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On Solutions of Emden–Fowler-Type Equations

V. S. Samovol*

National Research University Higher School of Economics, Moscow, Russia

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Abstract—The paper deals with solutions to Emden–Fowler-type equations of any arbitrary order. The asymptotic properties of solutions to these equations are studied, and a systematic survey of numerous uncoordinated results of analysis of continuable and noncontinuable solutions is given.

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1. INTRODUCTION

We consider an equation of the form

$$\begin{gathered} y^{(n)} = \frac{d^n y}{dx^n} = p(x)|y|^\sigma \operatorname{sgn} y, \qquad n \geq 2, \quad \sigma > 1, \\ y = y(x), \quad p(x) \in C^0, \quad x, y \in \mathbb{R}^1, \quad p(x) \neq 0. \end{gathered} \tag{1}$$

For $n = 2$ and $p(x) = \pm x^\beta$, $x > 0$, $\beta = \text{const}$, this is the well-known Emden–Fowler equation related to the studies of several physical processes. Equation (1) was considered by numerous authors (see, in particular, [1]–[6]).

The main goal in the present paper is to obtain a systematic description of solutions of Eq. (1) as $x \to \pm\infty$ depending on the growth parameter of the function $p(x)$. The solutions tending to infinity as $x \to a \neq \pm\infty$ are also considered.

Definition 1. A solution $y(x)$ of Eq. (1) is said to be *continuable to the right* if it is defined in a neighborhood of $+\infty$.

Definition 2. A solution $y(x)$ of Eq. (1) is said to be *continuable to the left* if it is defined in a neighborhood of $-\infty$.

Definition 3. A solution $y(x)$ of Eq. (1) is said to be *continuable on the entire axis* if it is defined for any $x \in \mathbb{R}^1$.

Definition 4. A nontrivial solution $y(x)$ of Eq. (1) is said to be *oscillating to the right* (*to the left*) if, for each x lying in its domain of definition, there is $\widetilde{x} > x$ ($\widetilde{x} < x$) such that $y(\widetilde{x}) = 0$.

The solutions that are not continuable in some direction are assumed to be noncontinuable in that direction.

We describe the solutions of Eq. (1) satisfying the following condition imposed on the function $p(x)$:

$$|p(x)| \geq c|x|^{-n}, \qquad c = \text{const} > 0, \quad |x| \geq x_0 > 0. \tag{2}$$

We note that if condition (2) is satisfied, then the results obtained for even and odd n differ significantly. Moreover, we see that there is significant difference in the behavior of solutions of Eq. (1) for $p(x) > 0$ and $p(x) < 0$.

*E-mail: 555svs@mail.ru

First, we formulate a general theorem which proves the existence of noncontinuable solutions of Eq. (1) for $p(x) > 0$ independently of the growth characteristics of the function $p(x)$ and the evenness of n. This theorem was first stated and proved in the report delivered by V. A. Kondratiev and the author in 1980 at the Seminar on the Qualitative Theory of Differential Equations at Moscow State University [2].

Theorem 1. *If $p(x) > 0$, then, for any number a_1, there is a solution $y(x)$ of Eq.* (1) *which is noncontinuable to the right and satisfies the condition*

$$\lim_{x\to a_1-0} |y^{(i)}(x)| = +\infty, \qquad 0 \le i \le n-1. \tag{3}$$

Remark. It follows from the proof of the theorem that if the solution $y(x)$ of Eq. (1) (for $p(x) > 0$) with the initial data $y^{(i)}(x_0) = c_i$, $0 \le i \le n-1$, satisfies condition (3), then, for some $\varepsilon > 0$, any solution $y_1(x)$ such that

$$y_1^{(i)}(x_0) = c_{1i}, \quad 0 \le i \le n-1, \qquad \sum_{i=0}^{n-1} |c_{1i} - c_i| < \varepsilon,$$

also satisfies a condition of the form (3) possibly with another value of a_1.

The following theorems present the results related to the description of nonoscillating solutions of Eq. (1) under condition (2) and under different assumptions on the sign of the function $p(x)$ and the evenness of n.

Theorem 2. *If condition* (2)*, where n is even and $p(x) > 0$, is satisfied, then Eq.* (1) *has solutions $y(x)$ which are noncontinuable to the right and satisfy condition* (3) *for a finite $a_1 > x_0$.*

Moreover, this equation has solutions continuable to the right and noncontinuable to the left, these solutions satisfy the conditions $y^{(i)}(x)$, $0 \le i \le n-1$, and, on their domain of definition, these solutions are monotone functions such that

$$y^{(i)}(x)y^{(i+1)}(x) < 0, \qquad \lim_{x\to+\infty} y^{(i)}(x) = 0, \quad 0 \le i \le n-1, \tag{4}$$

and for some finite $a_2 < x_0$,

$$\lim_{x\to a_2+0} |y^{(i)}(x)| = +\infty, \qquad 0 \le i \le n-1. \tag{5}$$

If condition (2) *is satisfied, then, for a positive coefficient $p(x)$ and even n, Eq.* (1) *has no other solutions preserving sign for $x \ge x_0$.*

The assertion that, under the conditions of Theorem 2, Eq. (1) has solutions of the form (4) was proved in [3].

Theorem 3. *If condition* (2)*, where n is odd and $p(x) > 0$, is satisfied, then Eq.* (1) *has solutions which are noncontinuable to the right and satisfy condition* (3).

If condition (2) *is satisfied, then, for a positive coefficient $p(x)$ and odd n, Eq.* (1) *has no other sign preserving solutions for $x \ge x_0$.*

The assertion that Eq. (1), under the conditions of Theorem 3, has no sign preserving solutions continuable to the right was proved in [4].

Theorem 4. *If condition* (2)*, where n is even and $p(x) < 0$, is satisfied, then all nontrivial solutions of Eq.* (1) *are oscillating both to the right and to the left.*

The assertion that all continuable nontrivial solutions of Eq. (1), under condition (2) and for even n and $p(x) < 0$, are oscillating was proved in [4].

Theorem 5. *If condition* (2)*, where n is odd and $p(x) < 0$, is satisfied, then Eq.* (1) *has solutions $y(x)$ continuable to the right, noncontinuable to the left, preserving sign, and such that all $y^{(i)}(x)$, $0 \le i \le n-1$, are monotone functions on the domain of definition of the solutions and conditions* (4) *and* (5) *are satisfied.*

Under condition (2)*, for a negative coefficient $p(x)$ and odd n, Eq.* (1) *has no other solutions preserving sign for $x \ge x_0$.*

The main result of Theorem 5 can be obtained as a consequence of the assertions given in [3], [4].

