,f,E)/(n+1)\le\nu(n,f,E)/n$;

(b) *if a sequence of functions* $f_j:E\to\mathbb{R}$ *converges pointwise on* E *to* f *as* $j\to\infty$, *then* $\nu(n,f,E)\le\liminf_{j\to\infty}\nu(n,f_j,E)$;

(c) $\nu(n,g,E)\le\nu(n,f,E)+2n\sup_{x\in E}|f(x)-g(x)|$ *if* $g:E\to\mathbb{R}$;

(d) $|f(y)-f(x)|+\nu(n,f,E\cap[a,x])\le\nu(n+1,f,E\cap[a,y])$ *if* $x,\ y\in E$, $x\le y$;

(e) $\nu(n,f,E')\le\nu(n,f,E)$ *whenever* $\varnothing\ne E'\subset E\subset[a,b]$.

PROOF. (a) is equivalent to the inequality

$$\nu(n+1,f,E)\le\nu(n,f,E)+\frac{\nu(n+1,f,E)}{n+1}\,, \tag{7}$$

and so, following [4, Lemma], we prove (7). By the definition of $\nu(n+1,f,E)$, given $\varepsilon>0$ there exist $a_i,b_i\in E$ (depending on ε in general), $i=1,\dots,n+1$, such that $a_1\le b_1\le a_2\le b_2\le\dots\le a_n\le b_n\le a_{n+1}\le b_{n+1}$ and

$$\sum_{i=1}^{n+1}|f(b_i)-f(a_i)|\le\nu(n+1,f,E)\le\varepsilon+\sum_{i=1}^{n+1}|f(b_i)-f(a_i)|.$$

Let d_0 denote the minimal term in the sum on the left hand side. We have $d_0\le\nu(n+1,f,E)/(n+1)$. On the other hand, the right hand side of these inequalities implies $\nu(n+1,f,E)\le\varepsilon+\nu(n,f,E)+d_0$, which gives (7) due to the arbitrariness of $\varepsilon>0$.

(b) Given points $a_1\le b_1\le\dots\le a_n\le b_n$ from E and $j\in\mathbb{N}$, the definition of $\nu(n,f_j,E)$ implies

$$\sum_{i=1}^{n}|f_j(b_i)-f_j(a_i)|\le\nu(n,f_j,E).$$

Passing to the limit inferior as $j\to\infty$ in this inequality and taking into account the pointwise convergence of f_j to f, we get:

$$\sum_{i=1}^{n}|f(b_i)-f(a_i)|\le\liminf_{j\to\infty}\nu(n,f_j,E).$$

It remains to take the supremum over all a_i,b_i, $i=1,\dots,n$, as above.

(c) For points $a_1 \le b_1 \le a_2 \le b_2 \le \ldots \le a_n \le b_n$ from E, we have:

$$\begin{aligned}\sum_{i=1}^{n}|g(b_i)-g(a_i)| &\le \sum_{i=1}^{n}|g(b_i)-f(b_i)|+\sum_{i=1}^{n}|f(b_i)-f(a_i)|+\sum_{i=1}^{n}|f(a_i)-g(a_i)|\\ &\le n\sup_{x\in E}|g(x)-f(x)|+\nu(n,f,E)+n\sup_{x\in E}|f(x)-g(x)|,\end{aligned}$$

and it suffices to take the supremum over all $a_1,\ldots,a_n,b_1,\ldots,b_n$.

(d) and (e) are straightforward consequences of the definition of the modulus of variation. ■

4. – Functions with simple discontinuities

Let S be a fixed *dense* subset of $[a,b]$. We denote by $U(S)$ the set of all functions $f:[a,b]\to\mathbb{R}$ such that for every point $x\in(a,b]$ the left limit $f_{|S}(x-)=\lim_{S\ni y\to x-0}f(y)$ and for every $x\in[a,b)$ the right limit $f_{|S}(x+)=\lim_{S\ni y\to x+0}f(y)$ exist and are finite, where as usual the symbol $f_{|S}$ denotes the restriction of f to the set S. The set $U(S)$ is called *Jeffery's class* ([12]).

We have the following characterization of the class $U(S)$ in terms of the modulus of variation and at the same time a generalization of Theorem C:

Theorem 3. $U(S)=\bigl\{f:[a,b]\to\mathbb{R}\bigm| \lim\limits_{n\to\infty}\nu(n,f,S)/n=0\bigr\}$.

Proof. *Inclusion "$\supset$".* Let $f:[a,b]\to\mathbb{R}$ be such that $\nu(n,f,S)/n\to 0$ as $n\to\infty$. For $n\in\mathbb{N}$ and $s\in S$ we set $\nu_n(s)=\nu(n,f,S\cap[a,s])$. By Lemma 1(e), the function $\nu_n:S\to\mathbb{R}^+$ is nondecreasing; it is also bounded: in fact, there exists $n_0\in\mathbb{N}$ such that $\nu(n,f,S)\le n$ for all $n\ge n_0$, and so, by virtue of Lemma 1(e), (d), $\nu_n(s)\le\nu(n,f,S)\le\max\{n_0,n\}$ for all $n\in\mathbb{N}$ and $s\in S$. It follows that the limit $\nu_{n|S}(x-)$ exists at each $a<x\le b$. We will show that $f_{|S}(x-)$ exists as well (the existence of $f_{|S}(x+)$ for $a\le x<b$ is treated similarly). Given $a\le s\le t<x$ with $s,t\in S$, by Lemma 1(d), (7) and Lemma 1(e), we have:

$$\begin{aligned}|f(t)-f(s)| &\le \nu(n+1,f,S\cap[a,t])-\nu(n,f,S\cap[a,s])\\ &\le \nu(n,f,S\cap[a,t])+\frac{\nu(n+1,f,S\cap[a,t])}{n+1}-\nu(n,f,S\cap[a,s])\\ &\le \nu_n(t)+\frac{\nu(n+1,f,S)}{n+1}-\nu_n(s)\\ &\le |\nu_n(t)-\nu_{n|S}(x-)|+\frac{\nu(n+1,f,S)}{n+1}+|\nu_{n|S}(x-)-\nu_n(s)|.\end{aligned}$$

