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Weight function for the quantum affine algebra $U_q(A_2^{(2)})$

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Abstract

In this article, we give an explicit formula for the universal weight function of the quantum twisted affine algebra $U_q(A_2^{(2)})$. The calculations use the technique of projecting products of Drinfeld currents onto the intersection of Borel subalgebras of different types.

1 Introduction

A universal weight function of a quantum affine algebra is a family of functions with values in its Borel subalgebra satisfying certain coalgebraic properties. It can be used either to construct solutions of q-difference Knizhnik-Zamolodchikov equations [TV] or to construct the off-shell Bethe vectors, thus generalizing the nested Bethe ansatz procedure [KR]. A general construction of a weight function has been suggested in [EKP]. It uses the existence of Borel subalgebras of two different types in a quantum affine algebra. One type is related to the realization of $U_q(\widehat{\mathfrak{g}})$ as a quantized Kac-Moody algebra, and the other comes from the "current" realization of $U_q(\widehat{\mathfrak{g}})$ proposed by Drinfeld in [D]. It was proved in [EKP] that the weight function of the quantum affine algebra can be represented as the projection of a product of so called Drinfeld currents on the intersection of Borel subalgebras of $U_q(\widehat{\mathfrak{g}})$ of different types.

In the paper [T] the method of the algebraic Bethe ansatz was developed for the case of the quantum twisted affine algebra $U_q(A_2^{(2)})$. In particular there was suggested an inductive procedure for the construction of off-shell Bethe vectors for finite dimensional representations of $U_q(A_2^{(2)})$.

The main result of the present paper consists in the applying of the approach of [EKP] to obtain an explicit formula for the universal weight function of the quantum twisted affine algebra $U_q(A_2^{(2)})$. As a corollary we derive an integral presentation for the factors of the universal R-martix of $U_q(A_2^{(2)})$, connecting the usual and the Drinfeld comultiplications, as well as for the universal R-matrix itself. An analogous formula for the universal R-martix of $U_q(\widehat{\mathfrak{sl}}_2)$ is presented in [KP].

The paper is organized as follows. In Section 2, we describe two realizations of the quantum twisted affine algebra $U_q(A_2^{(2)})$. In Section 3, we adapt the general result of [DKP1] and [EKP] to

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the quantum twisted affine algebra $U_q(A_2^{(2)})$. In Section 4, we introduce the so called "composite currents" and derive their analytical properties, which are crucial for our investigations. Section 5 contains an exposition of the main result. Here combinatorial formulae for the universal weight function and other related objects are presented. Most of the proofs are collected in Section 6. Finally, Section 7 contains examples of the universal weight function.

2 Two descriptions of the quantum affine algebra $U_q(A_2^{(2)})$

2.1 The Drinfeld realization

Quantum affine algebra $U_q(A_2^{(2)})$[3] is an associative algebra generated by elements

$$e_n, f_n,\ n\in\mathbb{Z},\quad a_n,\ n\in\mathbb{Z}\setminus\{0\},\quad \text{and}\quad K_0^{\pm1},$$

subject to certain commutation relations. The relations are given as formal power series identities for the following generating functions (currents):

$$e(z)=\sum_{k\in\mathbb{Z}} e_k z^{-k},\quad f(z)=\sum_{k\in\mathbb{Z}} f_k z^{-k},\quad K^{\pm}(z)=K_0^{\pm1}\exp\left(\pm(q-q^{-1})\sum_{n>0} a_{\pm n} z^{\mp n}\right)$$

as follows:

$$\begin{aligned}
(z-q^2w)(qz+w)e(z)e(w) &= (q^2z-w)(z+qw)e(w)e(z), &(2.1)\\
(q^2z-w)(z+qw)f(z)f(w) &= (z-q^2w)(qz+w)f(w)f(z), &(2.2)\\
K^+(z)e(w)K^+(z)^{-1} &= \alpha(w/z)e(w), &(2.3)\\
K^+(z)f(w)K^+(z)^{-1} &= \alpha(w/z)^{-1}f(w), &(2.4)\\
K^-(z)e(w)K^-(z)^{-1} &= \alpha(z/w)^{-1}e(w), &(2.5)\\
K^-(z)f(w)K^-(z)^{-1} &= \alpha(z/w)f(w), &(2.6)\\
K^{\pm}(z)K^{\pm}(w) &= K^{\pm}(w)K^{\pm}(z), &(2.7)\\
K^-(z)K^+(w) &= K^+(w)K^-(z), &(2.8)\\
e(z)f(w)-f(w)e(z) &= \frac{1}{q-q^{-1}}\left(\delta(z/w)K^+(w)-\delta(z/w)K^-(z)\right). &(2.9)
\end{aligned}$$

where

$$\alpha(x)=\frac{(q^2-x)(q^{-1}+x)}{(1-q^2x)(1+q^{-1}x)},\tag{2.10}$$

and $\delta\left(\frac{z}{w}\right)$ is a formal Laurent series, given by

$$\delta(z/w)=\sum_{n\in\mathbb{Z}}(z/w)^n.\tag{2.11}$$

Generating functions $e(z)$, $f(z)$ satisfy the cubic Serre relations (see [D]):

$$\begin{aligned}
\mathop{\mathrm{Sym}}_{z_1,z_2,z_3}\left(q^{-3}z_1-(q^{-2}+q^{-1})z_2+z_3\right)e(z_1)e(z_2)e(z_3) &= 0, &(2.12)\\
\mathop{\mathrm{Sym}}_{z_1,z_2,z_3}\left(q^{-3}z_1^{-1}-(q^{-2}+q^{-1})z_2^{-1}+z_3^{-1}\right)f(z_1)f(z_2)f(z_3) &= 0, &(2.13)\\
\mathop{\mathrm{Sym}}_{z_1,z_2,z_3}\left(q^{3}z_1^{-1}-(q^{2}+q)z_2^{-1}+z_3^{-1}\right)e(z_1)e(z_2)e(z_3) &= 0, &(2.14)\\
\mathop{\mathrm{Sym}}_{z_1,z_2,z_3}\left(q^{3}z_1-(q^{2}+q)z_2+z_3\right)f(z_1)f(z_2)f(z_3) &= 0. &(2.15)
\end{aligned}$$

[3] with zero central charge and dropped grading element.

Define the *principal* grading on the algebra $U_q(A_2^{(2)})$ by the relations

$$\deg e_n = 3n+1, \qquad \deg f_n = 3n-1, \qquad \deg a_n = 3n, \qquad \deg K_0^{\pm 1} = 0. \tag{2.16}$$

The assignment (2.16) defines $U_q(A_2^{(2)})$ as a graded associative algebra.

The Hopf algebra structure on $U_q(A_2^{(2)})$ can be defined as follows:

$$\Delta^{(D)}(e(z)) = e(z)\otimes 1 + K^-(z)\otimes e(z), \tag{2.17}$$
$$\Delta^{(D)}(f(z)) = 1\otimes f(z) + f(z)\otimes K^+(z), \tag{2.18}$$
$$\Delta^{(D)}(K^\pm(z)) = K^\pm(z)\otimes K^\pm(z), \tag{2.19}$$
$$a^{(D)}(e(z)) = -(K^-(z))^{-1}e(z), \tag{2.20}$$
$$a^{(D)}(f(z)) = -f(z)(K^+(z))^{-1}, \tag{2.21}$$
$$a^{(D)}(K^\pm(z)) = (K^\pm(z))^{-1}, \tag{2.22}$$
$$\varepsilon^{(D)}(e(z)) = 0, \tag{2.23}$$
$$\varepsilon^{(D)}(f(z)) = 0, \tag{2.24}$$
$$\varepsilon^{(D)}(K^\pm(z)) = 1. \tag{2.25}$$

where $\Delta^{(D)}$, $\varepsilon^{(D)}$ and $a^{(D)}$ are the comultiplication, the counit and the antipode maps respectively. We will call $\Delta^{(D)}$ a *Drinfeld comultiplication*. The map $\Delta^{(D)}$ defines the structure of topological bialgebra on $\mathcal{A} = U_q(A_2^{(2)})$ which means that for any $n\in\mathbb{Z}$ this map sends an element x of the graded component $\mathcal{A}[n]$ to the sum of spaces $\sum_{k\geq 0}\mathcal{A}[m(n)-k]\otimes\mathcal{A}[-m(n)+k]$, where $m(n)$ is an integer such that for each k the corresponding summand on the right hand side of $\Delta^{(D)}(x)$ is finite. See [EKP, Section 2] for details.

2.2 Chevalley generators of $U_q(A_2^{(2)})$

Another realization of $U_q(A_2^{(2)})$ can be obtained using Chevalley generators. Quantum affine algebra $U_q(A_2^{(2)})$ is an associative algebra generated by elements $e_{\pm\alpha}$, $e_{\pm(\delta-2\alpha)}$, $k_\alpha^{\pm 1}$, $k_{\delta-2\alpha}^{\pm 1}$, satisfying the following relations:

$$k_\alpha e_{\pm\alpha}k_\alpha^{-1} = q^{\pm 1}e_{\pm\alpha}, \qquad k_\alpha e_{\pm(\delta-2\alpha)}k_\alpha^{-1} = q^{\mp 2}e_{\pm(\delta-2\alpha)},$$
$$k_{\delta-2\alpha}e_{\pm\alpha}k_{\delta-2\alpha}^{-1} = q^{\mp 2}e_{\pm\alpha}, \qquad k_{\delta-2\alpha}e_{\pm(\delta-2\alpha)}k_{\delta-2\alpha}^{-1} = q^{\pm 4}e_{\pm(\delta-2\alpha)},$$
$$k_\alpha^2 k_{\delta-2\alpha} = 1, \qquad \left[e_{\pm\alpha}, e_{\mp(\delta-2\alpha)}\right] = 0,$$
$$[e_\alpha, e_{-\alpha}] = \frac{k_\alpha - k_\alpha^{-1}}{q-q^{-1}}, \qquad \left[e_{\delta-2\alpha}, e_{-(\delta-2\alpha)}\right] = \frac{k_{\delta-2\alpha}-k_{\delta-2\alpha}^{-1}}{q-q^{-1}},$$
$$(\mathrm{ad}_q\, e_{\pm\alpha})^5 e_{\pm(\delta-2\alpha)} = 0, \qquad (\mathrm{ad}_q\, e_{\pm(\delta-2\alpha)})^2 e_{\pm\alpha} = 0,$$

where

$$(\mathrm{ad}_q\, e_{\pm\alpha})(x) = e_{\pm\alpha}x - k_\alpha^{\pm 1}xk_\alpha^{\mp 1}e_{\pm\alpha},$$
$$(\mathrm{ad}_q\, e_{\pm(\delta-2\alpha)})(x) = e_{\pm(\delta-2\alpha)}x - k_{\delta-2\alpha}^{\pm 1}xk_{\delta-2\alpha}^{\mp 1}e_{\pm(\delta-2\alpha)}.$$

The principal grading is given by relations

$$\deg e_{\pm\alpha} = \pm 1, \qquad \deg e_{\pm(\delta-2\alpha)} = \pm 1, \qquad \deg k_\alpha^{\pm 1} = \deg k_{\delta-2\alpha}^{\pm 1} = 0.$$

The Hopf algebra structure associated with this realization can be defined by

$$\begin{aligned}
\Delta(e_\alpha) &= e_\alpha \otimes 1 + k_\alpha \otimes e_\alpha, & \Delta(e_{\delta-2\alpha}) &= e_{\delta-2\alpha} \otimes 1 + k_{\delta-2\alpha} \otimes e_{\delta-2\alpha},\\
\Delta(e_{-\alpha}) &= 1 \otimes e_{-\alpha} + e_{-\alpha} \otimes k_\alpha^{-1}, & \Delta(e_{-(\delta-2\alpha)}) &= 1 \otimes e_{-(\delta-2\alpha)} + e_{-(\delta-2\alpha)} \otimes k_{\delta-2\alpha}^{-1},\\
\Delta(k_\alpha^{\pm 1}) &= k_\alpha^{\pm 1} \otimes k_\alpha^{\pm 1}, & \Delta(k_{\delta-2\alpha}^{\pm 1}) &= k_{\delta-2\alpha}^{\pm 1} \otimes k_{\delta-2\alpha}^{\pm 1},\\
\varepsilon(e_{\pm\alpha}) &= 0, & \varepsilon(e_{\pm(\delta-2\alpha)}) &= 0,\\
\varepsilon(k_\alpha^{\pm 1}) &= 1, & \varepsilon(k_{\delta-2\alpha}^{\pm 1}) &= 1,\\
a(e_\alpha) &= -k_\alpha^{-1} e_\alpha, & a(e_{\delta-2\alpha}) &= -k_{\delta-2\alpha}^{-1} e_{\delta-2\alpha},\\
a(e_{-\alpha}) &= -e_{-\alpha} k_\alpha, & a(e_{-(\delta-2\alpha)}) &= -e_{-(\delta-2\alpha)} k_{\delta-2\alpha},\\
a(k_\alpha^{\pm 1}) &= k_\alpha^{\mp 1}, & a(k_{\delta-2\alpha}^{\pm 1}) &= k_{\delta-2\alpha}^{\mp 1},
\end{aligned}$$

where Δ, ε and a are the comultiplication, the counit and the antipode maps respectively. We will call Δ "standard" comultiplication.

