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Social threshold aggregations

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Abstract A problem of axiomatic construction of a social decision function is studied for the case when individual opinions of agents are given as m-graded preferences with arbitrary integer $m \geq 3$. It is shown that the only rule satisfying the introduced axioms of Pairwise Compensation, Pareto Domination and Noncompensatory Threshold and Contraction is the threshold rule.

1 Introduction

The aim of this article is to investigate the following problem of construction of a social decision function. Given a set of n agents, each agent evaluates alternatives from a finite set X using complete and transitive preferences (rankings), and we look for a complete and transitive social preference over the alternatives. This kind of aggregation has been considered in many publications, beginning with the seminal work by Arrow (1963). In order to solve the problem, two ways have been proposed. Arrow's

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kind of axiomatics can be described as the local aggregation, cf. Aleskerov (1999); in other words, the aggregation is done on the basis of pairwise comparisons of alternatives. Another way is to use certain non-local procedures, e.g. positional rules, for which only a few works with very well-constructed axiomatics exist, cf. Austen-Smith and Banks (1999), May (1952), Moulin (1988), Smith (1973) and Young (1974a,b, 1975).

One of the non-local rules is the Borda voting rule (Young (1974b)). An application of Borda's rule is often not adequate, since any summation of ranks has a 'compensatory nature': a low evaluation of some alternative by an agent can be compensated by high evaluations of the other agents. Thus, if we would like to take carefully into account low evaluations of alternatives when the quality or perfectness of alternatives is important, the Borda rule or its counterparts cannot be applied.

Let us consider two examples (see also Sect. 3).

Example 1 [1] Suppose that a committee of four members 1, 2, 3 and 4 evaluates three candidates x, y and z to elect for a position. The commitee's evaluations of candidates are given by the following linear preferences:

1	2	3	4
x	x	z	z
y	y	y	y
z	z	x	x

The summation of ranks of the candidates gives the same number of scores 8 for every candidate, and it is impossible to make a choice. However, very often the compromise choice is the candidate y. Such decision is suggested, for instance, by the approval voting procedure, cf. Brams and Fishburn (2002). In what follows other rules describing this decision will be proposed.

Example 2 It is a common practice for scientific journals to accept or reject manuscripts submitted for publication on the basis of reports of two referees. If at least one referee evaluates the manuscript as 'bad' in a certain sense, the manuscript is rejected. The manuscript is usually accepted if the two referees provide 'positive' opinions. Clearly, this kind of a situation is of noncompensatory nature, and so, the question is: what rule(s) describe(s) the journal's choice to accept a manuscipt?

In a recent series of three articles by Aleskerov and Yakuba (2003, 2007) and Aleskerov et al. (2007), an axiomatic construction of the new aggregation procedure, called the *threshold rule*, has been presented for three-graded rankings, i.e. when the evaluations of alternatives are made by grades 1, 2 and 3 meaning 'bad', 'average' and 'good', respectively. The axioms used are Pairwise Compensation, Pareto Domination, Noncompensatory Threshold and Contraction.

The Pairwise Compensation axiom means that if all agents, but two, evaluate two alternatives equally, and the two agents put 'mutually inverse' grades, then the two alternatives have the same rank in the social decision (which may also be interpreted as 'anonymity of grades').

[1] The idea of this example was proposed by Professor P. Pattanaik.

The Pareto Domination axiom states that if the grades of all agents for one alternative are not less than for the second alternative and the grade of at least one agent for the first alternative is strictly greater than that of the second one, then in the social ranking the first alternative has a higher rank than the second alternative.

The Noncompensatory Threshold axiom reveals the main idea of the threshold aggregation: if at least one agent evaluates an alternative as 'bad', then, no matter how many 'good' grades it admits, in the social ranking this alternative is ranked lower than any alternative evaluated as 'average' by all agents.

In this context, the Contraction means that if for two alternatives the evaluations of some agent are equal, then the agent may be 'excluded' from the consideration when the social ranking is constructed, and the social decision is achieved by remaining agents' evaluations.

It was shown by Aleskerov and Yakuba (2003, 2007) and Aleskerov et al. (2007) that the threshold rule is the only rule satisfying the above axioms. In the context of three-graded rankings, the threshold rule aggregates individual preferences in the following way: if the number of 'bad' evaluations of the first alternative is greater than that of the second one, then the first alternative has lower rank in the social ranking, and if the numbers of 'bads' for both alternatives are equal and the number of 'average' evaluations of the first alternative is greater than that of the second alternative, then the second alternative is socially more preferable.

In this article, we extend the notion of the threshold rule to the case when the agents' evaluations are represented by the m-valued grades with an arbitrary integer $m \geq 3$ and show that the threshold rule is the only rule, which satisfies the abovementioned appropriately interpreted axioms. In this model, low evaluations of some agents are of main concern: they cannot be compensated by high grades of the other agents. This concerns the situation when the quality or perfectness of alternatives is of great value and interest. On the other hand, an aggregation procedure can be made taking carefully into account high grades of agents: this is the case when we are interested in at least one good feature of alternatives. It is exactly the dual model, and it has all advantages of the dual model including the axiomatic construction of a social decision function.

Yet, one more remark ought to be made concerning an interpretation of the Noncompensatory property. Under this property, any agent giving a low grade to an alternative puts it down in the social decision as compared to an alternative with average grades. Thus, marginal opinions may strongly influence the social decision.

The main results of this article have been presented in Aleskerov and Chistyakov (2008); Aleskerov et al. (2010) and part of them is published without proofs in Sections 1–3 from Chistyakov and Kalyagin (2008).

The article is organized as follows. In Sect. 2, we present necessary definitions and the main result, Theorem 1. In Sect. 3, we compare the threshold rule, the simple majority rule and Borda's rule and show that they produce in general different social rankings on the same individual profile. In Sect. 4, we show that the equivalence classes of the weak order P, generated by the threshold rule, and the indifference classes generated by P coincide and establish the key properties of monotone representatives of the indifference classes which play a crucial role in the proof of the main result. In Sect. 5, we develop the dual threshold aggregation axiomatics. Section 6 contains the proofs of all results including Theorem 1.

2 The main result

Let X be a finite set of alternatives of cardinality $|X| \geq 2$, $N = \{1, 2, \ldots, n\}$ be a set of $n \geq 2$ agents and $M = \{1, 2, \ldots, m\}$ be a set of ordered grades $1 < 2 < \cdots < m$ with $m \geq 3$. An evaluation procedure for alternatives from X is a map of the form $E : X \times N \to M$, which assigns to each alternative $x \in X$ and each agent $i \in N$ a grade $x_i = E(x, i) \in M$. As a result of the evaluation procedure E each alternative $x \in X$ is characterized by a collection of n grades $x_1, \ldots, x_n$, that is

$$X \ni x \longmapsto \widehat{x} = E(x, \cdot) = (x_1, \ldots, x_n) \in M^n,$$

where $M^n = \{(x_1, \ldots, x_n) : x_i \in M \text{for each } i \in N\}$ is the set of all n-dimensional vectors with components from M. In practice the vector-grades $\widehat{x} = (x_1, \ldots, x_n)$ for the alternative x may represent expert grades, questionnaire data, test data, etc.

The set $\widehat{X} = \{\widehat{x} : x \in X\} \subset M^n$ is an individual profile on X. The problem is to rank the elements of X making use of the individual profile $\widehat{X}$. By a *ranking* of X we mean a complete and transitive binary relation on X. Since $\widehat{X} \subset M^n$ and each alternative $x \in X$ is completely characterized by its profile vector $\widehat{x}$, with no loss of generality throughout the paper we assume that $X = \widehat{X} = M^n$, and so,

$$x \in X \quad \text{iff} \quad x = \widehat{x} = (x_1, \ldots, x_n) \in M^n \;\; \text{with} \;\; x_i \in M,$$

where 'iff' means as usual 'if and only if'.