We note that the above results also permit describing the solutions of Eq. (1) in a neighborhood of the left boundary of their continuability domain, which can be attained by the change $x = -u$. Then the equations of any even order preserve the form, and the function $p(x)$ in the equations of any odd order changes sign.

2. EXAMPLES

The first three examples illustrates the above theorems. The fourth example demonstrates how these theorems can be used to analyze a special equation of the form (1) and to describe all possible types of its solutions, both those preserving sign near the boundaries of their domain of definition and the oscillating ones.

Example 1 (Theorems 1 and 2)**.** The equation

$$y^{(n)} = |y|^{\sigma} \operatorname{sgn} y, \qquad n \ge 2, \quad \sigma > 1, \tag{6}$$

where n is an even number, has the solutions

$$y_1(x) = c(-x+1)^{\beta}, \quad x < 1, \qquad y_2 = c(x-1)^{\beta}, \quad x > 1, \quad \beta = \frac{n}{1-\sigma},$$
$$c = (\beta(\beta-1) \cdot \dots \cdot (\beta - n + 1))^{1/(\sigma-1)},$$

the first of which satisfies condition (3) with $a_1 = 1$ and the second satisfies conditions (4) and (5) with $a_2 = 1$.

Example 2 (Theorem 3)**.** The equation

$$y^{(3)} = |y|^{\sigma} \operatorname{sgn} y, \qquad \sigma > 1, \tag{7}$$

has the solution

$$y = y_1(x) = c(-x+1)^{\beta}, \qquad x < 1, \quad \beta = \frac{3}{1-\sigma}, \quad c = (-\beta(\beta-1)(\beta-2))^{1/(\sigma-1)},$$

satisfying condition (3), where $a_1 = 1$.

We show that any positive solution $y(x)$ of Eq. (7) satisfies condition (3). If a solution is noncontinuable to the right, then one can easily see that it has the form (3). We consider a solution continuable to the right. If $y(x) > 0$, then $y'''(x) > 0$ and the function $y''(x)$ increases. If $\lim_{x\to+\infty} y''(x) > 0$, then, for large x, $y^{(i)}(x) > 0$, $0 \le i \le 2$, and, by Lemma 1 (see the next section), $y(x)$ satisfies condition (3). Now we assume that this limit is equal to zero (it cannot be less than zero, because the solution is then negative for large x). The function $y'(x)$ is decreasing and positive. But then $y(x)$ increases and satisfies the condition $y(x) \ge A > 0$. Hence we have $y^{(3)} \ge A^{\sigma}$, which implies the limit $\lim_{x\to+\infty} y''(x) = +\infty$. The obtained contradiction proves the desired assertion.

Example 3 (Theorem 5)**.** The equation

$$y^{(n)} = -|y|^{\sigma} \operatorname{sgn} y, \qquad \sigma > 1, \tag{8}$$

where n is an odd number, has the solution

$$y = c(x-1)^{\beta}, \qquad \beta = \frac{n}{1-\sigma}, \quad c = (-\beta(\beta-1) \cdot \dots \cdot (\beta-n+1))^{1/(\sigma-1)},$$

satisfying conditions (4) and (5) for $a_2 = 1$.

Example 4. We consider the equation

$$y^{(4)} = |y|^{\sigma} \operatorname{sgn} y, \qquad \sigma > 1. \tag{9}$$

It follows from Theorem 2 that the solutions of this equation can a priori have the following form:

(1) they have a vertical asymptote on the right and tend to zero as $x \to -\infty$; the solutions preserve the sign on the entire domain of their definition;

(2) they have a vertical asymptote on the left and tend to zero as $x \to +\infty$; the solutions preserve the sign on the entire domain of their definition;

(3) they have vertical asymptotes on the right and on the left; the solutions can be either of the same sign or of opposite signs near the asymptotes;

(4) they have a vertical asymptote on the right (left) and oscillate to the left (to the right);

(5) they oscillate in both directions.

Now we show that solutions of all these types actually exist.

The positive solutions of the first and second types are considered in Example 1. We multiply them by -1 and obtain the corresponding negative solutions.

The solutions of the third type tending to $+\infty$ as $x \to \pm a$, $a = \text{const} > 0$, can be obtained for the initial conditions $y(0) = 1$, $y^{(i)}(0) = 0$, $1 \le i \le 3$. This solution is positive, is an even function, and has vertical asymptotes on the right and on the left (according to Lemma 1 in the next section).

The solutions of the third type tending to $+\infty$ as $x \to a$, $a = \text{const} > 0$, and to $-\infty$ as $x \to -a$ can be obtained for the initial conditions

$$y(0) = y''(0) = y'''(0) = 0, \qquad y'(0) = 1.$$

Now we construct a solution of the fourth type. We consider the solutions $y_\lambda(x)$ satisfying the following initial conditions:

$$y_\lambda(0) = y'_\lambda(0) = y''_\lambda(0) = \lambda, \quad y'''_\lambda(0) = \varepsilon, \qquad 0 \le \lambda \le 1, \quad \varepsilon > 0. \tag{10}$$

The solution $y_0(x)$ tends to $+\infty$ as $x \to a$, $a = \text{const} > 0$, and to $-\infty$ as $x \to -a$. For small ε, the solution $y_1(x)$ tends to $+\infty$ as $x \to a_1$, $a_1 = \text{const} > 0$, and as $x \to a_2$, $a_2 = \text{const} < 0$. We note that if the solution has a vertical asymptote, then the solutions with close initial conditions also have a vertical asymptote (see the remark to Theorem 1). It is easy to see that there is $0 < \lambda < 1$ such that the solution $y_\lambda(x)$ tends to $+\infty$ as $x \to a_3$, $a_3 = \text{const} > 0$, and oscillates to the left (here we bear in mind that, by Theorem 2, the continuable nonoscillating solutions of our equation must satisfy condition (4), which is impossible for $y_\lambda(x)$).