2.3 The isomorphism between two realizations

The isomorphism between two descriptions of the quantum affine algebra $U_q(A_2^{(2)})$ can be established by the following maps:

$$\begin{aligned}
k_{\delta-2\alpha} &\to K_0^{-2}, & k_\alpha &\to K_0,\\
e_{\delta-2\alpha} &\to a(qf_1 f_0 - f_0 f_1)K_0^{-2}, & e_\alpha &\to e_0,\\
e_{-(\delta-2\alpha)} &\to aK_0^2(q^{-1}e_0 e_{-1} - e_{-1}e_0), & e_{-\alpha} &\to f_0,
\end{aligned} \tag{2.26}$$

where $a = \frac{1}{\sqrt{q+q^{-1}}}$.

Remark. For generic q, the mapping above is an isomorphism of associative algebras. It preserves the counit, but does not respect the comultiplication maps Δ and $\Delta^{(D)}$. The relation between the two comultiplications is described in Section 3.4.

3 Orthogonal decomposition and twists of Drinfeld double

In this section, we adapt the theory of orthogonal decompositions and twists of Drinfeld double developed in [EKP] to the $U_q(A_2^{(2)})$ case.

3.1 Borel subalgebras in $U_q(A_2^{(2)})$

Recall that in any quantum affine algebra there exist Borel subalgebras of two types. Borel subalgebras of the first type come from the Drinfeld ("current") realization of $U_q(A_2^{(2)})$. Let U_F denote the subalgebra of $U_q(A_2^{(2)})$, generated by the elements $K_0^{\pm 1}$, $f_n, n \in \mathbb{Z}$; $a_n, n > 0$, and let U_E denote the subalgebra of $U_q(A_2^{(2)})$, generated by the elements $K_0^{\pm 1}$, $e_n, n \in \mathbb{Z}$; $a_n, n < 0$. They are Hopf subalgebras of $U_q(A_2^{(2)})$ with respect to comultiplication $\Delta^{(D)}$. The "current" Borel subalgebra U_F contains the subalgebra U_f generated by elements f_n, $n \in \mathbb{Z}$. The "current" Borel subalgebra U_E contains the subalgebra U_e, generated by elements e_n, $n \in \mathbb{Z}$.

Borel subalgebras of the second type are obtained via the Chevalley realization. Let $U_q(\mathfrak{b}_+)$ and $U_q(\mathfrak{b}_-)$ denote a pair of subalgebras of $U_q(A_2^{(2)})$ generated by elements

$$e_\alpha,\ e_{\delta-2\alpha},\ k_\alpha^{\pm 1} \qquad \text{and} \qquad e_{-\alpha},\ e_{-(\delta-2\alpha)},\ k_\alpha^{\pm 1}$$

respectively. In terms of the Drinfeld realization, these subalgebras are generated by elements

$$K_0^{\pm 1},\ e_0,\ qf_1f_0 - f_0f_1 \qquad \text{and} \qquad K_0^{\pm 1},\ f_0,\ q^{-1}e_0e_{-1} - e_{-1}e_0,$$

respectively. The algebras $U_q(\mathfrak{b}_\pm)$ are Hopf subalgebras of $U_q(A_2^{(2)})$ with respect to comultiplication Δ and, moreover, coideals with respect to the Drinfeld comultiplication. More precisely:

Proposition 3.1 *For any element $x \in U_q(\mathfrak{b}_+)$ we have an equality*

$$\Delta^{(D)}(x) = x \otimes 1 + \sum_{i=0}^{\infty} a_i \otimes b_i \quad \textit{for some} \quad b_i \in U_q(\mathfrak{b}_+), \quad \textit{such that} \quad \varepsilon(b_i) = 0. \tag{3.1}$$

In particular, subalgebra $U_q(\mathfrak{b}_+)$ is a left coideal of $U_q(A_2^{(2)})$ with respect to coproduct $\Delta^{(D)}$, i.e. $\Delta^{(D)}(U_q(\mathfrak{b}_+)) \subset U_q(A_2^{(2)}) \otimes U_q(\mathfrak{b}_+)$. Analogously, $U_q(\mathfrak{b}_-)$ is a right coideal of $U_q(A_2^{(2)})$ with respect to the same coproduct.

The proof of Proposition 3.1 is given in Section 6.1.

Let U_F^+, U_f^-, U_e^+ and U_E^- denote the following intersections of Borel subalgebras:

$$\begin{aligned} U_f^- &= U_F \cap U_q(\mathfrak{b}_-), \qquad & U_F^+ &= U_F \cap U_q(\mathfrak{b}_+),\\ U_e^+ &= U_E \cap U_q(\mathfrak{b}_+), \qquad & U_E^- &= U_E \cap U_q(\mathfrak{b}_-). \end{aligned}$$

Here the upper sign indicates the Borel subalgebra $U_q(\mathfrak{b}_\pm)$ containing the given algebra, and the lower letter indicates the "current" Borel subalgebra U_F or U_E that it is intersected with. These letters are capitals if the subalgebra contains imaginary root generators a_n and are lower case otherwise. The intersections have coideal properties with respect to both comultiplications. In particular, Proposition 3.1 and its analog for Borel subalgebra $U_q(\mathfrak{b}_-)$ imply the inclusions

$$\begin{aligned} \Delta^{(D)}(U_F^+) &\subset U_F \otimes U_F^+, \qquad & \Delta^{(D)}(U_f^-) &\subset U_f^- \otimes U_F,\\ \Delta^{(D)}(U_E^-) &\subset U_E^- \otimes U_E, \qquad & \Delta^{(D)}(U_e^+) &\subset U_E \otimes U_e^+. \end{aligned}$$

3.2 Projections

Let $\mathcal{A}$ be a bialgebra with multiplication map μ, comultiplication map δ, unit 1 and counit ε. We say that its subalgebras $\mathcal{A}_1$ and $\mathcal{A}_2$ determine an orthogonal decomposition of $\mathcal{A}$ (see [ER]), if:

(i) The algebra $\mathcal{A}$ admits a decomposition $\mathcal{A} = \mathcal{A}_1\mathcal{A}_2$, such that the multiplication map $\mu\colon \mathcal{A}_1 \otimes \mathcal{A}_2 \to \mathcal{A}$ establishes an isomorphism of linear spaces.

(ii) $\mathcal{A}_1$ is the left coideal of $\mathcal{A}$, $\mathcal{A}_2$ is the right coideal of $\mathcal{A}$:

$$\delta(\mathcal{A}_1) \subset \mathcal{A} \otimes \mathcal{A}_1\ , \qquad \delta(\mathcal{A}_2) \subset \mathcal{A}_2 \otimes \mathcal{A}.$$

There is a pair of biorthogonal decompositions of "current" Borel subalgebras, equipped with comultiplications $\Delta^{(D)}$ and its opposite, namely $U_E = U_e^+U_E^-$ and $U_F = U_f^-U_F^+$. The condition (i) is a corollary of the theory of Cartan-Weyl bases (see [KT]); the condition (ii) follows from Proposition 3.1.

Let P^+ and P^- denote the projection operators corresponding to the second decomposition, so that for any $f_+ \in U_F^+$ and for any $f_- \in U_f^-$

$$P^+(f_-f_+) = \varepsilon(f_-)f_+, \qquad P^-(f_-f_+) = f_-\varepsilon(f_+). \tag{3.2}$$

Projection operators related to the first decomposition are denoted by P^{*+} and P^{*-}, in such a way that for any $e_+ \in U_e^+$ and for any $e_- \in U_E^-$

$$P^{*+}(e_+e_-) = e_+\varepsilon(e_-), \qquad P^{*-}(e_+e_-) = \varepsilon(e_+)e_-. \tag{3.3}$$

3.3 Completions

Algebra $U_q(A_2^{(2)})$ admits a natural completion $\overline{U}_q(A_2^{(2)})$, which can be described as the minimal extension of $U_q(A_2^{(2)})$, that acts on every left highest weight and on every right lowest weight $U_q(A_2^{(2)})$-module with respect to the standard Borel subalgebras(see [DKP2]).

In a highest weight representation of $U_q(A_2^{(2)})$, any matrix coefficient of an arbitrary product of currents $f(z_1)\dots f(z_n)$ is a Laurent polynomial in $\{z_1,\dots,z_n\}$ over the ring of formal power series in variables $\{z_2/z_1, z_3/z_2, \dots, z_n/z_{n-1}\}$. It converges to a rational function in the domain $|z_1| \gg |z_2| \gg \dots \gg |z_n|$. The latter observation and the commutation relation (2.2), which dictate the way of taking an analytic continuation from the domain above, allow us to regard products of currents as operator-valued meromorphic functions with values in $\overline{U}_q(A_2^{(2)})$.

The projection operators $P^\pm$, defined in the previous subsection, extend to the completion $\overline{U}_F \subset \overline{U}_q(A_2^{(2)})$. The projection operators $P^{*\pm}$ extend to another completion $\overline{\overline{U}}_E \subset \overline{\overline{U}}_q(A_2^{(2)})$, where $\overline{\overline{U}}_q(A_2^{(2)})$ denotes the minimal extension of $U_q(A_2^{(2)})$ that acts in every left lowest weight and in every right highest weight $U_q(A_2^{(2)})$-module.

3.4 Relations between coproducts and the universal R-matrix

Let $\bar{\mathcal{R}}$ and $\mathcal{K}$ denote the following formal series of elements of the tensor product $U_E \otimes U_F$:

$$\begin{aligned}\bar{\mathcal{R}} &= \exp\left((q-q^{-1})\oint e(z)\otimes f(z)\frac{dz}{z}\right) = 1 + (q-q^{-1})\oint e(z)\otimes f(z)\frac{dz}{z} + \\ &\quad + \frac{(q-q^{-1})^2}{2!}\oint\oint e(z_1)e(z_2)\otimes f(z_1)f(z_2)\frac{dz_1}{z_1}\frac{dz_2}{z_2} + \cdots,\\ \mathcal{K} &= \exp\left(\sum_{n>0}\frac{n(q-q^{-1})^2}{(q^n-q^{-n})(q^n+(-1)^{n+1}+q^{-n})}a_{-n}\otimes a_n\right).\end{aligned}$$

By $\oint f(z)dz$ we mean a formal integral that is equal to the coefficient of z^{-1} in the formal series $f(z)$. We further set

$$\mathcal{R}_+ = \left(P^{*-}\otimes P^+\right)(\bar{\mathcal{R}}), \qquad \mathcal{R}_- = \left(P^{*+}\otimes P^-\right)(\bar{\mathcal{R}}), \qquad \text{and} \qquad \mathcal{R} = \mathcal{R}_+^{21} q^{h\otimes h}\mathcal{K}^{21}\mathcal{R}_-, \tag{3.4}$$

where $h = \log K_0$.

