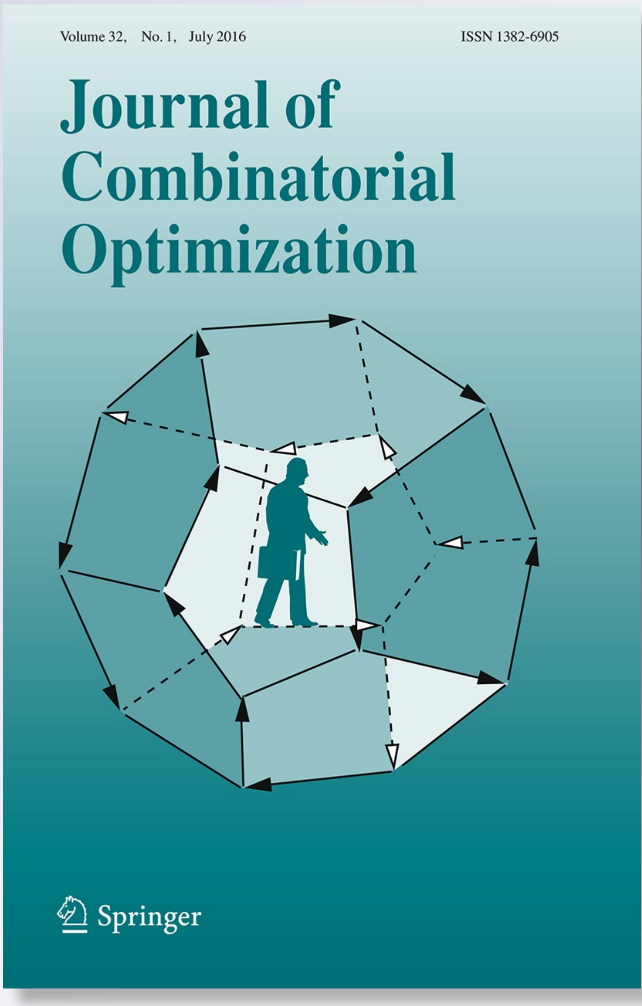
A complexity dichotomy and a new boundary class for the dominating set problem

D. S. Malyshev

Journal of Combinatorial Optimization

ISSN 1382-6905
Volume 32
Number 1

J Comb Optim (2016) 32:226-243
DOI 10.1007/s10878-015-9872-z



Springer

J Comb Optim (2016) 32:226–243
DOI 10.1007/s10878-015-9872-z

A complexity dichotomy and a new boundary class for the dominating set problem

D. S. Malyshev[1]

Published online: 3 April 2015

Abstract We study the computational complexity of the dominating set problem for hereditary graph classes, i.e., classes of simple unlabeled graphs closed under deletion of vertices. Every hereditary class can be defined by a set of its forbidden induced subgraphs. There are numerous open cases for the complexity of the problem even for hereditary classes with small forbidden structures. We completely determine the complexity of the problem for classes defined by forbidding a five-vertex path and any set of fragments with at most five vertices. Additionally, we also prove polynomial-time solvability of the problem for some two classes of a similar type. The notion of a boundary class is a helpful tool for analyzing the computational complexity of graph problems in the family of hereditary classes. Three boundary classes were known for the dominating set problem prior to this paper. We present a new boundary class for it.

Keywords Dominating set · Computational complexity · Polynomial-time algorithm · Boundary class

1 Introduction

We consider only simple unlabeled graphs in the paper. For a graph G, a subset $D \subseteq V(G)$ is said to be a *dominating set* if each element of $V(G) \setminus D$ has a neighbor in D. We also say that $V' \subseteq V(G)$ *dominates* $V'' \subseteq V(G)$ if each element of V'' has a neighbor in V'. The size of a minimum dominating set of G is called the *domination*

✉ D. S. Malyshev
dsmalyshev@rambler.ru; dmalishev@hse.ru

[1] National Research University Higher School of Economics, 25/12 Bolshaja Pecherskaja Ulitsa, Nizhny Novgorod 603155, Russia

number of G denoted by $\gamma(G)$. The *dominating set problem* is to recognize whether $\gamma(G) \geq k$ or not for given a simple graph G and natural k. It is NP-complete for the class of all graphs and remains intractable under its substantial restrictions (Bertossi 1984; Clark et al. 1990; Garey and Johnson 1979; Yannakakis and Gavril 1980).

A graph H is called an *induced subgraph* of G if H is obtained from G by deletion of vertices. If H is obtained from G by deletion of vertices and edges, then H is said to be a *subgraph* of G. A *class* is a set of graphs closed under isomorphism. A class of graphs is called *hereditary* if it is closed under deletion of vertices. A *strongly hereditary class* is a hereditary class that is additionally closed under deletion of edges. It is well known that any hereditary (and only hereditary) class $\mathcal{X}$ can be defined by a set of its forbidden induced subgraphs $\mathcal{Y}$. We write $\mathcal{X} = Free(\mathcal{Y})$ in this case, and graphs in $\mathcal{X}$ are said to be *$\mathcal{Y}$-free*. If $\mathcal{Y} = \{G\}$, then we write "G-free" instead of "$\{G\}$-free". There is a unique minimal under inclusion set $\mathcal{Y}$ such that $\mathcal{X} = Free(\mathcal{Y})$ denoted by $Forb(\mathcal{X})$. If $Forb(\mathcal{X})$ is finite, then $\mathcal{X}$ is said to be *finitely defined*.

There are several papers devoted to obtaining a complete complexity dichotomy within some subfamilies of the hereditary graph classes family (AbouEisha et al. 2014; Alekseev 2003; Broersma et al. 2012; Golovach and Paulusma 2014; Golovach et al. 2015, 2013; Korobitsyn 1992; Kral' et al. 2001; Kratsch and Schweitzer 2012; Lozin 2008; Lozin and Malyshev 2015; Malyshev 2013, 2014a, b; Schweitzer 2014). There are two natural restrictions to classify. The first of them is bounding the number of elements in $\mathcal{Y}$, the second one is bounding the sizes of these elements.

