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SOLUTION OF THE HOTELLING'S GAME IN SECURE STRATEGIES

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We show that the classic Hotelling's model of spatial competition between two players with linear transport costs (1929) has the price equilibrium solution for all locations under the assumption that duopolists secure themselves against being driven out of the market by ndercutting. In order to formalize this natural logic of player's behavior we employ the concept of the equilibrium in secure strategies (EinSS) as the generalization of the Nash-Cournot equilibrium. Existence and uniqueness of the equilibrium solution of the price setting subgame allows to obtain the complete solution of the two-stage location-price Hotelling's game. The obtained results are interpreted and further research is discussed.

Key words: Hotelling, Product differentiation, Equilibrium in Secure Strategies.
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Показано, что классическая игра пространственной конкуренции Г. Хотеллинга (1929) с линейными транспортными ценами имеет ценовое равновесие при любых положениях игроков при условии, что они гарантируют свою безопасность от угрозы демпингового вытеснения с рынка. Чтобы формализовать эту естественную логику поведения используется Равновесие в безопасных стратегиях (РБС). РБС является обобщением равновесия Нэша – Курно. Существование и единственность решения ценовой игры позволяет получить полное решение двухшаговой игры расположений Хотеллинга. Полученные результаты обсуждаются с экономической точки зрения.

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Content

1. Introduction and review

The model of the spatial competition is the model of interaction of firms (players) selling homogeneous product for the customers distributed along the line (or in a more complicated domain). This line can be either physical distance (street in the city, coastline, highway) or amount of product characteristic preferred by customers (horizontal product differentiation in the model of Monopolistic Competition or even the political spectrum in the Downs Model of Political Competition). There are several alternative models of spatial competitions, in particular [Downs, 1957], [Prescott, Visher, 1977], [Lancaster, 1979] which are beyond the scope of this article. Here we study the Hotelling's model in its original formulation. Numerous customers are evenly distributed along an interval and buy product from the firm who quotes the least delivered price. No customer has any preference for any firm except on the ground of product price plus transportation cost which is assumed linear with the distance. There are three stages in the game (or two stages for the firms). At the first stage firms choose the locations of their shops. At the second stage they quote prices for their products. And at the third stage customers choose the firm.

The model was formulated by H.Hotelling in 1929 for two players selling product. In his article an equilibrium point in the price-setting subgame was considered as a point of simultaneous local maximum of profit functions for both players. He obtained the equilibrium prices and quantities as the functions of the shop locations. However his analysis was incomplete since he did not examine when the founded local maximum would be a global maximum. Based on his analysis of the first stage of the game H.Hotelling concluded that the tendency towards agglomeration dominates the incentive to differentiate which was called *the* *minimum differentiation principle.*

Fifty years later C.d'Aspremont et al. in [d'Aspremont et al., 1979] showed that the price equilibrium found by Hotelling does not always exist and they obtained the corresponding necessary and sufficient existence conditions. The price equilibrium existence problem turned out to be the principal theoretical problem for the Hotelling's model.

The base game setting contains several questions mentioned already by Hotelling himself which must be solved in order to make model more adequate for practical applications. First of all there are strategies through which one player undercuts the delivered price of the other, and attracts to himself the whole market. These strategies are particularly advantageous when both players choose locations close enough to each other. In particular condition found in [d'Aspremont et al., 1979] determines the limit after which these strategies must be taken into account.

In fact the profit functions of players are discontinuous and two-peaked. Nash-Cournot equilibrium exists only if both players choose the second peak provided that it is higher than the first one. It means that in the state of equilibrium no player can benefit by undercutting and pressing the competitor out of the market. However if the mentioned condition found in [d'Aspremont et al., 1979] is violated then for at least one player the first peak of the profit function becomes higher than the second one and it is profitable for this player to undercut the competitor throughout the whole market. From the other hand in the resulting monopolistic state the player driven out of the market can always quote some low positive price and get some positive profit. Therefore any monopolistic state of the market is also unstable. Hence there are no Nash-Cournot equilibria when players choose locations too close to each other. Since no price equilibrium exist for some locations then the location-setting subgame and Hotelling's game as a whole become incorrect. This situation is the principal difficulty of the Hotelling's model which prevent its satisfactory solution up to now.

After the article [d'Aspremont et al., 1979] had appeared a wide stream of papers on the subject is not diminishing up to now. This fact confirms the topicality of the model even in spite of the appeared difficulties and some doubts about its practical usefulness. The literature published by 2001 was reviewed in [Brenner, 2001]. The approaches suggested to tackle the price equilibrium existence problem can be divided into three groups. The first approach is related to modifications of the transport cost functions so that the price Nash-Cournot equilibrium would exist for all location pairs. This approach was initiated in [d'Aspremont et al., 1979] by using a quadratic transportation cost function instead of the linear one. In this case the existence of a price equilibrium is ensured for any pair of locations and Nash-Cournot equilibrium in the location stage of the game implies maximum differentiation, an incentive of players to move away as far as possible from each other. The discussion on relation between principles of differentiation and unification in the Hotelling's model continues up to now and different authors suggest different answers. A considerable part of papers on spatial competition is devoted to different functional forms of the transport cost as it allows to get round the existence problem of subgame perfect equilibrium. However this approach can not answer the question what happens when one player can undercut the competitor throughout the whole market. Another approach for the first time suggested in [Dasgupta and Maskin, 1986] is to solve the problem in terms of mixed strategies. The first study of the Hotelling's model in mixed strategies was carried out in [Osborn, 1987]. However the complete solution in explicit form is not obtained up to now because of its complexity. The third approach is to modify the model in a more sophisticated way. For example in [Ahlin, 2006] the preferences of customers depend upon 'the congestion', i.e. the customer face on an additional cost proportional to the number of other customers purchasing from the same firm. Although such models can be solved completely but simplicity and transparency of the original Hotelling's setting is somewhat lost. Thus the problem of driving the competitor out of the market by undercutting mentioned already by Hotelling himself is still the principal unsolved obstacle for the complete analysis of the model.