Proposition 3.2 *(i) The tensor $\mathcal{R}_-^{21}$ is a cocycle for $\Delta^{(D)}$ such that for any $x \in U_q(A_2^{(2)})$*

$$\Delta(x) = \mathcal{R}_-^{21}\Delta^{(D)}(x)\left(\mathcal{R}_-^{21}\right)^{-1},$$

(ii) The tensor $\mathcal{R}$ is the universal R-matrix for $U_q(A_2^{(2)})$ with opposite comultiplication Δ^{op}:

$$\Delta(x) = \mathcal{R}\Delta^{op}(x)\mathcal{R}^{-1} \qquad \textit{for any} \quad x \in U_q(A_2^{(2)}).$$

Proof: For any element $X \in U_E \otimes U_F$, let $:X:$ denote the same element presented as a sum $\sum_i a_i b_i \otimes c_i d_i$, where $a_i \in U_e^+$, $b_i \in U_E^-$, $c_i \in U_f^-$, $d_i \in U_F^+$. In the work [DKP1], it was proved that the element $\bar{\mathcal{R}}' = :\bar{\mathcal{R}}:\mathcal{K}$ is a well defined series in elements of the tensor product $\overline{\overline{U}}_E \otimes \overline{U}_F$, which represents the tensor of the Hopf[4] pairing $\langle\ ,\ \rangle : U_E \otimes U_F^{op} \to \mathbb{C}$, that is

$$\langle 1 \otimes a, \bar{\mathcal{R}}' \rangle = a, \qquad \langle b \otimes 1, \bar{\mathcal{R}}' \rangle = b$$

for any $a \in U_E$ and $b \in U_F$.[5] Due to Proposition 3.2 and to the theory of Cartan-Weyl bases for $U_q(A_2^{(2)})$ developed in [KT],[6] the pair (U_e^+, U_E^-) forms an orthogonal decomposition of the bialgebra U_E. Analogously, the dual pair (U_f^-, U_F^+) forms an orthogonal decomposition of the bialgebra U_F. This implies the decomposition $\bar{\mathcal{R}}' = \bar{\mathcal{R}}_1' \bar{\mathcal{R}}_2'$, where $\bar{\mathcal{R}}_1'$ is the tensor of the pairing of U_e^+ and U_f^-, $\bar{\mathcal{R}}_2'$ is the tensor of the pairing of U_E^- and U_F^+. They are given by the relations:

$$\begin{aligned} \bar{\mathcal{R}}_1' &= 1 \otimes P^-(\bar{\mathcal{R}}') = P^{*+} \otimes P^-(\bar{\mathcal{R}}') = P^{*+} \otimes P^-(\bar{\mathcal{R}}), \\ \bar{\mathcal{R}}_2' &= 1 \otimes P^+(\bar{\mathcal{R}}') = P^{*-} \otimes P^+(\bar{\mathcal{R}}') = \left(P^{*-} \otimes P^+(\bar{\mathcal{R}})\right)\mathcal{K}. \end{aligned} \tag{3.5}$$

The last equalities in both lines of (3.5) are implied by the following observation: the application of projection operators to both tensor components automatically performs the normal ordering of the tensor $\bar{\mathcal{R}}$. The factorization of the tensor $\mathcal{K}$ follows from the definitions of the projections.

Now, part (i) of the proposition becomes a direct generalization of Proposition 3.8 of [EKP]. One can check that the conditions (H1)–(H6) of Section 2.3 of [EKP] are satisfied for the algebras $A_1 = U_e^+$, $A_2 = U_E^-$, $B_1 = U_F^+$, $B_2 = U_f^-$ due to the theory of Cartan-Weyl bases, Proposition 3.2 and the relations (3.5). Under the same settings part (ii) is the direct generalization of the results of [EKP, Section 2.3]. □

Remark. Note that all the results of this and of previous sections could be easily modified for the central extended algebra $U_q(A_2^{(2)})$ with an added grading element. See e.g. [DK].

3.5 The universal weight function

Let V be a representation of the algebra $U_q(A_2^{(2)})$ and v be a vector in V. We call v a highest weight vector with respect to the current Borel subalgebra U_E if

$$\begin{aligned} e(z)v &= 0, \\ K^{\pm}(z)v &= \lambda(z)v, \end{aligned}$$

where $\lambda(z)$ is a meromorphic function, decomposed in series in z^{-1} for $K^+(z)$ and in series in z for $K^-(z)$. Representation V is called a representation with highest weight vector $v \in V$ with respect to U_E if it is generated by v over $U_q(A_2^{(2)})$. Suppose that for every finite ordered set $I = \{i_1, \ldots, i_n\}$, an element $W(z_{i_1}, \ldots, z_{i_n})$ is chosen. It is a Laurent polynomial in $\{z_{i_1}, \ldots, z_{i_n}\}$ over the ring $U_q(A_2^{(2)})\left[\left[z_{i_2}/z_{i_1}, z_{i_3}/z_{i_2}, \ldots, z_{i_n}/z_{i_{n-1}}\right]\right]$ satisfying the following conditions:

[4]which means that $\langle a_1 a_2, b\rangle = \langle a_1 \otimes a_2, \Delta^{op}(b)\rangle$ and $\langle a, b_1 b_2\rangle = \langle \Delta(a), b_1 \otimes b_2\rangle$ for any $a, a_1, a_2 \in U_E$ and $b, b_1, b_2 \in U_F$.

[5]In [DKP1] this result was derived for $U_q(\widehat{\mathfrak{sl}}_2)$. However, the arguments therein are general and work for any quantum affine algebra.

[6]To adapt the result of [KT] to our case, one should replace q-commutators with q^{-1}-commutators in the construction of the Cartan-Weyl basis.

(1) for any representation V that is a highest weight representation with respect to U_E with highest weight vector v the function

$$w_V(z_{i_1},\ldots,z_{i_n}) = W(z_{i_1},\ldots,z_{i_n})v$$

converges in the domain $|z_{i_1}| \gg \cdots \gg |z_{i_n}|$ to a meromorphic V-valued function;

(2) $W = 1$ if $I = \varnothing$;

(3) let $V = V_1 \otimes V_2$ be a tensor product of the highest weight representations with highest weight vectors v_1 and v_2 and the highest weight series $\{\lambda^{(1)}(z)\}$ and $\{\lambda^{(2)}(z)\}$; then for an ordered set I we have

$$w_V\left(\{z_{i|i\in I}\}\right) = \sum_{I=I_1\bigsqcup I_2} w_{V_1}\left(\{z_{i|i\in I_1}\}\right)\otimes w_{V_2}\left(\{z_{i|i\in I_2}\}\right)\times$$
$$\times\prod_{i\in I_1}\lambda^{(2)}(z_i)\quad\times\prod_{\substack{i<j,\\ i\in I_1, j\in I_2}}\frac{(q^{-2}-z_j/z_i)(q+z_j/z_i)}{(1-q^{-2}z_j/z_i)(1+qz_j/z_i)}. \qquad (3.6)$$

A collection $W(z_{i_1},\ldots,z_{i_n})$ for all n is called the universal weight function. Using Proposition 3.2, we get the following direct generalization of [EKP, Theorem 3]:

Proposition 3.3 *The collection $P^+\big(f(z_{i_1}),\ldots,f(z_{i_n})\big)$ is a weight function for $U_q(A_2^{(2)})$:*

$$W(z_{i_1},\ldots,z_{i_n}) = P^+\big(f(z_{i_1}),\ldots,f(z_{i_n})\big). \qquad (3.7)$$

Our main goal from now on is to calculate the latter projection.

4 Composite currents

In the following sections we use the abbreviated notation P for the projection P^+.

4.1 Definitions

We introduce a pair of new currents $s(z)$ and $t(z)$. The first one plays a crucial role in the following discussion since the main result is formulated in terms of the projection $P(s(z))$. Thus $s(z)$ and $t(z)$ are a pair of Laurent series with coefficients in the completed algebra $\overline{U}_F$ given by

$$s(z) = \operatorname*{res}_{w=-q^{-1}z} f(z)f(w)\frac{dw}{w}, \qquad (4.1)$$

$$t(z) = \operatorname*{res}_{w=q^2z} f(z)f(w)\frac{dw}{w}. \qquad (4.2)$$

We call generating functions $s(z)$, $t(z)$ the *composite currents*. We treat them as meromorphic operator-valued functions in the category of left highest weight $U_q(A_2^{(2)})$-modules.

Proposition 4.1 *In $\overline{U}_F$ the composite currents $s(z)$ and $t(z)$ can be represented as the following products:*

$$s(z) = \frac{(q+q^2)(q^{-1}-q)}{1+q^3}f(-q^{-1}z)f(z), \qquad (4.3)$$

$$t(z) = \frac{(q+q^2)(q^2-q^{-2})}{1+q^3}f(q^2z)f(z). \qquad (4.4)$$

Proof:

$$\underset{w=-q^{-1}z}{\mathrm{res}} f(z)f(w)\frac{dw}{w} = \lim_{w\to -q^{-1}z}\left((w+q^{-1}z)f(z)f(w)\frac{1}{w}\right) =$$
$$= \lim_{w\to -q^{-1}z}\left(\frac{(z-q^2w)(qz+w)}{(q^2z-w)w}f(w)f(z)\right) = \frac{(q+q^2)(q^{-1}-q)}{1+q^3}f(-q^{-1}z)f(z).$$

The second equality is proved in a similar way. □

4.2 Analytic properties

We use the following vanishing conditions on products of Drinfeld currents: *diagonal conditions* (i) follow from relation (2.2) and *Serre conditions* (ii) follow from analytic Serre relations (2.12) (see [DK]):

(i) the product $f(z)f(w)$ has a simple zero on the hyperplane $z = w$;

(ii) the product $f(z_1)f(z_2)f(z_3)$ has a simple zero on the following lines: $z_1 = -qz_3 = q^2z_2$, $z_2 = -qz_1 = q^2z_3$ and $z_3 = -qz_2 = q^2z_1$.