Now let us construct solutions oscillating in both directions (the fifth type of solutions). To this end, we consider the solutions $y_\lambda(x)$ satisfying the following initial conditions:

$$y_\lambda(0) = \lambda, \quad y'_\lambda(0) = 0, \quad y''_\lambda(0) = \lambda - 1, \quad y'''_\lambda(0) = 0, \qquad 0 \le \lambda \le 1. \tag{11}$$

The solution $y_0(x)$ tends to $-\infty$ as $x \to \pm a_1$, $a_1 = \text{const} > 0$, and the solution $y_1(x)$ tends to $+\infty$ as $x \to \pm a_2$, $a_2 = \text{const} > 0$. There is $0 < \lambda < 1$ such that the solution $y_\lambda(x)$ oscillates to the right. But $y_\lambda(x)$ is an even function, because $y'_\lambda(0) = y'''_\lambda(0) = 0$. Therefore, this solution oscillates in both directions.

In conclusion, we show that any oscillating solution of the fourth type is continuable on the left semiaxis, and any solution of the fifth type is continuable on the entire axis.

We multiply both sides of (9) by $y'(x)$ and integrate the obtained expression from zero to x_k, where x_k is an extremum point of the considered oscillating solution $y(x)$ which lies between two neighboring zeros of this solution. As a result, we obtain

$$-y'(0)y'''(0) + \frac{(y''(0))^2}{2} \ge \left|\frac{y^{\sigma+1}(x_k)}{\sigma+1}\right| - \left|\frac{y^{\sigma+1}(0)}{\sigma+1}\right|.$$

This implies that if this solution is noncontinuable to the right, then the inequality $|y(x)| \le D$, $D = \text{const} > 0$, holds near the point $\widetilde{x}$ which is the right boundary of its domain of definition, and in turn, this implies that $|y^{(i)}(x)| \le D_1|x|^i \le D_2$, $D_1, D_2 = \text{const} > 0$, $0 \le i \le 3$. But in this case, the point $\widetilde{x}$ cannot be a boundary of the domain of definition of the solution. Now it is obvious that the solution can oscillate on an infinite interval. Therefore, the solution of the fourth type is continuable to the left, and the solution of the fifth type is continuable on the entire axis.

3. PROOFS OF THE THEOREMS

It seems that the already published proofs of several assertions close to those formulated above are overloaded by technical details hiding the main idea. In such cases, we present simpler proofs which reveal the relationship between the solutions of Eq. (1) and the characteristics of the function $p(x)$.

We begin with the proof of Theorem 1. Under the additional assumption that the function $p(x)$ satisfies the Hölder condition, this theorem can be proved by the methods of power geometry. We return to these methods (see [6], [7]) for studying the solutions of Eq. (1) in our subsequent works.

Proof of Theorem 1. We consider the solution $y(x)$ of Eq. (1) satisfying the initial conditions

$$y^{(i)}(x_0) = y_0^{(i)} \ge 0, \quad 1 \le i \le n-1, \qquad y(x_0) = A > 0. \tag{12}$$

We fix an arbitrary number $\varepsilon > 0$ and show that, for a given ε, there is a number $A_0 = A_0(\varepsilon, x_0)$ such that if $A \ge A_0$, then the solution $y(x)$ satisfying the initial conditions (12) also satisfies condition (3) for $x_0 < a_1 < x_0 + \varepsilon$.

Assume the converse. Then the increasing function $y(x)$ is defined for all $x_0 \le x \le x_0 + \varepsilon$.

Without loss of generality, we assume that $p(x) \ge 1$ for $x_0 \le x \le x_0 + \varepsilon$ (this can be obtained by the transformation $y = cz$, $c = \text{const} > 0$). For convenience, we assume that $x_0 = 0$.

From (12), for $0 \le x \le \varepsilon$, we obtain the chain of inequalities

$$y^{(n)}(x) \ge A^\sigma, \qquad y \ge \frac{A^\sigma x^n}{n!} \ge \frac{A^\sigma x^n}{n^n},$$
$$y^{(n)}(x) \ge \frac{A^{\sigma^2} x^{n\sigma}}{n^{n\sigma}}, \qquad y \ge \frac{A^{\sigma^2} x^{n(\sigma+1)}}{n^{n\sigma+n}(\sigma+1)^n}.$$

Now we assume that

$$y \ge \frac{A^{\sigma^k} x^{n(\sigma^{k-1}+\dots+1)}}{n^{n(\sigma^{k-1}+\dots+1)}(\sigma+1)^{n\sigma^{k-2}}(\sigma^2+\sigma+1)^{n\sigma^{k-3}}\cdots(\sigma^{k-1}+\dots+1)^n},$$

and it is easy to see that

$$y \ge \frac{A^{\sigma^{k+1}} x^{n(\sigma^{k}+\dots+1)}}{n^{n(\sigma^{k}+\dots+1)}(\sigma+1)^{n\sigma^{k-1}}(\sigma^2+\sigma+1)^{n\sigma^{k-2}}\cdots(\sigma^{k}+\dots+1)^n}.$$

This implies that

$$y \ge \frac{A^{\sigma^k} x^{n(\sigma^k-1)/(\sigma-1)}}{D^{\sigma^k}}, \qquad D = D(n,\sigma) > 0, \quad 0 \le x \le \varepsilon, \quad k = 1, 2, \dots .$$

We find the logarithm of the last inequality for $x = \varepsilon$ and obtain

$$\ln y(\varepsilon) \ge \sigma^k \biggl(\ln\biggl(\frac{A}{D}\biggr) + \frac{n(\sigma^k-1)}{(\sigma-1)\sigma^k}\ln\varepsilon\biggr), \qquad k = 1, 2, \dots . \tag{13}$$

Now we choose a large A_0 such that

$$\ln\biggl(\frac{A_0}{D}\biggr) + \frac{n(\sigma^k-1)}{(\sigma-1)\sigma^k}\ln\varepsilon > 0.$$

Then we have $A \ge A_0$ and (13) leads to a contradiction as $k \to \infty$.

Let an arbitrary point a_1 be given. We consider the set of all numbers A such that the solutions $y_A(x)$ of Eq. (1) satisfying the initial conditions

$$y_A^{(i)}(x_0) = 0, \quad 1 \le i \le n-1, \qquad y_A(x_0) = A > 0, \tag{14}$$

are defined on the interval $[0, a_1]$. This set is not empty and is bounded below as follows from the above proof. Let A_1 be the least upper bound of this set. We prove that the solution $y_1(x)$ with initial conditions (14), where $A = A_1$, satisfies condition (3). It is clear that this solution cannot be continued for $x \ge a_1$, otherwise, this would contradict the definition of the number A_1.