Another essential parameter of the model is the price elasticity of demand. In the base setting it is supposed that each customer consumes a unit of the product per unit time in each unit of length of line. The demand is thus at the extreme of inelasticity. In the simplest way the elasticity of demand can be introduced in the model by the condition of non-negativity of the customer profit function, i.e. if the customer purchasing the product gets negative profit, he or she abstains from the purchasing. Different modifications of the transport cost function have been already mentioned. Among the other important parameters extending the model are the number of players (more than 2), different distributions of customers, the uncertainty, the incentive to collude, other equilibrium concepts such as solution in mixed strategies or Stackelberg equilibria when players take moves sequentially etc. A brief comparative list of papers selected by authors are given in the Table 1.

The organization of the paper is as follows. In the next section the Hotelling's setting of the model is presented. In

Section 3 we introduce the concepts of the Equilibrium in Secure Strategies (EinSS) and of the Best Secure Response (BSR) and investigate their properties which provide foundation for our method. In Section 4 we define secure strategies for the price-setting Hotelling's subgame. In Sections 5 and 6 we present the complete solution of the Hotelling's game in its original formulation in secure strategies. Finally Section 7 concludes paper with a summary and interpretation of results.

Table 1. Comparative list of the selected papers on Hotelling's model.

Article	cost function	players	game	elasticity	modification	finding
Hotelling 1929	linear	2	price	no	base model	price equilibrium (PE) solution
D'Aspremont Gabszewicz Thisse 1979	linear quadratic	2	price	no	1) base model 2) quadratic cost function	PE existence condition PE always exists
Economides 1986	x^α $1 \leqslant \alpha \leqslant 2$	2	2-stage	no	different cost functions	PE existence condition
Osborn Pitchik 1987	linear	2	2-stage, mixed strateg.	no	strategy as a price distributed on the interval	PE always exists
Tabuchi Thisse 1995	quadratic	2	2-stage	no	non-uniform distribution of customers	solutions for different distributions

Article	cost function	players	game	elasticity	modification	finding
Lambertini 1997	quadratic	2	2-stage, Stackelberg	yes	players make simultaneous or sequential moves	solutions is heavily effected by the sequence of the decisions
Brenner 2001	quadratic	>2	2-stage	yes	several players with symmetric equidistant locations	location solution for three players, no location solutions for n>3
Mazalov Sakaguchi 2003	quadratic on a plane	2	2-stage	no	customers are distributed inside the circle	optimal prices and locations inside the circle
Benassi Chirco 2008	quadratic	2	2-stage	no	non-uniform distribution of customers	existence of assymmetric equilibria for symmetric distributions

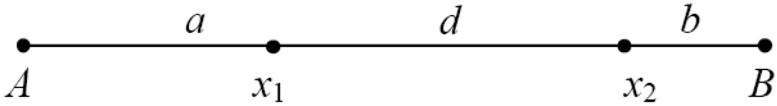

Fig. 1. Location of the two players on the interval.

2. Hotelling's Model

In the original Hotelling's setting customers are distributed with a constant density along the interval $[A, B]$ of length l. Without loss of generality the distribution density can be taken as unit. Two firms (players) are selling an identical product at prices p_1 and p_2 in the points x_1 and x_2 $(x_1 \leqslant x_2)$ located at respective distances a and b from the ends of the interval $(a + b \leqslant l, a \geqslant 0, b \geqslant 0$ - see Fig.1). Sometimes we will also use below the index notation: $a_1 = a, a_2 = b$. Let $d = l - a - b$ denote the distance between the firms. Each customer transports his purchases home at a certain cost per unit distance. Also without loss of generality this transportation cost per unit distance can be taken as unit. The unit quantity of the product is consumed in each unit of time in each unit of length of interval. The demand is thus at the extreme of inelasticity. No customer has any preference for either firm except on the ground of price plus transportation cost. Therefore the sold quantities q_1 and q_2 are equal respectively to the lengths of intervals with the customers choosing the corresponding firm.

One seeks for the subgame perfect equilibrium in the three-stage dynamical game.
Stage 1. Firms choose their locations $x_1, x_2 (x_1 \leqslant x_2)$.
Stage 2. Firms quote prices for their product $p_1, p_2 \in [0, \infty]$.
Stage 3. Customers choose the firm they buy from.

The profit functions of the firms are:

$$
(1)\qquad
\begin{aligned}
u_1(p_1,p_2) &= \begin{cases} p_1(a+b+d), & p_1 < p_2 - d \\ p_1(a+\frac{d+p_2-p_1}{2}), & |p_1-p_2| \leqslant d \\ 0, & p_1 > p_2 + d \end{cases} \\
u_2(p_1,p_2) &= \begin{cases} p_2(b+a+d), & p_2 < p_1 - d \\ p_2(b+\frac{d+p_1-p_2}{2}), & |p_1-p_2| \leqslant d \\ 0, & p_2 > p_1 + d \end{cases}
\end{aligned}
$$

Further we will also use the following notation.

$$
(2)\qquad u_i(p_1,p_2) = \begin{cases} u_i^{I}, & p_i < p_{-i} - d \\ u_i^{II}, & |p_i - p_{-i}| \leqslant d \\ u_i^{III}, & p_i > p_{-i} + d \end{cases}
$$

Depending on the quoted prices there are three possibilities for the first firm plotted in Fig.2. In the domain I it captures the whole market. There is a price competition between two firms in the domain II. And finally in the domain III the first firm fails the competition and retires from the market.