We collect data derived from the vanishing conditions in the following table:

$s(z)f(w)$		$f(w)s(z)$	
zeros	poles	zeros	poles
$w = z$	$w = q^2z$	$w = z$	$w = z$
$w = -q^{-1}z$	$w = -q^{-1}z$	$w = -q^{-1}z$	$w = -q^{-3}z$
$w = -qz$	$w = -qz$	$w = -qz$	$w = -qz$
$w = q^{-2}z$	$w = q^{-2}z$	$w = q^{-2}z$	$w = q^{-2}z$

where the first pair of zeros is a consequence of the diagonal conditions while the second pair is obtained from the Serre relations. Cancelling zeros and poles at the same points we derive that $s(z)f(w)$ has just one simple zero surviving on $w = z$ and just one simple pole on $w = q^2z$. Analogously, $f(w)s(z)$ has a simple zero on $w = -q^{-1}z$ and a simple pole on $w = -q^{-3}z$. Therefore the following equality of holomorphic functions holds:

$$\frac{(q^2z-w)}{(z-w)}s(z)f(w) = \frac{(z+q^3w)}{(z+qw)}f(w)s(z). \tag{4.5}$$

In a similar way, we obtain one more table:

$s(z)s(w)$		$s(w)s(z)$	
zeros	poles	zeros	poles
$w = z$	$w = q^2z$	$w = z$	$w = q^{-2}z$
$w = -qz$	$w = -q^3z$	$w = -q^{-1}z$	$w = -q^{-3}z$
$w = z$	$w = z$	$w = z$	$w = z$
$w = -q^{-1}z$	$w = -q^{-1}z$	$w = -qz$	$w = -qz$
$w = q^2z$	$w = q^2z$	$w = q^{-2}z$	$w = q^{-2}z$
$w = -q^{-1}z$	$w = -q^{-1}z$	$w = -qz$	$w = -qz$
$w = -qz$	$w = -qz$	$w = -q^{-1}z$	$w = -q^{-1}z$
$w = q^{-2}z$	$w = q^{-2}z$	$w = q^2z$	$w = q^2z$

In this case the first four zeros are derived from diagonal conditions and the rest from Serre relations. Thus the product $s(z)s(w)$ has simple zeros at the points $w = z$ and $w = -qz$ and simple poles at the points $w = q^2 z$ and $w = -q^3 z$. Once again we obtain the following equality of holomorphic functions:

$$\frac{(q^2 z - w)(q^3 z + w)}{(z - w)(qz + w)} s(z)s(w) = \frac{(z - q^2 w)(z + q^3 w)}{(z - w)(z + qw)} s(w)s(z). \tag{4.6}$$

5 Calculation of the weight function

5.1 Required notation

In this section we introduce some notation that is frequently used below. The following rational functions are formal power series, converging in the domain $|z_1| \gg \cdots \gg |z_n|$.

$$\rho_k(z_1, \dots, z_{n-1}; z_n) = \prod_{\substack{i=1 \\ i \neq k}}^{n-1} \frac{z_n - z_i}{z_k - z_i} \prod_{i=1}^{n-1} \frac{z_k - q^2 z_i}{z_n - q^2 z_i}, \tag{5.1}$$

$$\lambda_k(z_1, \dots, z_{n-1}; z_n) = \frac{z_k}{q z_n + z_k} \prod_{i=1}^{n-1} \frac{(z_n - z_i)(z_k + q^3 z_i)}{(z_k + q z_i)(z_n - q^2 z_i)}, \tag{5.2}$$

$$\mu_k(z_1, \dots, z_{n-1}; z_n) = \prod_{\substack{i=1 \\ i \neq k}}^{n-1} \frac{z_n - z_i}{z_k - z_i} \prod_{i=1}^{n-1} \frac{(z_n + q z_i)(z_k - q^2 z_i)(z_k + q^3 z_i)}{(z_k + q z_i)(z_n - q^2 z_i)(z_n + q^3 z_i)}, \tag{5.3}$$

$$\nu_k(z_1, \dots, z_{n-1}; z_n) = -q^n \prod_{\substack{i=1 \\ i \neq k}}^{n-1} \frac{z_n + q z_i}{z_k - z_i} \prod_{i=1}^{n-1} \frac{(z_n - z_i)(z_k + q z_i)(z_k - q^2 z_i)}{(q z_k + z_i)(z_n - q^2 z_i)(z_n + q^3 z_i)}. \tag{5.4}$$

Let $\mathcal{F}(z_1, \dots, z_{n-1}; z_n)$ and $\mathcal{S}(z_1, \dots, z_{n-1}; z_n)$ denote the following combinations of projections:

$$\mathcal{F}(z_1, \dots, z_{n-1}; z_n) = P(f(z_n)) - \sum_{k=1}^{n-1} \rho_k(z_1, \dots, z_{n-1}; z_n) P(f(z_k)), \tag{5.5}$$

$$\begin{aligned} \mathcal{S}(z_1, \dots, z_{n-1}; z_n) = P(s(z_n)) &- \sum_{k=1}^{n-1} \mu_k(z_1, \dots, z_{n-1}; z_n) P(s(z_k)) - \\ &- \sum_{k=1}^{n-1} \nu_k(z_1, \dots, z_{n-1}; z_n) P(s(-q z_k)). \end{aligned} \tag{5.6}$$

Let $I = \{i_1, \dots, i_r\}$ and $J = \{j_1, \dots, j_r\}$ be ordered subsets of an ordered set $\{1, \dots, n\}$. We call an ordered pair of subsets $\{I, J\}$ a P^+-*admissible pair* of cardinality r if the following hold:

- $I \cap J = \varnothing$;
- $j_1 > j_2 > \cdots > j_r$;
- $j_\ell > i_\ell$, for $\ell = 1, \dots, r$.

For each P^+-admissible pair $\{I,J\}$ of cardinality r, and for any $k=1,\dots,r$, we introduce a formal power series $\tau^k_{I,J}(z_1,\dots,z_n)$ that is the decomposition in the domain $|z_1|\gg|z_2|\gg\cdots\gg|z_n|$ of the rational function

$$\tau^k_{I,J}(z_1,\dots,z_n)=-\lambda_{i_k}\left(z_{i_1},\dots,z_{i_{k-1}},z_1,\dots,z_{j_k-1};z_{j_k}\right)\prod_{\substack{\ell=1\\ \ell\neq i_1,\dots,i_{k-1}}}^{i_k-1}\alpha\left(\frac{z_\ell}{z_{i_k}}\right)\prod_{\substack{\ell=1\\ \ell\neq i_1,\dots,i_k}}^{j_k-1}\alpha\left(\frac{-qz_\ell}{z_{i_k}}\right), \tag{5.7}$$

where $z_{i_1},\dots,z_{i_{k-1}}$ are skipped in the row $z_1,\dots,z_{j_k-1}$ of variables of the function λ_{i_k} above. In other words, each of $z_{i_1},\dots,z_{i_{k-1}}$ occur in the variables of function λ_{i_k}, but only once.

Now, for each P^+-admissible pair $\{I,J\}$ of cardinality r and for any $k=1,\dots,n$ such that $k\notin I\cup J$, we introduce a generating function $\mathcal{F}^k_{I,J}(z_1,\dots,z_n)$. Namely, for such a k there exists a unique $p=1,\dots,r+1$ for which $j_p<k<j_{p-1}$, with $j_0=n+1$, $j_{r+1}=0$. Then we set

$$\mathcal{F}^k_{I,J}(z_1,\dots,z_n)=\mathcal{F}\left(z_{i_1},\dots,z_{i_{p-1}},z_1,\dots,z_{k-1};z_k\right), \tag{5.8}$$

where again $z_{i_1},\dots,z_{i_{p-1}}$ are skipped in the row $z_1,\dots,z_{k-1}$.

Finally, we define a q-commutator [7]

$$[a,b]_{q^{\pm1}}=ab-q^{\pm1}ba \tag{5.9}$$

and a pair of ordered products

$$\prod_{k=1}^{\overrightarrow{r}}G_k=G_1G_2\dots G_r,\qquad \prod_{k=1}^{\overleftarrow{r}}G_k=G_r\dots G_2G_1.$$

5.2 Main results

Theorem 1 *Projections of currents $f(z)$ and $s(z)$ can be written as follows:*

$$P(f(z)) = \sum_{n>0}f_nz^{-n}, \tag{5.10}$$

$$P(s(z)) = -\frac{1}{q+q^{-2}}\left(q\,[P(f(z)),f_0]_{q^{-1}}+\left[f_1z^{-1},f_0+P(f(z))\right]_{q^{-1}}\right). \tag{5.11}$$

Theorem 2 *The projection $P(f(z_1)\cdots f(z_n))$ can be expressed by the following explicit formula:*

$$P\left(f(z_1)\cdots f(z_n)\right)=\sum_{r=0}^{[n/2]}\sum_{\substack{\{I,J\}\\ |I|=|J|=r}}\prod_{m=1}^{r}\tau^m_{I,J}(z_1,\dots,z_n)\times$$
$$\times\prod_{k=1}^{\overrightarrow{r}}\mathcal{S}\left(z_{i_1},\dots,z_{i_{k-1}};z_{i_k}\right)\prod_{\substack{\ell=1\\ \ell\notin I,J}}^{\overrightarrow{n}}\mathcal{F}^\ell_{I,J}(z_1,\dots,z_n), \tag{5.12}$$

where $\mathcal{S}(z_1,\dots,z_{n-1};z_n)$, $\tau^k_{I,J}(z_1,\dots,z_n)$, $\mathcal{F}^k_{I,J}(z_1,\dots,z_n)$ are defined by (5.6), (5.7), (5.8) respectively. The second sum in the formula (5.12) is taken over P^+-admissible pairs $\{I,J\}$ of cardinality r, and in the case $I=J=\varnothing$ we have only one summand

$$\prod_{\ell=1}^{\overrightarrow{n}}\mathcal{F}\left(z_1,\dots,z_{\ell-1};z_\ell\right).$$

[7] please note that the definition differs from the one given in [KT].

We illustrate Theorem 2 by giving examples in Section 7.

5.3 Calculation of other projections

In this section we obtain some formulae for projections P^-, P^{*+}, P^{*-}. The proofs almost literally reproduce the ones of Theorem 2, and we skip them.

Let us introduce a composite current

$$\tilde{s}(z) = \operatorname*{res}_{w=-qz} f(w)f(z)\frac{dw}{w}.$$

As a matter of fact $\tilde{s}(z) = -s(-qz)$, but $\tilde{s}(z)$ is more convenient for calculating projection P^-. The following theorem holds:

Theorem 3 *Projections $P^-(f(z))$ and $P^-(\tilde{s}(z))$ can be written as follows:*

$$P^-(f(z)) = \sum_{n\leqslant 0} f_n z^{-n}, \tag{5.13}$$

$$P^-(\tilde{s}(z)) = \frac{1}{1+q^3}\Big(\left[f_0, P^-(f(z))\right]_q + q\left[P^-(f(z)) - f_0, f_1 z^{-1}\right]_q\Big). \tag{5.14}$$

Now we introduce some more notation:

$$\tilde{\rho}_k(z_1; z_2, \ldots, z_n) = \prod_{\substack{i=2\\ i\neq k}}^{n} \frac{z_1 - z_i}{z_k - z_i} \prod_{i=2}^{n} \frac{q^2 z_k - z_i}{q^2 z_1 - z_i}, \tag{5.15}$$

$$\tilde{\lambda}_k(z_1; z_2, \ldots, z_n) = -\frac{q z_k}{z_1 + q z_k} \prod_{i=2}^{n} \frac{(z_1 - z_i)(q^3 z_k + z_i)}{(q z_k + z_i)(q^2 z_1 - z_i)}, \tag{5.16}$$

$$\tilde{\mu}_k(z_1; z_2, \ldots, z_n) = \prod_{\substack{i=2\\ i\neq k}}^{n} \frac{z_1 - z_i}{z_k - z_i} \prod_{i=2}^{n} \frac{(q z_1 + z_i)(q^2 z_k - z_i)(q^3 z_k + z_i)}{(q z_k + z_i)(q^2 z_1 - z_i)(q^3 z_1 + z_i)}, \tag{5.17}$$

$$\tilde{\nu}_k(z_1; z_2, \ldots, z_n) = -q^{n-1} \prod_{\substack{i=2\\ i\neq k}}^{n} \frac{q z_1 + z_i}{z_k - z_i} \prod_{i=2}^{n} \frac{(z_1 - z_i)(q z_k + z_i)(q^2 z_k - z_i)}{(z_k + q z_i)(q^2 z_1 - z_i)(q^3 z_1 + z_i)}. \tag{5.18}$$

Define the following combinations of projections:

$$\tilde{\mathcal{F}}(z_1; z_2, \ldots, z_n) = P^-(f(z_1)) - \sum_{k=2}^{n} \tilde{\rho}_k(z_1; z_2, \ldots, z_n) P^-(f(z_k)), \tag{5.19}$$

$$\begin{aligned} \tilde{\mathcal{S}}(z_1; z_2, \ldots, z_n) = P^-(\tilde{s}(z_1)) &- \sum_{k=2}^{n} \tilde{\mu}_k(z_1; z_2, \ldots, z_n) P^-(\tilde{s}(z_k)) - \\ &- \sum_{k=2}^{n} \tilde{\nu}_k(z_1; z_2, \ldots, z_n) P^-(\tilde{s}(-q^{-1} z_k)), \end{aligned} \tag{5.20}$$

Now, let $I = \{i_1, \ldots, i_r\}$ and $J = \{j_1, \ldots, j_r\}$ be ordered subsets of an ordered set $\{1, \ldots, n\}$. We call an ordered pair of subsets $\{I, J\}$ a P^--*admissible pair* of cardinality r if the following hold:

- $I \cap J = \varnothing$;

- $i_1 < i_2 < \cdots < i_r$;
- $i_\ell < j_\ell$, for $\ell = 1, \ldots, r$.