An *independent set* in a graph is a subset of its pairwise nonadjacent vertices. The size of a maximum independent set of a graph G is denoted by $\alpha(G)$. A *clique* in a graph is a subset of its pairwise adjacent vertices. The size of a maximum clique of a graph G is denoted by $\omega(G)$. A *vertex cover* of a graph is a subset of its vertices such that any edge is incident to a vertex of the set. The size of a minimum vertex cover of a graph G is denoted by $\beta(G)$. The *independent set, vertex cover*, and *clique problems* are to verify for a given graph G and a natural number k whether $\alpha(G) \geq k$, $\beta(G) \leq k$, and $\omega(G) \geq k$, respectively. These three problems are related, since for any graph G we have $\alpha(G)+\beta(G) = |V(G)|$ and $\alpha(G) = \omega(\overline{G})$, where $\overline{G}$ is the complement of G. So, all three problems are polynomially equivalent. There is known a complete complexity dichotomy for the independent set problem and the family of hereditary classes defined by forbidden induced subgraphs with at most five vertices followed from the papers of Alekseev (1983, 1999), Lozin and Mosca (2004), Lokshtantov et al. (2014). Namely, the problem is polynomial for $Free(\{G_1, \ldots, G_k\})$ with $\max |V(G_i)| \leq 5$ if one of the graphs $G_1, \ldots, G_k$ is the disjoint sum of paths, and it is NP-complete for all other classes in the subfamily. Thus, there exists a dichotomy within the those subfamily for all three problems.

The *vertex k-colorability problem* is to recognize whether the set of vertices of a given graph can be partitioned into at most k independent sets. The *edge k-colorability problem* for a graph is the vertex k-colorability problem for its line graph. The *chromatic number problem* for a given graph and a number k is to recognize whether the set of its vertices can be partitioned into at most k independent sets. The chromatic number problem is polynomial for $Free(\{G\})$ if G is an induced subgraph of a four-vertex path or a three-vertex path plus a vertex, and the problem is NP-complete for all other G (Kral' et al. 2001). The vertex 3-colorability problem is polynomial-time solvable for a class in $\{Free(\{G\}) : |V(G)| \leq 6\}$ if G is the disjoint union of simple paths and NP-complete in all other choices of G (Broersma et al. 2012). A similar result is known for the vertex 4-colorability problem and the five-vertex barrier for G (Golovach et al. 2013). The complexity of the vertex 3-colorability problem for all pairs of forbidden induced subgraphs with at most five vertices was discovered by Malyshev (2013). A similar result was obtained for the edge 3-colorability problem in (Malyshev 2014a). For all but three cases, the complexity of the chromatic number problem was discovered for hereditary classes having forbidden induced structures with at most four vertices by Lozin and Malyshev (2015).

A complete complexity dichotomy for the dominating set problem is known within the family of hereditary graph classes defined by a single forbidden induced subgraph (Korobitsyn 1992). Namely, the problem is polynomial-time solvable for $Free(\{G\})$ if G is a path with at most four vertices plus an arbitrary number of isolated vertices, and it is NP-complete for all other choices of G. But, there are numerous "blind-spots" for the complexity of the problem even for classes defined by two small forbidden induced subgraphs (Information system on graph classes and their inclusions 2015a). A dichotomy for the dominating set problem within the subfamily of hereditary classes defined by forbidding a five-vertex path and any set of fragments with at most five vertices is the main result of this paper.

The notion of a boundary graph class is a helpful tool for analyzing the computational complexity of graph problems within the family of finitely defined graph classes. Namely, a graph problem is NP-complete for a finitely defined class if and only if it includes a boundary class for the problem, unless $P = NP$. There are known three boundary classes for the dominating set problem. We present a fourth class for it in the paper.

2 Notation

We use the standard notation P_n, C_n, O_n, K_n for the simple path, the chordless cycle, the empty and complete graph with n vertices, respectively. A graph $K_{p,q}$ is a complete bipartite graph with p vertices in the first part and q in the second. A graph $fork$ is obtained from a $K_{1,3}$ by subdividing an arbitrary its edge. A graph orb is obtained from a K_4 by adding a new vertex and an edge connecting the added vertex to one vertex of a K_4. Similarly, a graph $sinker$ is obtained by adding a vertex and two edges incident to the new vertex and two vertices of K_4. A graph $bull$ is obtained from a P_5 by connecting the second and fourth its vertices by an edge. A graph $cricket$ is obtained from a K_3 by adding two vertices and two edges incident to the new vertices and the same vertex of a K_3. Graphs $dart$ and $kite$ are obtained from a K_4 minus an edge by adding a vertex and an edge incident to the new vertex and to a degree three or a degree two vertex, respectively. A graph gem is obtained from a P_4 by adding a new vertex and four edges incident to the new vertex and all vertices of P_4. A graph $hammer$ is obtained from a $fork$ by adding a new edge incident to two leaves adjacent to the degree three vertex.

For a graph G, a subset $V' \subseteq V(G)$, and a vertex $x \in V(G) \setminus V'$, $N_{V'}(x)$ is the set $V' \cap N(x)$. Notice that writing $N_{V'}(x)$ we mean $x \notin V'$ without specifying of that. Two vertices $x \in V(G) \setminus V'$ and $y \in V(G) \setminus V'$ are said to be V'-*equivalent* if $N_{V'}(x) = N_{V'}(y)$. A formula $N(x)$ denotes the neighborhood of a vertex x, $N[x] \triangleq N(x) \cup \{x\}$ is the *closed neighborhood* of x.

A *sum* $G_1 + G_2$ is the disjoint union of G_1 and G_2 with non-intersected sets of vertices. A graph kG is the disjoint union of k copies of G. For a graph G, a vertex $x \in V(G)$ and an edge $e \in E(G)$, the formulae $G \setminus \{x\}$ and $G \setminus \{e\}$ denote the subgraphs of G obtained by deleting x or e, respectively.

3 Boundary graph classes for the dominating set problem

The notion of a boundary graph class is a helpful tool for analyzing the computational complexity of graph problems within the family of hereditary graph classes. This notion was originally introduced by Alekseev for the independent set problem (Alekseev 2003). It was applied for the dominating set problem later (Alekseev et al. 2004). A study of boundary graph classes for some graph problems was proceeded in the paper of Alekseev et al. (2007), where the notion was stated in its most general form. Let us give necessary definitions.

Let Π be an NP-complete graph problem. The term "graph problem" is not defined here, and it is understood intuitively as a question on some input graph. A hereditary graph class $\mathcal{X}$ is called Π-*easy* if Π can be solved in polynomial time for its graphs. If the problem Π is NP-complete for graphs in a hereditary class, then this class is called Π-*hard*. A class of graphs is said to be Π-*limit* if this class is the intersection of an infinite monotonically decreasing sequence of Π-hard classes. In other words, $\mathcal{X}$ is Π-limit if there is an infinite sequence $\mathcal{X}_1 \supseteq \mathcal{X}_2 \supseteq \ldots$ of Π-hard classes such that $\mathcal{X} = \bigcap_{k=1}^{\infty} \mathcal{X}_k$. Each Π-hard class is Π-limit. A minimal under inclusion Π-limit class is called Π-*boundary*.

The following theorem certifies the significance of the boundary class notion (Alekseev et al. 2007).