The following notation will be used throughout the article. Given $x, y \in X$, we write $x \succcurlyeq y$ to denote the condition $x_i \geq y_i$ for all $i \in N$, and we write $x \succ y$ to mean that $x \succcurlyeq y$ and there is an $i_0 \in N$ such that $x_{i_0} > y_{i_0}$. Note that the partial order relations $\succcurlyeq$ and $\succ$ on X do not solve the problem of ranking of X, because not all profile vectors from X are comparable using these relations. Also, given $x \in X$ and $j \in M$, we denote by $v_j(x)$ the number of grades j in the vector $x = (x_1, \ldots, x_n)$:

$$v_j(x) = |\{i \in N : x_i = j\}|. \tag{1}$$

Note that $0 \leq v_j(x) \leq n$ for all $x \in X$ and $j \in M$ and

$$\sum_{j=1}^{m} v_j(x) = v_1(x) + v_2(x) + \cdots + v_m(x) = n \quad \text{for all} \quad x \in X. \tag{2}$$

Finally, given $x \in X$, we set

$$V_k(x) = \sum_{j=1}^{k} v_j(x) \quad \text{if} \quad 1 \leq k \leq m \quad \text{and} \quad V_0(x) = 0, \tag{3}$$

so that equality (2) can be simply written as $V_m(x) = n$, $x \in X$.

By a *social decision function* on X we mean a function $\varphi : X \to \mathbb{R}$ satisfying the following properties: given $x, y \in X$, we have: (a) the inequality $\varphi(x) > \varphi(y)$ holds iff the alternative x is socially (strictly) more preferable than the alternative y (in the sense to be made precise below), and (b) $\varphi(x) = \varphi(y)$ iff the alternatives x and y are socially indifferent.

We look for a social decision function $\varphi : X \to \mathbb{R}$, which satisfies the following three axioms (A.1), (A.2) and (A.3).

(A.1) (Pairwise Compensation): if $x, y \in X$ and $v_j(x) = v_j(y)$ for all $1 \le j \le m-1$, then $\varphi(x) = \varphi(y)$.
(A.2) (Pareto Domination): if $x, y \in X$ and $x \succ y$, then $\varphi(x) > \varphi(y)$.
(A.3) (Noncompensatory Threshold and Contraction): for each natural number $3 \le k \le m$ the following condition holds:
(A.3.k) if $x, y \in X$, $v_j(x) = v_j(y)$ for all $1 \le j \le m-k$ (if $k = m$, this condition is omitted), $v_{m-k+1}(x) + 1 = v_{m-k+1}(y) \neq n - V_{m-k}(y)$, $V_{m-k+2}(x) = n$ and $V_{m-k+1}(y) + v_m(y) = n$, then $\varphi(x) > \varphi(y)$.

Recall that the binary relation $\angle = \angle_k$ on the set $\mathbb{R}^k$ of all k-dimensional vectors with real components is said to be the *lexicographic ordering* if, given $u = (u_1, \ldots, u_k)$ and $v = (v_1, \ldots, v_k)$ from $\mathbb{R}^k$, we have: $u \angle v$ in $\mathbb{R}^k$ iff there exists an $1 \le i \le k$ such that $u_j = v_j$ for all $1 \le j \le i-1$ (with no condition if $i = 1$) and $u_i < v_i$. It is well known (e.g. Fishburn (1973)) that $\angle$ is a linear order on $\mathbb{R}^k$; more precisely, $\angle$ is transitive (i.e. if $u \angle v$ and $v \angle w$, then $u \angle w$), the negation of $\angle$ is of the form: $\neg(u \angle v)$ iff $v \angle u$ or $v = u$, and $\angle$ is trichotomous (i.e. either $u = v$, or $u \angle v$, or $v \angle u$).

Setting

$$v(x) = (v_1(x), \ldots, v_{m-1}(x)) \in \{0, 1, \ldots, n\}^{m-1} \quad \text{for} \quad x \in X, \tag{4}$$

the property $v(x) \angle v(y)$ in $\mathbb{R}^{m-1}$ will be called the *threshold rule* for the comparison of alternatives x and y (with respect to the number of low grades). We say that a binary relation P on X is *generated* by the threshold rule if $P = \{(x, y) \in X \times X : v(x) \angle v(y)\}$. In other words, given $x, y \in X$, we have $(x, y) \in P$ iff $v(x) \angle v(y)$, which can be interpreted in the sense that the alternative x is socially (strictly) more preferable than the alternative y.

The main properties of P are straightforward consequences of the properties of the lexicographic ordering: given $x, y, z \in X$, we have:

(P.1) if $(x, y) \in P$ and $(y, z) \in P$, then $(x, z) \in P$ (transitivity of P);
(P.2) $(x, y) \notin P$ is equivalent to $(y, x) \in P$ or $v(y) = v(x)$ (negation of P);
(P.3) either $v(x) = v(y)$, or $(x, y) \in P$, or $(y, x) \in P$ (generalized 'connectedness' of P);
(P.4) $(x, x) \notin P$ (irreflexivity of P);
(P.5) if $(x, y) \notin P$ and $(y, z) \notin P$, then $(x, z) \notin P$ (negative transitivity of P);
(P.6) $(x, y) \notin P$ or $(y, x) \notin P$ (completeness of $P^c = X^2 \setminus P$).

A binary relation P satisfying properties (P.1), (P.4) and (P.5) is commonly known as a *weak order* on X. It is also known (cf. Aleskerov (1999)) that any weak order P on X is characterized by the family of its equivalence classes, whose construction is recalled now. Set $X_1' = \pi(X)$ where, given nonempty $A \subset X, \pi(A) = \{x \in A :$

$(y, x) \notin P$ for all $y \in A\}$ is the choice function for P (cf. (Aizerman and Aleskerov, 1995, Section 2.3)). Inductively, if $\ell \geq 2$ and nonempty subsets $X'_1, \ldots, X'_{\ell-1}$ of X such that $\bigcup_{k=1}^{\ell-1} X'_k \neq X$ are already defined, we put $X'_\ell = \pi\left(X \setminus (\bigcup_{k=1}^{\ell-1} X'_k)\right)$. Since X is finite, there exists a unique positive integer $s = s(X)$ such that $X = \bigcup_{\ell=1}^{s} X'_\ell$. Now, setting $X_\ell = X'_{s-\ell+1}$ for $\ell = 1, 2, \ldots, s$, the disjoint collection $\{X_\ell\}_{\ell=1}^{s}$ is said to be the *family of equivalence classes* of the weak order P, and has the following property: given $x, y \in X$, $(x, y) \in P$ iff there exist two integers k and ℓ with $1 \leq k < \ell \leq s$ such that $x \in X_\ell$ and $y \in X_k$. This property shows that the alternative x is more preferable than the alternative y iff x lies in an equivalence class with a greater ordinal number, and so, this defines the canonical (strict) ranking of X. The value $s = s(X)$ for the relation P generated by the threshold rule will be calculated below in Lemma 1(b).

We say that a function $\varphi : X \to \mathbb{R}$ is *coherent* with the family $\{X_\ell\}_{\ell=1}^{s}$ of equivalence classes of the weak order P on X if, given $x, y \in X$, the inequality $\varphi(x) > \varphi(y)$ holds iff there exist $1 \leq k < \ell \leq s$ such that $x \in X_\ell$ and $y \in X_k$.

The main result of this article is the following.

Theorem 1 *A social decision function $\varphi : X \to \mathbb{R}$ satisfies axioms* (A.1), (A.2) *and* (A.3) *iff it is coherent with the family of equivalence classes of the weak order P on X generated by the threshold rule $v(x) \angle v(y)$ in $\mathbb{R}^{m-1}$.*

This theorem will be proved in Sect. 6. A certain interpretation of it is in order. Given a binary relation P on X and a function $\varphi : X \to \mathbb{R}$, if for all $x, y \in X$ we have

$$(x, y) \in P \quad \text{iff} \quad \varphi(x) > \varphi(y), \tag{5}$$

then P is said to be representable by means of φ or, shortly, *φ-representable*, and φ is said to be a *preference function for P*. Taking this into account as well as the definitions preceding Theorem 1, we can reformulate Theorem 1 as follows: *a social decision function on X satisfies the axioms Pairwise Compensation, Pareto Domination, Noncompensatory Threshold and Contraction iff it is a preference function for the binary relation on X generated by the threshold rule.*

3 A comparison with the known rules

In this section, we construct an example, for which the simple majority rule, the Borda voting rule and the threshold rule produce different social decisions.

