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ON THE KUROSH PROBLEM IN VARIETIES OF ALGEBRAS

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Abstract. We consider a couple of versions of the classical Kurosh problem (whether there is an infinite-dimensional algebraic algebra) for varieties of linear multioperator algebras over a field. We show that, given an arbitrary signature, there is a variety of algebras of this signature such that the free algebra of the variety contains polylinear elements of arbitrarily large degree, while the clone of every such element satisfies some nontrivial identity. If, in addition, the number of binary operations is at least 2, then each such clone may be assumed to be finite-dimensional. Our approach is the following: we cast the problem in the language of operads and then apply the usual homological constructions in order to adopt Golod's solution to the original Kurosh problem. This paper is expository, so that some proofs are omitted. At the same time, the general relations of operads, algebras, and varieties are widely discussed.

1. Introduction

1.1. The Kurosh Problem and Multioperator Algebras. In his prognosis of the future of general algebra, A. G. Kurosh predicted in 1970 that the main interests of general algebra would move, in the next decades, to the neutral territory between universal and general algebra [7, p. 9]. Following this tendency, in this note we try to extend some classical algebraic ideas of the time of Kurosh to a territory very close to universal algebra.

We start with a classical question of Kurosh: whether there can be a finitely generated, infinite-dimensional, algebraic algebra. The first solution was found in 1964 by E. S. Golod [4], who constructed a number of analogous examples for different algebraic systems (such as associative algebras, Lie algebras, and p-groups). Here we will discuss the Kurosh problem in the setting of linear universal algebra, for algebraic systems in place of algebras in the formulation. To construct suitable examples, we extend the original Golod technique, including the Golod–Shafarevich theorem [5].

We consider the varieties of multioperator linear Ω-algebras over a field k (the term from [7, Sec. 13]). Every multilinear element of the countably generated free algebra F of a variety W may be identified with a family of multilinear operators (or operations) acting on the algebras of W. The linear combinations of finite compositions of a multilinear operator with itself form a linear subspace of F, called the *clone* of the operator. We will see that the notion of clone of a single operator is analogous to the concept of one-generator subalgebra of an associative algebra. Therefore, one can consider two natural versions of the above Kurosh problem. The strong version (we will refer to it as the Burnside problem) is the following: Does there exist a variety W (of a given finite signature Ω) such that it admits nonzero n-linear operations for all arbitrarily large n while every clone of a single operation is finite-dimensional? A weaker version of the problem is the following: Does there exist a variety W, again with infinite-dimensional space of multilinear operations, such that every such operation satisfies a nontrivial "identity," i.e., some nontrivial linear combination of compositions of the operation with itself is zero?

One can consider also relative versions of the above problems, i.e., with an additional restriction that the variety W must be a subvariety of a given variety $\mathcal{V}$. In this paper, we focus on the nonrelative versions, but our method (based on Theorem 4.1) gives also some examples of the relative case.

1.2. The Language of Operads. In linear universal algebra (when the algebras are modules over a ring or a field and the operations are multilinear homomorphisms), there are two main languages for definitions and theorems, which are used for different purposes.

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The first one is the classical language of *varieties* and *identities*; it is the most popular one in ring theory and in general algebra as well. The theory of algebras with polynomial identities is usually described in this language. The works of Kurosh are also written in it.

Another language is based on the concept of *operad*. It is used mostly in algebraic topology and mathematical physics. This language seems more appropriate for discussions on the homological properties of algebraic systems, including Massey operations and Kontsevich formality.

Our first step here is to give a brief dictionary of these two languages in Sec. 2. In this dictionary, we define operads and related notions in terms of varieties; dually, we define the varieties, identities, etc. in operadic terms. Therefore, it is assumed that the reader understands at least one of these two languages. We hope that this brief dictionary will help both kinds of universal algebraists to understand each other or, at least, to recognize the well-known objects in foreign-language descriptions. We use this dictionary in order to get a more natural formulation of our versions of the universal Kurosh problem.