Notice here that Hotelling considered absolutely inelastic demand. In the simplest way the elasticity of demand can be introduced in the model by the condition of non-negativity of the customer utility function taken in the form:

$$
(3)\qquad u(x) = 1 - \min_{i\in\{1,2\}} (p_i + |x_i - x|),
$$

where x is a customer location and product utility is taken as unit without loss of generality. If the customer purchasing the product gets negative profit, he or she abstains from the

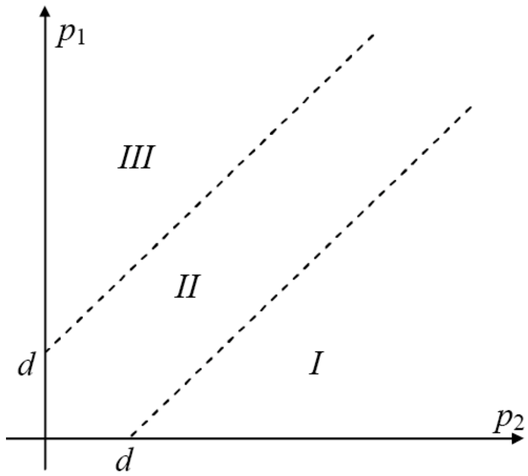

Fig. 2. Three domains for the profit function of the first player.

purchasing. Under this condition the firm prices do not exceed the unit: $p_1, p_2 \in [0, 1]$. In this paper however we restrict our study only to inelastic demand.

For the described setting of the model with inelastic demand the classical results were obtained. In 1929 H.Hotelling found the equilibrium prices and quantities for the price subgame.

$$
\begin{gathered}
p_1 = l + \frac{a-b}{3}, \; p_2 = l - \frac{a-b}{3} \\
q_1 = \frac{1}{2}\left(l + \frac{a-b}{3}\right), \; q_2 = \frac{1}{2}\left(l - \frac{a-b}{3}\right)
\end{gathered} \tag{4}
$$

For this Hotelling's solution d'Aspremont, Gabszewicz and Thisse in 1979 proved the following existence condition:

Theorem 1. *For $a+b=l$, the unique equilibrium point is given by $p_1^* = p_2^* = 0$. For $a+b<l$, there is an equilibrium point if and only if*

$$(5) \quad \left(l + \frac{a-b}{3}\right)^2 \geqslant \frac{4}{3} l(a+2b), \; \left(l + \frac{b-a}{3}\right)^2 \geqslant \frac{4}{3} l(b+2a),$$

and whenever it exists an equilibrium point is uniquely determined by (4).

These results are the starting point for our study.

3. An Equilibrium in Secure Strategies

An Equilibrium in Secure Strategies (EinSS) postulates the incentive of players to maximize their profit under the condition of security against the actions of other players. This logic entirely corresponds with the natural behavior of duopolists in the Hotelling's model. Below we provide the formal definitions of the simple EinSS of the game $G = (S_i, u_i, i \in N)$ according to [Iskakov, 2005]:

Definition 1. ***A threat*** *of player j to player i is a pair of strategy profiles $\{s, (s'_j, s_{-j})\}$ such that $u_j(s'_j, s_{-j}) > u_j(s)$ and $u_i(s'_j, s_{-j}) < u_i(s)$. The strategy profile s is said to* ***contain the threat*** *to player i. The profile (s'_j, s_{-j}) and the strategy s'_j of player j is said to* ***threaten*** *to player i.*

Definition 2. *A strategy s_i of player i is* ***a secure strategy*** *at a given complement s_{-i} if profile s does not contain any threats for player i. A strategy profile s is* ***a secure profile*** *if all its strategies are secure strategies.*

Definition 3. ***A set*** $W_i(s)$ ***of preferable strategies secured against threats*** *is a set of strategies* s'_i *of player* i *at a given* s *such that* $u_i(s'_i, s_{-i}) \geqslant u_i(s)$ *and provided that* $u_i(s'_i, s'_j, s_{-ij}) \geqslant u_i(s)$ *for any threat* $\{(s'_i, s_{-i}), (s'_i, s'_j, s_{-ij})\}$ *of player* $j \neq i$ *to player* i*.*

Definition 4. *A strategy profile* s^* *is* ***an Equilibrium in Secure Strategies*** *(EinSS) if and only if for all* i *we have that*

$$W_i(s^*) \neq \varnothing, \quad s_i^* \in \arg \max_{s_i \in W_i(s^*)} u_i(s_i, s_{-i}^*).$$

From the above definitions it follows that $W_i(s^*)$ consists of the secure strategies of player i which do not change his or her equilibrium profit at a given complement s_{-i}^*:

$$s_i \in W_i(s^*) \ <=> \ \begin{cases} u_i(s_i, s_{-i}^*) = u_i(s^*) \\ s_i \text{ is a secure strategy at } s_{-i}^* \end{cases}$$

In particular it implies that all strategies in the EinSS are secure strategies. However it is essential that in the definition of EinSS we use the set $W_i(s^*)$ rather than simply the set of secure strategies. Otherwise EinSS might include some meaningless points such as the profiles which correspond to the minimum possible in the game payoffs.

In the EinSS no player can securely benefit by changing his or her strategy while the other player keep theirs unchanged. In contrast to Nash equilibrium their choice is limited by narrower set of strategies secured against threats. Therefore the EinSS postulates the incentive of players to maximize their profit under the condition of security against the actions of other players.