Let $X = \{x, y, z\}$ be a set of three different alternatives, $N = \{1, \ldots, 13\}$ a set of $n = 13$ voters and $M = \{1, 2, 3\}$ the set of grades (i.e. $m = 3$). Consider the following linear orders of voters from N:

3 voters	4 voters	6 voters	rank
x	x	y	3
y	z	z	2
z	y	x	1

This means that for the first three voters x is the most preferable alternative, y is the next one and z is the less preferable alternative, and likewise for the other voters. The problem of voting is to construct a (linear) binary relation on X corresponding to the social decision of the society N.

(a) According to the *simple majority rule* the pair of alternatives (x, y) is included into the social decision (relation) if the preference of the form 'x is more preferable than y' occurs among the simple majority of voters. In our example we have: for $3 + 4 = 7$ voters x is more preferable than y, for $3 + 6 = 9$ voters y is more preferable than z and for $3 + 4 = 7$ voters x is more preferable than z (and there is no simple majority among the other possibilities). Thus, the social decision is $x \succ y \succ z$, where $\succ$ means 'is preferred to'.

(b) In the Borda voting procedure to each alternative x from X each voter $i \in N$ associates some rank $\rho_i(x)$ in such a way that the more preferable the alternative the higher the rank. In our example for the first voter among the first three voters we have: $\rho_1(x) = 3$, $\rho_1(y) = 2$ and $\rho_1(z) = 1$, and likewise for the remaining voters. Then we set $\rho(x) = \sum_{i\in N} \rho_i(x)$ for all $x \in X$. According to the *Borda voting rule* an alternative x is socially more preferable than an alternative y if $\rho(x) > \rho(y)$. For the example above we have:

$$\begin{aligned}\rho(y) = 3\cdot 2 + 4\cdot 1 + 6\cdot 3 = 28 > \rho(x) = (3+4)\cdot 3 + 6\cdot 1 = 27 > \\ > \rho(z) = 3\cdot 1 + (4+6)\cdot 2 = 23.\end{aligned}$$

Thus, the social decision is $y \succ x \succ z$.

(c) Interpreting the ranks of alternatives from (b) as the grades, for the example above we have (the asterisk denotes the ordered vector grades):

$$v_1(x)=6,\ v_2(x)=0,\ v_3(x)=7,\ \text{or}\ \ x^*=(\underbrace{1,1,1,1,1,1}_{6},\underbrace{3,3,3,3,3,3,3}_{7}),$$

$$v_1(y)=4,\ v_2(y)=3,\ v_3(y)=6,\ \text{or}\ \ y^*=(\underbrace{1,1,1,1}_{4},\underbrace{2,2,2}_{3},\underbrace{3,3,3,3,3,3}_{6}),$$

$$v_1(z)=3,\ v_2(z)=10,\ v_3(z)=0,\ \text{or}\ \ z^*=(\underbrace{1,1,1}_{3},\underbrace{2,2,2,2,2,2,2,2,2,2}_{10}).$$

Since $v_1(z) = 3 < v_1(y) = 4 < v_1(x) = 6$, then $v(z) \angle_2 v(y) \angle_2 v(x)$, and so, according to the *threshold rule* the social decision is $z \succ y \succ x$.

4 Monotone representatives and indifference classes

Since the binary relation P on X generated by the threshold rule is a weak order, the *indifference* relation I is defined as

$$I = \{(x, y) \in X \times X : (x, y) \notin P \text{ and } (y, x) \notin P\}. \quad (6)$$

Clearly, I is an equivalence relation on X (i.e. it is reflexive, symmetric and transitive) and, by virtue of the properties (P.2) and (P.3), we have:

$$I = \{(x, y) \in X \times X : v(x) = v(y)\}. \tag{7}$$

Then the *indifference class* of an alternative $x \in X$ is the set

$$I_x = \{y \in X : (y, x) \in I\} = \{y \in X : v(y) = v(x)\}, \tag{8}$$

and, as usual, given $x, y \in X$, we find: $I_x = I_y$ iff $(x, y) \in I$, $I_x \cap I_y = \varnothing$ iff $(x, y) \notin I$, and $X = \bigcup_{x \in X} I_x$ (disjoint union). We denote by X/I the quotient set $\{I_x : x \in X\}$ of all the indifference classes with respect to I.

In this way the binary relation $R = P \cup I$ is a canonical *ranking* of X: R is transitive ($(x, y) \in R$ and $(y, z) \in R$ imply $(x, z) \in R$) and complete (given $x, y \in X$, $(x, y) \in R$ or $(y, x) \in R$). However, throughout the paper we prefer to deal with the strict preference relation P.

Given $I_x \in X/I$ for some $x \in X$, by virtue of (8), the vector $v(I_x) = v(y) = v(x)$ is well defined for any $y \in I_x$. Then the quotient binary relation P/I given by

$$P/I = \{(I_x, I_y) \in (X/I) \times (X/I) : v(I_x) \angle v(I_y) \text{ in } \mathbb{R}^{m-1}\}$$

is a *linear order* on X/I. In fact, since the transitivity and irreflexivity of P/I are clear, it suffices to verify only the connectedness of P/I, i.e. if $I_x, I_y \in X/I$ and $I_x \neq I_y$, then $(I_x, I_y) \in P/I$ or $(I_y, I_x) \in P/I$. Indeed, $I_x \neq I_y$ implies $I_x \cap I_y = \varnothing$ and $(x, y) \notin I$. Thus, $v(x) \neq v(y)$, which gives $v(I_x) \neq v(I_y)$, and by the completeness of the lexicographic ordering $\angle = \angle_{m-1}$ we obtain $v(I_x) \angle v(I_y)$ or $v(I_y) \angle v(I_x)$.

We note that, by virtue of (4) and (2), the equality $v(y) = v(x)$ in (8) actually means that $v_j(y) = v_j(x)$ for all $j \in M$, that is, the vector y can be obtained from the vector x (and vice versa) by a permutation of its coordinates:

$$I_x = \{y \in X : \exists \text{ a permutation } \sigma \text{ of } N \text{ such that } y = x \circ \sigma\},$$

where the equality $y = x \circ \sigma$ involving the composition $x \circ \sigma$ means as usual that $y_i = x_{\sigma(i)}$ for all $i \in N$.

In order to facilitate the treatment of indifference classes I_x from X/I, in each class I_x we select a 'principal' representative $x^* = (x_1^*, x_2^*, \ldots, x_n^*) \in I_x$, whose coordinates are ordered in ascending order: $x_1^* \leq x_2^* \leq \cdots \leq x_n^*$ or

$$x^* = (\overbrace{\underbrace{1, \ldots, 1}_{v_1(x)}, \underbrace{2, \ldots, 2}_{v_2(x)}, \ldots, \underbrace{m-1, \ldots, m-1}_{v_{m-1}(x)}, \underbrace{m, \ldots, m}_{v_m(x)}}^{n}), \tag{9}$$

where the numbers $v_j(x)$ under the braces denote the lengths of the corresponding underbraced subvectors. The alternative x^*, called the *monotone representative* of the class I_x (or simply of the vector x), is uniquely determined, although it can be obtained from x by different permutations of its coordinates. It is clear from the above that $v_j(x^*) = v_j(x)$ for all $j \in M$, or $v(x^*) = v(x)$, and so, $I_x = I_{x^*}$ for all $x \in X$. We denote by $X^* = \{x^* : x \in X\}$ the subset of X of all monotone representatives and by P^*—the restriction of P to $X^* \times X^*$.