Now let the right boundary $\widetilde{a}$ of the domain of definition of the solution $y_1(x)$ be less than the number a_1. We denote $\varepsilon = (a_1 - \widetilde{a})/2$ and fix the number $A_0(\varepsilon, \widetilde{a})$. Since $\lim_{x\to\widetilde{a}} y_1(x) = +\infty$, we can choose $A_2 < A_1$ such that $y_{A_2}(\widetilde{a}) > A_0(\varepsilon, \widetilde{a})$. But then the point $a_1 > \widetilde{a} + \varepsilon$ cannot belong to the domain of definition of the solution $y_{A_2}(x)$. The obtained contradiction completes the proof of Theorem 1. □

To prove Theorems 2–5, we first prove the following two lemmas.

Lemma 1. *The solution $y = y(x)$ of the equation*

$$\begin{gathered} y^{(n)} = p(x)|y|^{\sigma} \operatorname{sgn} y, \qquad n \ge 2, \quad \sigma > 1, \\ p(x) \in C^0, \quad x, y \in \mathbb{R}^1, \qquad p(x) \ge c x^{-(n-1)\sigma - 1}, \quad x \ge x_0 > 0, \\ c = \text{const} > 0, \end{gathered} \tag{15}$$

satisfying the conditions $y(x) > 0$, $x \ge x_0$, and $y^{(n-1)}(x_0) > 0$ is noncontinuable to the right and satisfies condition (3).

Proof. We let $\widetilde{x}$ denote the right boundary of the domain of definition of the solution. We note that if $\widetilde{x} < +\infty$, then $y(x)$ satisfies condition (3). Indeed, since $y^{(n)}(x) > 0$, it follows that $y^{(n-1)}(x)$ is an increasing function. Then if $\lim_{x\to\widetilde{x}-0} y^{(n-1)}(x) = D_1 < +\infty$, then $|y(x)| \le D_2 < +\infty$, and hence $|y^{(n)}(x)| \le D_3 < +\infty$ and

$$|y^{(i)}(x)| \le D_4 < +\infty, \qquad 0 \le i \le n-1,$$

for $x_0 \le x < \widetilde{x}$. But then the point $\widetilde{x}$ cannot be the boundary of the domain of definition of the solution. Therefore, we have $\lim_{x\to\widetilde{x}-0} y^{(n-1)}(x) = +\infty$. Then $y^{(n-2)}(x)$ is an increasing function near the point $\widetilde{x}$ and $\lim_{x\to\widetilde{x}-0} y^{(n-2)}(x) = +\infty$ as follows from the above. Similarly, we can conclude that $\lim_{x\to\widetilde{x}-0} y^{(i)}(x) = +\infty$, $0 \le i \le n-1$, i.e., the solution $y(x)$ satisfies condition (3).

Now let $\widetilde{x} = +\infty$. It follows from the conditions $y(x) > 0$, $x \ge x_0$, and $y^{(n-1)}(x_0) > 0$ that $y(x) > dx^{n-1}$, where $d = \text{const} > 0$, for $x \ge x_0$. Substituting this inequality into (15) and integrating, we obtain

$$y^{(n-1)}(x) \ge c d^{\sigma} \ln \frac{x}{x_0}, \qquad x \ge x_0. \tag{16}$$

Moreover, without loss of generality, we assume that $y^{(i)}(x) \ge 0$, $x \ge x_0$, $1 \le i \le n-1$, and (performing the change $y = c^{1-\sigma} z$ if necessary) $c = 1$ in (15).

In what follows, we need the inequality

$$\begin{gathered} \int_a^x t^p \ln^q \frac{t}{a}\, dt \ge \frac{1}{1 + p + q^2}\, x^{p+1} \ln^q \frac{x}{\widetilde{a}}, \\ x \ge \widetilde{a} = a\sqrt[q]{e}, \quad a > 0, \quad p \ge -1, \quad q \ge 1. \end{gathered} \tag{17}$$

This inequality follows from the fact that the difference of the functions in its left- and right-hand sides is positive at the point $\widetilde{a}$, while the difference of their derivatives is nonnegative for $x \ge a\sqrt[q]{e}$.

We proceed by induction. We assume that the inequality

$$y^{(n-1)}(x) \ge \frac{1}{c_k} \ln^{\omega_k} \frac{x}{x_k}, \qquad x \ge x_k \ge x_0, \quad c_k, \omega_k = \text{const} \ge 1. \tag{18}$$

holds for $x \ge x_k > 0$. We integrate expression (18) $n-1$ times, apply (17), and obtain

$$y(x) \ge \frac{1}{c_k(n-1+\omega_k^2)^{n-1}}\, x^{n-1} \ln^{\omega_k} \frac{x}{x_{k+1}}, \qquad x \ge x_{k+1} = x_k e^{(n-1)/\omega_k}.$$

Substituting this inequality into Eq. (15) and integrating the result, we obtain

$$\begin{gathered} y^{(n-1)}(x) \ge \frac{1}{c_{k+1}} \ln^{\omega_{k+1}} \frac{x}{x_{k+1}}, \qquad x \ge x_{k+1} = x_k e^{(n-1)/\omega_k}, \\ \omega_{k+1} = \omega_k\sigma + 1, \quad c_{k+1} = A c_k^{\sigma} \sigma_1^k, \quad A = \left(\frac{2n\sigma}{(\sigma-1)^2}\right)^{n\sigma}, \quad \sigma_1 = \sigma^{2n\sigma}. \end{gathered} \tag{19}$$

Taking the relations $\omega_1 = 1$ and $c_1 = d^{-\sigma}$ into account, we obtain

$$\begin{gathered} y^{(n-1)}(x) \ge \frac{1}{B^{\sigma^k}} \ln^{\beta_k} \frac{x}{\widetilde{x}_0}, \qquad x \ge \widetilde{x}_0, \\ B = B(n,\sigma,c_1) = \text{const} > 0, \quad \beta_k = \frac{\sigma^k - 1}{\sigma - 1}, \quad \widetilde{x}_0 = \mu x_0, \quad \mu = \sum_{i=1}^{\infty} \frac{\sigma - 1}{\sigma^i - 1}, \end{gathered}$$

from the above. But the obtained inequality leads to a contradiction for a sufficiently large fixed x (for example, for $\ln\ln(x/\widetilde{x}_0) > 2(\sigma-1)\ln B$) and as $k \to \infty$. The proof of Lemma 1 is complete. □

Lemma 1 easily implies the following assertion.