Theorem 1 *A finitely defined class $\mathcal{X}$ is Π-hard if and only if it contains some Π-boundary class.*

So, by the theorem, the notion of a boundary class expresses the computational complexity of Π in "topological" terms by the inclusion relation. Theorem 1 can be reformulated by an incidence table if $Forb(\mathcal{X}) = \{G_1, G_2, \ldots, G_k\}$ is known. Rows of the table correspond to $G_1, G_2, \ldots, G_k$, columns to Π-boundary classes. We write one in (i, j)-cell if G_i belongs to jth Π-boundary class and zero otherwise. The problem Π is polynomial for $\mathcal{X}$ if and only if each column of the table is nonzero, otherwise $P = NP$. One more interesting result is a dichotomy claiming that any finitely defined class is either Π-easy or Π-hard, unless $P = NP$.

Three boundary graph classes are known for the dominating set problem (Alekseev et al. 2004). One of them is $\mathcal{S}$, which consists of all forests with at most three leaves in each connected component. The next one is $\mathcal{T}$ consisting of line graphs of graphs in $\mathcal{S}$. The third class is $\mathcal{Q}$, whose graphs are obtained by acting some mapping on elements of $\mathcal{S}$. For a graph $G = (V, E)$, a graph $Q(G)$ has vertex set $V \cup E$ and edge set $\{(v_i, v_j) : v_i, v_j \in V\} \cup \{(v, e) : v \in V, e \in E, v \text{ is incident to } e\}$. The class $\mathcal{Q}$ is the set $\{G : \exists H \in \mathcal{S}, G = Q(H)\}$ plus the set of all induced subgraphs of its graphs.

4 Polynomial-time solvability of the dominating set problem for some graph classes

4.1 The class $Free(\{P_5, dart\})$

Lemma 1 *Every connected $\{P_5, dart\}$-free graph is either gem-free or its domination number is at most four.*

Proof Let G be a connected graph that contains a *gem* as an induced subgraph. We consider an induced *gem* dominating a maximum possible number of vertices. Its vertices are denoted by x_1, x_2, x_3, x_4, y, where (x_1, x_2, x_3, x_4) is an induced path of the copy. We will show that $\{x_1, x_2, x_3, x_4\}$ is a dominating set of G, and this fact yields the lemma. Assume, there is a vertex $z_1 \in \bigcup_{i=1}^{4} N(x_i)$ having a neighbor $z_2 \notin \bigcup_{i=1}^{4} N(x_i)$. As G is $\{P_5, dart\}$-free, any neighbor of y must belong to $\bigcup_{i=1}^{4} N(x_i)$ and $N(x_1) \subseteq \bigcup_{i=2}^{4} N(x_i)$, $N(x_4) \subseteq \bigcup_{i=1}^{3} N(x_i)$. For the same reason, z_1 has two or three neighbors in $\{x_1, x_2, x_3, x_4\}$. If there are three such neighbors, then (z_1, x_1) and (z_1, x_4) are edges of G (as G is *dart*-free). To avoid an induced P_5, z_1 must be adjacent to y. Hence, G contains a *dart* as an induced subgraph. If $N(z_1) \cap \{x_1, x_2, x_3, x_4\} = \{x', x''\}$, then z_1 also must be adjacent to y. If $(x', x'') \notin E(G)$ or $\{x', x''\} = \{x_2, x_3\}$, then G has an induced *dart*. If $\{x', x''\} = \{x_1, x_2\}$ or $\{x', x''\} = \{x_3, x_4\}$, then y, z_1, x_2, x_3, x_4 or y, x_1, x_2, x_3, z_1 induce a *gem*, and they dominate more vertices than y, x_1, x_2, x_3, x_4. We have a contradiction with the choice above. □

Theorem 2 *The dominating set problem is polynomial in the class $Free(\{P_5, dart\})$.*

Proof By Lemma 1, the dominating set problem for $Free(\{P_5, dart\})$ can be polynomially reduced to the same problem for $Free(\{P_5, gem\})$. The last class is easy for the problem (Brandstädt et al. 2005). Hence, the theorem holds. □

4.2 The classes $Free(\{P_5, orb\})$, $Free(\{P_5, sinker\})$, $Free(\{P_5, kite\})$, $Free(\{P_5, cricket\})$, $Free(\{P_5, fork\})$

It is well-known that each connected P_5-free graph has a dominating P_3 or a dominating clique (Bacsó and Tuza 1990).

Lemma 2 *The domination number of any connected $\{P_5, orb\}$-free graph is at most three.*

Proof Let $G \in Free(\{P_5, orb\})$ be a connected graph. If it has a dominating P_3, then $\gamma(G) \leq 3$. Hence, we may assume that G has a dominating clique Q. Let

$\{x_1, x_2, \ldots, x_k\} \subseteq Q$ be a minimal dominating set of G. For every i, there is a vertex $y_i \in V(G) \setminus Q$ such that $y_i \in N(x_i) \setminus \bigcup_{j=1, j \neq i}^{n} N(x_j)$ (due to the minimality). Therefore $k \leq 3$, as y_1, x_1, x_2, x_3, x_4 induce an *orb* otherwise. So, $\gamma(G) \leq 3$. □

The dominating set problem for P_5-free graphs can be polynomially reduced to the same problem for connected P_5-free graphs having dominating cliques with at least four vertices. Moreover, for a P_5-free graph, a dominating clique can be computed in polynomial time if one exists (Bacsó and Tuza 1990). In the next lemmas, we assume that G is a P_5-free graph with known such a clique, which is denoted by Q. We also assume that Q is maximal under inclusion, as each clique can be fulfilled to a maximal one in polynomial time.

Two vertices $x \notin Q$ and $y \notin Q$ are said to be *non-congruent* if $N_Q(x) \neq N_Q(y)$ and $\max(|N_Q(x)|, |N_Q(y)|) \leq |Q| - 2$. A maximum set of pairwise non-congruent vertices is denoted by Φ. A formula Q_1 denotes $\{x \in Q : \exists y \notin Q[(x, y) \in E(G), |N_Q(y)| = 1]\}$, $N'(x) \triangleq \{y : (y, x) \in E(G), |N_Q(y)| = 1\}$ for $x \in Q_1$.

Lemma 3 *For every connected* $\{P_5, sinker\}$*-free graph* G *we have* $\gamma(G) \leq 4$ *or* $\gamma(G) = |\Phi| = |Q_1|$.