For each P^--admissible pair $\{I, J\}$ of cardinality r, and for any $k = 1, \ldots, r$, we introduce a formal power series $\tilde{\tau}^k_{I,J}(z_1, \ldots, z_n)$ that is the decomposition in the domain $|z_1| \gg |z_2| \gg \cdots \gg |z_n|$ of the rational function

$$\tilde{\tau}^k_{I,J}(z_1, \ldots, z_n) = \tilde{\lambda}_{j_k}\left(z_{i_k}; z_{i_k+1}, \ldots, z_n, z_{j_{k-1}}, \ldots, z_{j_1}\right) \prod_{\substack{\ell=j_k+1 \\ \ell \neq j_1, \ldots, j_{k-1}}}^{n} \alpha\left(\frac{z_{j_k}}{z_\ell}\right) \prod_{\substack{\ell=i_k+1 \\ \ell \neq j_1, \ldots, j_k}}^{n} \alpha\left(\frac{-q z_{j_k}}{z_\ell}\right), \tag{5.21}$$

where $z_{j_1}, \ldots, z_{j_{k-1}}$ are skipped in the row $z_{i_k+1}, \ldots, z_n$ of variables of the function $\tilde{\lambda}_{i_k}$ above. In other words each of $z_{j_1}, \ldots, z_{j_{k-1}}$ occur in the variables of function $\tilde{\lambda}_{i_k}$, but only once.

Now, for each P^--admissible pair $\{I, J\}$ of cardinality r and for any $k = 1, \ldots, n$ such that $k \notin I, J$, we introduce a generating function $\tilde{\mathcal{F}}^k_{I,J}(z_1, \ldots, z_n)$. Namely, for such a k there exists a unique $p = 1, \ldots, r+1$ for which $i_{p-1} < k < i_p$, with $i_0 = 0$, $i_{r+1} = n+1$. Then we set

$$\tilde{\mathcal{F}}^k_{I,J}(z_1, \ldots, z_n) = \tilde{\mathcal{F}}\left(z_k; z_{k+1}, \ldots, z_n, z_{j_{p-1}}, \ldots, z_{j_1}\right), \tag{5.22}$$

where again $z_{j_1}, \ldots, z_{j_{p-1}}$ are skipped in the row $z_{k+1}, \ldots, z_n$.

Then we have the following theorem:

Theorem 4 *The projection $P^-(f(z_1)\cdots f(z_n))$ can be expressed by the following explicit formula:*

$$P^-\left(f(z_1)\cdots f(z_n)\right) = \sum_{r=0}^{[n/2]} \sum_{\substack{\{I,J\} \\ |I|=|J|=r}} \prod_{m=1}^{r} \tilde{\tau}^m_{I,J}(z_1, \ldots, z_n) \times$$
$$\times \prod_{\substack{\ell=1 \\ \ell \notin I,J}}^{\overrightarrow{n}} \tilde{\mathcal{F}}^\ell_{I,J}(z_1, \ldots, z_n) \prod_{k=1}^{\overleftarrow{r}} \tilde{\mathcal{S}}\left(z_{j_k}; z_{j_{k-1}}, \ldots, z_{j_1}\right), \tag{5.23}$$

where $\tilde{\mathcal{S}}(z_1; z_2 \ldots, z_n)$, $\tilde{\tau}^k_{I,J}(z_1, \ldots, z_n)$, $\tilde{\mathcal{F}}^k_{I,J}(z_1, \ldots, z_n)$ are defined by (5.20), (5.21), (5.22) respectively. The second sum in the formula (5.23) is taken over P^--admissible pairs $\{I, J\}$, and in the case $I = J = \varnothing$ we have only one summand

$$\prod_{\ell=1}^{\overrightarrow{n}} \tilde{\mathcal{F}}\left(z_\ell; z_{\ell+1}, \ldots, z_n\right).$$

To obtain the formulae for projections P^{*+} and P^{*-} we introduce an involution ι of the algebra $U_q(A_2^{(2)})$ such that

$$\iota(e_n) = f_{-n}, \qquad \iota(f_n) = e_{-n}, \qquad \iota(a_n) = a_{-n}, \qquad \iota(K_0) = K_0^{-1}. \tag{5.24}$$

One can see that

$$\iota P^{\pm} = P^{*\mp}\iota. \tag{5.25}$$

Using this identity we can compute $P^{*\pm}(e(z_1)\ldots e(z_n))$ as $\iota P^{\mp}(f(z_1^{-1})\ldots f(z_n^{-1}))$.

Corollary 5.1 *An integral presentation for the factors of the universal $\mathcal{R}$-matrix of $U_q(A_2^{(2)})$ can be obtained by applying formulae (5.12), (5.23) and (5.25) to relations (3.4).*

6 Proofs

6.1 Proof of Proposition 3.2

The statement is multiplicative with respect to x; hence it is sufficient to prove it for generators e_α, $e_{\delta-2\alpha}$, $k_\alpha^{\pm 1}$ of the subalgebra $U_q(\mathfrak{b}_+)$. For elements e_α and $k_\alpha^{\pm 1}$ the property (3.1) follows directly from the isomorphism between the realizations (2.26) and the formulae of the comultiplication $\Delta^{(D)}$ (2.17)–(2.25). Thus we only need to check the property for $e_{\delta-2\alpha}$.

Let d be a grading on the commutative algebra, generated by the elements $a_i, i \in \mathbb{N}$ and K_0, such that $d(a_i) = i$ and $d(K_0) = 0$. Using Drinfeld comultiplication formulae we obtain

$$\Delta^{(D)}(f_0) = 1 \otimes f_0 + \sum_{n \geqslant 0} f_{-n} \otimes K_0 I_n,$$

$$\Delta^{(D)}(f_1) = 1 \otimes f_0 + \sum_{n \geqslant 0} f_{-n+1} \otimes K_0 J_n,$$

where $I_n, J_n \in \mathbb{C}(q)[a_1, a_2, \dots]$ are polynomials in variables $a_i, i \in \mathbb{N}$ of degree n over the quotient field $\mathbb{C}(q)$. According to (2.26) it only remains to check the statement of the lemma for

$$x = q f_1 f_0 - f_0 f_1.$$

The equality

$$\Delta^{(D)}(q f_1 f_0 - f_0 f_1) = 1 \otimes (q f_1 f_0 - f_0 f_1) + \sum_{n \geqslant 0} f_{-i+1} \otimes (q J_n f_0 - f_0 J_n)$$

holds modulo $U_q(A_2^{(2)}) \otimes U_q(\mathfrak{b}_+)$. Formulae (2.26) imply that

$$1 \otimes (q f_1 f_0 - f_0 f_1) \in U_q(A_2^{(2)}) \otimes U_q(\mathfrak{b}_+).$$

To prove the same for the second summand we use equality

$$K^+(z) f_0 = q^{-1} f_0 K^+(z) \mod U_q(\mathfrak{b}_+),$$

which one can obtain from relation (2.4). □

6.2 Proof of Theorem 1

The first part of the theorem immediately follows from the definition of the projection P^+. The proof of the second part is a little bit more complicated. We start with an equality

$$\operatorname*{res}_{w=-q^{-1}z} \left(f(z) f(w) \frac{dw}{w} \right) = \frac{1}{(q^2 + q^{-1})z} \operatorname*{res}_{w=-q^{-1}z} \left((q^2 z - w) f(z) f(w) \frac{dw}{w} \right).$$

Using the Cauchy Residue Theorem we rewrite the latter residue as the following sum:

$$\operatorname*{res}_{w=-q^{-1}z} \left((q^2 z - w) f(z) f(w) \frac{dw}{w} \right) = \\ = - \left(\operatorname*{res}_{w=\infty} \left((q^2 z - w) f(z) f(w) \frac{dw}{w} \right) + \operatorname*{res}_{w=0} \left((q^2 z - w) f(z) f(w) \frac{dw}{w} \right) \right).$$

It remains to calculate the residues at $w = 0$ and $w = \infty$.

$$\operatorname*{res}_{w=0}\left((q^2 z - w)f(z)f(w)\frac{dw}{w}\right) = q^2 z f(z) f_0 - f(z) f_1.$$

Since $f(z)f(w)$ is well defined only in the domain $|z| \gg |w|$, we evaluate the residue of its analytic continuation at $w = \infty$:

$$\operatorname*{res}_{w=\infty}\left((q^2 z - w)f(z)f(w)\frac{dw}{w}\right) = \operatorname*{res}_{w=\infty}\left(\frac{(z/w - q^2)(z/w + q^{-1})}{(1 + q^{-1}z/w)} f(w)f(z)dw\right),$$

which after putting $u = \dfrac{1}{w}$ equals

$$\operatorname*{res}_{u=0}\left(\frac{\left(q^2 - uz\right)\left(q^{-1} + uz\right)}{(1 + q^{-1}uz)} f\left(\frac{1}{u}\right) f(z)\frac{du}{u^2}\right) =$$
$$= z\left(q f_1 z^{-1} f(z) - q f_0 f(z) + \left(q^2 + q - 1 - q^{-1}\right)\sum_{n\geqslant 0} f_{-n}(-q^{-1}z)^n f(z)\right).$$

Previous relations imply

$$s(z) = -\frac{1}{q^2 + q^{-1}}\Big(q^2 \left[f(z), f_0\right]_{q^{-1}} + q\left[f_1 z^{-1}, f(z)\right]_{q^{-1}} + \\ + \left(q^2 + q - 1 - q^{-1}\right) P^{-}\left(f(-q^{-1}z)\right) f(z)\Big), \quad (6.1)$$

where $[a, b]_{q^{-1}}$ is defined by (5.9). Recalling projection properties (3.2),

$$P(s(z)) = -\frac{1}{q^2 + q^{-1}} P\left(q^2 \left[f(z), f_0\right]_{q^{-1}} + q\left[f_1 z^{-1}, f(z)\right]_{q^{-1}}\right).$$

To finish the proof we rewrite relation (2.2) in coordinates:

$$q^2 f_{n+2} f_m - q f_m f_{n+2} + q^2 f_{m+2} f_n - q f_n f_{m+2} = (1 - q^3)(f_{m+1} f_{n+1} + f_{n+1} f_{m+1}). \qquad (6.2)$$

Using relation (6.2), the projection properties (3.2) and the Borel subalgebras description, given in the Section 3, we obtain that

$$q f_1 f_n - f_n f_1 \in U_F^+, \quad \text{for } n \geqslant 0,$$
$$q f_m f_0 - f_0 f_m \in U_F^+, \quad \text{for } m > 0.$$

For $n > 0$ the relation above is obvious, for $n = 0$, $m = 1$ it follows from the formulae (2.26), and finally for $n < 0$, $m \neq 1$ equality (6.2) provides us with the desired relation. Therefore we obtain

$$P(s(z)) = -\frac{1}{q + q^{-2}}\left(q\left[P(f(z)), f_0\right]_{q^{-1}} + \left[f_1 z^{-1}, f_0 + P(f(z))\right]_{q^{-1}}\right).$$