Let us note that, given $x, y \in X$, we have:

$$(x, y) \in P \quad \text{iff} \quad (I_x, I_y) \in P/I \quad \text{iff} \quad (x^*, y^*) \in P^*$$

and

$$(x, y) \in I \quad \text{iff} \quad I_x = I_y \quad \text{iff} \quad x^* = y^*. \tag{10}$$

It follows from (P.1), (P.4) and (P.3) that P^* is a linear order on X^* and that the bijection $b : X/I \to X^*$, defined by $b(I_x) = x^*$ for all $x \in X$, is linear order preserving in the sense that $(I_x, I_y) \in P/I$ iff $(b(I_x), b(I_y)) \in P^*$; in other words, the pairs $(X/I, P/I)$ and (X^*, P^*) are *linear order isomorphic*.

Thus, we can work with the set X^* equipped with the linear order P^* instead of the quotient linear order set $(X/I, P/I)$.

In Lemma 1 we evaluate the number of elements in X^*, i.e. the number of monotone representatives of classes from the quotient set X/I.

Lemma 1 (a) $|X/I| = |X^*| = C^{m-1}_{n+m-1} = C^n_{n+m-1}$, *where* $C^k_n = \frac{n!}{k!(n-k)!}$ *is the usual binomial coefficient and* $|A|$ *denotes the number of elements in the set* A *under consideration.*

(b) $\{X_\ell\}^s_{\ell=1} = X/I$, *i.e. the family of all equivalence classes of the weak order* P *coincides with the quotient set of all the indifference classes with respect to* I*; hence,* $s = s(X) = C^{m-1}_{n+m-1}$.

5 The dual threshold aggregation

If the utmost perfection (quality) of alternatives is of main concern, we can apply the threshold rule to rank the set of alternatives. However, if we are interested in at least one good feature of alternatives, we should employ a different, but related, aggregation procedure, which will be called the *dual threshold aggregation*. Such a dual model for three-graded rankings had already been mentioned by Aleskerov and Yakuba (2007). In this section, we develop an axiomatic theory of the dual threshold aggregation in the general case.

Given an alternative $x \in X = M^n$, we set

$$\overline{v}(x) = (v_m(x), v_{m-1}(x), \ldots, v_2(x)) \in \{0, 1, \ldots, n\}^{m-1}.$$

The property $\overline{v}(y) \angle \overline{v}(x)$ in $\mathbb{R}^{m-1}$ will be called the *dual threshold rule* for the comparison of alternatives $x, y \in X$ (with respect to the number of high grades), and a binary relation on X of the form

$$\overline{P} = \{(x, y) \in X \times X : \overline{v}(y) \angle \overline{v}(x) \text{ in } \mathbb{R}^{m-1}\}$$

is said to be *generated* by the dual threshold rule. In other words, given two alternatives $x, y \in X$, we have $(x, y) \in \overline{P}$ iff $\overline{v}(y) \angle \overline{v}(x)$, and we say that x is (dually) strictly more preferable than y.

We are going to reduce the dual aggregation theory to the aggregation theory developed above. In order to do this, we introduce a permutation r of the set M as follows:

$$r(j) = m - j + 1 \quad \text{for all} \quad j \in \{1, 2, \dots, m\}.$$

Note that r is a bijection between $\{1, 2, \dots, m-1\}$ and $\{m, m-1, \dots, 2\}$, reversing the order of the numbers, and so, its self composition $r^2 = r \circ r$ is the identity on $\{1, 2, \dots, m-1\}$ and on $\{m, m-1, \dots, 2\}$: $r(r(j)) = j$ for all j. Given $x = (x_1, \dots, x_n) \in X = \{1, 2, \dots, m\}^n$, we set

$$\mathbf{r}(x) = (r(x_1), r(x_2), \dots, r(x_n)) = (m - x_1 + 1, m - x_2 + 1, \dots, m - x_n + 1),$$

and note that $\mathbf{r}(\mathbf{r}(x)) = x$, i.e. $\mathbf{r}(x') = x$ iff $x' = \mathbf{r}(x)$.

The following two properties (11) and (12) of $\mathbf{r}$ will be of significance:

$$v_j(\mathbf{r}(x)) = v_{r(j)}(x) \;\text{ for all }\; x \in X \;\text{ and }\; 1 \leq j \leq m. \tag{11}$$

In fact, we have:

$$\begin{aligned} v_j(\mathbf{r}(x)) &= |\{i \in N : r(x_i) = j\}| = |\{i \in N : m - x_i + 1 = j\}| \\ &= |\{i \in N : x_i = m - j + 1\}| = |\{i \in N : x_i = r(j)\}| = v_{r(j)}(x). \end{aligned}$$

It follows that $v_j(x) = v_{r(j)}(\mathbf{r}(x))$ and

$$\overline{v}(x) = v(\mathbf{r}(x)) \quad \text{and} \quad \overline{v}(\mathbf{r}(x)) = v(x) \quad \text{for all} \quad x \in X, \tag{12}$$

because

$$\begin{aligned} \overline{v}(x) &= (v_m(x), v_{m-1}(x), \dots, v_2(x)) = (v_{r(1)}(x), v_{r(2)}(x), \dots, v_{r(m-1)}(x)) \\ &= (v_1(\mathbf{r}(x)), v_2(\mathbf{r}(x)), \dots, v_{m-1}(\mathbf{r}(x))) = v(\mathbf{r}(x)). \end{aligned}$$

Now, given $x, y \in X$, we have:

$$(x, y) \in \overline{P} \;\text{ iff }\; \overline{v}(y) \angle \overline{v}(x) \;\text{ iff }\; v(\mathbf{r}(y)) \angle v(\mathbf{r}(x)) \;\text{ iff }\; (\mathbf{r}(y), \mathbf{r}(x)) \in P \tag{13}$$

or, equivalently, $(x, y) \in P$ iff $(\mathbf{r}(y), \mathbf{r}(x)) \in \overline{P}$.

By virtue of (13), the relation $\overline{P}$ on X satisfies the properties (P.1)–(P.6) (if we replace P in these properties by $\overline{P}$), and so, $\overline{P}$ is a weak order on X. For instance, the negation of $\overline{P}$ is of the form: given $x, y \in X$, $(x, y) \notin \overline{P}$ iff $(y, x) \in \overline{P}$ or $v(y) = v(x)$; in fact, it follows from (13) that

$$\begin{aligned} (x, y) \notin \overline{P} \text{ iff } (\mathbf{r}(y), \mathbf{r}(x)) \notin P \;\text{ iff }\; &[(\mathbf{r}(x), \mathbf{r}(y)) \in P \text{ or } v(\mathbf{r}(x)) = v(\mathbf{r}(y))] \\ \text{iff } &[(y, x) \in \overline{P} \text{ or } \overline{v}(x) = \overline{v}(y)], \end{aligned}$$

and it remains to note that, in view of (2), the condition '$v_j(x) = v_j(y)$ for all $2 \leq j \leq m$' is equivalent to the condition '$v_j(x) = v_j(y)$ for all $1 \leq j \leq m-1$'. This observation also shows that the indifference relation $\overline{I}$ on X generated by $\overline{P}$ coincides with the indifference relation I:

$$\overline{I} = \{(x, y) : (x, y) \notin \overline{P} \text{ and } (y, x) \notin \overline{P}\} = \{(x, y) : v(x) = v(y)\} = I.$$

In order to treat the axiomatics of preference functions for the relation $\overline{P}$, we note that if φ is a preference function for P and ψ is a preference function for $\overline{P}$, then, given $x, y \in X$, we have:

$$\psi(x) > \psi(y) \text{ iff } (x, y) \in \overline{P} \text{ iff } (\mathbf{r}(y), \mathbf{r}(x)) \in P \text{ iff } \varphi(\mathbf{r}(y)) > \varphi(\mathbf{r}(x)) \\ \text{iff } [-\varphi(\mathbf{r}(x)) > -\varphi(\mathbf{r}(y))]. \tag{14}$$