Proof Since G is $sinker$-free, any element of $V(G) \setminus Q$ has $|Q| - 1$ or 1 neighbors in Q. Hence, $\gamma(G) \leq 2$ if Q_1 is empty. Suppose that Q_1 is not empty, and let $Q_1 = \{x_1, x_2, \ldots, x_k\}$. Clearly, $|Q_1| = |\Phi|$. At first, we will show if there are two adjacent vertices $y_i \in N'(x_i)$ and $y_j \in N'(x_j)$, then $\{x_i, x_j, y_i, y_j\}$ is a dominating set and $\gamma(G) \leq 4$. Indeed, if there is a non-dominated vertex $z' \notin Q$, then it must have only one neighbor $z \in Q \setminus \{x_i, x_j\}$, since G is $sinker$-free. But y_i, y_j, x_j, z, z' induce a P_5, which contradicts the condition.

Let $\gamma(G) \geq 5$, y' and y'' be arbitrary elements of Φ. We will prove that there is no a vertex simultaneously adjacent to y' and y''. If z is such a vertex, then the vertex z cannot have only one neighbor in Q, since it would mean $\gamma(G) \leq 4$ yielding a contradiction with the assumption. Then y' or y'', z, and three elements of $N_Q(z)$ induce a $sinker$ otherwise. Thus, z does not exist. This fact implies that Φ has some k elements such that any two of them cannot be simultaneously dominated. Hence, $\gamma(G) \geq k$. The set Q_1 is a dominating set. So, $\gamma(G) = k$. □

Lemma 4 *For every connected* $\{P_5, kite\}$*-free graph* G *we have* $\gamma(G) \leq 4$ *or* $\gamma(G) = |Q_1|$.

Proof Firstly, we will show that if $\min |N_Q(x)| > 1$, then $\gamma(G) \leq 3$. Let x^* be a vertex such that there is no a vertex x satisfying the inclusion $N_Q(x) \subset N_Q(x^*)$, y_1 and y_2 be arbitrary distinct neighbors of x^* belonging to Q. There is no a vertex of G that is not dominated by an element of $\{x^*, y_1, y_2\}$. Indeed, if z' is such a vertex, then $(z', x^*) \notin E(G)$, $(z', y_1) \notin E(G)$, $(z', y_2) \notin E(G)$, and there is a vertex $z \in Q$ such that $(z', z) \in E(G)$ and $(x^*, z) \notin E(G)$. The vertex z exists in accordance with the choice of x^*. The graph G contains a $kite$ induced by x^*, y_1, y_2, z, z'.

Secondly, if there are adjacent vertices $y' \in N'(x')$ and $y'' \in N'(x'')$ for $x' \neq x''$, then $\{x', x'', y', y''\}$ is a dominating set of G. It can be proved in a similar way to those in the previous lemma.

Suppose that $Q_1 \neq \emptyset$ and there are no adjacent vertices $y' \in N'(x')$ and $y'' \in N'(x'')$ for any distinct vertices x' and x''. It is easy to check if z is a vertex with $|N_Q(z)| > 1$, then, for any $x \in Q_1$ and $y \in N'(x)$, the vertex z is dominated by x or y, since G is $kite$-free. Hence, $\gamma(G) \leq 4$ if $|Q_1| \leq 2$. Suppose $|Q_1| \geq 3$. If z is adjacent to $y' \in N'(x')$ and $y'' \in N'(x'')$, then $(z, x') \in E(G)$ or $(z, x'') \in E(G)$, since G is $kite$-free. Without loss of generality, $(z, x') \in E(G)$. The vertex z is adjacent to each vertex $y \in N'(x)$ for any $x \notin \{x', x''\}$, otherwise y'', z, x', x, y induce a P_5 or y, x, x', z, y' induce a $kite$. Therefore, $\{x', y', z\}$ is a dominating set. So, we can suppose that any two elements of $\bigcup_{x \in Q_1} \{y_x\}$, where $y_x \notin Q$ is an arbitrary neighbor of $x \in Q_1$ with $|N_Q(y_x)| = 1$, do not have a common neighbor. Hence, $\gamma(G) \geq |Q_1|$. Since Q_1 is a dominating set of G, then $\gamma(G) = |Q_1|$. □

Lemma 5 *For every connected $\{P_5, cricket\}$-free graph G we have $\gamma(G) \leq 7$ or $\gamma(G) = |\Phi|$.*

Proof We will show that if there are adjacent non-congruent vertices x and y, then $\gamma(G) \leq 7$. Since x and y are non-congruent, there is a vertex $v \in Q$ such that $v \in N(x) \setminus N(y)$ or $v \in N(y) \setminus N(x)$. There is no an element of $V(G) \setminus Q$ having a neighbor in $Q \setminus (N_Q(x) \cup N_Q(y))$ and non-adjacent to x, y, v simultaneously, since G is P_5-free. Recall that $\max(|N_Q(x)|, |N_Q(y)|) \leq |Q| - 2$. Let z be an element of $V(G) \setminus Q$ having a neighbor $z' \in N_Q(x) \cup N_Q(y)$, $x', x'' \in Q \setminus N(x)$ and $y', y'' \in Q \setminus N(y)$. At least one of the vertices x, y, x', x'', y', y'' is adjacent to z, since x, z, z', y', y'' or y, z, z', x', x'' induce a $cricket$ otherwise. Hence, $\{x, y, x', x'', y', y'', v\}$ is a dominating set of G and $\gamma(G) \leq 7$.

Suppose that G has no adjacent non-congruent vertices. If there are non-congruent vertices x and y having a common neighbor in Q, then $\gamma(G) \leq 5$. The set $N_Q(x) \cup N_Q(y)$ must contain at least $|Q| - 1$ elements, otherwise G is not $cricket$-free. Let V' be a set containing exactly two elements of $Q \setminus N(x)$, exactly two elements of $Q \setminus N(y)$, and an element of $Q \setminus (N_Q(x) \cup N_Q(y))$ if one exists. The set V' must be dominating, otherwise there is a vertex z having a neighbor $z' \in N_Q(x) \cup N_Q(y)$ such that z, z', x or y, and some two elements of V' induce a $cricket$.