Corollary. *The solution $y = y(x)$ of Eq.* (15) *with the initial conditions*

$$y^{(i)}(x_0) \ge 0, \quad x \ge x_0, \quad 0 \le i \le n-1, \qquad \sum_{i=0}^{n-1} y^{(i)}(x_0) > 0,$$

is noncontinuable to the right and satisfies condition (3).

Obviously, this assertion follows from Lemma 1 if we see that $y(x) > 0$ for $x > x_0$.

Lemma 2. *If the condition $(-1)^n p(x) > 0$ is satisfied, then Eq.* (1) *has a solution $y(x)$ that is continuable to the right and satisfies the condition*

$$(-1)^i y^{(i)}(x) > 0, \qquad 0 \le i \le n-1, \tag{20}$$

for all $x_0 \le x < +\infty$.

Moreover, if condition (2) *is satisfied, then conditions* (4) *and* (5) *are also satisfied, and if the condition*

$$|p(x)| \le c x^{-n-\delta}, \qquad c, \delta = \text{const} > 0, \quad x \ge x_0 > 0, \tag{21}$$

is valid, then

$$\lim_{x\to+\infty} y(x) = A, \qquad 0 < A < +\infty. \tag{22}$$

Proof. In what follows, we denote

$$z_{i+1}(x) = y^{(i)}(x), \quad 0 \le i \le n-1, \qquad z(x, z^0) = (y^{(0)}(x), \dots, y^{(n-1)}(x)),$$

where $y^{(0)}(x) = y(x)$ is the solution of Eq. (1) which satisfies the following initial conditions:

$$(y^{(0)}(x_0), \dots, y^{(n-1)}(x_0)) = z^0.$$

To prove the existence of solutions of the form (20), we use the construction idea described in [5].

In the n-dimensional space of points $z = (z_1, \dots, z_n)$, we consider the domain Ω defined by the system of inequalities

$$\begin{cases} (-1)^{i+1} z_i \ge 0, \quad 1 \le i \le n, \\ \sum\limits_{1 \le i \le n} (-1)^{i+1} z_i \le 1. \end{cases}$$

By $\partial\Omega$ we denote the boundary of Ω.

Now we consider the solutions of Eq. (1) with the initial conditions $y^{(i-1)}(x_0) = z_i^0$, $1 \le i \le n$, $z^0 = (z_1^0, \dots, z_n^0) \in S$, where S is the domain defined by the system

$$\begin{cases} (-1)^{i+1} z_i \ge 0, \quad 1 \le i \le n, \\ \sum\limits_{1 \le i \le n} (-1)^{i+1} z_i = 1. \end{cases}$$

We also let S_1 denote the domain defined by the system

$$\begin{cases} (-1)^{i+1} z_i > 0, \quad 1 \le i \le n, \\ \sum\limits_{1 \le i \le n} (-1)^{i+1} z_i = 1. \end{cases}$$

We prove that, in the set of points $z^0 \in S_1$, there is a point such that $z(x, z^0) \in \Omega \setminus \partial\Omega$ for all $x > x_0$, which implies that condition (20) is satisfied. Assume the converse. This means that, for each point $z^0 \in S_1$, there is a unique $x = x(z^0) > x_0$ and $1 \le i_0 \le n$ such that

$$z_{i_0}(x(z^0), z^0) = 0,$$
$$(-1)^{i+1} z_i(x(z^0), z^0) \ge 0, \quad (-1)^{i+1} z_i(x, z^0) > 0, \quad x_0 \le x < x(z^0), \qquad 1 \le i \le n.$$

For $z^0 \in \partial S$, we set $x(z^0) = x_0$. Here $\partial S = S \setminus S_1$ is the boundary of S.

We introduce the mapping

$$H_1(z^0) = z(x(z^0), z^0), \qquad z^0 \in S.$$

We also note that, for $z^0 \in S_1$ and $x_0 \le x < x(z^0)$, the function

$$Q(z(x, z^0)) = \sum_{1 \le i \le n} (-1)^{i+1} z_i(x, z^0)$$

strictly decreases and $0 < \mu(z^0) < 1$, where $\mu(z^0) = Q(z(x(z^0), z^0))$. We also note that $\mu(z^0) = 1$ for $z^0 \in \partial S_1$.

Let us consider the mapping

$$H_2 \colon S \to \partial S, \qquad H_2(z^0) = \frac{1}{\mu(z^0)} H_1(z^0), \quad z^0 \in S.$$

The mapping H_2 is continuous and identical on ∂S. But S is homeomorphic to an $(n-1)$-dimensional ball V, and ∂S is homeomorphic to its boundary sphere ∂V. The mapping H_2 induces a continuous mapping $H_3 \colon V \to \partial V$ that is identical on ∂V. We assume that the ball V is centered at the origin and introduce the mapping

$$H_4 \colon \nu \to -\nu, \qquad \nu \in \partial V$$

on ∂V. Then the mapping $H_4 H_3 \colon V \to \partial V$ is a continuous mapping of the ball V into itself without fixed points, which contradicts the well-known Brouwer theorem. The obtained contradiction proves that there is a trajectory satisfying condition (20).

Now we show that if condition (2) is satisfied, then the obtained solution also satisfies condition (4). We consider only the case of even n. If n is an odd number, then the reasoning is similar.

Obviously, if condition (4) is not satisfied, then there is a number $0 \le i_0 \le n-2$ such that

$$\lim_{x\to+\infty} y^{(i)}(x) = 0, \quad i_0 < i \le n-1, \qquad y^{(i_0)}(x) \ge \widetilde{c} > 0, \quad \widetilde{c} = \mathrm{const} > 0, \quad x \ge x_1, \tag{23}$$

where $x_1 \ge x_0$ is a certain number (the case $i_0 = n-1$ is impossible, because this contradicts Lemma 1).

First, we assume that $i_0 > 0$. Then we have $\lim_{x\to+\infty} y^{(i_0-1)}(x) = +\infty$ and hence, we have $y^{(i_0)}(x)y^{(i_0-1)}(x) > 0$ for large x, which contradicts (20).