For the reader's convenience, let us give a rough and brief translation table in phrase-book style.

A phrase-book

variety	—	operad
subvariety	—	quotient operad
clone	—	suboperad
signature	—	set of generators
identities	—	relations
free algebra	—	free algebra
codimension series	—	generating function
T-space	—	right ideal
T-ideal	—	ideal
Specht properties	—	Noether properties

Now our two versions of the Kurosh problem are formulated as follows. The Burnside problem (or the strong Kurosh problem) for operads: Does there exist an infinite finitely generated operad P such that each of its one-generated suboperads is finite? The Kurosh problem (in a weak version) for operads: Does there exist an infinite finitely generated operad P such that each of its one-generated suboperads is nonfree? The relative version reads as follows: Given an operad $\mathcal{S}$, can one choose the operad P above to be a quotient of $\mathcal{S}$?

In this paper, we give partial answers to both of these problems. We show that such varieties and operads do exists (in the stronger version, with a restriction on the signature). The relative versions are in general open.

Let us give a "bi-lingual" formulation (some details are given in Sec. 4).

Theorem 1.1 (Corollaries 4.3, 4.4)**.**

(1) *Let Ω be a finite signature. Then there exists a variety of algebras of the signature Ω such that there are nonzero multilinear operations of arbitrarily high order but the clone of each operation in a free algebra satisfies a nontrivial identity. If $\Omega(2)$ has at least two elements, then one can assert, in addition, that the clone of each operation in a free algebra is finite-dimensional.*

(2) *Let $X = X(2), X(3), \ldots$ be an $\mathbb{S}$-module, i.e., a sequence of representations of symmetric groups $\mathrm{S}_2, \mathrm{S}_3, \ldots$. Then there exists an operad $\mathcal{P}$ generated by X such that no one-generated suboperad in it is absolutely free. If $\dim X_2 \geq 3$, then we can assert, in addition, that each of its one-generated suboperads is finite.*

1.3. Organization of the Paper. This paper is expository; most proofs are omitted and will be published in a subsequent paper [9].

The dictionary mentioned before is given in Sec. 2. In Sec. 3, we discuss the (well-known) analogy between operads and graded associative algebras. This leads to an analogy of linear universal algebra and graded ring theory. So, we describe the notions of ideals, modules, generators, and other algebraic terms in operad theory.

This analogy allows one to develop a version of classical homological algebra, including free resolutions and derived functor, in the category of modules of given operads. We use it in order to transfer the Golod–Shafarevich theorem to operads (see Sec. 4). This leads, by the Golod method, to the construction of infinite operads with finite one-generated sub-operads. This gives the solutions of weak and strong Kurosh problems for varieties (operads) for algebras of almost arbitrary signature (see Corollaries 4.3 and 4.4).

1.4. Assumptions. We consider varieties of multioperator linear algebras over a field k. To avoid technical details, we assume that the signature Ω of a variety is always locally finite and does not contain constants and unary operations, i.e., $\Omega = \Omega_2 \cup \Omega_3 \cup \ldots$, where the subsets Ω_n of n-ary operations (i.e., n-linear operators) are finite. To simplify the notation, we assume that the identical operator (which does not belong to Ω) is also applicable to any algebra of the variety. In Sec. 2 we assume, unless otherwise stated, that the ground field k has zero characteristic.

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2. Operads vs Varieties: a Dictionary

2.1. A Definition of Operad. Let W be a variety of k-linear algebras (without constants, with identity and without other unary operations) of some signature Ω. Consider the free algebra $F^W(X)$ on a countable set of indeterminates $X = \{x_1, x_2, \ldots\}$. Let $\mathcal{P}(n) \subset F$ be the subspace consisting of all multilinear generalized homogeneous polynomials in the variables $x_1, \ldots, x_n$.