□

6.3 Proof of Theorem 2

Assume $0 \leqslant p < k \leqslant n-1$. We introduce the formal power series $\tau_{k,p}(z_1,\dots,z_{n-1};z_n)$, which is the decomposition in the domain $|z_1| \gg |z_2| \gg \dots \gg |z_n|$ of the rational function

$$\tau_{k,p}(z_1,\dots,z_{n-1};z_n) = -\lambda_k(z_1,\dots,z_{n-1};z_n) \prod_{\ell=p+1}^{k-1} \alpha\left(\frac{z_\ell}{z_k}\right) \prod_{\substack{\ell=p+1 \\ \ell\neq k}}^{n-1} \alpha\left(\frac{-qz_\ell}{z_k}\right). \tag{6.3}$$

Now, we use iterative application of:

Lemma 6.1 *The projection* $P\big(s(z_1)\dots s(z_p)f(z_{p+1})\dots f(z_n)\big)$ *can be written as*

$$\begin{aligned} P\big(s(z_1)\dots s(z_p)f(z_{p+1})\dots f(z_n)\big) &= \\ &= P\big(s(z_1)\dots s(z_p)f(z_{p+1})\dots f(z_{n-1})\big)\mathcal{F}(z_1,\dots,z_{n-1};z_n) + \\ + \sum_{k=p+1}^{n-1} \tau_{k,p}(z_1,\dots,z_{n-1};z_n) &P\big(s(z_1)\dots s(z_p)s(z_k)f(z_{p+1})\dots f(z_{k-1})f(z_{k+1})\dots f(z_{n-1})\big), \end{aligned} \tag{6.4}$$

where $\mathcal{F}(z_1,\dots,z_{n-1};z_n)$ *is defined by (5.5).*

we reduce evaluation of $P(f(z_1)\dots f(z_n))$ to the problem of calculating $P(s(z_1)\dots s(z_m))$ or $P(s(z_1)\dots s(z_m)f(z_{m+1}))$. Applying Lemma 6.1 to $P(s(z_1)\dots s(z_m)f(z_{m+1}))$ once more we reduce the case to the evaluation of $P(s(z_1)\dots s(z_m))$. The latter is achieved by using:

Lemma 6.2 *The projection* $P\big(s(z_1)\dots s(z_n)\big)$ *admits a decomposition*

$$P\big(s(z_1)\dots s(z_n)\big) = P\big(s(z_1)\dots s(z_{n-1})\big)\mathcal{S}(z_1,\dots,z_{n-1};z_n), \tag{6.5}$$

where $\mathcal{S}(z_1,\dots,z_{n-1};z_n)$ *is defined by (5.6).*

One can see that on every step of the iterative application of Lemma 6.1, we either factor projection $P\big(s(z_1)\dots s(z_p)f(z_{p+1})\dots f(z_n)\big)$ into the product $P\big(s(z_1)\dots s(z_p)f(z_{p+1})\dots f(z_{n-1})\big)\times$ $\times\mathcal{F}(z_1,\dots,z_{n-1};z_n)$ or replace product $f(z_i)f(z_n)$ with the current $s(z_i)$. Thus, sets I and J in the formula (5.12) indicate that on the $(n+1-j_1)$-th step, we replaced $f(z_{i_1})f(z_{j_1})$ with $s(z_{i_1})$, on the $(n+1-j_2)$-th step we replaced $f(z_{i_2})f(z_{j_2})$ with $s(z_{i_2})$ etc. □

6.4 Proof of Lemma 6.1

For any current $a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n}$ let $a^{\pm}(z)$ denote currents

$$a^+(z) = \oint \frac{a(w)}{1-w/z}\frac{dw}{z} = \sum_{n>0} a_n z^{-n},$$
$$a^-(z) = -\oint \frac{a(w)}{1-z/w}\frac{dw}{w} = -\sum_{n\leqslant 0} a_n z^{-n}.$$

One can see, that

$$P(f(z)) = f^+(z), \qquad P^-(f(z)) = -f^-(z).$$

Let us introduce notation that is used in the proof below. Let M_c be a square matrix of order $n-1$, and F, S_c and V_c be $(n-1)$-dimensional vectors as follows:

$$\mathrm{F} = \big(P(f(z_1)), \dots, P(f(z_{n-1}))\big)\,, \tag{6.6}$$

$$\mathrm{S}_c = \big(P(s(cz_1)), \dots, P(s(cz_{n-1}))\big)\,, \tag{6.7}$$

$$\mathrm{V}_c = \left(\frac{1}{1-c^{-1}z_n/z_1}, \dots, \frac{1}{1-c^{-1}z_n/z_{n-1}}\right), \tag{6.8}$$

$$\mathrm{M}_c = \begin{pmatrix} \dfrac{1}{1-c^{-1}z_1/z_1} & \dots & \dfrac{1}{1-c^{-1}z_1/z_{n-1}} \\ \vdots & \ddots & \vdots \\ \dfrac{1}{1-c^{-1}z_{n-1}/z_1} & \dots & \dfrac{1}{1-c^{-1}z_{n-1}/z_{n-1}} \end{pmatrix}, \tag{6.9}$$

with $c=\pm q^k$. Next, we introduce the following rational functions of x:

$$\beta(x) = \frac{(1-x)(q^3+x)}{(1-q^2x)(q+x)}, \tag{6.10}$$

$$\gamma(x) = \frac{(q^2-x)(q^3+x)(1+qx)}{(1-q^2x)(1+q^3x)(q+x)}, \tag{6.11}$$

Let also

$$\alpha_c(w/z) = \operatorname*{Res}_{u=cz} \frac{\alpha(z/u)}{u-w}, \qquad \beta_c(w/z) = \operatorname*{Res}_{u=cz} \frac{\beta(z/u)}{u-w}, \qquad \gamma_c(w/z) = \operatorname*{Res}_{u=cz} \frac{\gamma(z/u)}{u-w}, \tag{6.12}$$

for $c=\pm q^k$. One can see that $\alpha_c(x) = A(c-x)^{-1}$, $\beta_c(x) = B(c-x)^{-1}$, $\gamma_c(x) = C(c-x)^{-1}$, for some constants A, B, C.

Now, by the definition of projection P we have $f(z) = f^+(z) - f^-(z)$. Using projection properties we obtain

$$\begin{aligned} &P\big(s(z_1)\dots s(z_p)f(z_{p+1})\dots f(z_n)\big) = \\ &\qquad = P\big(s(z_1)\dots s(z_p)f(z_{p+1})\dots f(z_{n-1})\big)P\big(f(z_n)\big) - \\ &\qquad\qquad - P\big(s(z_1)\dots s(z_p)f(z_{p+1})\dots f(z_{n-1})f^-(z_n)\big). \end{aligned}$$

To calculate the second summand on the right hand side we need to move current $f^-(z_n)$ in front of the product $s(z_1)\dots s(z_p)f(z_{p+1})\dots f(z_{n-1})$. This can be done by inductive application of:

Proposition 6.3 *The following commutation relations hold:*

a) let $\alpha(x)$ and $\alpha_c(w/z)$ be as in (2.10) and (6.12); then

$$\begin{aligned} f(z)f^-(w) &= \alpha(z/w)f^-(w)f(z) + \\ &\quad + \alpha_{q^2}(w/z)f^+(q^2z)f(z) + \alpha_{-q^{-1}}(w/z)f^+(-q^{-1}z)f(z); \end{aligned} \tag{6.13}$$

b) let $\beta(x)$ and $\beta_c(w/z)$ be as in (6.10) and (6.12); then

$$\begin{aligned} s(z)f^-(w) &= \beta(z/w)f^-(w)s(z) + \\ &\quad + \beta_{q^2}(w/z)f^-(q^2z)s(z) + \beta_{-q^{-1}}(w/z)f^+(-q^{-1}z)s(z). \end{aligned} \tag{6.14}$$

Informally, the proposition above states that on pushing $f^-(z_n)$ to the left in the product $s(z_1)\dots s(z_p)f(z_{p+1})\dots f(z_{n-1})f^-(z_n)$ some rational functions of z_n arise, which after decomposition into partial fractions appear to have only simple poles on the hyperplanes $z_n = q^2 z_i$ and $z_n = -q^{-1}z_i$. Thus the following formula holds:

$$P\big(s(z_1)\dots s(z_p)f(z_{p+1})\dots f(z_n)\big) = \\ = P\big(s(z_1)\dots s(z_p)f(z_{p+1})\dots f(z_{n-1})\big)P(f(z_n)) - \mathrm{V}_{q^2}\mathrm{X}^{\mathrm{T}} - \mathrm{V}_{-q^{-1}}\mathrm{Y}^{\mathrm{T}}, \quad (6.15)$$

where X and Y are $(n-1)$-dimensional vectors with coordinates

$$X_i = q^{-2}z_i^{-1} \operatorname*{Res}_{z_n=q^2z_i} P\big(s(z_1)\dots s(z_p)f(z_{p+1})\dots f(z_n)\big),$$
$$Y_i = -qz_i^{-1} \operatorname*{Res}_{z_n=-q^{-1}z_i} P\big(s(z_1)\dots s(z_p)f(z_{p+1})\dots f(z_n)\big),$$

and V_c is given by (6.8). Notice that X_i and Y_i do not depend on z_n. Since we have $n-1$ vanishing conditions, $s(z_1)\cdots s(z_p)f(z_{p+1})\cdots f(z_n)$ has simple zeros on the hyperplanes $z_n = z_j$ for $j = 1,\dots,n-1$; we can compose a system of linear equations

$$\mathrm{M}_{q^2}\mathrm{X}^{\mathrm{T}} + \mathrm{M}_{-q^{-1}}\mathrm{Y}^{\mathrm{T}} = P\big(s(z_1)\dots s(z_p)f(z_{p+1})\dots f(z_{n-1})\big)\mathrm{F}^{\mathrm{T}},$$

with F and M_q defined by (6.6), (6.9). Expressing X from the system above and substituting it into equation (6.15), we get

$$P\big(s(z_1)\dots s(z_p)f(z_{p+1})\dots f(z_n)\big) = \\ = P\big(s(z_1)\dots s(z_p)f(z_{p+1})\dots f(z_{n-1})\big)\left(P(f(z_n)) - \mathrm{V}_{q^2}\mathrm{M}_{q^2}^{-1}\mathrm{F}^{\mathrm{T}}\right) - \\ - \left(\mathrm{V}_{-q^{-1}} - \mathrm{V}_{q^2}\mathrm{M}_{q^2}^{-1}\mathrm{M}_{-q^{-1}}\right)\mathrm{Y}^{\mathrm{T}}.$$

Proposition 6.4 *We have the following equalities:*

a) let $\mathcal{F}(z_1,\dots,z_{n-1};z_n)$ be as in (5.5); then

$$P(f(z_n)) - \mathrm{V}_{q^2}\mathrm{M}_{q^2}^{-1}\mathrm{F}^{\mathrm{T}} = \mathcal{F}(z_1,\dots,z_{n-1};z_n); \quad (6.16)$$

b) let $\lambda_i(z_1,\dots,z_{n-1};z_n)$ be as in (5.2); then

$$\left(\mathrm{V}_{-q^{-1}} - \mathrm{V}_{q^2}\mathrm{M}_{q^2}^{-1}\mathrm{M}_{-q^{-1}}\right)\mathrm{Y}^{\mathrm{T}} = \sum_{i=1}^{n-1}\lambda_i(z_1,\dots,z_{n-1};z_n)Y_i. \quad (6.17)$$