In the further discussion we will use the following intuitively obvious definitions.

Definition 5. ***A set of secure strategies*** *of player i at a given complement s_{-i} is denoted as $V_i(s_{-i})$.*

Definition 6. ***The best secure response function*** *of player i at a given complement s_{-i} is a multifunction $BSR_i = \arg\max_{s_i \in V_i(s_{-i})} u_i(s_i, s_{-i})$.*

Definition 7. ***The best secure response set of player*** *i is a set*

$$M_{BSR_i} = \{s | s_i = BSR_i(s_{-i}), \forall s_{-i} \in S_i\}$$

Definition 8. ***The best secure response set*** *is a set*

$$M_{BSR} = \bigcup_{i \in N} M_{BSR_i} = \{s | s_i = BSR_i(s_{-i}), \forall i \in N\}$$

Let us denote the set of Nash equilibria and the set of EinSS as M_{NE} and M_{SSE} respectively. Then the set of M_{ESS} can be characterized by the following statement.

Theorem 2. *$M_{NE} \subseteq M_{SSE} \subseteq M_{BSR}$. The inverse inclusions are not valid.*

Proof. $M_{NE} \subseteq M_{ESS}$: Let s^* be a Nash equilibrium. Then for every player i s_i^* is a secure strategy, i.e. $s_i^* \in W_i(s^*)$. And s_i^* is the best strategy of all possible ones. Therefore by definition s^* is an EinSS.
$M_{ESS} \not\subseteq M_{NE}$:
Counterexample: $K_1 = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 0 & 3 \\ 2 & 3 & 0 \end{pmatrix}$, $K_2 = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 3 & 0 \\ 2 & 0 & 3 \end{pmatrix}$,

$M_{NE} = \emptyset$, $M_{ESS} = \{(1,1)\}$.
$M_{ESS} \subseteq M_{BSR}$:
Let s^* be an EinSS $<=>$ $s_i^* \in \arg\max_{s_i \in W_i(s^*)} u_i(s_i, s_{-i}^*)$ $=>$ $s_i^* \in W_i(s^*)$ $=>$ s_i^* is a secure strategy, i.e. $s_i^* \in V_i(s_{-i}^*)$. Let us consider s_i such as $u_i(s_i, s_{-i}^*) > u_i(s^*)$. Then $s_i \notin W_i(s^*)$, i.e. $\exists j \neq i$, $\exists s_j \neq s_j^* : u_j(s_i, s_j, s_{-ij}^*) > u_j(s_i, s_{-i}^*)$, $u_i(s_i, s_j, s_{-ij}^*) < u_i(s^*) < u_i(s_i, s_{-i}^*)$ $=>$ $s_i \notin V_i(s_{-i}^*)$ $=>$ $s_i \notin \arg\max_{s_i' \in V_i(s^*)} u_i(s_i', s_{-i}^*)$ $=>$ $s^* \in M_{BSR}$.
$M_{BSR} \nsubseteq M_{ESS}$:
Counterexample: $K_1 = \begin{pmatrix} 1 & 8 \\ 5 & 4 \\ 6 & 3 \end{pmatrix}$, $K_2 = \begin{pmatrix} 1 & 0 \\ 1 & 2 \\ 1 & 2 \end{pmatrix}$, $M_{ESS} = \emptyset$, $M_{BSR} = \{(1,1)\}$. $\square$

The closest to the EinSS concept in the Game Theory is the concept of the solution in terms of objections and counter objections employed in the coalition theory. For the first time terms 'objections' and 'counter objections' was introduced in [Aumann, 1964] for the analysis of coalition stability. Subsequently several more similar approaches were suggested. We shall consider here only those concepts which in modified way can be applied to the non-cooperative games without coalitions. Among these concepts are the objection and counter objection equilibrium in differential games described in [Vaisbord, 1980] and V-solutions described in [Vilkas, 1990]. Below we provide brief comparison of the EinSS with these concepts.

1. The EinSS applies to individual players rather than coalitions which simplifies analysis considerably. Indeed the extension of the EinSS concept to the cooperative games seems to be a non-trivial problem. Therefore the comparison is possible

only in cases when all coalitions consist of a single player.

2. In accordance with our definition a 'threat' implies that there is a particular player who can decrease his profit. In the mentioned alternative concepts an 'objection' implies an objection to a game profile but not to a particular player.

3. In our approach a 'threat' is defined only in relation to a given game profile whereas the definition in [Vilkas, 1990] employs much stronger requirement that the coalition strategy must be profitable at arbitrary strategies of players outside of coalition.

These differences do not allow to apply the mentioned alternative approaches to the Hotelling's problem.

4. Secure Strategies in the Hotelling's Game

In order to determine the secure strategy set let us consider the profit function of the first player $u_1(p_1, p_2)$ at a fixed price of the second player p_2. For $p_2 \leqslant d$ the profile (p_1, p_2) lies in the domains II and III (see Fig.2) and the profit function $u_1(p_1, p_2)$ is a one-peaked function of p_1 as plotted in Fig.3. The peak is reached either inside the segment $[0, p_2 + d]$ or at its right endpoint. For $p_2 > d$ the profile (p_1, p_2) lies in the three successive domains I, II and III and the profit function $u_1(p_1, p_2)$ is a two-peaked function of p_1 as plotted in Fig.4 with the highest peak being either the first or the second one.