Now let $i_0 = 0$ in (23). Then, for large x, we successively obtain

$$(-1)^n y^{(n)}(x) \ge d_n x^{-n}, \quad \dots, \quad y'(x) \le -d_1 x^{-1}, \qquad d_i = \mathrm{const} > 0.$$

It follows from the last inequality that, $y(x) < 0$ for large x, which is impossible. The case $i_0 = 0$ is considered.

After the change $z(t) = y(-t)$, condition (5) is obviously satisfied, which follows from the corollary of Lemma 1, because Eq. (1) finally becomes

$$z^{(n)} = \frac{d^n z}{dt^n} = \widetilde{p}(t)|z|^\sigma \operatorname{sgn} z, \qquad \widetilde{p}(t) = (-1)^n p(-t) > 0$$

and condition (20) implies that $z^{(i)}(-x_0) > 0$, $0 \le i \le n-1$, for both even and odd n.

Now we assume that condition (21) is satisfied. Then we can set $c = 1$ in (21) (performing the change $y = zc^{1/(1-\sigma)}$ if necessary). We again consider only the case of even n. If (22) is not satisfied, then $\lim_{x\to+\infty} y(x) = 0$. Now, for any $0 < \varepsilon < 1$ and $x \ge x_1 \ge 1$, where $x_1 = x_1(\varepsilon)$ is a number, we have $0 < y(x) < \varepsilon$. Moreover, we also have $\lim_{x\to+\infty} y^{(i)}(x) = 0$, $1 \le i \le n-1$. With this taken into account, we n times integrate the inequality $y^{(n)}(x) \le \varepsilon^\sigma x^{-n-\delta}$ on the interval $[x, +\infty)$ (where $x \ge x_1$) and obtain $y(x) < \varepsilon^\sigma x^{-\delta} \le \varepsilon^\sigma$. Substituting this inequality into Eq. (1) and integrating, we obtain $y(x) < \varepsilon^{\sigma^2}$ for $x \ge x_1$. Proceeding in this way, we prove that $0 < y(x_1) < \varepsilon^{\sigma^k}$, where $\lim_{k\to+\infty} \varepsilon^{\sigma^k} = 0$, which implies that $y(x_1) = 0$, but this is impossible. The proof of Lemma 2 is complete. □

Several assertions close to Lemmas 1 and 2 can be found in [3].

Proof of Theorem 2. It follows from Theorem 1 and Lemma 2 that there exist solutions satisfying condition (3) and solutions satisfying conditions (4) and (5) under the assumption that condition (2) is also satisfied. It remains to show that, in the case of even n, Eq. (1) under condition (2) and for $p(x) > 0$ has no other solutions preserving sign for $x \ge x_0$. Clearly, we can restrict our consideration to the case of positive solutions (performing the change $y = -z$ if necessary). Let $y(x) > 0$ for $x \ge x_0$. By $\widetilde{x}$ we denote the right boundary of the domain of definition of this solution. Just as at the beginning of the proof of Lemma 1, we must show that if $\widetilde{x} < +\infty$, then $y(x)$ satisfies condition (3).

We consider the positive solution $y(x)$ defined for $x_0 \le x < +\infty$. Since $y^{(n)}(x) > 0$, it follows that $y^{(n-1)}(x)$ is an increasing function. Then we have $\lim_{x\to+\infty} y^{(n-1)}(x) = 0$ and $y^{(n-1)}(x) < 0$, since otherwise, either the condition that $y(x)$ is positive is violated or (if $\lim_{x\to+\infty} y^{(n-1)}(x) > 0$) all $y^{(i)}(x) > 0$, $0 \le i \le n-1$, for large x and the solution satisfies condition (3) according to Lemma 1. Considering the functions $y^{(i)}(x)$, $0 \le i \le n-1$, we conclude that either condition (4) is satisfied (and hence (5) as was shown in the proof of Lemma 2) or there is a number $0 \le i_0 \le n-2$ such that

$$\begin{gathered} \lim_{x\to+\infty} y^{(i)}(x) = 0, \quad i_0 < i \le n-1, \qquad y^{(i_0)}(x) \ge \widetilde{c} > 0, \\ \widetilde{c} = \mathrm{const} > 0, \quad x \ge x_1 \ge x_0. \end{gathered} \tag{24}$$

First, let $i_0 > 0$. Then, for large x, we have $y(x) \ge c_1 x^{i_0}$ and $c_1 = \mathrm{const} > 0$. This implies that $y^{(n)}(x) \ge c_2 x^{i_0\sigma - n}$, where $c_2 = \mathrm{const} > 0$. Integrating this inequality on the interval $[x, +\infty)$ and taking (24) into account, we obtain $y^{(i_0)}(x) \ge c_3 x^{i_0(\sigma-1)}$, $c_3 = \mathrm{const} > 0$, which implies that $y(x) \ge c_4 x^{i_0\sigma}$ and $c_4 = \mathrm{const} > 0$ for large x. Proceeding in this way, we, for large x, obtain the inequality

$$y^{(n)}(x) \ge c_5 x^{i_0\sigma^k - n}, \qquad c_5 = \mathrm{const} > 0, \tag{25}$$

where $k > 0$ is an arbitrary integer. But this is impossible, because, for $i_0\sigma^k > n-1$, it follows from (25) that $y^{(n-1)}(x) > 0$ for large x and it follows from Lemma 1 that the solution satisfies condition (3).

If $i_0 = 0$, then since $y(x) \ge \widetilde{c} > 0$, we successively obtain

$$y^{(n)}(x) \ge c_n x^{-n}, \quad \dots, \quad y'(x) \le -c_1 x^{-1}, \qquad c_i = \text{const} > 0,$$

for large x. But it follows from the last inequality that $\lim_{x\to+\infty} y(x) = -\infty$, which contradicts the condition $y(x) \ge \widetilde{c} > 0$. The obtained contradiction completes the proof of Theorem 2. □