Let $\varepsilon>0$ be arbitrary. Choose $n=n(\varepsilon)\in\mathbb{N}$ such that

$$\frac{\nu(n+1,f,S)}{n+1}\le\varepsilon.$$

Let $\delta = \delta(\varepsilon) \in (0, x-a)$ satisfy the property:

$$|\nu_n(y) - \nu_{n|S}(x-)| \le \varepsilon \quad \text{for all} \quad y \in S \cap [x-\delta, x).$$

Then for all $s,\ t \in S \cap [x-\delta, x)$ we have from the above calculation: $|f(t)-f(s)| \le 3\varepsilon$, and so, the Cauchy criterion of the existence of the limit $f_{|S}(x-)$ is satisfied.

Inclusion "$\subset$". Given $f \in U(S)$, by virtue of the existence of one-sided limits $f_{|S}(x-)$ at each $x \in (a, b]$ and $f_{|S}(x+)$ at each $x \in [a, b)$, there exists a sequence $\{f_j\}$ of step functions on $[a, b]$ such that $f_{j|S}$ converges *uniformly on* S to $f_{|S}$ as $j \to \infty$. The construction of $\{f_j\}$ is similar (with obvious suitable changes) to the one given in the necessity part in [9, (7.6.1)]. We recall only that f_j is said to be a *step function* on $[a, b]$ if for some partition $a = c_0 < c_1 < \ldots < c_{m-1} < c_m = b$ of $[a, b]$ the function f_j takes a constant value on each open interval (c_{i-1}, c_i), $i = 1, \ldots, m$.

Clearly, each step function f_j is in $\mathrm{BV}[a, b]$, and so, $\lim_{n\to\infty} \nu(n, f_j)/n = 0$ according to (2). By Lemma 1(e), $\nu(n, f_j, S) \le \nu(n, f_j)$, which implies $\lim_{n\to\infty} \nu(n, f_j, S)/n = 0$. Applying Lemma 1(c), we get:

$$\frac{\nu(n, f, S)}{n} \le \frac{\nu(n, f_j, S)}{n} + 2\sup_{x\in S} |f_j(x) - f(x)|, \qquad n,\ j \in \mathbb{N}.$$

Given $\varepsilon > 0$, choose $j = j(\varepsilon) \in \mathbb{N}$ such that $\sup_{x\in S} |f_j(x) - f(x)| \le \varepsilon$ and then choose $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that $\nu(n, f_j, S)/n \le \varepsilon$ for all $n \ge n_0$. It follows that $\nu(n, f, S)/n \le 3\varepsilon$ for all $n \ge n_0$. Q. E. D. ■

5. – Proofs of Theorems 1 and 2

PROOF OF THEOREM 1. We set $\mu(n) = \limsup_{j\to\infty} \nu(n, f_j)$, $n \in \mathbb{N}$. Condition (6) implies that $\mu(n)$ is finite for all $n \in \mathbb{N}$: in fact, there exists $n_0 \in \mathbb{N}$ such that $\limsup_{j\to\infty} \nu(n, f_j) \le n$ whenever $n \ge n_0$, and Lemma 1(d) with $E = [a, b]$ and $x = y = b$ yields

$$\limsup_{j\to\infty} \nu(n, f_j) \le \limsup_{j\to\infty} \nu(n_0, f_j) \le n_0 \quad \text{if} \quad 1 \le n \le n_0.$$

Step 1. There is a subsequence $\{f_j^\circ\}$ of our original sequence f_j such that

$$\gamma(n) = \lim_{j\to\infty} \nu(n, f_j^\circ) \ \text{ exists and } \ \gamma(n) \le \mu(n) \ \text{ for all } \ n \in \mathbb{N}. \tag{8}$$

In fact, from the definition of $\mu(1)$ we find a subsequence $\{f_j^1\}$ of $\{f_j\}$ such that $\nu(1, f_j^1) \to \mu(1)$ as $j \to \infty$. We set $\gamma(1) = \mu(1)$ and $\gamma(2) = \limsup_{j\to\infty} \nu(2, f_j^1)$. Since $\gamma(2) \le \mu(2)$, we choose a subsequence $\{f_j^2\}$ of $\{f_j^1\}$ such that $\nu(2, f_j^2) \to \gamma(2)$ as $j \to \infty$. Given $n \in \mathbb{N}$, $n \ge 3$, and the subsequence $\{f_j^{n-1}\}$ of $\{f_j\}$, we set $\gamma(n) = \limsup_{j\to\infty} \nu(n, f_j^{n-1})$ and, noting that $\gamma(n) \le \mu(n)$, we pick a subsequence $\{f_j^n\}$ of $\{f_j^{n-1}\}$ such that $\nu(n, f_j^n) \to \gamma(n)$ as $j \to \infty$. Then the diagonal sequence $\{f_j^j\}_{j=1}^\infty$, which we denote by $\{f_j^\circ\}$, satisfies conditions (8) (note that f_j° with $j \ge n$ is a subsequence of $f_1^n, f_2^n, \dots, f_j^n, \dots$).