Any EinSS is a secured profile. Moreover according to theorem 2 the strategy of each player in the EinSS is the best secured response against the strategies of competitors. Therefore in order to find EinSS in the price Hotelling's subgame one has to analyze threats existing in the subgame and identify the secured profiles. Note that the player can threaten the competitor only

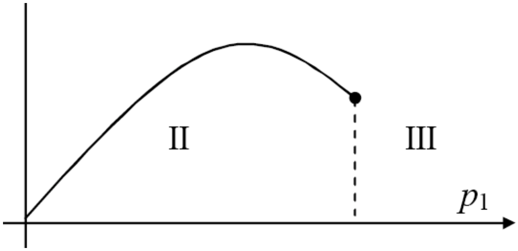

Fig. 3. The profit function $u_1(p_1, p_2)$ of player 1 at the fixed price $p_2 \leqslant d$ of player 2.

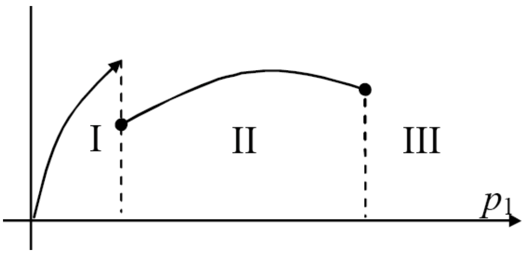

Fig. 4. The profit function $u_1(p_1, p_2)$ of player 1 at the fixed price $p_2 > d$ of player 2.

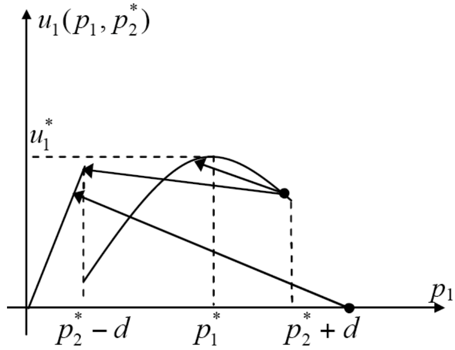

Fig. 5. Three types of threats in the price Hotelling's subgame.

by decreasing his price and there are three types of threats. Firstly the player entering the market can threaten monopolistic competitor. Secondly the player by decreasing his price can threaten competitor to drop his market share. And finally the player can threaten to drive the competitor out of the market by undercutting his price. All three cases are illustrated in Fig.5. This simple observation can be formalized in the following proposition.

Theorem 3. *The profile* (p_1, p_2) *in the price-setting subgame* $(P_i = \mathbb{R}^+, u_i(p_1, p_2), i \in \{1, 2\})$ *with the profit functions (1) is a secure strategy profile if and only if*

$$\begin{cases} p_i \leqslant \arg \max\limits_{|p-p_{-i}| \leqslant d} u_i^{II}(p, p_{-i}), i \in \{1, 2\} & \text{(6a)} \\ \text{if } p_{-i} > d, \quad u_i^I(p_{-i} - d) \leqslant u_i^{II}(p_i, p_{-i}), i \in \{1, 2\} & \text{(6b)} \end{cases}$$

Proof. Let us identify the secure profiles in the price-setting subgame.

If $p_1 < p_2 - d$ then the second player gets zero profit. And there is always threat for the first player that the second player will drop his price to $p_2' < p_1 + d$ and will get positive benefit. In this case the market share and the profit of the first player will decrease. The profile (p_1, p_2) is not secure. The case of $p_2 < p_1 - d$ is symmetrical. Therefore all secure profiles lies in the domain $|p_1 - p_2| \leqslant d$.

Let us now consider threats to player 2 if $|p_1 - p_2| \leqslant d$. According to (1) the profit function $u_1(p_1)$ at the fixed price p_2 is concave in the domains I and II and equals zero in the domain III, i.e. in general $u_1(p_1)$ is a two peaked function. Therefore the player 1 can benefit either by shifting price to the domain I or by moving price nearer to the peak of u_1 in the domain II (see Fig.5). The first case is possible when $p_2 > d$ and $\max\limits_{p\in[0,p_2-d]} u_1^I(p) > u_1^{II}(p_1, p_2)$ and there is always threat for player 2 to be driven out of the market. In the second case the threat to player 2 exists if and only if the player 1 can benefit by decreasing his or her price, i.e. when $p_1 > \arg\max\limits_{|p-p_2|\leqslant d} u_1^{II}(p, p_2)$. Therefore the security condition for the player 2 can be written in the form.

$$\begin{cases} |p_1 - p_2| \leqslant d \\ \text{if } p_2 > d, \quad \max\limits_{p\in[0,p_2-d]} u_1^I(p) = u_1^I(p_2 - d) \leqslant u_1^{II}(p_1, p_2) \\ p_1 \leqslant \arg\max\limits_{|p-p_2|\leqslant d} u_1^{II}(p, p_2) \end{cases}$$

The security condition for the player 1 we get by symmetry. These conditions are equivalent to (6). Notice that the condition $|p_1 - p_2| \leqslant d$ follows automatically from the conditions (6a). □

Inequalities in (6) has an obvious economic interpretation. The meaning of the first two inequalities (6a) was explained by Hotelling. They exclude losses of players in the process of competition when one of the players gets benefit by decreasing price and increasing his market share. The second conditions (6b) exclude for players the situation of pressing out of the market. These conditions were employed in the proving of the proposition 1 in [d'Aspremont et al., 1979] in order to find the limitation for the Hotelling's solution. However the question on what shall be the appropriate solution when the conditions (6b) become critical is open up to now.

In order to find the complete solution of the price subgame one has to take into account both conditions as they all have the substantial meaning in the given problem. By adding the second conditions in (6) we postulate in the frame of the EinSS concept that the players secure themselves against being driven out of the market by undercutting. From the other hand Nash equilibrium does not allow to analyze this kind of threat which implies sharp change of strategy by players and discontinuity of the best response function of at least one player. In this case functions of the best response do not intersect any more on the profile plane (p_1, p_2) and Nash equilibrium exists no longer.