Applying Proposition 6.4 we arrive at the equation

$$P\big(s(z_1)\dots s(z_p)f(z_{p+1})\dots f(z_n)\big) = \\ = P\big(s(z_1)\dots s(z_p)f(z_{p+1})\dots f(z_{n-1})\big)\mathcal{F}(z_1,\dots,z_{n-1};z_n) - \sum_{i=1}^{n-1}\lambda_i(z_1,\dots,z_{n-1};z_n)Y_i.$$

Now let us notice that $Y_i = 0$ for $i = 1,\dots,p$, since the product $s(z_1)\dots s(z_p)f(z_{p+1})\dots f(z_n)$ does not have poles on the hyperplanes $z_n = -q^{-1}z_i$ for $i = 1,\dots,p$. Taking into account that the action of projection $P(f(z)f(w))$ is well defined on every highest weight $U_q(A_2^{(2)})$-module for any arbitrary z and w, we obtain the following commutation relations:

$$P(f(z)f(w)) = \alpha(z/w)\, P(f(w)f(z)), \tag{6.18}$$
$$P(s(z)f(w)) = \beta(z/w)\, P(f(w)s(z)), \tag{6.19}$$
$$P(s(z)s(w)) = \gamma(z/w)\, P(s(w)s(z)), \tag{6.20}$$

with $\alpha(x)$, $\beta(x)$, $\gamma(x)$ defined by (2.10), (6.10) and (6.11) respectively. Notice, that in the relations above one should take an analytic continuation of the left hand or the right hand side when necessary. Using relation (6.18), we rewrite Y_k as follows:

$$Y_k = \prod_{i=p+1}^{n-1} \alpha(-qz_i/z_k) \times$$
$$\times (-qz_i^{-1}) \operatorname*{Res}_{z_n=-q^{-1}z_i} P\big(s(z_1)\dots s(z_p)f(z_k)f(z_{p+1})\dots f(z_{k-1})f(z_{k+1})\dots f(z_n)\big),$$

where $p < k < n$, and $\alpha(x)$ is defined by (2.10). Finally, using relation (6.19) and mutual commutativity of the projection P and of the operator of taking a residue, we obtain

$$Y_k = \prod_{i=p+1}^{k-1} \alpha(z_i/z_k) \prod_{\substack{i=p+1\\ i\neq k}}^{n-1} \alpha(-qz_i/z_k) \times$$
$$\times P\big(s(z_1)\dots s(z_p)s(z_k)f(z_{p+1})\dots f(z_{k-1})f(z_{k+1})\dots f(z_{n-1})\big).$$

Hence the statement of the lemma holds. □

6.5 Proof of Lemma 6.2

We start with:

Proposition 6.5 *Current $s(z)$ admits the following expansion:*

$$s(z) = P^+(s(z)) + \left(\frac{(q-q^{-1})}{(q-1+q^{-1})} f^-\left(-q^{-1}z\right) f(z)\right)^+ - s^-(z). \tag{6.21}$$

Hence the projection $P\big(s(z_1)\dots s(z_n)\big)$ can be evaluated via pushing $f^-\left(-q^{-1}z_n\right)$ and $s^-(z_n)$ in front of the product $s(z_1)\dots s(z_n)$. We use Proposition 6.3 and:

Proposition 6.6 *The following commutation relation holds:*

$$s(z)s^-(w) = \gamma(z/w)s^-(w)s(z) +$$
$$+ \gamma_{q^2}(w/z)s^+(q^2z)s(z) + \gamma_{-q^3}(w/z)s^+(-q^3z)s(z) + \gamma_{-q^{-1}}(w/z)s^-(-q^{-1}z)s(z), \tag{6.22}$$

where $\gamma(x)$ and $\gamma_c(w/z)$ are given by (6.11) and (6.12).

Then we arrive at the following equation:

$$P\big(s(z_1)\dots s(z_n)\big) = P\big(s(z_1)\dots s(z_{n-1})\big)P(s(z_n)) - \mathrm{V}_{q^2}\mathrm{X}^{\mathrm{T}} - \mathrm{V}_{-q^{-1}}\mathrm{Y}^{\mathrm{T}} - \mathrm{V}_{-q^3}\mathrm{Z}^{\mathrm{T}}, \qquad (6.23)$$

where X, Y and Z are $(n-1)$-dimensional vectors with coordinates

$$\begin{aligned} X_i &= q^{-2}z_i^{-1} \operatorname*{Res}_{z_n=q^2 z_i} P\big(s(z_1)\dots s(z_n)\big), \\ Y_i &= -qz_i^{-1} \operatorname*{Res}_{z_n=-q^{-1} z_i} P\big(s(z_1)\dots s(z_n)\big), \\ Z_i &= -q^{-3}z_i^{-1} \operatorname*{Res}_{z_n=-q^3 z_i} P\big(s(z_1)\dots s(z_n)\big), \end{aligned}$$

and V_q is defined by (6.8). Notice that $Y_i = 0$ for $i = 1,\dots,n-1$ since the product $s(z_1)\dots s(z_n)$ does not have poles on the hyperplanes $z_n = -q^{-1}z_i$, while X_i and Z_i are functions independent of z_n. Regarding $2n-2$ vanishing conditions, namely $s(z_1)\dots s(z_n) = 0$ on the hyperplanes $z_n = z_i$ and $z_n = -qz_i$, we can compose a system of $2n-2$ linear equations with $2n-2$ variables X_i and Z_i for $i = 1,\dots,n-1$. Let

$$\mathrm{M} := \begin{pmatrix} \mathrm{M}_{q^2} & \mathrm{M}_{-q^3} \\ \mathrm{M}_{-q} & \mathrm{M}_{q^2} \end{pmatrix}, \qquad \mathrm{V} := \big(\mathrm{V}_{q^2},\ \mathrm{V}_{-q^3}\big), \qquad \text{and} \qquad \mathrm{S} := \big(\mathrm{S}_1^{\mathrm{T}},\ \mathrm{S}_{-q}^{\mathrm{T}}\big),$$

where M_c, V_c and S_c are defined by (6.9), (6.8) and (6.7) respectively. Thus we can present our system of equations as

$$\mathrm{M}\begin{pmatrix} \mathrm{X}^{\mathrm{T}} \\ \mathrm{Z}^{\mathrm{T}} \end{pmatrix} = P\big(s(z_1)\dots s(z_{n-1})\big)\mathrm{S}^{\mathrm{T}}.$$

Solving the system and substituting X and Z into the equation (6.23) we get

$$P\big(s(z_1)\dots s(z_n)\big) = P\big(s(z_1)\dots s(z_{n-1})\big)\left(P(s(z_n)) - \mathrm{V}\mathrm{M}^{-1}\mathrm{S}^{\mathrm{T}}\right).$$

Now we finish the proof by applying:

Proposition 6.7 *We have the following equality:*

$$P(s(z_n)) - \mathrm{V}\mathrm{M}^{-1}\mathrm{S}^{\mathrm{T}} = \mathcal{S}(z_1,\dots,z_{n-1};z_n), \qquad (6.24)$$

where $\mathcal{S}(z_1,\dots,z_{n-1};z_n)$ is given by (5.6).

□

6.6 Proof of Propositions 6.3 and 6.6

Since the proofs are almost the same we present only the proof of Proposition 6.3 part *a*).
Let us recall relation (2.2):

$$(q^2z-u)(z+qu)f(z)f(u) = (z-q^2u)(qz+u)f(u)f(z).$$

Since product $f(z)f(u)$ has simple poles on the planes $u = q^2z$ and $u = -q^{-1}z$, we derive the following equality of two holomorphic functions

$$\begin{aligned} f(z)f(u) + \frac{1}{1-q^{-2}u/z}t(z) + \frac{1}{1+qu/z}s(z) = \\ = \alpha(z/u)f(u)f(z) - \frac{q^2z}{u}\frac{1}{1-q^2z/u}t(z) + \frac{z}{qu}\frac{1}{1+q^{-1}z/u}s(z). \end{aligned}$$

As a matter of fact, the coefficients of $t(z)$ and $s(z)$ on the left hand side and the right hand side are the same rational functions, but they are represented as power series in u/z and z/u respectively for the reasons mentioned in Section 3.3. Hence we get

$$f(z)f(u) = \frac{(q^2 - z/u)(q^{-1} + z/u)}{(1 - q^2 z/u)(1 + q^{-1} z/u)} f(u)f(z) - \delta(q^{-2}u/z)t(z) - \delta(-qu/z)s(z).$$

Under the action of $-\oint \frac{1}{1-w/u}\frac{du}{u}$ the previous equality turns into

$$f(z)f^-(w) = -\oint \alpha(z/u)\frac{1}{1-w/u} f(u)f(z)\frac{du}{u} + \frac{1}{1-q^{-2}w/z}t(z) + \frac{1}{1+qw/z}s(z).$$

Since the integral above can be rewritten as

$$\alpha(z/w)f^-(w)f(z) + \alpha(w/z)f^-(q^2 z)f(z) + \beta(w/z)f^-(-q^{-1}z)f(z),$$

using relations (4.3), (4.4), we derive the statement of the proposition.

□

6.7 Proof of Propositions 6.4 and 6.7

To prove part a) of the proposition we only need to find out that $\mathrm{V}_q\mathrm{M}_q^{-1}$ is an $(n-1)$-dimensional vector W_q with

$$W_q^{(k)} = \prod_{\substack{i=1\\ i\neq k}}^{n-1} \frac{z_n - z_i}{z_k - z_i} \prod_{i=1}^{n-1} \frac{z_k - qz_i}{z_n - qz_i}$$

for $k = 1, \ldots, n-1$. Regarding $W_q^{(k)}$ as a rational function over z_n, we observe that it has simple poles only on the planes $z_n = qz_i$ for $i = 1, \ldots, n-1$. We also know that $W_q^{(k)}$ is a rational function of degree -1 in z_n. Let us notice that V_q is equal to the i-th row of matrix M_q if $z_n = z_i$. Therefore $W_q^{(k)}$ equals 1 on the hyperplane $z_n = z_k$ and equals 0 on the hyperplanes $z_n = z_i$ for $i = 1, \ldots, n-1,\ i \neq k$. These conditions totally determine $W_q^{(k)}$. Now, replacing q by q^2 we derive statement a) of the proposition.

To prove the statement of part b) we need to show that

$$\lambda_k(z_n; z_1, \ldots, z_{n-1}) = \frac{1}{1 + qz_n/z_k} - \mathrm{V}_{q^2}\mathrm{M}_{q^2}^{-1}\mathrm{M}_{-q^{-1}}^{(k)}, \qquad (6.25)$$

where $\mathrm{M}_{-q^{-1}}^{(k)}$ stands for the k-th column of matrix $\mathrm{M}_{-q^{-1}}$. Regarded as a rational function of z_n, $\lambda_i(z_1, \ldots, z_{n-1}; z_n)$ has simple poles only on the hyperplanes $z_n = q^2 z_i$ for $i = 1, \ldots, n-1$ and $z_n = -q^{-1}z_k$. Since vector V_q is equal to the i-th row of the matrix M_q for $z_n = z_i$, $\lambda_k(z_1, \ldots, z_{n-1}; z_n)$ has simple zeros on the hyperplanes $z_n = z_i,\ i = 1, \ldots, n-1$. Considering the fact that $\lambda_k(z_1, \ldots, z_{n-1}; z_n)$ is a rational function of degree -1 in z_n, we can present it in the following form:

$$\lambda_i(z_1, \ldots, z_{n-1}; z_n) = \frac{1}{qz_n + z_i} \prod_{j=1}^{n-1} \left(\frac{z_n - z_j}{z_n - q^2 z_j} \right) \cdot C(z_1, \ldots, z_{n-1}).$$

Equality (6.25) implies that

$$\lim_{z_n \to -q^{-1}z_k} \big((1 + qz_n/z_k)\lambda_k(z_1, \ldots, z_{n-1}; z_n) \big) = 1.$$

We finish the proof by obtaining $C(z_1,\dots,z_{n-1})$ from the latter condition.