Proof of Theorem 3. In Theorem 1, it was proved that there are noncontinuable solutions satisfying condition (3). Now we show that if condition (2) is satisfied, then, for a positive coefficient $p(x)$ and odd n, Eq. (1) has no other sign preserving solutions for $x \ge x_0$. Assume the converse. Let $y(x)$ be a positive solution for $x \ge x_0$ such that condition (3) is not satisfied for it. Then we have $y^{(n)}(x) > 0$ and hence $y^{(n-1)}(x)$ is an increasing function. Just as in the proof of Lemma 1, we must prove that the solution under study is defined for $x_0 \le x < +\infty$. It is clear that $\lim_{x\to+\infty} y^{(n-1)}(x) = 0$; otherwise, either the positiveness condition is violated or, by Lemma 1, the solution satisfies condition (3). Therefore, we have $y^{(n-1)}(x) < 0$. Considering the functions $y^{(i)}(x)$, $0 \le i < n-1$, just as in the proof of Theorem 2, we can conclude that either $(-1)^{i+1}y^{(i)}(x) > 0$, $0 \le i \le n$, which contradicts the fact that $y(x)$ is positive, or there is a number i_0, $0 \le i_0 \le n-2$, such that

$$\lim_{x\to+\infty} y^{(i)}(x) = 0, \quad i_0 < i \le n-1, \qquad y^{(i_0)}(x) \ge \widetilde{c} > 0, \quad \widetilde{c} = \text{const} > 0, \quad x \ge x_0.$$

If $i_0 > 0$, then the same reasoning as in the proof of Theorem 2 leads to the conclusion that the solution under study satisfies condition (3).

Now we consider the case $i_0 = 0$. Since $y(x) \ge \widetilde{c} > 0$, it follows that $y^{(n)}(x) \ge c\widetilde{c}^{\sigma} x^{-n}$; therefore,

$$y^{(n-1)}(x) - y^{(n-1)}(x_1) \ge \frac{c\widetilde{c}^{\sigma}}{n-1}(x_1^{-n+1} - x^{-n+1}), \qquad x \ge x_1 \ge x_0.$$

We let x tend to $+\infty$ and obtain

$$y^{(n-1)}(x_1) \le \frac{-c\widetilde{c}^{\sigma}}{n-1} x_1^{-n+1}$$

for $x_1 \ge x_0$. Continuing this reasoning, we obtain the estimate

$$y'(x) \ge \frac{c\widetilde{c}^{\sigma}}{(n-1)!} x^{-1}, \qquad x \ge x_0,$$

which implies

$$y(x) \ge \frac{c\widetilde{c}^{\sigma}}{n^n} \ln \frac{x}{x_0}, \qquad x \ge x_0.$$

Let us proceed by induction. We assume that

$$y(x) \ge \frac{1}{c_k} \ln^{\omega_k} \frac{x}{x_0}, \qquad x \ge x_0, \quad c_k, \omega_k > 0. \tag{26}$$

Then we have

$$y^{(n)}(x) \ge \frac{1}{c_k^{\sigma} x^n} \ln^{\sigma\omega_k} \frac{x}{x_0}, \qquad x \ge x_0.$$

Just as above, we perform successive integrations and obtain

$$y(x) \ge \frac{1}{c_k^{\sigma} n^n (\sigma\omega_k + 1)} \ln^{\sigma\omega_k + 1} \frac{x}{x_0}, \qquad x \ge x_0.$$

Thus, the estimate (26) is proved for all integer $k > 0$, where

$$\omega_{k+1} = \sigma\omega_k + 1, \quad c_{k+1} = c_k^{\sigma} n^n (\sigma\omega_k + 1), \qquad \omega_1 = 1, \quad c_1 = (c\widetilde{c}^{\sigma} n^{-n})^{-1}.$$

This implies

$$\omega_k = \frac{\sigma^k - 1}{\sigma - 1}, \qquad 0 < c_k \le D^{\sigma^k}, \quad D = D(n,\sigma) = \text{const} > 1.$$

Therefore, from (26) we obtain the estimate

$$\ln y(x) \ge \frac{\sigma^k - 1}{\sigma - 1} \ln\ln \frac{x}{x_0} - \sigma^k \ln D, \qquad x \ge x_0, \quad k \ge 1,$$

which, for sufficiently large fixed x (for example, for $\ln x > \ln x_0 + D^\sigma$) and as $k \to +\infty$, implies that $\ln y(x) = +\infty$. This contradiction completes the proof of Theorem 3. $\square$

Proof of Theorem 4. We show that if condition (2) is satisfied, then, for a negative coefficient $p(x)$ and even n, any solution of Eq. (1) oscillates both to the right and to the left. Clearly, it suffices to prove the oscillation to the right, because the change $x = -t$ does not vary the form of the equation for even n.

We assume that, for $x \ge x_0$, the solution $y(x)$ preserves the sign for $x_0 \le x < \widetilde{x}$, where $\widetilde{x} \le +\infty$ is the right boundary of the domain of definition of the solution. For definiteness, let $y(x) > 0$ (otherwise, we can use the change $y = -z$). First, we show that $\widetilde{x} = +\infty$. Assume the converse; namely, let $\widetilde{x} < +\infty$. Since $y^{(n)}(x) < 0$, it follows that $y^{(n-1)}(x)$ is a decreasing function. If

$$\lim_{x \to \widetilde{x}-0} y^{(n-1)}(x) > -\infty,$$

then we obtain $|y(x)| \le D < +\infty$ for $x_0 \le x < \widetilde{x}$; hence

$$|y^{(n)}(x)| \le D_1 < +\infty, \qquad |y^{(i)}(x)| \le D_1 < +\infty, \quad 0 \le i \le n-1.$$

But then the point $\widetilde{x}$ cannot be the boundary of the domain of definition of the solution. But if $\lim_{x\to\widetilde{x}-0} y^{(n-1)}(x) = -\infty$, then we see that $y^{(n-2)}(x)$ is a decreasing function near the point $\widetilde{x}$. Proceeding as above, we then obtain $\lim_{x\to\widetilde{x}-0} y^{(n-2)}(x) = -\infty$. Continuing this argument in a similar way, we conclude that

$$\lim_{x \to \widetilde{x}-0} y^{(i)}(x) = -\infty, \qquad 0 \le i \le n-1,$$

which contradicts the condition that the solution $y(x)$ is positive. So we have proved that the sign preserving solution must be defined for all x, $x_0 \le x < +\infty$. Now we show that such solutions do not exist.