5. Solution of the price-setting subgame

Theorem 4. *The price-setting subgame* $(P_i = \mathbb{R}^+, u_i(p_1, p_2), i \in \{1, 2\})$ *with the profit functions (1) has the following unique* *solution in secure strategies for arbitrary location pair (a, b):*

1) $0 \leqslant a \leqslant 3l + b - 6\sqrt{bl}, \quad 0 \leqslant b \leqslant 3l + a - 6\sqrt{al},$
$p_1^* = l + (a-b)/3, \quad p_2^* = l + (b-a)/3,$
$u_1^* = (p_1^*)^2/2, \quad u_2^* = (p_2^*)^2/2$

2) $3l + b - 6\sqrt{bl} \leqslant a \leqslant \dfrac{\sqrt{l}-\sqrt{b}}{\sqrt{l}+\sqrt{b}}(4\sqrt{bl} - l - b), a \geqslant 0,$
$p_1^* = 2l - 2\sqrt{bl}, \quad p_2^* = 3l + b - a - 4\sqrt{bl},$
$u_1^* = (p_1^*)^2/2, \quad u_2^* = p_2^*(l - a + b + p_1^* - p_2^*)/2$

3) $3l + a - 6\sqrt{al} \leqslant b \leqslant \dfrac{\sqrt{l}-\sqrt{a}}{\sqrt{l}+\sqrt{a}}(4\sqrt{al} - l - a), b \geqslant 0,$
$p_1^* = 3l + a - b - 4\sqrt{al}, \quad p_2^* = 2l - 2\sqrt{al},$
$u_1^* = p_1^*(l - b + a + p_2^* - p_1^*)/2, \quad u_2^* = (p_2^*)^2/2$

4) $\dfrac{\sqrt{l}-\sqrt{b}}{\sqrt{l}+\sqrt{b}}(4\sqrt{bl} - l - b) \leqslant a \leqslant l - b,$
$\dfrac{\sqrt{l}-\sqrt{a}}{\sqrt{l}+\sqrt{a}}(4\sqrt{al} - l - a) \leqslant b \leqslant l - a,$
$p_i^* = 2(l - y_i), \quad u_i^* = l(p_{-i}^* - l + a + b), \quad i \in \{1, 2\},$

(7) $y_i = \sqrt[3]{-r_i/2 + \sqrt{R_i}} + \sqrt[3]{-r_i/2 - \sqrt{R_i}} + g_i/6,$

$R_i = (s_i/3)^3 + (r_i/2)^2,$
$s_i = -g_i^2/12 + f_i h_i/2,$
$r_i = -g_i^3/108 + f_i g_i h_i/12 - f_i^2 l,$
$g_1 = l + a + 3b, \quad h_1 = 3l - a + b, \quad f_1 = b,$
$g_2 = l + 3a + b, \quad h_2 = 3l - b + a, \quad f_2 = a$

Proof. *Existence.* The first solution in (7) is equivalent to (4, 5). This case was considered and proved in [Hotelling, 1929] and

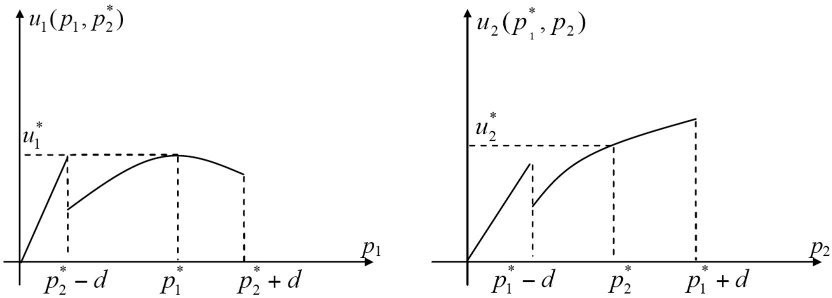

Fig. 6. Prices and profits of players at unilateral EinSS.

[d'Aspremont et al., 1979].

The second solution in (7) is the unique solution of the following system.

$$(8)\qquad \begin{cases} p_1^* = \arg\max\limits_{|p_1 - p_2^*| \leqslant d} u_1^{II}(p_1, p_2^*) \\ u_1^I(p_2^* - d) = u_1^{II}(p_1^*, p_2^*), \quad \text{if } p_2^* > d \\ p_2^* \leqslant \arg\max\limits_{|p_2 - p_1^*| \leqslant d} u_2^{II}(p_1^*, p_2) \\ u_2^I(p_1^* - d) = u_2^{II}(p_1^*, p_2^*), \quad \text{if } p_1^* > d \end{cases}$$

According to the equations of this system the strategy of player 1 is the best response and at the same time his profit equals the profit he would get if he would drive the competitor out of the market by undercutting (see Fig.6). Inequalities in this system state that the profit function of player 2 increases with his price and driving the competitor out is not profitable for him.

Let us prove that the profile (p_1^*, p_2^*) is the EinSS according to definition 4. According to theorem 3 the strategies of both players are secure. The price p_1^* is the best response of player 1 and satisfies the definition of EinSS. Let us consider the change in strategy of player 2 with a new price $p_2 \neq p_2^*$. If $p_2 < p_2^*$ then the profit of the player 2 decreases and price p_2 is not a preferable strategy for him $p_2 \notin W_2(p_1^*, p_2^*)$ by definition 3. Let us consider the case of $p_2 > p_2^*$. If we substitute p_2 in the second equation of (8) then the left hand side will be more than the right hand side, i.e. it will be more profitable for player 1 to drive the competitor out of the market. Therefore p_2 by definition 3 is not preferable strategy of player 2 secured against threats $p_2 \notin W_2(p_1^*, p_2^*)$. The price p_2^* satisfies the definition of EinSS. The profile (p_1^*, p_2^*) is the EinSS. This type of EinSS we will call *the unilateral EinSS* for the threats in it limit strategies of only one player (in a given case prices of player 2).