Suppose that there are no adjacent non-congruent vertices and non-congruent vertices having a common neighbor in Q, and the domination number of G is at least eight. Each element $a \in \Phi$ has a neighbor $b_a \in Q$. The union $\bigcup_{a \in \Phi} \{b_a\}$ plus at most two vertices of Q (they need to dominate possible elements of $V(G) \setminus Q$ having $|Q| - 1$ neighbors in Q) is a dominating set of G. Hence, $|\Phi| + 2 \geq \gamma(G) \geq 8$, i.e., $|\Phi| \geq 6$. If z is a vertex adjacent to two non-congruent vertices x and y, then $|N_Q(z)| = |Q| - 1$. Since $|\Phi| \geq 6$, the set $N_Q(z)$ contains vertices z_1 and z_2 that are not adjacent to x and y simultaneously. The vertices z, z_1, z_2, x, y induce a $cricket$. So, no one vertex can be adjacent to two non-congruent vertices. Hence, $\gamma(G) \geq |\Phi| = \phi(G)$. The set $\bigcup_{a \in \Phi} \{b_a\}$ is a dominating set of G . Hence, $\gamma(G) = |\Phi|$. □

Lemma 6 *Let $Q' = \{x_1, x_2, \ldots, x_l\} \subseteq Q$ be an arbitrary minimal under inclusion dominating set of a connected $\{P_5, fork\}$-free graph G. If $\gamma(G) \geq 5$, then $\gamma(G) = l$.*

Proof Clearly, Q' is a dominating set of G. Hence, $\gamma(G) \leq l$ and $l \geq 5$. Let $V_i \triangleq N(x_i) \setminus \bigcup_{j=1, j \neq i}^{l} N(x_j)$. For each i, the set V_i is not empty, due to the minimality of Q'. We will show that V_i is a clique for every i. Indeed, if V_i contains two nonadjacent vertices y_i^1 and y_i^2, then any vertex $y_j \in V_j$ must be adjacent to y_i^1 or y_i^2, otherwise $y_i^1, y_i^2, x_i, x_j, y_j$ induce a $fork$. If (y_i^1, y_j) and (y_i^2, y_j) are edges of G, then $y_i^1, y_i^2, y_j, x_j, x_k$ induce a $fork$. If $(y_i^1, y_j) \in E(G)$, $(y_i^2, y_j) \notin E(G)$ or vice versa, then $y_i^2, x_k, x_i, y_i^1, y_j$ or $y_i^1, x_k, x_i, y_i^2, y_j$ induce a $fork$.

Secondly, let us show that the subgraph induced by $\hat{V} \triangleq \bigcup_{j=1}^{l} V_j$ is complete or the disjoint union of the cliques $V_1, V_2, \ldots, V_l$. To this end, it is sufficient to prove if $y' \in V_i$ and $y'' \in V_j$ are adjacent, then y' is adjacent to all elements of $\hat{V} \setminus \{y'\}$. If y' is adjacent to a vertex $y_s \in V_s$, then it is adjacent to all elements of V_s, otherwise some element of V_s, y_s, y', x_i and $x_k (k \notin \{i, j, s\})$ induce a P_5. Hence, if y' is not adjacent to a vertex $y_{s'} \in V_{s'}$, then $s' \neq i$ and $s' \neq j$. If $(y'', y_{s'}) \notin E(G)$, then $y'', y', x_i, x_{s'}, y_{s'}$ induce a P_5, otherwise $y', y'', y_{s'}, x_{s'}, x_k (k \notin \{i, j, s'\})$ induce a P_5.

If $\hat{V}$ is a clique, then $\{x_1, x_2, y_1, y_2\}$ is a dominating set of G, where $y_1 \in V_1$ and $y_2 \in V_2$. Indeed, if there is a vertex y nonadjacent to any element of $\{x_1, x_2, y_1, y_2\}$, then y, any element of $N_{Q'}(y)$, x_1, y_1 and y_2 induce a P_5. Hence, $\hat{V}$ cannot be a clique, since $\gamma(G) \geq 5$. Thus, this set induces the disjoint union of cliques.

Suppose that z is a vertex that is adjacent to $y_i \in V_i$ and $y_j \in V_j$ simultaneously. Clearly, $z \notin \hat{V} \cup Q'$. We will show that z is adjacent to all elements of $\hat{V}$. Let $z \in Q \setminus Q'$. Indeed, z is adjacent to all elements of $V_i \cup V_j$, otherwise an element of $V_i \cup V_j$, y_i, y_j, z and x_k induce a $fork$. If there is a vertex $y_s \in V_s$, $s \notin \{i, j\}$ such that $(y_s, z) \notin E(G)$, then y_i, y_j, z, x_s and y_s induce a $fork$. Let $z \in V(G) \setminus (Q \cup \hat{V})$ now. Then z must be adjacent to at least two elements of Q'. Let $N_{Q'}(z) \neq \{x_i, x_j\}$. Then, to avoid an induced $fork$, $Q' \setminus \{x_i, x_j\} \subseteq N_{Q'}(z)$. Hence, to avoid an induced $fork$, z is adjacent to each element of $\hat{V} \setminus (V_i \cup V_j)$. One of the vertices Q does not belong to $N_Q(z)$. Arbitrary elements of $V_{i'}$ and $V_{j'}$, z, x_k and an element of $Q \setminus N_Q(z)$ induce a $fork$, where i', j', k are distinct, $\{i', j'\} \cap \{i, j\} = \emptyset$ and $k \neq i, k \neq j$. Hence, $N_{Q'}(z) = \{x_i, x_j\}$. To avoid an induced P_5, the vertex z must be adjacent to to each element of $\hat{V} \setminus (V_i \cup V_j)$. Moreover, to avoid an induced $fork$, z must be adjacent to all elements of $V_i \cup V_j$.

Let y be a vertex in $V(G) \setminus (Q \cup \hat{V})$ having neighbors in at most one of the sets $V_1, V_2, \ldots, V_l$. The vertex y must have at least two neighbors in Q', as $y \in \hat{V}$ otherwise. It must be adjacent to at least $l - 1$ elements of Q', otherwise there are vertices $x_i \in N_{Q'}(y)$, $x_j \in Q' \setminus N_{Q'}(y)$, $y_i \in V_i$, $y_j \in V_j$ such that $(y_i, y) \notin E(G)$ and $(y_j, y) \notin E(G)$, and the vertices y, y_i, x_i, x_j, y_j induce a $fork$.

If there is a vertex v that has neighbors in at least two of the sets $V_1, V_2, \ldots, V_l$, then, by both previous paragraphs, $\{v, x_1, x_2\}$ is a dominating set of G. Hence, there is no such a vertex, as $\gamma(G) \geq 5$. Therefore, $\gamma(G) \geq l$. So, $\gamma(G) = l$, as Q' is a dominating set. □

Theorem 3 *The dominating set problem can be solved in polynomial time for each of the graph classes* $Free(\{P_5, orb\})$, $Free(\{P_5, sinker\})$, $Free(\{P_5, kite\})$, $Free(\{P_5, cricket\})$, $Free(\{P_5, fork\})$

Proof By Lemmas 2–6, the dominating set problem can be polynomially reduced to computing one of the sets Φ, Q_1, and Q'. It can be done in polynomial time. Hence, the problem is polynomial for each of the classes. □

4.3 The class $Free\{P_5, K_{1,4}\}$

Let G be a connected $\{P_5, K_{1,4}\}$-free graph containing a $K_{1,3}$ induced by vertices x_0, x_1, x_2, x_3, where (x_0, x_1), (x_0, x_2), (x_0, x_3) are edges of G. We associate the following notations with G : $C_1 \triangleq \{x_1, x_2, x_3\}$, $C_2 \triangleq \bigcup_{i=1}^{3} N(x_i) \setminus N[x_0]$, A_1 is a subset of $N(x_0) \setminus C_1$ containing vertices that have at least one neighbor outside $N[x_0] \cup C_2$, A_2 consists of all neighbors of all elements in A_1 that do not belong to $N[x_0] \cup C_2$, $B \triangleq N(x_0) \setminus (A_1 \cup C_1)$. Clearly, C_1, C_2, A_1, A_2, B are pairwise non-intersected, and no one element of A_2 is adjacent to an element of $C_1 \cup B$.