Step 2. There exists a subsequence $\{f_j^*\}$ of the sequence $\{f_j^\circ\}$ from (8) and for each $n \in \mathbb{N}$ there exists a nondecreasing function $\nu_n : [a,b] \to \mathbb{R}^+$ such that

$$\lim_{j\to\infty} \nu(n, f_j^*, [a,x]) = \nu_n(x) \quad \text{for all} \quad n \in \mathbb{N} \quad \text{and} \quad x \in [a,b]. \tag{9}$$

First, we note that, given $n \in \mathbb{N}$, for each $j \in \mathbb{N}$ the function $x \mapsto \nu(n, f_j^\circ, [a,x])$ is nondecreasing on $[a,b]$ by Lemma 1(e), and according to (8) $\sup_{j\in\mathbb{N}} \nu(n, f_j^\circ) \le C(n)$ for some constant $C(n) \ge 0$.

The sequence of nondecreasing functions $x \mapsto \nu(1, f_j^\circ, [a,x])$, $j \in \mathbb{N}$, is uniformly bounded on $[a,b]$ by $C(1)$ (see Lemma 1(e)), and so, by Helly's Theorem A, the sequence $\{f_j^\circ\}$ contains a subsequence $\{f_j^{\circ 1}\}$ such that $\nu(1, f_j^{\circ 1}, [a,x])$ tends to $\nu_1(x)$ as $j \to \infty$ for all $x \in [a,b]$, where $\nu_1 : [a,b] \to [0, C(1)]$ is a nondecreasing function. Applying Helly's Theorem A one more time to the sequence of functions $x \mapsto \nu(2, f_j^{\circ 1}, [a,x])$, $j \in \mathbb{N}$, we find a subsequence $\{f_j^{\circ 2}\}$ of $\{f_j^{\circ 1}\}$ and a nondecreasing function $\nu_2 : [a,b] \to [0, C(2)]$ such that $\nu(2, f_j^{\circ 2}, [a,x]) \to \nu_2(x)$ as $j \to \infty$ for all $x \in [a,b]$. Inductively, if $n \ge 3$ and the subsequence $\{f_j^{\circ n-1}\}$ of $\{f_j^\circ\}$ is already constructed, we apply Helly's Theorem A to the sequence of nondecreasing functions $x \mapsto \nu(n, f_j^{\circ n-1}, [a,x])$, $j \in \mathbb{N}$, which is uniformly bounded on $[a,b]$ by $C(n)$: there exist a subsequence $\{f_j^{\circ n}\}$ of $\{f_j^{\circ n-1}\}$ and a nondecreasing function $\nu_n : [a,b] \to [0, C(n)]$ such that $\nu(n, f_j^{\circ n}, [a,x]) \to \nu_n(x)$ as $j \to \infty$ for all $x \in [a,b]$. It follows that the diagonal sequence $\{f_j^{\circ j}\}_{j=1}^\infty$, which we denote by $\{f_j^*\}$, satisfies conditions (9) and (8).

Step 3. For each $n \in \mathbb{N}$ the function ν_n from step 2 is monotone, and so, the set $Q_n \subset [a,b]$ of its points of discontinuity is at most countable. Setting $S = ([a,b] \cap \mathbb{Q}) \cup \bigcup_{n=1}^\infty Q_n$, where $\mathbb{Q}$ is the set of all rationals, we have: S is a countable dense subset of $[a,b]$ such that

$$\nu_n \quad \text{is continuous on} \quad [a,b] \setminus S \quad \text{for all} \quad n \in \mathbb{N}. \tag{10}$$

Since the sequence $\{f_j^*\}$ is uniformly bounded on $[a,b]$ and $S \subset [a,b]$ is countable, we may assume with no loss of generality (by again applying the standard diagonal process and passing to a subsequence of $\{f_j^*\}$ if necessary) that, for all $s \in S$, $f_j^*(s)$ converges to a point $f(s)$ as $j \to \infty$.

We are going to show that for any $t \in [a,b] \setminus S$ the sequence $\{f_j^*(t)\}$ also converges. Let $\varepsilon > 0$ be arbitrary. Since, by (6), $\mu(n)/n \to 0$ as $n \to \infty$, there is a number $n = n(\varepsilon) \in \mathbb{N}$, depending on ε, such that

$$\frac{\mu(n+1)}{n+1} \le \varepsilon. \tag{11}$$

The first condition in (8) implies the existence of a number $J_0 = J_0(\varepsilon) \in \mathbb{N}$, depending on ε, such that

$$\nu(n+1, f_j^*) \leq \gamma(n+1) + \varepsilon \quad \text{for all} \quad j \geq J_0. \tag{12}$$

By the density of S in $[a,b]$ and (10), there is a point $s = s(\varepsilon, t) \in S$, depending on ε and t, such that

$$|\nu_n(t) - \nu_n(s)| \leq \varepsilon. \tag{13}$$

From (9) we find a number $J_1 = J_1(\varepsilon, t) \in \mathbb{N}$, depending on ε and t, with the property that if $j \geq J_1$, then

$$|\nu(n, f_j^*, [a,t]) - \nu_n(t)| \leq \varepsilon \quad \text{and} \quad |\nu(n, f_j^*, [a,s]) - \nu_n(s)| \leq \varepsilon. \tag{14}$$