We conclude that φ is a preference function for P iff the function $\overline{\varphi}$, defined by $\overline{\varphi}(x) = -\varphi(\mathbf{r}(x))$ for all $x \in X$, is a preference function for $\overline{P}$, and vice versa: $\overline{\varphi}$ is a preference function for $\overline{P}$ iff the function φ, defined for $x \in X$ by $\varphi(x) = -\overline{\varphi}(\mathbf{r}(x))$, is a preference function for P. It follows from Theorem 1 that a function $\overline{\varphi} : X \to \mathbb{R}$ is a preference function for $\overline{P}$ iff the function $\varphi(x) = -\overline{\varphi}(\mathbf{r}(x))$ satisfies axioms (A.1)–(A.3), and by virtue of (14) with ψ replaced by $\overline{\varphi}$, given $x, y \in X$, we have:

$$\overline{\varphi}(x) > \overline{\varphi}(y) \quad \text{iff} \quad \varphi(x') > \varphi(y'), \quad \text{where } x' = \mathbf{r}(y) \text{ and } y' = \mathbf{r}(x).$$

So, replacing x by $\mathbf{r}(y)$ and y by $\mathbf{r}(x)$ in axioms (A.1)–(A.3) and taking into account equalities (11) and (12), we obtain the following (dual) axioms for function $\overline{\varphi}$. Axioms (A.1) and (A.2) remain the same, because conditions '$\overline{v}(x) = \overline{v}(y)$' and '$v(x) = v(y)$' are equivalent, and if $x \succ y$, then $\mathbf{r}(y) \succ \mathbf{r}(x)$, and so, $\varphi(\mathbf{r}(y)) > \varphi(\mathbf{r}(x))$ implying $\overline{\varphi}(x) > \overline{\varphi}(y)$. The third dual axiom assumes the following form:

($\overline{\text{A}}$.3) (Noncompensatory Dual Threshold and Contraction): for each integer $3 \leq k \leq m$ the following condition holds:

($\overline{\text{A}}$.3.k) if $x, y \in X$, $v_j(x) = v_j(y)$ for all $k+1 \leq j \leq m$ (if $k = m$, this condition is absent), $v_k(y) + 1 = v_k(x) \neq V_k(x)$, $V_{k-2}(y) = 0$ and $V_{k-1}(x) = v_1(x)$, then $\overline{\varphi}(x) > \overline{\varphi}(y)$.

The observations above lead to the following corollary of Theorem 1.

Theorem 2 *A social decision function* $\overline{\varphi} : X \to \mathbb{R}$ *satisfies axioms* (A.1), (A.2) *and* ($\overline{\text{A}}$.3) *iff it is coherent with the family of equivalence classes of the weak order* $\overline{P}$ *on* X *generated by the dual threshold rule* $\overline{v}(y) \angle \overline{v}(x)$ *in* $\mathbb{R}^{m-1}$.

6 Proofs of the results

Proof of Lemma 1. (a) Since the sets X/I and X^* are bijective, we have $|X/I| = |X^*|$. Let us show that

$$|X^*_{n,m}| = C^{m-1}_{n+m-1}, \tag{15}$$

where $X^*_{n,m} = \{(x^*_1, \ldots, x^*_n) \in M^n : x^*_1 \leq x^*_2 \leq \cdots \leq x^*_n\}$ is a more refined notation (just for the sake of this proof) of the set X^*. We apply induction argument on n for each natural m. If $n = 1$, for each natural m there are $m = C_m^{m-1}$ possibilities to choose an element $x^* = x^*_1 \in M = \{1, 2, \ldots, m\}$. If $n = 2$ and $x^* = (x^*_1, x^*_2) \in X^*_{2,m}$, then we have m possibilities to choose $x^*_1 = i \in M$ and, by virtue of the inequality $x^*_1 \leq x^*_2$, we have $m - i + 1$ possibilities to choose $x^*_2 \in \{i, i+1, \ldots, m\}$, and it follows that

$$|X^*_{2,m}| = \sum_{i=1}^{m}(m - i + 1) = \frac{m(m+1)}{2} = C_{m+1}^{m-1} = C_{2+m-1}^{m-1}$$

for each natural m. Now suppose that (15) holds for some natural n and each natural m. Consider an element $x^* = (x^*_1, x^*_2, \ldots, x^*_{n+1}) \in X^*_{n+1,m}$. There are m possibilities to choose the first coordinate $x^*_1 = i \in M$ and, by the induction hypothesis and (15), there are $C_{n+(m-i+1)-1}^{(m-i+1)-1} = C_{n+m-i}^{m-i}$ possibilities to choose the rest n coordinates $(x^*_2, \ldots, x^*_{n+1}) \in \{i, i+1, \ldots, m\}^n$ such that $x^*_2 \leq x^*_3 \leq \cdots \leq x^*_{n+1}$. By the summation by both indices formula, (see, e.g. Graham, Knuth and Patashnik (Graham et al., 1994, Section 5.1)), we obtain

$$|X^*_{n+1,m}| = \sum_{i=1}^{m} C_{n+m-i}^{m-i} = C_{n+m}^{m-1} = C_{(n+1)+m-1}^{m-1}$$

for each natural m, which is to be proved.

(b) First, let us verify that $\{X_\ell\}_{\ell=1}^{s} \subset X/I$. Given $1 \leq \ell \leq s$, we fix an $x \in X_\ell$ and show that $X_\ell = I_x$. In fact, by virtue of the definition of the set $X_\ell = X'_{s-\ell+1}$, for each $y \in X_\ell$ we have $x, y \in X \backslash (X'_1 \cup \cdots \cup X'_{s-\ell})$, where $X'_1 \cup \cdots \cup X'_{s-\ell} = \varnothing$ for $\ell = s$, and

$$(z, x) \notin P \text{ and } (z, y) \notin P \text{ for all } z \in X \backslash (X'_1 \cup \cdots \cup X'_{s-\ell}). \tag{16}$$

Setting $z = y$ and $z = x$ in (16), we get $(y, x) \notin P$ and $(x, y) \notin P$. It follows from (6) and (7) that $v(y) = v(x)$, which according to (8) means that $y \in I_x$ and proves that $X_\ell \subset I_x$. To prove the inverse inclusion, let $y \in I_x$. Since $y \in X$, we have $y \in X_k$ for some $1 \leq k \leq s$. If we suppose that $k \neq \ell$, then the property of the family of equivalence classes of the weak order P characterizing the relation P implies $(x, y) \in P$ or $(y, x) \in P$, and so, by property (P.3) we have $v(y) \neq v(x)$ or, by (8), $y \notin I_x$. This contradiction shows that actually $y \in X_\ell$ for all $y \in I_x$ implying $I_x \subset X_\ell$. Thus, we have proved that $X_\ell = I_x$, and so, $X_\ell \in X/I$ for all $1 \leq \ell \leq s$.
The reverse inclusion $X/I \subset \{X_\ell\}_{\ell=1}^{s}$ follows from the fact that if I_x is in X/I for some $x \in X$, then the representative x of the indifference class I_x lies in X_ℓ for some $1 \leq \ell \leq s$, which by the arguments above implies that $I_x = X_\ell$.
Finally, equality $s = C_{n+m-1}^{m-1}$ follows from equality $s = |X/I|$ and (a). □

The following two lemmas are of fundamental importance for the whole subsequent material. In Lemma 2 we show that the operation of taking the monotone representative $X \ni x \mapsto x^* \in X^*$ preserves the natural partial order relations $\succcurlyeq$ and $\succ$ on X, and in Lemma 3 we show that the relations $x^* \succcurlyeq y^*$ and $x^* \succ y^*$ can be characterized in terms of quantities from (1) and (3).