Definition 2.1. Given such a variety W, the sequence $\mathcal{P}_W = \mathcal{P} := \{\mathcal{P}(1), \mathcal{P}(2), \ldots\}$ of vector subspaces of $F^W(X)$ is called an *operad* (more precisely, symmetric, connected, k-linear operad with identity). The signature Ω is called a *generation set* of the operad $\mathcal{P}_W$.

The nth component $\mathcal{P}(n)$ may be identified with the set of all derived n-linear operations on the algebras of W; in particular, $\mathcal{P}(n)$ carries a natural structure of a representation of the symmetric group S_n. Such a sequence $Q = \{Q(n)\}_{n\in\mathbb{Z}}$ of representations $Q(n)$ of the symmetric groups S_n is called an $\mathbb{S}$-module, so that an operad carries the structure of an $\mathbb{S}$-module. Also, the composition of operations (i.e., substitution of an argument x_i by the result of another polylinear operation, with subsequent re-numeration of the variables) gives natural equivariant maps of S_*-modules $\circ_i \colon \mathcal{P}(n) \otimes \mathcal{P}(m) \to \mathcal{P}(n+m-1)$. Note that the axiomatization of these operations gives an abstract definition of operads (see [8] for a discussion of the different definitions).

A morphism of operads $f \colon \mathcal{P} \to \mathcal{P}'$ is a sequence of maps $f(n) \colon \mathcal{P}(n) \to \mathcal{P}'(n)$ of S_n-modules compatible with the compositions. For example, any inclusion $W \subset W'$ of varieties of the same signature gives a surjective operadic map $\mathcal{P}_W \to \mathcal{P}_{W'}$.

2.2. A Definition of Variety. Let $\mathcal{P}$ be an operad (with $\mathcal{P}(0)$ equal to zero and one-dimensional $\mathcal{P}(1)$ spanned by the identity element) with some discrete generating set Ω. Recall that an *algebra* over $\mathcal{P}$ (=$\mathcal{P}$-algebra) is a (nongraded) right $\mathcal{P}$-module, i.e., a vector space V with $\mathcal{P}$-action $\mathcal{P}(n) \colon V^{\otimes n} \to V$ compatible with compositions of operations and the $k[\mathrm{S}_*]$-module structures on $\mathcal{P}$. The class (or an additive category) of all algebras over $\mathcal{P}$ is denoted by $\mathcal{P}$-Alg.

Definition 2.2. Let $\mathcal{P}$ be an operad with generating set Ω. Then the category $\mathcal{P}$-Alg is called a *variety* of algebras of the signature Ω.

From now, let us fix a variety W and a corresponding operad $\mathcal{P}$, so that $W = W_{\mathcal{P}}$ and $\mathcal{P} = \mathcal{P}_W$. We also fix a minimal generating set (or signature) Ω. The implications $(W = W_{\mathcal{P}}) \Longleftrightarrow (\mathcal{P} = \mathcal{P}_W)$ will be discussed later in Proposition 2.4.

2.3. (Co)dimensions. The nth codimension of a variety W is just the dimension of the respective operad component: $\mathrm{c}_n(W) = \dim_k \mathcal{P}_W(n)$. The *codimension series* of the variety W, or the *generating function* of the operad $\mathcal{P}$, is a formal power series

$$\mathcal{P}(z) := \sum_{n\geq 1} \frac{\mathrm{c}_n(W)}{n!} z^n = \sum_{n\geq 1} \frac{\dim_k \mathcal{P}_W(n)}{n!} z^n.$$

An analogous generation function

$$Q(z) = \sum_{n\geq 1} \frac{\dim_k Q(n)}{n!} z^n$$

is defined for every $\mathbb{S}$-module Q with $\dim_k Q(n) < \infty$ for all $n \geq 1$. For more general versions of this generating function (which involve, in particular, the characters of the representations $\mathcal{P}(n)$ of groups S_n) we refer the reader to the well-known paper of Ginzburg and Kapranov [3].