Suppose $s < t$ (the case when $t < s$ is treated similarly). Applying successively inequalities of Lemma 1(d), (7), (13) and (14), and then Lemma 1(e), (12), (8) and (11), we have for all $j \geq \max\{J_0, J_1\}$:

$$\begin{aligned} |f_j^*(t) - f_j^*(s)| &\leq \nu(n+1, f_j^*, [a,t]) - \nu(n, f_j^*, [a,s]) \\ &\leq \nu(n+1, f_j^*, [a,t]) - \nu(n, f_j^*, [a,t]) \\ &\qquad + |\nu(n, f_j^*, [a,t]) - \nu_n(t)| \\ &\qquad + |\nu_n(t) - \nu_n(s)| \\ &\qquad + |\nu_n(s) - \nu(n, f_j^*, [a,s])| \\ &\leq \frac{\nu(n+1, f_j^*, [a,t])}{n+1} + 3\varepsilon \\ &\leq \frac{\nu(n+1, f_j^*)}{n+1} + 3\varepsilon \leq 5\varepsilon. \end{aligned} \tag{15}$$

Being convergent, the sequence $\{f_j^*(s)\}$ is Cauchy, and so, there is a number $J_2 = J_2(\varepsilon) \in \mathbb{N}$, depending on ε, such that

$$|f_j^*(s) - f_k^*(s)| \leq \varepsilon \quad \text{for all} \quad j \geq J_2 \quad \text{and} \quad k \geq J_2. \tag{16}$$

The number $J = \max\{J_0, J_1, J_2\}$ depends only on ε and, by virtue of (15) and (16), for all $j \geq J$ and $k \geq J$ we have:

$$|f_j^*(t) - f_k^*(t)| \leq |f_j^*(t) - f_j^*(s)| + |f_j^*(s) - f_k^*(s)| + |f_k^*(s) - f_k^*(t)| \leq 11\varepsilon.$$

Thus, $\{f_j^*(t)\}$ is a Cauchy sequence, and so, it is convergent as $j \to \infty$ to a point denoted by $f(t)$. Since the point $t \in [a,b] \setminus S$ is arbitrary, taking into account the arguments at the beginning of step 3, we have:

$$\lim_{j\to\infty} f_j^*(x) = f(x) \quad \text{for all} \quad x \in [a,b].$$

Applying Lemma 1(b) we conclude that, given $n \in \mathbb{N}$,

$$\nu(n,f) \le \liminf_{j\to\infty} \nu(n,f_j^*) \le \limsup_{j\to\infty} \nu(n,f_j) = \mu(n).$$

In particular, (6) implies $\lim_{n\to\infty} \nu(n,f)/n = 0$, and condition $\nu(1,f) \le \mu(1)$ means that the function $f : [a,b] \to \mathbb{R}$ defined above is bounded. ■

A number of remarks/comments on Theorem 1 are in order.

Remark 3. (a) The condition of the uniform boundedness of the sequence $\{f_j\}$ in Theorem 1 cannot be significantly lightened: we may suppose only that $|f_j(x_0)| \le C_0$ at some point $x_0 \in [a,b]$ for some constant $C_0 \ge 0$ and all $j \in \mathbb{N}$, but, in view of (6), this will imply the uniform boundedness of the sequence $\{f_j\}$. In fact, since $\mu(1)$ is finite, we have $\nu(1,f_j) \le C_1$ for some number $C_1 \ge 0$ and all $j \in \mathbb{N}$, and so, for all $x \in [a,b]$ and $j \in \mathbb{N}$ we find

$$|f_j(x)| \le |f_j(x_0)| + |f_j(x) - f_j(x_0)| \le C_0 + \nu(1,f_j) \le C_0 + C_1.$$

Moreover, we cannot do without the condition of the uniform boundedness of $\{f_j\}$ in Theorem 1: if $f_j : [0,1] \to \mathbb{R}$ is defined by $f_j(x) = j$ for all $x \in [0,1]$ and $j \in \mathbb{N}$, then $\nu(n,f_j) = 0$ for all n, $j \in \mathbb{N}$, and so, $\lim_{j\to\infty} \nu(n,f_j) = 0$, implying condition (6), but no subsequence of $\{f_j\}$ is convergent.

(b) Since the sequence $\{f_j\}$ in Theorem 1 is uniformly bounded, say, $|f_j(x)| \le C$ for all $x \in [a,b]$, $j \in \mathbb{N}$, and some $C \ge 0$, it is straightforward that $\sup_{j\in\mathbb{N}} \nu(n,f_j) \le 2Cn$, and so, $\mu(n)$ is finite for all $n \in \mathbb{N}$. In the proof of Theorem 1 we have chosen to deduce the finiteness of $\mu(n)$ via the condition (6) in order to include condition $|f_j(x_0)| \le C_0$ from Remark 3(a) as a virtual generalization.

(c) The outer limit $\lim_{n\to\infty}$ in condition (6) is quite natural, since, by virtue of Lemma 1(a), the sequences $\{\frac{1}{n}\nu(n,f_j)\}_{n=1}^{\infty}$ and $\{\frac{1}{n}\limsup_{j\to\infty} \nu(n,f_j)\}_{n=1}^{\infty}$ are nonincreasing (and bounded from below).

Remark 4. (a) Theorem 1 is *false* without condition (6). Let $\alpha : \mathbb{N} \to \mathbb{R}$ be a bounded sequence with $\liminf_{j\to\infty} |\alpha(j)| > 0$ and $f_j : [0,2\pi] \to \mathbb{R}$ be given by $f_j(x) = \alpha(j)|\sin(jx)|$, $x \in [0,2\pi]$, $j \in \mathbb{N}$. The modulus of variation $\nu(n,f_j)$ is equal to

$$\nu(n,f_j) = \begin{cases} n|\alpha(j)| & \text{if} \quad 1 \le n < 4j, \\ 4j|\alpha(j)| & \text{if} \quad n \ge 4j, \end{cases} \qquad n,\, j \in \mathbb{N}, \tag{17}$$

(where we note that $V(f_j) = 4j|\alpha(j)|$), and so,

$$\lim_{n\to\infty}\Big(\frac{1}{n}\limsup_{j\to\infty} \nu(n,f_j)\Big) = \limsup_{j\to\infty} |\alpha(j)| > 0.$$

At the same time no subsequence of f_j is convergent everywhere on $[0,2\pi]$.