The proof of Proposition 6.7 is an easy modification of the proof of Proposition 6.4 part *a*). □

6.8 Proof of Proposition 6.5

The statement of the proposition follows from equality (6.1), projection properties (3.2) and the statement *b)* of Theorem 1. □

7 Examples

In this section we work out particular cases of the formula (5.12) for $n=2,3,4$. Recall that all rational functions are considered as formal power series converging in the region $|z_1|\gg\cdots\gg|z_n|$. First of all let us recall that

$$P(f(z))=f^+(z)=\sum_{n>0}f_nz^{-n},$$

$$P(s(z))=-\frac{1}{q+q^{-2}}\sum_{n>0}\left(qf_nf_0-f_0f_n+f_1f_{n-1}-q^{-1}f_{n-1}f_1\right)z^{-n}.$$

Now, for $n=2$ we have

$$P(f(z_1)f(z_2))=\mathcal{F}^1_{\{\varnothing\},\{\varnothing\}}(z_1,z_2)\mathcal{F}^2_{\{\varnothing\},\{\varnothing\}}(z_1,z_2)+\tau^1_{\{1\},\{2\}}(z_1,z_2)\mathcal{S}(z_1).$$

Since $\mathcal{S}(z)=P(s(z))$ and $\mathcal{F}^k_{\{\varnothing\},\{\varnothing\}}(z_1,\dots,z_n)=\mathcal{F}(z_1,\dots,z_{n-1};z_n)$, we get

$$\begin{aligned}\mathcal{S}(z_1)&=P(s(z_1)),\\ \mathcal{F}^1_{\{\varnothing\},\{\varnothing\}}(z_1,z_2)&=f^+(z_1),\\ \mathcal{F}^2_{\{\varnothing\},\{\varnothing\}}(z_1,z_2)&=f^+(z_2)-\frac{q^2-1}{q^2-z_2/z_1}f^+(z_1).\end{aligned}$$

Here

$$\tau^1_{\{1\},\{2\}}(z_1,z_2)=-\frac{(1+q^3)(1-z_2/z_1)}{(1+q)(q^2-z_2/z_1)(1+qz_2/z_1)}.$$

For $n=3$, formula (5.12) turns into

$$\begin{aligned}P(f(z_1)f(z_2)f(z_3))=&\ \mathcal{F}^1_{\varnothing,\varnothing}(z_1,z_2,z_3)\mathcal{F}^2_{\varnothing,\varnothing}(z_1,z_2,z_3)\mathcal{F}^3_{\varnothing,\varnothing}(z_1,z_2,z_3)+\\ &+\tau^1_{\{1\},\{2\}}(z_1,z_2,z_3)\mathcal{S}(z_1)\mathcal{F}^3_{\{1\},\{2\}}(z_1,z_2,z_3)+\\ &+\tau^1_{\{1\},\{3\}}(z_1,z_2,z_3)\mathcal{S}(z_1)\mathcal{F}^2_{\{1\},\{3\}}(z_1,z_2,z_3)+\\ &+\tau^1_{\{2\},\{3\}}(z_1,z_2,z_3)\mathcal{S}(z_2)\mathcal{F}^1_{\{2\},\{3\}}(z_1,z_2,z_3).\end{aligned}$$

Here again

$$\mathcal{S}(z_i)=P(s(z_i)),$$

and $\mathcal{F}^k_{\{\varnothing\},\{\varnothing\}}(z_1,\dots,z_n)=\mathcal{F}(z_1,\dots,z_{n-1};z_n)$. Therefore we obtain

$$\begin{aligned}
\mathcal{F}^1_{\varnothing,\varnothing}(z_1,z_2,z_3) &= f^+(z_1),\\
\mathcal{F}^2_{\varnothing,\varnothing}(z_1,z_2,z_3) &= f^+(z_2)-\frac{q^2-1}{q^2-z_2/z_1}f^+(z_1),\\
\mathcal{F}^3_{\varnothing,\varnothing}(z_1,z_2,z_3) &= f^+(z_3)-\frac{(1-z_3/z_2)(q^2-1)(1-q^2z_2/z_1)}{(1-z_2/z_1)(q^2-z_3/z_1)(q^2-z_3/z_2)}f^+(z_1)-\\
&\quad -\frac{(1-z_3/z_1)(q^2-z_2/z_1)(q^2-1)}{(1-z_2/z_1)(q^2-z_3/z_1)(q^2-z_3/z_2)}f^+(z_2).
\end{aligned}$$

One can also see that

$$\begin{aligned}
\mathcal{F}^3_{\{1\},\{2\}}(z_1,z_2,z_3) &= \mathcal{F}^3_{\varnothing,\varnothing}(z_1,z_2,z_3),\\
\mathcal{F}^2_{\{1\},\{3\}}(z_1,z_2,z_3) &= \mathcal{F}^2_{\varnothing,\varnothing}(z_1,z_2,z_3),
\end{aligned}$$

and

$$\mathcal{F}^1_{\{2\},\{3\}}(z_1,z_2,z_3)=f^+(z_1)-\frac{z_2}{z_1}\frac{1-q^2}{1-q^2z_2/z_1}f^+(z_2).$$

The following expressions are obtained from corresponding rational functions by expansion in power series in the domain $|z_1|\gg|z_2|\gg|z_3|$:

$$\begin{aligned}
\tau^1_{\{1\},\{2\}}(z_1,z_2,z_3) &= -\frac{(1+q^3)(1-z_2/z_1)}{(1+q)(q^2-z_2/z_1)(1+qz_2/z_1)},\\
\tau^1_{\{1\},\{3\}}(z_1,z_2,z_3) &= -\frac{(1+q^3)(1-z_3/z_1)(1-z_3/z_2)}{(1+q)(1+qz_3/z_1)(q^2-z_3/z_1)(q^2-z_3/z_2)}\times\\
&\quad\times\frac{(q^2-z_2/z_1)(1+q^3z_2/z_1)}{(1-z_2/z_1)(q^3+z_2/z_1)},\\
\tau^1_{\{2\},\{3\}}(z_1,z_2,z_3) &= -\frac{(1+q^3)(1-z_3/z_1)(1-z_3/z_2)}{(1+q)(1+qz_3/z_1)(q^2-z_3/z_1)(q^2-z_3/z_2)}\times\\
&\quad\times\frac{(1-q^2z_2/z_1)(q+z_2/z_1)(1+q^3z_2/z_1)}{(1-z_2/z_1)(1+qz_2/z_1)(q^3+z_2/z_1)}.
\end{aligned}$$

Finally, we obtain formula (5.12) in the case $n=4$:

$$\begin{aligned}
&P(f(z_1)f(z_2)f(z_3)f(z_4))=\\
&\quad=\mathcal{F}^1_{\varnothing,\varnothing}(z_1,z_2,z_3,z_4)\mathcal{F}^2_{\varnothing,\varnothing}(z_1,z_2,z_3,z_4)\mathcal{F}^3_{\varnothing,\varnothing}(z_1,z_2,z_3,z_4)\mathcal{F}^4_{\varnothing,\varnothing}(z_1,z_2,z_3,z_4)+\\
&\quad+\sum_{a,b,i,j}\tau^1_{\{i\},\{j\}}(z_1,z_2,z_3,z_4)\mathcal{S}(z_i)\mathcal{F}^a_{\{i\},\{j\}}(z_1,z_2,z_3,z_4)\mathcal{F}^b_{\{i\},\{j\}}(z_1,z_2,z_3,z_4)+\\
&\quad+\tau^1_{\{1,2\},\{4,3\}}(z_1,z_2,z_3,z_4)\tau^2_{\{1,2\},\{4,3\}}(z_1,z_2,z_3,z_4)\mathcal{S}(z_1)\mathcal{S}(z_1;z_2)+\\
&\quad+\tau^1_{\{2,1\},\{4,3\}}(z_1,z_2,z_3,z_4)\tau^2_{\{2,1\},\{4,3\}}(z_1,z_2,z_3,z_4)\mathcal{S}(z_2)\mathcal{S}(z_2;z_1)+\\
&\quad+\tau^1_{\{3,1\},\{4,2\}}(z_1,z_2,z_3,z_4)\tau^2_{\{3,1\},\{4,2\}}(z_1,z_2,z_3,z_4)\mathcal{S}(z_3)\mathcal{S}(z_3;z_1),
\end{aligned}$$

where the sum is taken over all permutations $\{a, b, i, j\}$ of an ordered set $\{1, 2, 3, 4\}$ such that $i < j$ and $a < b$. In the latter example

$$
\begin{aligned}
\mathcal{S}(z) &= P(s(z)),\\
\mathcal{S}(z_1; z_2) &= P(s(z_2)) - \frac{(q-1)(q^3+1)(q+z_2/z_1)}{(q^2-z_2/z_1)(q^3+z_2/z_1)} P(s(z_1)) - \\
&\quad - \frac{q^2(q^2-1)(1-z_2/z_1)}{(q^2-z_2/z_1)(q^3+z_2/z_1)} P(s(-qz_1)),\\
\mathcal{F}^4_{\{i\},\{j\}}(z_1, z_2, z_3, z_4) &= \mathcal{F}^4_{\varnothing,\varnothing}(z_1, z_2, z_3, z_4) = \mathcal{F}(z_1, z_2, z_3; z_4),\\
\mathcal{F}^3_{\{i\},\{j\}}(z_1, z_2, z_3, z_4) &= \mathcal{F}^3_{\varnothing,\varnothing}(z_1, z_2, z_3, z_4) = \mathcal{F}(z_1, z_2; z_3),\\
\mathcal{F}^2_{\{1\},\{j\}}(z_1, z_2, z_3, z_4) &= \mathcal{F}^2_{\varnothing,\varnothing}(z_1, z_2, z_3, z_4) = \mathcal{F}(z_1; z_2),\\
\mathcal{F}^1_{\{2\},\{j\}}(z_1, z_2, z_3, z_4) &= \mathcal{F}(z_2; z_1),\\
\mathcal{F}^2_{\{3\},\{4\}} &= \mathcal{F}(z_3, z_1; z_2),\\
\mathcal{F}^1_{\{3\},\{4\}} &= \mathcal{F}(z_3; z_1),
\end{aligned}
$$

for all $1 \leqslant i < j \leqslant 4$, and

$$
\begin{aligned}
\tau^1_{\{1,2\},\{4,3\}}(z_1, z_2, z_3, z_4) &= -\lambda_1(z_1, z_2, z_3; z_4)\alpha(-qz_2/z_1)\alpha(-qz_3/z_1),\\
\tau^2_{\{1,2\},\{4,3\}}(z_1, z_2, z_3, z_4) &= -\lambda_2(z_1, z_2; z_3),\\
\tau^1_{\{2,1\},\{4,3\}}(z_1, z_2, z_3, z_4) &= -\lambda_2(z_1, z_2, z_3; z_4)\alpha(z_1/z_2)\alpha(-qz_1/z_2)\alpha(-qz_3/z_2),\\
\tau^2_{\{2,1\},\{4,3\}}(z_1, z_2, z_3, z_4) &= -\lambda_1(z_2, z_1; z_3),\\
\tau^1_{\{3,1\},\{4,2\}}(z_1, z_2, z_3, z_4) &= -\lambda_3(z_1, z_2, z_3; z_4)\alpha(z_1/z_3)\alpha(z_2/z_3)\alpha(-qz_1/z_3)\alpha(-qz_2/z_3),\\
\tau^2_{\{3,1\},\{4,2\}}(z_1, z_2, z_3, z_4) &= -\lambda_1(z_3, z_1; z_2),
\end{aligned}
$$

where $\lambda_k(z_1, \ldots, z_{n-1}; z_n)$ and $\alpha(x)$ are given by (5.2) and (2.10) respectively.

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