If the set Ω is finite, then the series $\mathcal{P}(z)$ defines an analytical function in a neighborhood of zero. For example, the operad $\mathcal{A}ss$ of associative algebras has the generating function $\mathcal{A}ss(z) = \frac{z}{1-z}$. For every proper quotient operad $\mathcal{P}$ of $\mathcal{A}ss$, we have

$$\lim_{n\to\infty} \frac{\ln \dim \mathcal{P}(n)}{n} = \ln c(\mathcal{P}),$$

where $c(\mathcal{P}) \in \mathbb{Z}$ (due to Giambruno and Zaitsev, see [2] and references therein); in particular, the function $\mathcal{P}(z)$ is in this case analytical in the whole complex plane.

2.4. Free Variety. Recall that a sequence $M = \{M(1), M(2), \dots\}$, where each $M(i)$ is a $k[\mathrm{S}_i]$-module, is called an $\mathbb{S}$-module. Given a sequence $\Omega = \{\Omega(1), \Omega(2), \dots\}$ of discrete sets, we naturally define a free $\mathbb{S}$-module $\mathbb{S}\Omega = \{k[\mathrm{S}_1]\Omega(1), k[\mathrm{S}_2]\Omega(2), \dots\}$. In particular, every operad is an $\mathbb{S}$-module, and every subset of an $\mathbb{S}$-module generates an $\mathbb{S}$-submodule. If Ω is a minimal generating set of an operad $\mathcal{P}$, then the $\mathbb{S}$-submodule $\mathbb{S}\Omega$ is also called a generating space of $\mathcal{P}$.

Definition 2.3 (of free variety)**.** A variety W of a signature Ω is called free if the generating set Ω minimally generates a free $\mathbb{S}$-submodule in the operad $\mathcal{P} = \mathcal{P}_W$ and the operad $\mathcal{P}$ itself is free with the generating $\mathbb{S}$-submodule $\mathbb{S}\Omega$.

A free algebra in a free variety is called absolutely free (of a given signature).

We will call the operad of a free variety *absolutely free*.

2.5. Free Operad. Let $\mathcal{P}$ be an operad generated by a subset Ω. The operad $\mathcal{P}$ is called free (on the generating set Ω), if the T-ideal T of identities of the variety $W = W_{\mathcal{P}}$ consists of linear combinations of the generators, i.e., by elements of the $\mathbb{S}$-submodule X of $\mathcal{P}$ generated by Ω. Since the free operad $\mathcal{P}$ is uniquely determined by the $\mathbb{S}$-submodule X, it is denoted by $\Gamma(X)$ (notation from [8]).

For example, any absolutely free operad is free (since it has no relations). On the other hand, the operad $\mathcal{G}en\mathcal{C}om$ of general (nonassociative) commutative algebras is free with the multiplication μ as a generator ($\Omega = \{\mu\}$), but is not absolutely free because of the identity $[x_1, x_2] := \mu(x_1, x_2) - \mu \circ \tau(x_1, x_2) \in X$, where τ is the generator of the group S_2.

2.6. Relations of Operads. For any two $\mathbb{S}$-submodules A and B of an operad $\mathcal{P}$, one can define a new $\mathbb{S}$-submodule $A \circ B \subset \mathcal{P}$ generated by all compositions $a(b_1, \dots, b_n)$, where $a \in A \cap \mathcal{P}(n)$ and $b_i \in B$. An $\mathbb{S}$-submodule $I \subset \mathcal{P}$ is called a left (right, two-sided) *ideal* if $I = \mathcal{P} \circ I$ (respectively, $I = I \circ \mathcal{P}$, $I = \mathcal{P} \circ I \circ \mathcal{P}$). The generating sets of ideals are defined in the obvious way.

It follows that the two-sided ideals are exactly the kernels of operadic morphisms. If an operad $\mathcal{P}$ is represented as a quotient (="image of a surjective morphism") of a free operad $\mathcal{P}'$ by a two-sided ideal I, then the elements of I are called the *relations* of $\mathcal{P}$. Given a generating set Ω of $\mathcal{P}'$, all the relations become the identities of the variety $W_{\mathcal{P}'}$ in this signature.