(b) On the other hand, if $\alpha(j) \to 0$ and $j|\alpha(j)| \to \infty$ as $j \to \infty$, the classical Helly Theorem B (with $V_* = V$) is inapplicable whereas Theorem 1 successfully applies. This shows that Helly's principles having to do with uniform boundedness of variations rely on the second line in (17) expressing the total variation of f_j, which is too rough. In Theorem 1 we make use of the first line in (17), which is more precise (see also Remark 5(c) below).

(c) Let $f_j : [0,1] \to \mathbb{R}$ be given by $f_j(x) = 0$ if $0 \le x < 1$ and $f_j(1) = j$. We have: $\nu(n, f_j) = j$, condition (6) is not satisfied, and no subsequence of $\{f_j(1)\}_{j=1}^\infty$ converges.

REMARK 5. (a) Condition (6) in Theorem 1 is *not necessary*. Define the sequence $f_j : [0,1] \to \mathbb{R}$ by $f_j(x) = g_j(x)$ if $0 \le x \le 1/j$ and $f_j(x) = 1$ if $1/j \le x \le 1$, where $g_j : [0,1] \to \mathbb{R}$ is given by $g_j(x) = 1$ if $j!x$ is integer and $g_j(x) = 0$ otherwise. Clearly, g_j converges pointwise on $[0,1]$ to the characteristic function $1_{\mathbb{Q}}$ of the rationals $\mathbb{Q}$, while f_j converges pointwise to the constant function $f(x) \equiv 1$. Since for given n, $j \in \mathbb{N}$ we have $\nu(n, f_j) = n$ if $n < 2 \cdot (j-1)!$ and $\nu(n, f_j) = V(f_j) = 2 \cdot (j-1)!$ if $n \ge 2 \cdot (j-1)!$, then $\lim_{j\to\infty} \nu(n, f_j) = n$, and so, condition (6) is not satisfied.

(b) Another example of this type can be given as follows. Define $f_j : [0, 2\pi] \to \mathbb{R}$ by $f_j(x) = |\sin(j^2 x)|$ if $0 \le x \le 2\pi/j$ and $f_j(x) = 0$ if $2\pi/j \le x \le 2\pi$. Then f_j converges pointwise to $f(x) \equiv 0$, while $\nu(n, f_j)$ is expressed by (17) with $\alpha(j) \equiv 1$, and so, (6) does not hold.

(c) Let $f_j : [0,1] \to \mathbb{R}$ be defined by $f_j = 1_{\mathbb{Q}}/j$, where $1_{\mathbb{Q}}$ is the restriction to $[0,1]$ of the characteristic function of the rationals. Clearly, $V_*(f_j) = \infty$ for all $j \in \mathbb{N}$ with any interpretation of the generalized variation V_* as exposed above, so Theorem B cannot be applied. On the other hand, $\nu(n, f_j) = n/j$, and so, condition (6) is satisfied.

PROOF OF THEOREM 2. (a) By Lemma 1(c), given n, $j \in \mathbb{N}$, we get:

$$\nu(n, f_j) \le \nu(n, f) + 2n \sup_{x\in[a,b]} |f(x) - f_j(x)|,$$

which, by virtue of the uniform convergence of f_j to f, implies

$$\limsup_{j\to\infty} \nu(n, f_j) \le \nu(n, f) \quad \text{for all} \quad n \in \mathbb{N}.$$

By Lemma 1(b), $\nu(n, f) \le \liminf_{j\to\infty} \nu(n, f_j)$, and so, the limit $\lim_{j\to\infty} \nu(n, f_j)$ exists and is equal to $\nu(n, f)$. It remains to take into account that $\nu(n, f)/n \to 0$ as $n \to \infty$.

(b) By Egorov's Theorem and our assumptions, given $\varepsilon > 0$ there exists a measurable set $E \subset [a,b]$ (depending on ε in general) with measure not exceeding ε such that f_j converges to f uniformly on $[a,b] \setminus E$ as $j \to \infty$. By the (analogous) arguments as in (a), we have:

$$\lim_{j\to\infty} \nu(n, f_j, [a,b] \setminus E) = \nu(n, f, [a,b] \setminus E), \qquad n \in \mathbb{N}. \tag{18}$$

Since $\nu(n, f, [a, b] \setminus E) \le \nu(n, f)$ by Lemma 1(e) and $\nu(n, f)/n \to 0$ as $n \to \infty$ by the assumption, our assertion follows now from (18). ■

REMARK 6. Theorem 2(a) is false if we replace the uniform convergence of f_j to f by the pointwise convergence — see Remark 5(a), (b) above.

6. – Helly's principle for more general generalized variations

Here we treat Helly type selection principles of more general nature than exposed above and show that they are consequences of our Theorem 1.

6.1. Let us recall the notion of Φ-bounded variation introduced by Schramm in [16]. Let $\Phi = \{\varphi_i\}_{i=1}^{\infty}$ be a sequence of *φ-functions*, i.e., each $\varphi_i : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, nondecreasing, unbounded, and $\varphi_i(u) = 0$ if and only if $u = 0$. In what follows we will do *without the convexity* of each φ_i, assumed in [16], which will also give some generalization of Schramm's result. The sequence Φ is said to be a *Φ-sequence* if the following two conditions hold:

$$\varphi_{i+1}(u) \le \varphi_i(u) \quad \text{for all} \quad i \in \mathbb{N} \quad \text{and} \quad u \in \mathbb{R}^+ \tag{19}$$

and

$$\sum_{i=1}^{\infty} \varphi_i(u) = +\infty \quad \text{for all} \quad u > 0. \tag{20}$$

A function $f : [a, b] \to \mathbb{R}$ is said to be of *Φ-bounded variation* on $[a, b]$ in the sense of M. Schramm, in symbols $f \in \mathrm{BV}_\Phi[a, b]$, if its *total Φ-variation* defined by

$$V_\Phi(f) = \sup \sum_{i=1}^{m} \varphi_i\Big(|f(b_i) - f(a_i)|\Big)$$

is finite, the supremum being taken over all $m \in \mathbb{N}$ and all non-ordered collections of non-overlapping intervals $[a_k, b_k] \subset [a, b]$, $k = 1, \dots, m$.