Since $y^{(n-1)}(x)$ is a decreasing function, it follows that

$$\text{either} \quad \lim_{x\to+\infty} y^{(n-1)}(x) = 0 \qquad \text{or} \quad \lim_{x\to+\infty} y^{(n-1)}(x) = c_1 > 0;$$

this limit cannot be negative, because the considered solution becomes negative for large x in this case. If this limit is greater than zero, then we have $y(x) \ge c_2 x^{n-1}$, $c_2 > 0$, for large x. But then condition (2) implies that $y^{(n)}(x) < -c_3 x^{(n-1)\sigma - n}$, $c_3 > 0$. Integrating the last inequality (and taking into account that $(n-1)\sigma - n > -1$), we obtain $\lim_{x\to+\infty} y^{(n-1)}(x) = -\infty$. This contradicts our assumption that the solution is positive. So we have proved that $\lim_{x\to+\infty} y^{(n-1)}(x) = 0$. Then, for $x_0 \le x < +\infty$, we have $y^{(n-1)}(x) > 0$, and hence the function $y^{(n-2)}(x)$ increases. But then

$$\text{either} \quad \lim_{x\to+\infty} y^{(n-2)}(x) = 0 \qquad \text{or} \quad \lim_{x\to+\infty} y^{(n-2)}(x) = c_2 > 0.$$

Continuing this argument, we conclude that either

$$\lim_{x\to+\infty} y^{(i)}(x) = 0, \qquad (-1)^{i+1} y^{(i)}(x) > 0, \quad 0 \le i \le n-1, \tag{27}$$

or, for some $0 \le i_0 \le n-2$,

$$\lim_{x\to+\infty} y^{(i)}(x) = 0, \quad i_0 < i \le n-1, \qquad \lim_{x\to+\infty} y^{(i_0)}(x) > 0. \tag{28}$$

But (27) is impossible for $i = 0$, because this contradicts the condition that $y(x)$ is positive. Therefore, we have (28). We show that this is also impossible.

First, we assume that $i_0 > 0$. Then we have $y(x) \geq c_3 x^{i_0}$ and $c_3 = \text{const} > 0$ for large x. This implies $y^{(n)}(x) \leq -c_4 x^{i_0\sigma - n}$, $c_4 = \text{const} > 0$. Integrating this inequality on the interval $[x, +\infty)$ and taking into account (28), we obtain $y^{(i_0)}(x) \geq c_5 x^{i_0(\sigma-1)}$, $c_5 = \text{const} > 0$, which implies $y(x) \geq c_6 x^{i_0\sigma}$ and $c_6 = \text{const} > 0$ for large x. Continuing this reasoning, we conclude that the inequality

$$y^{(n)}(x) \leq -\widetilde{c}_k x^{i_0\sigma^k - n}, \qquad \widetilde{c}_k = \text{const} > 0, \tag{29}$$

where $k > 0$ is an arbitrary integer, holds for large x.

But this is impossible, because it follows from (29) that $\lim_{x\to+\infty} y^{(n-1)}(x) = -\infty$ for $i_0\sigma^k > n - 1$, which contradicts (28).

Now we consider the case $i_0 = 0$. Since $y(x) \geq \widetilde{c} > 0$, we have $y^{(n)}(x) \leq -c\widetilde{c}^{\sigma} x^{-n}$, and hence

$$y^{(n-1)}(x) - y^{(n-1)}(x_1) \leq \frac{-c\widetilde{c}^{\sigma}}{n-1}(x_1^{-n+1} - x^{-n+1}), \qquad x \geq x_1 \geq x_0.$$

We let x tend to $+\infty$ and obtain

$$y^{(n-1)}(x_1) \geq \frac{c\widetilde{c}^{\sigma}}{n-1} x_1^{-n+1}$$

for $x_1 \geq x_0$. Continuing this reasoning, we obtain the estimate

$$y'(x) \geq \frac{c\widetilde{c}^{\sigma}}{(n-1)!} x^{-1}, \qquad x \geq x_0,$$

which implies

$$y(x) \geq \frac{c\widetilde{c}^{\sigma}}{n^n} \ln \frac{x}{x_0}, \qquad x \geq x_0.$$

Proceeding in this way, just as in the proof of Theorem 3, we obtain the estimate

$$\ln y(x) \geq \frac{\sigma^k - 1}{\sigma - 1} \ln\ln \frac{x}{x_0} - \sigma^k \ln D, \qquad x \geq x_0, \quad k \geq 1,$$
$$D = D(n, \sigma) = \text{const} > 0.$$

For a sufficiently large fixed x, this implies that the expression in the right-hand side of the inequality tends to $+\infty$ as $k \to +\infty$, which is impossible. The obtained contradiction shows that Eq. (1) has no nonoscillating solutions under the conditions of the theorem. The proof of Theorem 4 is complete. □

Proof of Theorem 5. The existence of solutions of the required form was proved in Lemma 2. We show that, under the conditions of the theorem, any constant-sign solution of Eq. (1) satisfies conditions (4) and (5).

We assume that, for $x \geq x_0$, the solution $y(x)$ preserves sign for $x_0 \leq x < \widetilde{x}$, where $\widetilde{x} \leq +\infty$ is the right boundary of the domain of definition of the solution. The argument used at the beginning of the proof of Theorem 4 shows that $\widetilde{x} = +\infty$.

Let $y(x) > 0$. Following the same reasoning as in the proof of the preceding theorem, we conclude that this solution satisfies either the conditions

$$\lim_{x\to+\infty} y^{(i)}(x) = 0, \quad i_0 < i \leq n-1, \qquad \lim_{x\to+\infty} y^{(i_0)}(x) > 0 \tag{30}$$

for some $0 \leq i_0 \leq n-2$ and $x \geq x_0$ or the conditions

$$\lim_{x\to+\infty} y^{(i)}(x) = 0, \qquad (-1)^i y^{(i)}(x) > 0, \quad 0 \leq i \leq n-1, \quad x \geq x_0. \tag{31}$$

But (30) is impossible, which can be proved just as the impossibility of (28) was proved in the preceding theorem. Therefore, condition (31) is satisfied, and condition (4) is proved. We perform the change $\widetilde{y}(t) = y(-t)$. Since n is odd, the equation for $\widetilde{y}(t)$ has the form

$$\frac{d^n \widetilde{y}}{dt^n} = \widetilde{p}(t)|\widetilde{y}|^\sigma \operatorname{sgn} \widetilde{y}, \qquad \widetilde{p}(t) = -p(-t) > 0.$$

But the corollary of Lemma 1 can be applied to the solution $\widetilde{y}(t)$ of this equation, because it follows from (31) that $d^i\widetilde{y}/dt^i > 0$, $0 \le i \le n-1$, for $t = -x_0$, which proves that condition (5) is satisfied. The proof of Theorem 5 is complete. $\square$

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