For example, the operad $\mathcal{C}om$ of commutative associative algebras, as a quotient of the free operad $\mathcal{G}en\mathcal{C}om$ described above, has the associativity relation $\mathrm{Ass}(x_1, x_2, x_3) := (x_1 \cdot x_2) \cdot x_3 - x_1 \cdot (x_2 \cdot x_3)$ (where $a \cdot b := \mu(a, b)$), and all other relations belong to the two-sided ideal I in $\mathcal{G}en\mathcal{C}om$ generated by this relation $\mathrm{Ass}(x_1, x_2, x_3)$.

2.7. Identities of Varieties. Let F be a free algebra over an operad $\mathcal{P}$. Then one can consider every element of F as an operation on other algebras of the variety, where the generators of F are replaced by elements of another algebra. So, for every $p, q \in F$ one can define a composition $p \circ_i q \in F$ (it is equal to p if p does not really depend on the variable x_i). Note that, in contrast to the operad composition, there is no variable renumbering after composition, e.g., $p(x_1, x_2) \circ_1 x_2 = p(x_2, x_2)$ etc. Then for every subset $C \subset F$ one can define two composition subset $F \circ C = \{f \circ_i c\}$ and $C \circ F = \{c \circ_i f\}$ (where f, c, and i runs through F, C, and $\mathbb{Z}_{\geq 1}$, respectively). A linear subspace $C \subset F$ is called an ideal (T-space, subalgebra) if it is closed under composition of the first type (respectively, second type, both types).

Let W' be a free variety with signature Ω, and let $\mathcal{P}' = \mathcal{P}_{W'}$ be the corresponding operad. The ideal I of relations of the operad $\mathcal{P}$ in $\mathcal{P}'$ generates a T-ideal T in the absolutely free algebra $F = F^{W'}(X)$ with a countable generating set X. The elements of T are called *identities* of the variety W.

The standard linearization process gives a procedure to establish the following proposition.

Proposition 2.4. *Let F be a free algebra with countable generating set of a variety W over a field k of zero characteristic. Then every T-space or a T-ideal Y in F is generated by the subset $Y \cap \mathcal{P}_W$ of multilinear elements. Moreover, this subset $Y \cap \mathcal{P}_W$ forms an ideal* (*right or two-sided, respectively*) *in the operad $\mathcal{P}_W$.*

In particular, it follows that the T-ideal T of identities of an arbitrary variety W is generated by the relations of the operad $\mathcal{P} := \mathcal{P}_W$, hence $W = W_{\mathcal{P}}$.

The proof of this proposition (essentially, it is a description of the linearization process mentioned above) is essentially the same for all types of linear algebras (see, e.g., [6] or [2] for the case of associative PI-algebras). For example, the linearization of an identity x_1^2 in the free algebra of $W_{\mathcal{G}en\mathcal{C}om}$ leads to the identity $x_1 x_2$ (because $x_1 x_2 = \frac{1}{2}((x_1 + x_2)^2 - x_1^2 - x_2^2)$), i.e., the identity x_1^2 defines the variety of algebras with zero multiplication, i.e., the category of vector spaces.

Note that the linearization essentially depends on the assumption $\operatorname{char} k = 0$. If the characteristic of the field is positive, then only the implication $(W = W_{\mathcal{P}}) \implies (\mathcal{P} = \mathcal{P}_W)$ is valid, but the reverse implication fails.

3. Operads & Graded Algebras: an Analogy

The homological theory of operads is similar to the homological theory of associative algebras. There is a number of homological constructions, which are successively moved from rings to operads: (co)bar constructions, minimal models, (DG-)resolutions and (DG-)modules, Koszul duality, etc. [1, 3, 8]. Here we move to operads a part of classical homological algebra, namely, the theory of torsion functors. In a standard way, we will construct free resolutions of modules over operads and use them to define and calculate the derived functors of an operadic analogue of the tensor product. This will be used later in our version of the Golod–Shafarevich theorem.