Note that the spaces $\mathrm{BV}[a, b]$, $\mathrm{BV}_\varphi[a, b]$ and $\mathrm{BV}_\Lambda[a, b]$ alluded to above correspond to Φ-sequences $\Phi = \{\varphi_i\}_{i=1}^{\infty}$ with $\varphi_i(u) = u$, $\varphi_i(u) = \varphi(u)$ and $\varphi_i(u) = u/\lambda_i$, respectively, where $i \in \mathbb{N}$ and $u \in \mathbb{R}^+$.

An analogue of the Helly selection theorem presented in [16, Theorem 2.8] is exactly Theorem B from Section 1, where $* = \Phi$ is a given Φ-sequence. Let us show that for the sequence $\{f_j\}$ from Theorem B, for which $C = \sup_{j \in \mathbb{N}} V_\Phi(f_j)$ is finite, condition (6) is fulfilled.

Given $n \in \mathbb{N}$, let $a \le a_1 \le b_1 \le a_2 \le b_2 \le \ldots \le a_n \le b_n \le b$ be arbitrary. By the definition of V_Φ, for $j \in \mathbb{N}$ and any permutation $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$ we have:

$$\sum_{i=1}^{n} \varphi_i\Big(\big|f_j(b_{\sigma(i)}) - f_j(a_{\sigma(i)})\big|\Big) \le V_\Phi(f_j) \le C,$$

and so, the definition of $\nu(n, f_j)$ implies:

$$\sup_{j\in\mathbb{N}} \nu(n, f_j) \le \sup \sum_{i=1}^{n} u_i\,, \tag{21}$$

where the supremum is taken over all $\{u_i\}_{i=1}^n \subset \mathbb{R}^+$ such that

$$\sum_{i=1}^{n} \varphi_i\bigl(u_{\sigma(i)}\bigr) \le C \quad \text{for all permutations } \sigma \text{ of the set } \{1,\dots,n\}. \tag{22}$$

Let $\xi(n)$ designate the right hand side in (21). We claim that $\xi(n)/n \to 0$ as $n \to \infty$, and so, (6) is satisfied. In fact, given $\varepsilon > 0$, by virtue of (20), set

$$N_0 = N_0(\varepsilon) = \min\biggl\{ N \in \mathbb{N} \biggm| \sum_{i=1}^{N} \varphi_i\Bigl(\frac{\varepsilon}{2}\Bigr) > C \biggr\}.$$

Let $n \in \mathbb{N}$, $n \ge N_0$, and $\{u_i\}_{i=1}^n \subset \mathbb{R}^+$ be arbitrary numbers satisfying condition (22). We set $I_1(n) = \{1 \le i \le n \mid u_i \le \varepsilon/2\}$ and $I_2(n) = \{1 \le i \le n \mid u_i > \varepsilon/2\}$, and let $|I_1(n)|$ and $|I_2(n)|$ denote the number of elements in $I_1(n)$ and $I_2(n)$, respectively. Conditions (19) and (22) imply $\varphi_1(u_i) \le C$, and so, $u_i \le \varphi_1^{-1}(C) \equiv \sup\{u \in \mathbb{R}^+ \mid \varphi_1(u) \le C\}$ for all $i \in \{1,\dots,n\}$. Also, (22) yields $|I_2(n)| < N_0$; indeed, if $|I_2(n)| \ge N_0$, say, $I_2(n) = \{i_k\}_{k=1}^{K_0}$ with $N_0 \le K_0 \le n$ and $i_k \in \{1,\dots,n\}$ for $k = 1,\dots,K_0$, then we define the permutation $\sigma : \{1,\dots,n\} \to \{1,\dots,n\}$ as follows: $\sigma(k) = i_k$ if $k \in \{1,\dots,K_0\}$ and arbitrarily for $K_0 < k \le n$, so that

$$\sum_{i=1}^{n} \varphi_i(u_{\sigma(i)}) \ge \sum_{k\in I_2(n)} \varphi_k(u_{\sigma(k)}) \ge \sum_{k=1}^{N_0} \varphi_k(u_{i_k}) \ge \sum_{k=1}^{N_0} \varphi_k\Bigl(\frac{\varepsilon}{2}\Bigr) > C,$$

which contradicts (22). Thus, $|I_2(n)| < N_0$. Now, we have:

$$\begin{aligned}\sum_{i=1}^{n} u_i &= \sum_{i\in I_1(n)} u_i + \sum_{i\in I_2(n)} u_i \le |I_1(n)| \cdot \frac{\varepsilon}{2} + |I_2(n)| \cdot \varphi_1^{-1}(C)\\ &\le n \cdot \frac{\varepsilon}{2} + N_0 \cdot \varphi_1^{-1}(C) \quad \text{for all } n \ge N_0.\end{aligned}$$

Thanks to the arbitrariness of $\{u_i\}_{i=1}^n$ having property (22), the last estimate means that $\xi(n)/n \le (\varepsilon/2) + N_0\varphi_1^{-1}(C)/n$ for all $n \ge N_0$, and so, $\lim_{n\to\infty} \xi(n)/n = 0$.