Lemma 7 *The set* $V(G)$ *coincides with* $\{x_0\} \cup C_1 \cup C_2 \cup A_1 \cup A_2 \cup B$.

Proof Suppose that there is a vertex $x \in A_2 \cup C_2$ having a neighbor $y \notin C_1 \cup C_2 \cup A_1 \cup A_2 \cup B$. The vertex x has a neighbor $z \in A_1 \cup C_1 \cup B$, which cannot be adjacent to all elements of C_1, as G is $K_{1,4}$-free. But, an element of C_1, x_0, z, x, y induce a P_5. We have a contradiction with the assumption. □

Lemma 8 *If* $x \in A_2$, $y \in C_2 \cup A_2$, *and they are not* A_1*-equivalent, then* x *and* y *are not adjacent.*

Proof Suppose that $(x, y) \in E(G)$. If $y \in A_2$, then there is a vertex $z \in N_{A_1}(x) \otimes N_{A_1}(y)$. The vertex z cannot be adjacent to all vertices x_1, x_2, x_3, since G is $K_{1,4}$-free. Hence, one of the vertices x_1, x_2, x_3, the vertices x_0, z, x and y induce a P_5. If $y \in C_2$, then there are vertices $x' \in N_{C_1}(y)$ and $x'' \in C_1 \setminus N_{C_1}(y)$, since G is $K_{1,4}$-free. The vertices x, y, x', x_0, x'' induce a P_5. We have a contradiction with the assumption. □

Lemma 9 *For each vertex* $v \in A_1$, *there are no vertices* $u_1, u_2, u_3 \in A_2$ *adjacent to* v *such that any two of them are not* A_1*-equivalent.*

Proof Assume the opposite. By the previous lemma, $\{u_1, u_2, u_3\}$ is independent. Hence, v, x_0, u_1, u_2, u_3 induce a $K_{1,4}$. We have a contradiction. □

Let D be a minimum dominating set of G and D' be a minimal under inclusion subset of D dominating X, where $X \triangleq \{x_0\} \cup C_1 \cup C_2 \cup B$. By the minimality of D', $D' \cap A_2 = \emptyset$. Let $D'_1 \triangleq D' \cap X$ and $D'_2 \triangleq D' \setminus D'_1 = D' \cap A_1$. Let $D'' \triangleq D \setminus D'$.

Lemma 10 *The set* D' *contains at most* 21 *elements.*

Proof The set $\{x_0, x_1, x_2, x_3\}$ dominates X, and $D'' \cup D'_2 \cup \{x_0, x_1, x_2, x_3\}$ is a dominating set of G. Since D is minimum, $|D'_1| \leq 4$. Let D^* be a minimal under inclusion subset of D'_2 that dominates $\bigcup_{v \in D'_2} N_{A_2}(v)$. By the minimality, for each $x \in D^*$ and $y \in D^*$ we have $N_{A_2}(x) \nsubseteq N_{A_2}(y)$ and $N_{A_2}(y) \nsubseteq N_{A_2}(x)$. To avoid an induced P_5, the set D^* must be a clique. Since $D'' \cup D^* \cup \{x_0, x_1, x_2, x_3\}$ is a dominating set of G and D is minimum, $|D^*| \geq |D'_2| - 4$.

Since D' is minimal under inclusion, there is a set $N(D^*)$ such that $N(D^*) \subseteq X$, $|N(D^*)| = |D^*|$, and any element of $N(D^*)$ is dominated by only one element of D^*. If $|D^*| \leq 2$, then $|D'| = |D'_1| + |D'_2| \leq 4 + |D^*| + 4 \leq 10$. Let $|D^*| \geq 3$. The set $N(D^*)$ must be independent, as two adjacent vertices $u_1, u_2 \in N(D^*)$, the neighbor $v_1 \in D^*$ of u_1, a vertex $v_2 \in D^* \setminus (N_{D^*}(u_1) \cup N_{D^*}(u_2))$ and its neighbor in A_2 induce a P_5. Since G is $K_{1,4}$-free, $|B \cap N(D^*)| \leq 3$ and $|C_2 \cap N(D^*)| \leq 6$. Hence, $|D'| = |D'_1| + |D'_2| \leq 4 + |D^*| + 4 = 8 + |\{x_0\} \cap N(D^*)| + |C_1 \cap N(D^*)| + |C_2 \cap N(D^*)| + |B \cap N(D^*)| \leq 8 + 1 + 3 + 6 + 3 \leq 21$. □

The set $D'' \cap X$ has at most one element, otherwise elements of $D'' \cap X$ can be removed from D'' and x_0 can be added to obtain a smaller dominating set than D. Let $D'' \cap A_2$ contain two elements x and y. If $N_{A_1}(x) \cap N_{A_1}(y) \neq \emptyset$, then $(D \setminus \{x, y\}) \cup \{x_0, z\}$ is a minimum dominating set of G, where z is an arbitrary element of $N_{A_1}(x) \cap N_{A_1}(y)$. If $N_{A_1}(x) \cap N_{A_1}(y) = \emptyset$, then, to avoid an induced P_5, z_1 and z_2 must dominate $N_{A_1}(y)$ and $N_{A_1}(x)$ respectively, where $z_1 \in N_{A_1}(x)$ and $z_2 \in N_{A_1}(y)$ are arbitrary vertices.

Let $A'_1 \triangleq A_1 \setminus D'$, A'_2 be a part of A_2 non-dominated by $D' \cup D'' \cap A_2$. Let G' be the subgraph of G induced by $A'_1 \cup A'_2$ and G'' be the graph obtained from G' by adding edges to make A'_1 to be a clique. If there is a dominating set of G'' included in A'_1 with at most two elements, then $\gamma(G) \leq |D'| + |D'' \cap (X \cup A_2)| + |\{x_0\}| + 2 \leq 21 + 2 + 1 + 2 \leq 26$. We assume that any such a set has at least three vertices.