Note that for every operad $\mathcal{P}$ one can define graded right and left modules over it: they are $\mathbb{S}$-modules V with the structure of $\mathcal{P}$-algebras (right modules) or with the compositions $V(n) \circ_i \mathcal{P}(m) \to V(n + m - 1)$ (left modules), where in both cases the structure should be compatible with the operadic and $\mathbb{S}$-module structures. The composition functor $- \circ_{\mathcal{P}} L$ (where L is a graded left $\mathcal{P}$-module) from the category mod-$\mathcal{P}$ of graded right $\mathcal{P}$-modules to the category k-mod of graded vector spaces over k is analogous to the tensor product of modules over a graded algebra. It has left derived functors $\mathrm{Tor}_i^{\mathcal{P}}(R, L)$

that are analogous to the usual Tors of modules over graded algebras. These operadic torsion functors can be calculated using free resolutions (or cofibrant resolutions in the DG case) of the first argument [1].

A formal explanation of these ideas is given by the following chain of standard statements.

Proposition 3.1. *Let $\mathcal{P}$ be an operad. Then the category* mod-$\mathcal{P}$ *of all graded right $\mathcal{P}$-modules is Abelian* (*hence, it is an Abelian subcategory of the category of all $\mathbb{S}$-modules*).

Let V be an arbitrary $\mathbb{S}$-module. A composition right $\mathcal{P}$-module $V \circ \mathcal{P}$ is called *free* (and V is called its minimal $\mathbb{S}$-module of generators). As an $\mathbb{S}$-module, it is a composition product, so that its generation function is equal to $(V \circ \mathcal{P})(z) = V\big(\mathcal{P}(z)\big)$. For example, $\mathcal{P}$ itself is a free right module generated by the trivial $\mathbb{S}$-module $k\,\mathrm{id}$.

Suppose that M is a right graded $\mathcal{P}$-module minimally generated by an $\mathbb{S}$-module V' isomorphic to V. Then there is a (unique up to an isomorphism $V \to V'$) surjective map of $\mathcal{P}$-modules $p\colon V \circ \mathcal{P} \to M$ that isomorphically maps V to V'. The kernel $\ker p$ belongs to the "submodule of decomposables" $V \circ_{\mathcal{P}} \mathcal{P}_+ \subset V \circ \mathcal{P}$, where $\mathcal{P}_+ = \mathcal{P}(2) \oplus \mathcal{P}(3) \oplus \dots$ is the maximal ideal of $\mathcal{P}$. Iterating this construction, we get an exact sequence of right graded $\mathcal{P}$-modules

$$\dots F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \to 0,$$

where all modules F_i are free (that is why we refer to the subsequence $\mathbf{F}\colon \dots F_2 \to F_1 \to F_0$ as the *free resolution* of M). In addition, we have $\operatorname{im} d_i \subset F_{i-1} \circ \mathcal{P}_+$ for all $i \geq 0$. A resolution with the last property will be called *minimal*.

The second part of the following standard proposition can be proved in the same way as for graded modules over a connected graded associative algebra (where all graded projective modules are free).

Proposition 3.2.

(1) *Every right $\mathcal{P}$-module M admits a minimal free resolution* $\mathbf{F}$.
(2) *If $\mathbf{F}'$ is another free resolution of M, then the complex $\mathbf{F}$ of $\mathcal{P}$-modules is a direct summand of $\mathbf{F}'$.*
(3) *The minimal resolution $\mathbf{F}$ is unique up to isomorphism of complexes of right $\mathcal{P}$-modules.*

Let R and L be right and left graded $\mathcal{P}$-modules. Then one can define a composition k-module $R \circ_{\mathcal{P}} L$. It is a quotient k-module of $R \circ L$ by the relations induced by the action of $\mathcal{P}$.