Thus, according to Theorem 1 a subsequence of $\{f_j\}$, again denoted by $\{f_j\}$, converges pointwise on $[a,b]$ to a function $f : [a,b] \to \mathbb{R}$. We have to show that $f \in \mathrm{BV}_\Phi[a,b]$. Let $[a_k, b_k] \subset [a,b]$, $k = 1,\dots,m$, be an arbitrary collection of non-overlapping intervals. Since

$$\sum_{i=1}^{m} \varphi_i\Bigl(\bigl|f_j(b_i) - f_j(a_i)\bigr|\Bigr) \le V_\Phi(f_j) \quad \text{for all} \quad j \in \mathbb{N},$$

then passing to the limit inferior as $j \to \infty$ and taking into account the pointwise convergence of f_j to f, we get:

$$\sum_{i=1}^{m} \varphi_i\Big(|f(b_i) - f(a_i)|\Big) \le \liminf_{j\to\infty} V_\Phi(f_j).$$

Thus, $V_\Phi(f) \le \liminf_{j\to\infty} V_\Phi(f_j) \le C$, and so, $f \in \mathrm{BV}_\Phi[a,b]$.

REMARK 7. The assumption "$\varphi_i(u) = 0$ if and only if $u = 0$" is significant for the validity of Musielak-Orlicz's and Schramm's selection principles. In fact, let $\varphi_i = \varphi$ for all $i \in \mathbb{N}$, where $\varphi(u) = 0$ if $0 \le u \le 2$ and $\varphi(u) = u - 2$ if $u \ge 2$. For the sequence $f_j(x) = \sin(jx)$, $0 \le x \le 2\pi$, $j \in \mathbb{N}$, we have $\sup_{j\in\mathbb{N}} V_\Phi(f_j) = 0$, but $\{f_j\}$ does not contain a subsequence convergent everywhere on $[0, 2\pi]$.

REMARK 8. The main results of this work (Lemma 1, Theorems 1–3 and Section 6.1) are also valid for functions defined on an arbitrary nonempty subset of $\mathbb{R}$ and taking values from a metric space; see [8] for the appropriate setting.

6.2. Let S be a dense subset of $[a,b]$ and $U(S)$ be Jeffery's class from Section 4. Given $f \in U(S)$, we set $S(f) = \{x \in S \mid f_{|S}(x-) = f(x) = f_{|S}(x+)\}$. The set $S \setminus S(f)$ is at most countable (cf. [2, Lemma 2.1]).

Let $\varnothing \ne E \subset [a,b]$, $\alpha = \inf E$ and $\beta = \sup E$. A function $f \in U(S)$ is said to be of *bounded variation on E relative to S* in the sense of R. Jeffery [12] (see also P. Bhakta [2]), which is written as $f \in \mathrm{BV}_S(E)$, if the *total variation of f on E relative to S* given by

$$\begin{aligned} V_S(f;E) \;=\; \sup\Big\{ & |f_{|S}(x_0-) - f(\alpha)| + \sum_{i=1}^{m} |f_{|S}(x_i-) - f_{|S}(x_{i-1}+)| \\ & + |f(\beta) - f_{|S}(x_m+)| \Big\} \end{aligned}$$

is finite, where the supremum is taken over all $m \in \mathbb{N}$ and all $\{x_i\}_{i=0}^{m} \subset E$ such that $\alpha < x_0 < x_1 < \ldots < x_{m-1} < x_m < \beta$. It is known ([2]) that $\mathrm{BV}[a,b]$ is a proper subset of $\mathrm{BV}_S([a,b])$.

The following variant of Helly's selection principle is due to P. C. Bhakta ([2, Theorem 3.4]):

THEOREM D. *Let S be a dense, measurable subset of $[a,b]$ of full measure. Suppose a sequence of functions $\{f_j\} \subset U(S)$ is such that there is a positive constant C, for which $V_S(f_j;[a,b]) \le C$, $|f_j(a)| \le C$, $|f_j(b)| \le C$ and $|f_{j|S}(x\pm)| \le C$ for all $j \in \mathbb{N}$ and $a < x < b$. Then $\{f_j\}$ contains a subsequence which converges almost everywhere on $[a,b]$ to a function from* $\mathrm{BV}[a,b]$.

We show that Theorem D is a consequence of Theorem 1. The set $S_0 = \bigcap_{j=1}^{\infty} S(f_j)$ is of full measure on $[a,b]$ because $S \setminus S_0$ is at most countable. Given $j \in \mathbb{N}$, the Jordan variation of f_j on S_0, denoted by $V(f_j, S_0)$, is estimated by

$$V(f_j, S_0) \leq V_S(f_j; S_0) \leq V_S(f_j; [a,b]) \leq C,$$

and so,

$$\sup_{j \in \mathbb{N}} \nu(n, f_j, S_0) \leq \sup_{j \in \mathbb{N}} V(f_j, S_0) \leq C;$$

also, $|f_j(x)| = |f_{j|S}(x\pm)| \leq C$ for all $x \in S_0$ and $j \in \mathbb{N}$. By Theorem 1 and Remark 8, a subsequence of $\{f_j\}$ converges pointwise on S_0 to a function $f : S_0 \to \mathbb{R}$ of bounded Jordan variation: $V(f, S_0) < \infty$. Following [7, p. 10, below (2.4)] we extend f to a function $\widetilde{f} : [a,b] \to \mathbb{R}$ such that $V(\widetilde{f}) = V(f, S_0)$. Then the extracted subsequence converges almost everywhere on $[a,b]$ to the extension $\widetilde{f} \in \mathrm{BV}[a,b]$.