Lemma 2 *Given $x, y \in X$, we have*:
(a) *if $x \succcurlyeq y$, then $x^* \succcurlyeq y^*$*;
(b) *if $x \succ y$, then $x^* \succ y^*$.*

Proof of Lemma 2. (a) By the assumption, $x_i \geq y_i$ for all $i \in N$. If $x^* = x \circ \sigma$ and $y^* = y \circ \theta$ are monotone representatives of x and y (more precisely, of the classes I_x and I_y), respectively, where σ and θ are some permutations of N, then $x_1^* \leq x_2^* \leq \cdots \leq x_n^*$ and $y_1^* \leq y_2^* \leq \cdots \leq y_n^*$. We have to show that $x_i^* \geq y_i^*$ for all $i \in N$. On the contrary, suppose that $x_k^* < y_k^*$ for some $k \in N$. Then $x_1^* \leq \cdots \leq x_k^* < y_k^* \leq \cdots \leq y_n^*$ or, equivalently, $x_{\sigma(1)} \leq \cdots \leq x_{\sigma(k)} < y_{\theta(k)} \leq \cdots \leq y_{\theta(n)}$. Therefore,

$$x_i < y_\ell \quad \text{for all} \quad i \in \{\sigma(1), \ldots, \sigma(k)\} \quad \text{and} \quad \ell \in \{\theta(k), \ldots, \theta(n)\}. \tag{17}$$

Since σ and θ are permutations (i.e. bijections of N into itself), then

$$\{\sigma(1), \ldots, \sigma(k)\} \cap \{\theta(k), \ldots, \theta(n)\} = \{\sigma(1), \ldots, \sigma(k)\} \setminus \{\theta(1), \ldots, \theta(k-1)\} \tag{18}$$

is a nonempty set, where $\{\theta(1), \ldots, \theta(k-1)\} = \varnothing$ if $k = 1$. Taking an element $i = \ell$ from the intersection of the sets at the left hand side of (18), by virtue of (17) we get $x_\ell < y_\ell$, which contradicts the condition $x_\ell \geq y_\ell$.

(b) Suppose that $x \succ y$. Then $x_i \geq y_i$ for all $i \in N$ and there exists an $i_0 \in N$ such that $x_{i_0} > y_{i_0}$. Since x^* is obtained from x by a permutation of coordinates (and likewise for y^*), we find

$$\sum_{i \in N} x_i^* = \sum_{i \in N} x_i > \sum_{i \in N} y_i = \sum_{i \in N} y_i^*.$$

By (a), we have: $x \succcurlyeq y$ implies $x^* \succcurlyeq y^*$, i.e. $x_i^* \geq y_i^*$ for all $i \in N$. The inequality above shows that $x_k^* > y_k^*$ for some $k \in N$, and so, $x^* \succ y^*$. □

Lemma 3 *Given $x, y \in X$, we have*:
(a) *$x^* \succcurlyeq y^*$ iff $V_k(x) \leq V_k(y)$ for all $1 \leq k \leq m-1$;*
(b) *$x^* \succ y^*$ iff there exists a $1 \leq k \leq m-1$ such that $v_j(x) = v_j(y)$ for all $1 \leq j \leq k-1$ (no condition if $k = 1$), $v_k(x) < v_k(y)$ and $V_p(x) \leq V_p(y)$ for all $k+1 \leq p \leq m-1$ (with no last condition if $k = m-1$).*

Proof of Lemma 3. To start with, several remarks concerning monotone representatives x^* of vectors $x \in X$ are in order. Recall that $v_j(x^*) = v_j(x)$ for all $j \in M$, and (9) and condition $x_1^* \leq x_2^* \leq \cdots \leq x_n^*$ imply that, given $i \in N$ and $k \in M$, we have:

$$x_i^* = k \quad \text{iff} \quad \sum_{j=1}^{k-1} v_j(x) + 1 \leq i \leq \sum_{j=1}^{k} v_j(x), \tag{19}$$

or, more precisely,

$$x_i^* \geq k \quad \text{iff} \quad i \geq V_{k-1}(x) + 1, \quad \text{and} \quad x_i^* \leq k \quad \text{iff} \quad i \leq V_k(x). \tag{20}$$

(a) *Necessity* On the contrary, suppose that $V_k(x) > V_k(y)$ for some $1 \leq k \leq m-1$. If $1 \leq k \leq m-2$, we set $i = V_k(x)$. Then, by virtue of (19), $x_i^* = k$. On the other hand, the inequality $i \geq V_k(y) + 1$ and the assertion on the left in (20) imply $y_i^* \geq k+1$, and so, $x_i^* < y_i^*$. Now, if $k = m-1$, we set $i = V_{m-1}(y)+1$. Then (19) implies $y_i^* = m$, and the inequality $V_{m-1}(x) \geq i$ and the assertion on the right in (20) give $x_i^* \leq m-1$, and so, $x_i^* < y_i^*$. In both cases we get a contradiction with the assumption $x^* \succcurlyeq y^*$.
Sufficiency Let us fix $i \in N$ arbitrarily. Then for some uniquely defined number $1 \leq k \leq m$ we have the inequalities $V_{k-1}(x) + 1 \leq i \leq V_k(x)$, and so, according to (19), $x_i^* = k$. If $k = m$, then $x_i^* = m \geq y_i^*$. Now, if $1 \leq k \leq m-1$, then, by the assumption, $V_k(x) \leq V_k(y)$, and so, $i \leq V_k(y)$. It follows from the right hand side assertion in (20) that $y_i^* \leq k = x_i^*$. Thus, we have shown that $x_i^* \geq y_i^*$ for all $i \in N$, i.e. $x^* \succcurlyeq y^*$.

(b) *Sufficiency* The assumptions on the right hand side of the assertion imply $V_p(x) = V_p(y)$ for all $1 \leq p \leq k-1$ (or no condition if $k = 1$), $V_k(x) < V_k(y)$ and $V_p(x) \leq V_p(y)$ for all $k+1 \leq p \leq m-1$ (or no last condition if $k = m-1$). By item (a) of this lemma, $x^* \succcurlyeq y^*$. The equality $x^* = y^*$ cannot hold for, otherwise, by (10) we would have $v(x) = v(y)$, which contradicts the inequality $v_k(x) < v_k(y)$.
Necessity. Suppose $x^* \succ y^*$. Then $x^* \succcurlyeq y^*$, and by item (a) of this lemma we find that

$$V_p(x) \leq V_p(y) \quad \text{for all} \quad 1 \leq p \leq m-1. \tag{21}$$

However, since $x^* \succ y^*$, then $x^* \neq y^*$, and so, (10) implies $v(x) \neq v(y)$. It follows that the number $k = \min\{1 \leq j \leq m-1 : v_j(x) \neq v_j(y)\}$ is well defined. If $k = 1$, then $v_1(x) \neq v_1(y)$, and at the same time we know from (21) with $p = 1$ that $v_1(x) \leq v_1(y)$. Thus, $v_1(x) < v_1(y)$ and $V_p(x) \leq V_p(y)$ for all $k+1 = 2 \leq p \leq m-1$, as asserted in this case. If $2 \leq k \leq m-2$, then by the definition of k we find $v_j(x) = v_j(y)$ for all $1 \leq j \leq k-1$ and $v_k(x) \neq v_k(y)$. Setting $p = k$ in (21), we get

$$\sum_{j=1}^{k-1} v_j(x) + v_k(x) \leq \sum_{j=1}^{k-1} v_j(y) + v_k(y),$$

and so, $v_k(x) \leq v_k(y)$. Therefore, $v_k(x) < v_k(y)$ and $V_p(x) \leq V_p(y)$ for all $k+1 \leq p \leq m-1$, as desired. Finally, if $k = m-1$, then $v_j(x) = v_j(y)$ for all $1 \leq j \leq m-2 = k-1$ and $v_{m-1}(x) \neq v_{m-1}(y)$. At the same time (21) with $p = m-1$ implies $v_{m-1}(x) \leq v_{m-1}(y)$, and so, $v_{m-1}(x) < v_{m-1}(y)$. This argument completes the proof of Lemma 3. □

The interpretation following Theorem 1 shows that in the proof of this theorem we can argue in terms of the φ-representability (5) of the relation P generated by the threshold rule.

In order to prove Theorem 1, we need two more lemmas.

Lemma 4 *The relation P on X generated by the threshold rule is representable by means of a function $\varphi : X \to \mathbb{R}$ iff the function φ satisfies the following m conditions: given $x, y \in X$,*

(a$_1$) *if $v_j(x) = v_j(y)$ for all $1 \leq j \leq m-1$, then $\varphi(x) = \varphi(y)$;*
(a$_2$) *if $v_j(x) = v_j(y)$ for all $1 \leq j \leq m-2$ and $v_{m-1}(x) < v_{m-1}(y)$, then $\varphi(x) > \varphi(y)$;*

and for each integer $3 \leq k \leq m$ also the condition:

(a$_k$) *if $v_j(x) = v_j(y)$ for all $1 \leq j \leq m-k$ (no condition if $k = m$), $v_{m-k+1}(x) < v_{m-k+1}(y)$ and $V_{m-k+2}(x) = n$, then $\varphi(x) > \varphi(y)$.*

Lemma 4 in its turn relies on the following.