Lemma 11 *There exists a minimum dominating set of G containing x_0 or any dominating set of G'' included in A'_1 is a dominating set of G'.*

Proof Suppose that there is a dominating set $D^+ = \{v_1, \ldots, v_k\} \subseteq A'_1$ of G'' with a minimum possible number of elements that is not a dominating set of G'. Hence, there is a vertex $v \in A'_1$ such that $(v, v_i) \notin E(G')$ for each i. Due to the minimality of D^+, the graph G'' has a vertex $u_i \in N_{A'_2}(v_i) \setminus \bigcup_{j=1, j \neq i}^{k} N_{A'_2}(v_i)$ for each i. Hence, to avoid an induced P_5, the set D^+ is a clique of G. Taking into account that $k \geq 3$, it is easy to verify that v cannot be adjacent to an element of $\{u_1, \ldots, u_k\}$, otherwise G is not P_5-free. Since $v \in A'_1$, there is a vertex $v' \in N_{A_2}(v) \setminus \{u_1, \ldots, u_k\}$ in G. To avoid an induced P_5 and by Lemma 8, v' must be adjacent to all elements of D^+ and non-adjacent to u_1. To avoid an induced $K_{1,4}$, $N_{C_1}(v_1), \ldots, N_{C_1}(v_k)$ must have one element. Clearly, $N_{C_1}(v) \neq \emptyset$. Moreover, $N_{C_1}(v_i) = N_{C_1}(v)$ for each i, otherwise G is not P_5-free. Let $N_{C_1}(v) = \{x_3\}$.

The vertices x_1 and x_2 cannot have neighbors in C_2, otherwise such a neighbor must be adjacent to v_1 and non-adjacent to v', v_1, x_0, the neighbor, v', v_1 induce a $K_{1,4}$. There are vertices $u^* \in D'$ and $u^{**} \in D'$ dominating x_1 and x_2, respectively (perhaps, $u^* = u^{**}$). Hence, u^* and u^{**} belong to $\{x_0, x_1, x_2\} \cup (A_1 \setminus A_1') \cup B$. If u^* or u^{**} has no neighbors outside $N[x_0]$, then one of them can be changed to x_0 to keep the minimality of D. Similarly, one may assume that there is a neighbor $u' \in (A_2 \setminus A_2') \cup C_2$ of u^*. Since G is $\{P_5, K_{1,4}\}$-free, $\{v', u_1, u'\}$ is independent and $(u', v_1) \notin E(G)$, $(u', v) \notin E(G)$. Similarly, u^* must be adjacent to at least one element in each of the sets $\{v, v'\}$ and $\{v_1, u_1\}$, and it cannot be adjacent to all elements of each of the sets $\{v, v_1\}$, $\{v', u_1\}$, and $\{v, u_1\}$. Therefore, u^* is adjacent to v_1 and v'. Hence, $(u^*, x_2) \notin E(G)$ and $u^* \neq u^{**}$. The vertices u^* and x_3 must be adjacent, otherwise x_1, u^*, v_1, x_3, v induce a P_5. The vertices x_3 and u' must be adjacent, since u^*, x_1, x_3, v', u' induce a $K_{1,4}$ otherwise.

One may assume that $N_{A_2}(u^*) \nsubseteq N_{A_2}(u^{**})$ and $N_{A_2}(u^{**}) \nsubseteq N_{A_2}(u^*)$, otherwise one of the vertices u^* and u^{**} can be changed to x_0. Let $u' \in N_{A_2 \setminus A_2'}(u^*) \setminus N_{A_2}(u^{**})$ and $u'' \in N_{A_2 \setminus A_2'}(u^{**}) \setminus N_{A_2}(u^*)$ now. Therefore, $\{v', v_1, u', u''\}$ is independent, u^{**} is adjacent to v_1 and v', and u'' is adjacent to x_3. But, the vertices x_3, v, v_1, u', u'' induce a $K_{1,4}$ in G. We have a contradiction. Hence, D^+ is a dominating set of G'. □

Let A_2'' be a maximal under inclusion subset of A_2' that does not contain a pair of A_1'-equivalent vertices. By Lemma 8, A_2'' is independent. Moreover, by Lemma 9, each element of A_1' is adjacent to at most two elements of A_2''. Delete from G'' all vertices in A_1' that have no a neighbor in A_2'', and add a new vertex adjacent to all vertices of A_1' having only one neighbor in A_2'' if they exist. We denote the resultant graph by G'''. Clearly, D^+ is a minimum dominating set of G'' and G'''. Hence, $\gamma(G''') = \gamma(G'') = |D^+|$.

The set $V(G''')$ can be uniquely split into two parts A^+ and B^+, where A^+ is a clique and B^+ is independent. Each element of A^+ has two neighbors in B^+. We construct a graph H as follows. Its vertices are elements of B^+, and two vertices are adjacent if and only if G''' has their common neighbor in A^+. An *edge cover* of a graph is a set of edges such that every vertex of the graph is incident to at least one edge of the set. Any dominating set of G''' included in A^+ is an edge cover of H, and any edge cover of H corresponds to a dominating set of G''' included in A^+. Hence, $\gamma(G''')$ is equal to the size of a minimum edge cover of H. A minimum edge cover of a graph can be found in polynomial time. Hence, some minimum dominating sets of G'' and G''' can be found in polynomial time.

Theorem 4 *The dominating set problem can be solved in polynomial time for graphs in* $Free(\{P_5, K_{1,4}\})$.

Proof For a connected graph $G \in Free(\{P_5, K_{1,4}\})$, one can verify whether $\gamma(G) \leq 26$ or not and find $\gamma(G)$ if the answer is "yes". If G is $K_{1,3}$-free, then one can apply the previous theorem. Otherwise, we enumerate all subsets of $V(G)$ with at most 23 elements and consider among them only dominating X. For every such a subset V', we construct the corresponding graphs G' (notice, $A_1' \triangleq A_1 \setminus V'$, A_2' be a part of A_2 non-dominated by V') and G'', G''' and determine a minimum dominating set D^+ of G''' included in A^+. If $V' \cup D^+$ is a dominating set of G, then calculate $|V'| + |D^+|$. Minimal of these sums is $\gamma(G)$, by Lemmas 10 and 11. □

5 Main result

Lemma 12 *The dominating set problem for* $Free(\{G + O_1\})$ *can be polynomially reduced to the same problem for* $Free(\{G\})$.

Proof Let $H \in Free(\{G + O_1\})$ be a graph containing G as an induced subgraph. This subgraph G must dominate all vertices of H, i.e., $\gamma(H) \leq |V(G)|$. Hence, any $\{G + O_1\}$-free graph is either G-free or its domination number is at most $|V(G)|$. An induced copy of G can be found in H in polynomial time if one exists. So, there is a polynomial-time reduction. □

The following theorem is the main result of this paper.