The third solution in (7) is obtained from the second solution by interchanging players.

The fourth solution in (7) is the unique solution of the following system.

$$(9) \qquad \begin{cases} u_1^I(p_2^* - d) = u_1^{II}(p_1^*, p_2^*), & \text{if } p_2^* > d \\ u_2^I(p_1^* - d) = u_2^{II}(p_1^*, p_2^*), & \text{if } p_1^* > d \\ p_1^* \leqslant \arg \max\limits_{|p_1 - p_2^*| \leqslant d} u_1^{II}(p_1, p_2^*) \\ p_2^* \leqslant \arg \max\limits_{|p_2 - p_1^*| \leqslant d} u_2^{II}(p_1^*, p_2) \end{cases}$$

According to the equations of this system the profits of both players (in the right hand sides) are equal to the profit they would get if they would drive the competitor out of the market by undercutting (in the left hand sides). Inequalities of this system state that the profit functions of both players increase with increasing their prices (see Fig.7).

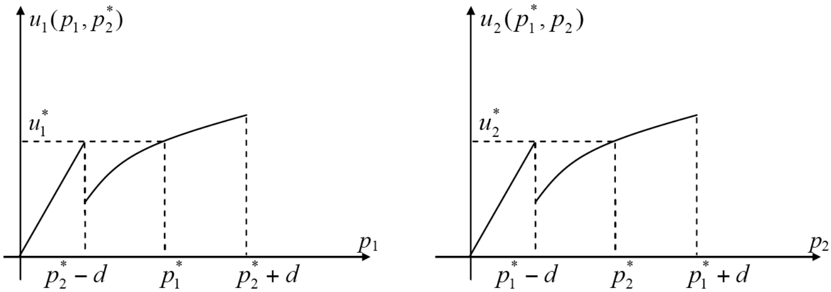

Fig. 7. Prices and profits of players at bilateral EinSS.

Let us prove that the profile (p_1^*, p_2^*) is the EinSS according to definition 4. According to theorem 3 the strategies of both players are secure. Let us consider the change in strategy of player 2 with a new price $p_2 \neq p_2^*$. If $p_2 < p_2^*$ then the profit of the player 2 decreases and price p_2 is not a preferable strategy for him by definition 3 $p_2 \notin W_2(p_1^*, p_2^*)$. Let us consider the case of $p_2 > p_2^*$. If we substitute p_2 in the first equation of (9) then the left hand side will be more than the right hand side, i.e. it will be more profitable for player 1 to drive the player 2 out of the market. Therefore p_2 by definition 3 is not preferable strategy for player 2 secured against threats. The price p_2^* satisfies the definition of EinSS. In a similar way the price p_1^* also satisfies the definition of EinSS. This type of EinSS we will call *the bilateral EinSS* since each player in it is limited by the threat from the competitor.

Uniqueness. Let us prove that there are no other solutions. Let (p_1, p_2) be an EinSS. Then (p_1, p_2) is a profile of secure strategies and according to theorem 3 must satisfy (6). Furthermore, if for at least one player both inequalities (6a) and (6b) are strict then this player can slightly increase his price and profit without being exposed to any threats, i.e. the profile (p_1, p_2) can not be an EinSS. Therefore for each player either (6a) or (6b) must turn into equality and we obtain only four possible cases of EinSS considered above. □

The four domains of the location pairs $(a/l,\ b/l)$ which correspond to the four cases in (7) are plotted in Fig.8. They cover all possible location pairs and intersect only on the boundaries where equilibrium prices and profit functions join continuously. Contours of the solutions p_1^*/l and u_1^*/l^2 in these domains are plotted in Fig.9. The solution in the first domain coincides with the solution found in [Hotelling, 1929] and [d'Aspremont et al., 1979].

6. Solution of Hotelling's location-then-price game

Existence and uniqueness of equilibrium in the price-setting subgame allows to obtain the correct solution of the two-stage Hotelling game in its original formulation.

Theorem 5. *The Hotelling's location-then-price game* $(x_1 = a,\ x_2 = l - b,\ u_i(p_1^*(a,b), p_2^*(a,b)),\ i \in \{1,2\})$ *with the profit functions (1) and the equilibrium prices* $p_i^*(a,b)$ *defined in (7) reaches the following Nash equilibria* (a^*, b^*)*:*

$$a^* = l/4,\ \ b^* = l/4; \tag{10a}$$

$$\left(l + \frac{a^* - b^*}{3}\right)^2 = \frac{4}{3}(a^* + 2b^*)l,\ \ 0 \leqslant a^* < l/4; \tag{10b}$$

$$\left(l + \frac{b^* - a^*}{3}\right)^2 = \frac{4}{3}(b^* + 2a^*)l,\ \ 0 \leqslant b^* < l/4. \tag{10c}$$

There are no other Nash equilibria in the game.

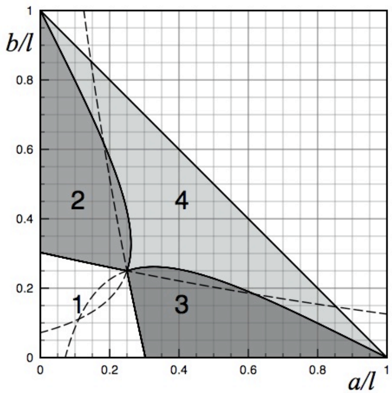

Fig. 8. Four domains of the location pairs $(a/l,\, b/l)$ in the price subgame.