Lemma 5 *If the function $\varphi : X \to \mathbb{R}$ satisfies conditions* (a$_1$)–(a$_m$) *of Lemma 4, then for each $3 \leq k \leq m$ it also satisfies the following condition:*

(b$_k$) *given $x, y \in X$, if $v_j(x) = v_j(y)$ for all $1 \leq j \leq m-k$ (with no condition if $k = m$) and $v_{m-k+1}(x) < v_{m-k+1}(y)$, then $\varphi(x) > \varphi(y)$,*

as well as one more condition:

(c) *given $x, y \in X$, if $\varphi(x) = \varphi(y)$, then $v_j(x) = v_j(y)$ for all $1 \leq j \leq m-1$.*

Proof of Lemma 5 *Step 1* We start with establishing condition (b$_3$). In order to do this, suppose that $v_j(x) = v_j(y)$ for all $1 \leq j \leq m-3$ (no condition if $m = 3$) and $v_{m-2}(x) < v_{m-2}(y)$. It follows from (2), where x is replaced by y, that

$$v_1(x)+\cdots+v_{m-3}(x)+v_{m-2}(x) < v_1(y)+\cdots+v_{m-3}(y)+v_{m-2}(y) \leq n.$$

Consider an auxiliary vector $z \in X$ such that

$$v_j(z) = v_j(x) = v_j(y) \text{ for all } 1 \leq j \leq m-3 \text{ (no condition if } m = 3),$$
$$v_{m-2}(z) = v_{m-2}(x) \quad \text{and} \quad v_{m-1}(z) = n - V_{m-2}(x).$$

Let us compare the values $\varphi(x)$ and $\varphi(z)$. By virtue of (2), we have $v_1(x) + \cdots + v_{m-1}(x) \leq n$, and so, $v_{m-1}(x) \leq n - V_{m-2}(x) = v_{m-1}(z)$ implying

$$v_j(x) = v_j(z) \text{ for all } 1 \leq j \leq m-2 \quad \text{and} \quad v_{m-1}(x) \leq v_{m-1}(z).$$

If $v_{m-1}(x) = v_{m-1}(z)$, then, by virtue of the assumption (a_1) from Lemma 4, we find that $\varphi(x) = \varphi(z)$, and if $v_{m-1}(x) < v_{m-1}(z)$, then according to (a_2) we have $\varphi(x) > \varphi(z)$. Thus, $\varphi(x) \geq \varphi(z)$. Now we compare the values $\varphi(z)$ and $\varphi(y)$. Since

$$v_j(z) = v_j(y) \text{ for all } 1 \leq j \leq m-3 \text{ (no condition if } m = 3\text{)},$$
$$v_{m-2}(z) = v_{m-2}(x) < v_{m-2}(y) \quad \text{and} \quad V_{m-1}(z) = n,$$

then applying assumption (a_3) from Lemma 4, where x is replaced by z, we arrive at the inequality $\varphi(z) > \varphi(y)$. Therefore, $\varphi(x) \geq \varphi(z) > \varphi(y)$, as was asserted in ($b_3$).

Step 2 Now suppose that the assertion (b_k) is already established for some $3 \leq k \leq m-1$ and show that the assertion (b_{k+1}) holds as well. For this, given $x, y \in X$, assume that $v_j(x) = v_j(y)$ for all $1 \leq j \leq m-k-1$ (no condition if $m = k+1$) and $v_{m-k}(x) < v_{m-k}(y)$. This and (2) imply $V_{m-k}(x) = \sum_{j=1}^{m-k} v_j(x) < \sum_{j=1}^{m-k} v_j(y) \leq n$. Then a vector $z \in X$ with the following properties is well defined:

$$v_j(z)=v_j(x)=v_j(y) \text{ for all } 1 \leq j \leq m-k-1 \text{ (no condition if } m=k+1\text{)},$$
$$v_{m-k}(z)=v_{m-k}(x) \quad \text{and} \quad v_{m-k+1}(z)=n-V_{m-k}(x).$$

In order to compare the values $\varphi(x)$ and $\varphi(z)$, we note that the inequality $\sum_{j=1}^{m-k+1} v_j(x) \leq n$ implies $v_{m-k+1}(x) \leq n - V_{m-k}(x) = v_{m-k+1}(z)$, and so,

$$v_j(x) = v_j(z) \text{ for all } 1 \leq j \leq m-k \text{ and } v_{m-k+1}(x) \leq v_{m-k+1}(z).$$

If $v_{m-k+1}(x) = v_{m-k+1}(z)$, then $\sum_{j=1}^{m-k+1} v_j(x) = \sum_{j=1}^{m-k+1} v_j(z) = n$, and so, $v_j(x) = v_j(z) = 0$ for all $m-k+2 \leq j \leq m$. By (a_1) of Lemma 4, we get $\varphi(x) = \varphi(z)$. Now, if $v_{m-k+1}(x) < v_{m-k+1}(z)$, then applying the (already established!) assertion (b_k) with y replaced by z we find $\varphi(x) > \varphi(z)$. Thus, $\varphi(x) \geq \varphi(z)$. In order to compare $\varphi(z)$ and $\varphi(y)$, note that

$$v_j(z) = v_j(y) \text{ for all } 1 \leq j \leq m-k-1 \text{ (no condition if } m = k+1\text{)},$$
$$v_{m-k}(z) = v_{m-k}(x) < v_{m-k}(y) \quad \text{and} \quad V_{m-k+1}(z) = n.$$

Applying the assumption (a_{k+1}) of Lemma 4 with x replaced by z, we get $\varphi(z) > \varphi(y)$. It follows that $\varphi(x) \geq \varphi(z) > \varphi(y)$ implying (b_{k+1}).
Setting successively $k = 3, k = 4, \ldots, k = m-1$ and applying Step 2 we establish all assertions (b_3)–(b_m).

Step 3 Assertion (c) follows by contradiction from the properties presented in Lemma 4 (a_2) and Lemma 5 (b_3)–(b_m). □

Proof of Lemma 4 Necessity Suppose that P is φ-representable, i.e. condition (5) holds, and let $x, y \in X$. Taking into account (4), property (P.3) and (5), we find

$$\varphi(x) \neq \varphi(y) \text{ iff } [\varphi(x) > \varphi(y) \text{ or } \varphi(y) > \varphi(x)] \text{ iff } [(x, y) \in P \text{ or } (y, x) \in P]$$
$$\text{iff } v(x) \neq v(y),$$

and so,

$$v(x) = v(y) \text{ iff } [\neg(v(x) \neq v(y))] \text{ iff } [\neg(\varphi(x) \neq \varphi(y))] \text{ iff } \varphi(x) = \varphi(y),$$

which proves (a_1). Now, let $2 \leq k \leq m$ be arbitrary. If the assumptions in (a_k) hold, then it follows from the definition of P that $(x, y) \in P$, and so, by virtue of (5), we have $\varphi(x) > \varphi(y)$.