Proposition 3.3. *Let L be a left graded $\mathcal{P}$-module.*

(1) *The functor $C_L\colon X \mapsto X \circ_{\mathcal{P}} L$ is right exact on $\mathcal{P}$-*mod.
(2) *There exist derived functors $L_* C_L(M)$ whose value $\operatorname{Tor}_i^{\mathcal{P}}(M, L) := L_i C_L(M)$ can be calculated, for each $i > 0$, as the ith homology $\mathbb{S}$-module of the complex*

$$\mathbf{F} \circ_{\mathcal{P}} L\colon \dots F_2 \circ_{\mathcal{P}} L \to F_1 \circ_{\mathcal{P}} L \to F_0 \circ_{\mathcal{P}} L.$$

Note that the above derived functors have been introduced by Fresse [1, 2.2.4] in a more general context of DG-operads. The second part of the above Proposition 3.3 follows from [1, Proposition 2.2.5]. For example, the minimal $\mathbb{S}$-submodules that generate the modules F_i from the minimal free resolution can be calculated as

$$F_i / F_i \circ \mathcal{P}_+ = \operatorname{Tor}_i^{\mathcal{P}}(M, \mathcal{P}/\mathcal{P}_+).$$

4. A Criterion for Infinite Operads and the Kurosh Problem

Here we give an operadic version of the famous Golod–Shafarevich theorem that gives a criterion for an associative algebra to be infinite-dimensional. Although the original version of the proof of Golod and Shafarevich [5] can almost directly be cast in the language of operads (with the Shafarevich complex replaced by the first step of the construction of a minimal model of the operad $\mathcal{P}$, see [8]), we prefer to adopt another approach (explained by Ufnarovski [10]) based on a direct construction of the minimal free resolution of the trivial module.

Theorem 4.1. *Let $\mathcal{P}$ be an operad minimally generated by an $\mathbb{S}$-module $X \subset \mathcal{P}$ with a minimal $\mathbb{S}$-module of relations $R \subset \Gamma(X)$. Here we assume that both these $\mathbb{S}$-modules are locally finite, i.e., all their graded component are of finite dimension. Suppose that the formal power series*

$$\left(1 - \frac{X(z)}{z} + \frac{R(z)}{z}\right)^{-1}$$

has nonnegative coefficients. Then the operad $\mathcal{P}$ is infinite.

Sketch of proof. Consider the trivial bimodule $I = \mathcal{P}/\mathcal{P}_+$ (where $\mathcal{P}_+ = \mathcal{P}(2) \oplus \mathcal{P}(3) \oplus \dots$ is the maximal ideal of $\mathcal{P}$, as before). For the generators of the beginning of its minimal free resolution, we have $\mathrm{Tor}_0^{\mathcal{P}}(I,I) \cong I$, $\mathrm{Tor}_1^{\mathcal{P}}(I,I) \cong X$, and $\mathrm{Tor}_2^{\mathcal{P}}(I,I) \cong R$. This means that the beginning of the resolution looks like

$$0 \to \Omega^3 \to R \circ \mathcal{P} \xrightarrow{d_2} X \circ \mathcal{P} \to \mathcal{P} \to I \to 0, \tag{1}$$

where Ω^3 is the kernel of d_2.

Taking the Euler characteristics of the exact sequence (1), we get the equality of formal power series

$$\Omega^3(z) = (R \circ \mathcal{P})(z) - (X \circ \mathcal{P})(z) + \mathcal{P}(z) - I(z).$$

Since the formal power series $\Omega^3(z)$ has nonnegative coefficients, we obtain the following coefficient-wise inequality

$$R\big(\mathcal{P}(z)\big) - X\big(\mathcal{P}(z)\big) + \mathcal{P}(z) - z \geq 0.$$

Manipulations with formal power series (including the Lagrangian inverse) complete the proof. □

For operads generated by binary operations, one can simplify the above condition.