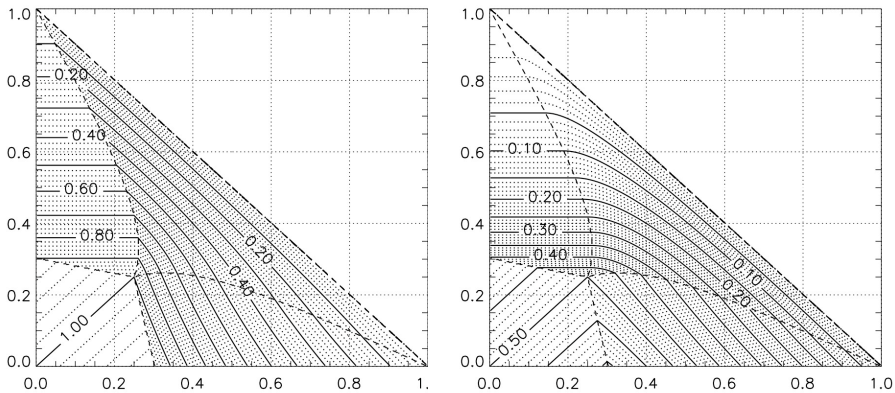

Fig. 9. Solution of the price subgame p_1^*/l (left) and u_1^*/l^2 (right).

Proof. In the plane of the location pairs (a,b) let us consider a set of strategy profiles M^{BR_1}, where the strategy of the first player is the best response against the strategy of the second one, and a set of strategy profiles M^{BR_2}, where the strategy of the second player is the best response against the strategy of the first one. From the theorem 4 it follows that

$$M^{BR_1} = \Big\{(a,b):\ 3l + b - 6\sqrt{bl} \leqslant a \leqslant \frac{\sqrt{l}-\sqrt{b}}{\sqrt{l}+\sqrt{b}}(\sqrt{bl} - l - b), b \geqslant l/4, a \geqslant 0\Big\} \cup$$
$$\cup \Big\{(a,b):\ b = 3l + a - 6\sqrt{al}, 0 \leqslant b \leqslant l/4\Big\};$$
$$M^{BR_2} = \Big\{(a,b):\ 3l + a - 6\sqrt{al} \leqslant b \leqslant \frac{\sqrt{l}-\sqrt{a}}{\sqrt{l}+\sqrt{a}}(\sqrt{al} - l - a), a \geqslant l/4, b \geqslant 0\Big\} \cup$$
$$\cup \Big\{(a,b):\ a = 3l + b - 6\sqrt{bl}, 0 \leqslant a \leqslant l/4\Big\}.$$

The set of Nash equilibria in the location subgame is the intersection of these two sets which gives the statement of the theorem. □

Surprisingly that the boundary of nonexistence of the price Nash-Cournot equilibrium found in [d'Aspremont et al., 1979] are the equilibrium solutions of the location subgame in terms of secure strategies. However all found equilibrium locations except the symmetrical one (10a) are not strict equilibria and they can be unstable.

7. Conclusions

We considered the classical model proposed in 1929 by H.Hotelling. The considerable limitation of this model is that for a great variety of transport functions no price equilibrium exist. In these cases the model is not amenable to further analysis. In particular the price Nash-Cournot equilibrium does not exist when one player can undercut his rival's price and take away his entire business with profit to himself. On the one hand this threat of 'pressing out' is an essential factor of the Hotelling's game. On the other hand Nash-Cournot equilibrium does not allow to analyze this kind of threat which implies sharp change of strategy by players and discontinuity of the best response functions. We have demonstrated that the concept of equilibrium could be restored for these cases if we take an assumption that duopolists secure themselves against being driven out of the market by undercutting. For the first time we suggest the complete solution of the price-setting Hotelling's game in its original setting with linear transport costs and inelastic demand. Our solution coincides with the solution found by Hotelling (1929) and d'Aspremont et al. (1979) for those locations that permit Nash equilibria. Our approach allows to overcome the problem of the existence of the price equilibrium in the Hotelling's model and open possibility for its further study and practical applications.

Equilibrium existence and uniqueness in the price-setting subgame allowed to obtain the correct solution of the two-stage location-price Hotelling's game. The equilibrium locations lie on the boundary between the price Nash-Cournot equilibria found by Hotelling and the 'pressing out' price equilibria in secure strategies. This is a natural boundary where the incentive of players to minimize differentiation asserted by Hotelling gives place to the incentive to maximize differentiation under the threat of being undercut throughout the whole market by the rival. We consider the threat of 'pressing out' as a key factor which balances the tendency towards unification. Incorporating this factor into the model provides the mechanism to balance the degree of differentiation depending on the parameters of the market. The original Hotelling's model however is very simple and in fact does not have any such parameters. Therefore the differentiation study requires considering more complicated model.

In our paper we provided basic definitions and simple properties of the Equilibrium in Secure Strategies (EiSS). The proposed concept of equilibrium offers several desired properties. First of all it allows to take into account existing in the game threats from the actions of other players. Secondly the EiSS coincides with the Nash Equilibrium when Nash Equilibrium exists. And finally it allows to exclude sharp changes in the strategies of players and replace discontinuous best response functions by the continuous best secure response functions. Successful application of the EiSS to the Hotelling's game allowed to reveal and formalize an essential factor in the classical problem which confirms the practical value and adequacy of the proposed concept.

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