Sufficiency. We show that if $\varphi : X \to \mathbb{R}$ satisfies conditions (a_1)–(a_m), then (5) holds. Given $x, y \in X$, the implication '⇒' (necessity in (5)) is a straightforward consequence of the definition of P, items (b_m), (b_{m-1}), …, (b_3) of Lemma 5 and item (a_2) of Lemma 4. In order to prove the reverse implication in (5), we apply property (P.2), (4), the just established implication '⇒' and the assumption (a_1):

$$(x, y) \notin P \text{ iff } [(y, x) \in P \text{ or } v(y) = v(x)] \text{ implies}$$
$$[\varphi(y) > \varphi(x) \text{ or } \varphi(y) = \varphi(x)] \text{ iff } \varphi(y) \geq \varphi(x),$$

and so,

$$\varphi(x) > \varphi(y) \text{ iff } [\neg(\varphi(y) \geq \varphi(x))] \text{ implies } [\neg((x, y) \notin P)] \text{ iff } (x, y) \in P,$$

which was to be proved. □

Proof of Theorem 1 Sufficiency Suppose that $\varphi : X \to \mathbb{R}$ is a preference function for the binary relation P on X generated by the threshold rule, and $x, y \in X$. Then axiom (A.1) follows in the same way as condition (a_1) of Lemma 4. In order to verify axiom (A.2), assume that $x \succ y$. Then Lemma 2(b) implies $x^* \succ y^*$, and so, by virtue of Lemma 3(b) there exists a $1 \leq k \leq m - 1$ such that $v_j(x) = v_j(y)$ for all $1 \leq j \leq k - 1$ (no condition if $k = 1$) and $v_k(x) < v_k(y)$. It follows from the definition of P that $(x, y) \in P$, which together with (5) gives $\varphi(x) > \varphi(y)$ and establishes (A.2). Now, if $3 \leq k \leq m$ and the assumptions of condition (A.3.k) in axiom (A.3) hold, then $v_j(x) = v_j(y)$ for all $1 \leq j \leq m - k$ (with no condition if $k = m$) and $v_{m-k+1}(x) < v_{m-k+1}(y)$, and so, once again $(x, y) \in P$. The desired inequality $\varphi(x) > \varphi(y)$ is now a consequence of (5).

Necessity It suffices to verify that, given a social decision function φ from X into $\mathbb{R}$, axioms (A.1), (A.2) and (A.3) imply conditions (a_1)–(a_m) of Lemma 4; in fact, Lemma 4 implies the φ-representability of P, which is equivalent to the coherence of φ with the family of equivalence classes $\{X_\ell\}_{\ell=1}^{s}$ of the weak order P on X generated by the threshold rule $v(x) \angle v(y)$ in $\mathbb{R}^{m-1}$.

Clearly, axiom (A.1) coincides with condition (a_1) of Lemma 4.

Let us show that (A.1) and (A.2) imply (a_2). For this, we suppose that $x, y \in X$ are such that $v_j(x) = v_j(y)$ for all $1 \le j \le m-2$ and $v_{m-1}(x) < v_{m-1}(y)$. By Lemma 3 (b) (with $k = m-1$), we have $x^* \succ y^*$. Then axiom (A.2) implies $\varphi(x^*) > \varphi(y^*)$. Noting that $v_j(x) = v_j(x^*)$ and $v_j(y) = v_j(y^*)$ for all $1 \le j \le m-1$, and so, by virtue of axiom (A.1), $\varphi(x) = \varphi(x^*)$ and $\varphi(y) = \varphi(y^*)$, which gives $\varphi(x) > \varphi(y)$.

Now we show that, for each $3 \le k \le m$, axioms (A.1) and (A.2) and condition (A.3.k) of axiom (A.3) imply property (a_k) of Lemma 4. Before we do it, let us note that the conclusion in (A.3.k) is valid also in the case when $v_{m-k+1}(y) = n - V_{m-k}(y)$: in fact, if this is the case, we have

$$\begin{aligned} x^* &= (\underbrace{1,\ldots,1}_{v_1(x)},\ldots,\underbrace{m-k,\ldots,m-k}_{v_{m-k(x)}},\underbrace{m-k+1,\ldots,m-k+1}_{v_{m-k+1}(x)},\underbrace{m-k+2}_{1}) \\ &\succ (\underbrace{1,\ldots,1}_{v_1(x)},\ldots,\underbrace{m-k,\ldots,m-k}_{v_{m-k(x)}},\underbrace{m-k+1,\ldots,m-k+1}_{v_{m-k+1}(x)+1}) = y^*, \end{aligned}$$

and so, as above, (A.1) and (A.2) imply $\varphi(x) = \varphi(x^*) > \varphi(y^*) = \varphi(y)$.

Suppose that the assumptions in (a_k) hold, i.e. $x, y \in X$ are such that $v_j(x) = v_j(y)$ for all $1 \le j \le m-k$ (this condition is absent if $k = m$), $v_{m-k+1}(x) < v_{m-k+1}(y)$ and $V_{m-k+2}(x) = n$. We have to show that $\varphi(x) > \varphi(y)$. For this, we consider two auxiliary vectors $x', y' \in X$ such that

$$\begin{gathered} v_j(y') = v_j(y) = v_j(x) \ \text{ for all } \ 1 \le j \le m-k \text{ (no condition if } k = m), \\ v_{m-k+1}(y') = v_{m-k+1}(y) \quad \text{and} \quad v_m(y') = n - V_{m-k+1}(y), \\ v_j(x') = v_j(y') \ \text{ for all } \ 1 \le j \le m-k \text{ (again no condition if } k = m), \\ v_{m-k+1}(x') = v_{m-k+1}(y') - 1 = n - V_{m-k}(y') - v_m(y') - 1 \\ \text{and} \quad v_{m-k+2}(x') = v_m(y') + 1. \end{gathered}$$

First, by Lemma 3 (a) we have $x^* \succcurlyeq x'^*$; in fact, the inequalities $V_p(x) \le V_p(x')$ for all $1 \le p \le m-1$ are consequences of the following: $v_j(x) = v_j(y) = v_j(x')$ for all $1 \le j \le m-k$ (no condition if $k = m$),

$$v_{m-k+1}(x) \le v_{m-k+1}(y) - 1 = v_{m-k+1}(y') - 1 = v_{m-k+1}(x'),$$

$V_{m-k+2}(x) = n$ and $V_{m-k+2}(x') = n$, and so, $v_l(x) = v_l(x') = 0$ for all $m-k+3 \le l \le m$. So, the inequality $x^* \succcurlyeq x'^*$ and axioms (A.1) and (A.2) imply (in the already *standard* manner) that $\varphi(x) = \varphi(x^*) \ge \varphi(x'^*) = \varphi(x')$.

Second, since $v_j(x') = v_j(y')$ for all $1 \le j \le m-k$ (with no condition if $k = m$), $v_{m-k+1}(x') + 1 = v_{m-k+1}(y')$,

$$V_{m-k+2}(x') = n \quad \text{and} \quad V_{m-k+1}(y') + v_m(y') = n,$$

then the assumptions of condition (A.3.k) are satisfied for x' and y', and so, we get $\varphi(x') > \varphi(y')$.

Third, $y'^* \succcurlyeq y^*$, because (cf. Lemma 3(a)) $v_j(y') = v_j(y)$ for all $1 \le j \le m-k+1$, $v_j(y') = 0$ for all $m-k+2 \le j \le m-1$ and

$$v_m(y') = n - V_{m-k+1}(y) = \sum_{j=m-k+2}^{m} v_j(y) \ge v_m(y).$$

By the *standard* procedure as above, (A.1) and (A.2), we have $\varphi(y') = \varphi(y'^*) \ge \varphi(y^*) = \varphi(y)$.

Thus, $\varphi(x) \ge \varphi(x') > \varphi(y') \ge \varphi(y)$, and so, condition ($a_k$) follows.

This completes the proof of Theorem 1. □

Finally, we note that axioms (A.1), (A.2) and (A.3) are logically independent. In fact, Theorem 1 can also be interpreted as follows: a function $\varphi : X \to \mathbb{R}$ satisfies axioms (A.1)–(A.3) iff the φ-generated binary relation $P_\varphi = \{(x, y) \in X \times X : \varphi(x) > \varphi(y)\}$ is generated by the threshold rule (i.e. $P_\varphi = P$). However, if φ satisfies only either 1) (A.1) and (A.2), or 2) (A.1) and (A.3), or 3) (A.2) and (A.3), then $P_\varphi \ne P$, in general. This shows also that all axioms (A.1)–(A.3) are essential for the validity of Theorem 1. For instance, if $\varphi : X \to \mathbb{R}$ is given by $\varphi(x) = \sum_{j=1}^{m} j v_j(x) = \sum_{i=1}^{m} x_i$ for $x \in X$, then φ satisfies (A.1) and (A.2) and does not satisfy (A.3), and $P_\varphi \ne P$. The verification of possibilities 2) or 3) is left to the interested reader.

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