Remark 9. Recall that a sequence $\{f_j\}$ of real valued functions on $[a,b]$ is said to be *equicontinuous* if for each $\varepsilon > 0$ there exists $u_0(\varepsilon) > 0$ such that $|f_j(x) - f_j(y)| \leq \varepsilon$ for all x, $y \in [a,b]$, $|x - y| \leq u_0(\varepsilon)$, and all $j \in \mathbb{N}$; in other words, $\lim_{u \to +0} \sup_{j \in \mathbb{N}} \omega(u, f_j) = 0$, where $\omega(\cdot, f_j)$ is the modulus of continuity of f_j. This definition, estimate (5) and Theorem 1 give the well known Ascoli's Theorem: *A uniformly bounded equicontinuous sequence of real valued functions* $\{f_j\}$ *on* $[a,b]$ *contains a pointwise convergent subsequence.*

Remark 10. Pointwise selection principles of different kind, also not having to do with uniform boundedness of variations and basing on the notion of the oscillation, are contained in the works of K. Schrader [15] and L. Di Piazza and C. Maniscalco [10]. These principles are valid only for *real valued* functions (cf. Remark 8) and the pointwise limit function of the extracted subsequence may be highly irregular, e.g., may have no simple discontinuities. At present a complete relationship between these principles and Theorem 1 (or its extensions [8]) is not known to the author, and further investigation is needed to clarify this relationship.

7. – Appendix: proofs of estimates (1)–(5)

Proof of (1) and (2). Let $f \in \mathrm{BV}[a,b]$. Clearly, $\nu(n,f) \leq V(f)$ for all $n \in \mathbb{N}$, and so, $\sup_{n \in \mathbb{N}} \nu(n,f) \leq V(f)$. On the other hand, if $a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$ for any given $n \in \mathbb{N}$, we have:

$$\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \leq \nu(n,f) \leq \sup_{n \in \mathbb{N}} \nu(n,f),$$

which gives $V(f) \le \sup_{n\in\mathbb{N}} \nu(n,f)$, and so, (2) follows. Now, if f is monotone, then for all $n \in \mathbb{N}$ we have:

$$|f(b)-f(a)| = \nu(1,f) \le \nu(n,f) \le V(f) = |f(b)-f(a)|,$$

which proves (1). ■

Proof of (3) (cf. [3, Theorem 3]). Given $f \in \mathrm{BV}_\varphi[a,b]$, we set $M = \max\{1, V_\varphi(f)\}$. If $n \in \mathbb{N}$ and $a \le a_1 \le b_1 \le a_2 \le b_2 \le \ldots \le a_n \le b_n \le b$, then by virtue of the convexity of φ, we have:

$$\begin{aligned}\varphi\Big(\frac{1}{Mn}\sum_{i=1}^{n}|f(b_i)-f(a_i)|\Big) &\le \frac{1}{M}\,\varphi\Big(\frac{1}{n}\sum_{i=1}^{n}|f(b_i)-f(a_i)|\Big)\\ &\le \frac{1}{Mn}\sum_{i=1}^{n}\varphi\Big(|f(b_i)-f(a_i)|\Big) \le \frac{1}{Mn}V_\varphi(f) \le \frac{1}{n},\end{aligned}$$

and so, $\sum_{i=1}^{n}|f(b_i)-f(a_i)| \le Mn\varphi^{-1}(1/n)$, which implies (3). ■

Proof of (4) (cf. [1]). If $f \in \mathrm{BV}_\Lambda[a,b]$, $n \in \mathbb{N}$, $a \le a_1 < b_1 \le a_2 < b_2 \le \ldots \le a_n < b_n \le b$ and $\sigma : \{1,\ldots,n\} \to \{1,\ldots,n\}$ is a permutation, we have $\sum_{i=1}^{n}|f(b_i)-f(a_i)|/\lambda_{\sigma(i)} \le V_\Lambda(f)$. Substituting into this inequality the n permutations $\sigma = \sigma_k$, $k = 0,1,\ldots,n-1$, defined by $\sigma_k(i) = n-k+i$ if $1 \le i \le k$ and $\sigma_k(i) = i-k$ if $k+1 \le i \le n$, and summing the n resulting inequalities, we get

$$\Big(\sum_{i=1}^{n}\frac{1}{\lambda_i}\Big)\cdot\Big(\sum_{i=1}^{n}|f(b_i)-f(a_i)|\Big) \le nV_\Lambda(f),$$

from which the inequality (4) follows. ■

Proof of (5) (cf. [4, Theorem 1]). Let $g : [0,2\pi] \to \mathbb{R}$ be a continuous function defined by

$$g(\tau) = f\Big(a + \tau\,\frac{b-a}{2\pi}\Big), \qquad 0 \le \tau \le 2\pi.$$

By inequality (3) from [4], there exists an absolute constant $C > 0$ such that the following estimate holds:

$$\nu(n,g,[0,2\pi]) \le C\,n\,\omega\Big(\frac{1}{n}, g, [0,2\pi]\Big), \qquad n \in \mathbb{N}.$$

Since $\nu(n,g,[0,2\pi]) = \nu(n,f)$ for all $n \in \mathbb{N}$ and

$$\omega(u,g,[0,2\pi]) = \omega\Big(\frac{b-a}{2\pi}\,u, f\Big), \qquad 0 \le u \le 2\pi,$$

we have:

$$\nu(n,f) \le C\,n\,\omega\Big(\frac{b-a}{2\pi}\cdot\frac{1}{n}, f\Big) \le C\,n\,\omega\Big(\frac{b-a}{n}, f\Big). \quad ■$$

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