Corollary 4.2. *Let $\mathcal{P}$ be an operad and let X and R be as above. Suppose that $\mathcal{P}$ is generated by binary operations (i.e., $X = X(2)$). Suppose that the function*

$$\phi(z) = 1 - \frac{X(z)}{z} + \frac{R(z)}{z}$$

is analytical in a neighborhood of zero (this is always the case if X is finitely generated) and has a positive real root z_0 in this neighborhood such that $\phi(z_0)' \neq 0$. Then the operad $\mathcal{P}$ is infinite.

Corollary 4.3 (the weak Kurosh problem for multi-operational algebras)**.** *Suppose that the ground field k is countable. Let Ω be an arbitrary nonempty countable signature. Then there is a variety W of algebras of signature Ω such that the clone of every polylinear operation in this variety satisfies some nontrivial identity while there are multilinear elements of the free algebra $F^W(x_1, x_2, \dots)$ of arbitrary high degrees.*

Proof. It is sufficient to prove that the suboperad in $\mathcal{P}_W$ generated by an arbitrary single multilinear operation is not absolutely free. Let us enumerate by positive integers all elements (=operations) of degree (=arity) ≥ 2 in the absolutely free operad $\mathcal{F}$ generated by Ω. If the suboperad P in $\mathcal{F}$ generated by such an operation p_i (where $i \in \mathbb{Z}_+$) is not absolutely free (e.g., if the $\mathbb{S}$-submodule X generated by p_i in $\mathcal{F}$ is not free), then its image in $\mathcal{P}_W$ is not absolutely free either.

Suppose that the operad P is absolutely free. Let us denote by R_i the sum of all multilinear compositions of $N = N_i$ copies of p_i, where the numbers N_i are chosen so that the degrees t_i of the elements R_i increase: $t_1 < t_2 < \dots$ Then every element R_i is invariant under the action of the symmetric group S_{t_i}. Therefore, the generating function of the $\mathbb{S}$-module R generated by all these elements R_i is

$$R(z) \leq \sum_{n\geq 1} \frac{z^{t_n}}{t_n!} \leq \sum_{n \geq t_1} \frac{z^n}{n!}$$

(the last coefficient-wise inequality follows from the inequalities $\dim R(n) \leq 1$ for all $n \geq 1$). If the numbers t_i are chosen so that they are sufficiently large ($0 \ll t_1 \ll t_2 \ll \dots$), then Theorem 4.1 and the above estimate imply that the generating function of the operad $\mathcal{P}$ generated by Ω with the $\mathbb{S}$-module of relations R is infinite. □

The next claim gives a stronger version of the Kurosh problem, i.e., the Burnside problem.

Corollary 4.4 (the Burnside problem for multi-operational algebras)**.** *Suppose that the ground field k is countable. Let $X = X(2) \cup X(3) \cup \dots$ be an $\mathbb{S}$-module such that $\dim X(2) \geq 3$. Then there is an infinite operad $\mathcal{P}$ generated by X such that every element $x \in \mathcal{P}$ is strongly nilpotent, i.e., the suboperad generated by x is finite.*

Remark 4.5. In the language of varieties, the above corollary looks as follows: there exist a variety of algebras with two binary operations such that every operation in these algebras is nilpotent, i.e., for every operation (i.e., homogeneous element of free algebra) there is a number N such that every composition of N operations of this kind is zero for all possible substitutions of variables.

Idea of proof. For every multilinear operation p (say, n-ary, where $n \geq 2$) and every sufficiently large number d, we define a suitable set of relations $S(p,d)$ as the set of all possible compositions of the operation p with itself of the following kind: in each of d copies of p in the compositions, at least $n-1$ of the inputs of p are replaced by variables, i.e., every element of $S(p,d)$ looks like a "branch" of length d whose nodes are marked by p. Then we choose d sufficiently large for each p and use Corollary 4.2. □

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