Theorem 5 *Let* $\mathcal{Y}$ *be an arbitrary set of graphs having at most five vertices. The dominating set problem for* $\mathcal{X} = Free(\{P_5\} \cup \mathcal{Y})$ *is NP-complete if* $\mathcal{Y} \cap \mathcal{Q} = \emptyset$. *If* $\mathcal{Y} \cap \mathcal{Q} \neq \emptyset$, *then the problem can be solved in polynomial time for* $\mathcal{X}$.

Proof Recall that $\mathcal{Q}$ is a boundary class for the dominating set problem. Hence, if $\mathcal{X}$ includes $\mathcal{Q}$ (equivalently, $\mathcal{Y} \cap \mathcal{Q} = \emptyset$), then the problem is NP-complete for $\mathcal{X}$. Let $\mathcal{Y} \cap \mathcal{Q} \neq \emptyset$ and G be a graph in $\mathcal{Q}$ containing at most five vertices. The graph G cannot contain $C_4, C_5, 2K_2$, and $\overline{K_2 + O_3}, \overline{K_3 + O_2}$. Taking into account a list of all five-vertex graphs (Information system on graph classes and their inclusions 2015b), it is easy to check that G is an induced subgraph of one of the graphs $P_4 + O_5, K_5 + O_5, orb + O_5, sinker + O_5, kite + O_5, dart + O_5, cricket + O_5, fork + O_5, K_{1,4} + O_5, gem + O_5, bull + O_5$. The classes $Free(\{P_5, gem\}), Free(\{P_4\}), Free(\{P_5, bull\}), Free(\{P_5, K_5\})$ are easy for the problem (Brandstädt et al. 2005; Courcelle et al. 2000; Kratsch 2000; Zverovich 2003). Hence, by Lemma 12 and Theorems 2–4, the dominating set problem is polynomial-time solvable for $\mathcal{X}$. □

6 The classes $Free(\{fork, bull\})$ and $Free(\{fork, hammer\})$

Lemma 13 *If* x *and* y *are adjacent and* $N(x) \setminus \{y\} = N(y) \setminus \{x\}$, *then* $\gamma(G) = \gamma(G \setminus \{x\})$.

Proof Let $H \triangleq G \setminus \{x\}$. Since each dominating set of H must have an element of $N[y]$, each dominating set of H is a dominating set of G. Hence, $\gamma(G) \leq \gamma(H)$. A minimum dominating set D of G must have an element of $N[x]$, and it cannot contain x and y simultaneously. Moreover, if $x \in D$, then $D \setminus \{x\} \cup \{y\}$ is also a minimum dominating set of G. Therefore, there is a minimum dominating set of G that is a dominating set of H. Hence, $\gamma(H) \leq \gamma(G)$. So, $\gamma(G) = \gamma(H)$. □

Three vertices of a graph form an *asteroidal triple* if every two of them are connected by a path avoiding the neighborhood of the third. A graph is called *AT-free* if it does not contain asteroidal triples.

Lemma 14 *The dominating set problem for* $\{fork, bull\}$*-free graphs can be polynomially reduced to the same problem for* $AT - free$ *graphs.*

Proof Every graph in $Free(\{fork, bull, C_6, C_7, \ldots\})$ is AT-free (Information system on graph classes and their inclusions 2015c). Let G be a connected $\{fork, bull\}$-free graph that contains an induced cycle C_n, where $n \geq 6$. Suppose that $V(G) \neq V(C_n)$. Any vertex outside C_n having a neighbor on it has three neighbors on C_n or all elements of $V(C_n)$ are adjacent to the vertex, as G is $\{fork, bull\}$-free. Moreover, these three neighbors must be consecutive if $n \geq 6$ or pairwise non-adjacent if $n = 6$. Similarly, there are no vertices x_1, x_2, x_3 outside C_n such that x_1 and x_2 have no neighbors on C_n, x_3 has them, (x_1, x_2) and (x_2, x_3) are edges of G, $(x_1, x_3) \notin E(G)$. Moreover, if a vertex x has three neighbors on C_n, then any element of $N(x)$ belongs to the cycle or has a neighbor on it.

Each vertex x adjacent to all vertices of C_n must be adjacent to all elements of $\{x : x \notin V(C_n), |N_{V(C_n)}| = 3\}$, since G is $bull$-free. Let V' be a maximal under inclusion set of vertices adjacent to all vertices of C_n such that for any its members x and y the sets $\{z : z \in N(x) \setminus V(C_n), N_{V(C_n)}(z) = \emptyset\}$ and $\{z : z \in N(y) \setminus V(C_n), N_{V(C_n)}(z) = \emptyset\}$ are nonempty and non-included to each other. Clearly, V' is independent (since G is $bull$-free), and it contains at most two elements, as G is $fork$-free. If $V' \neq \emptyset$, then V' and any vertex of C_n constitute a dominating set of G. If $V' = \emptyset$ and there exist vertices adjacent to all vertices of C_n, then any vertex of such type and any vertex of the cycle dominate all vertices of G. Moreover, if $n = 6$ and there are no vertices adjacent all elements of $V(C_6)$, then $V(C_6)$ is dominating set of G.

Suppose that $n > 6$ and there are no vertices adjacent to all vertices of C_n. Let $x \notin V(C_n)$ have neighbors y_1, y_2, y_3 on C_n listed in the clockwise order. Let $y_0 \in V(C_n)$ be the right neighbor of y_1 and $y_4 \in V(C_n)$ be the left neighbor of y_3. We will show that $N(x) \setminus \{y_2\} = N(y_2) \setminus \{x\}$. If $y \in N(y_2) \setminus \{x, y_1, y_3\}$, then it must be adjacent to three consecutive vertices of C_n. Hence, $N_{V(C_n)}(y) = \{y_4, y_3, y_2\}$ or $N_{V(C_n)}(y) = \{y_3, y_2, y_1\}$ or $N_{V(C_n)}(y) = \{y_2, y_1, y_0\}$. In all three cases y must be adjacent to x to avoid an induced $fork$ or $bull$. If $y \in N(x) \setminus N[y_2]$, then $\{y_3, y_4\} \cap N(y) \neq \emptyset$ and $\{y_1, y_0\} \cap N(y) \neq \emptyset$, to avoid an induced $bull$. The vertex y must be adjacent to three consecutive vertices of C_n. These three conditions are inconsistent, as $n > 6$. We have a contradiction. So, $N(x) \setminus \{y_2\} = N(y_2) \setminus \{x\}$ and the vertex x can been removed preserving the domination number of G by the previous lemma. This finishes the lemma. □

Lemma 15 *The dominating set problem for $\{fork, hammer\}$-free graphs can be polynomially reduced to the same problem for $\{P_5, fork, hammer\}$-free graphs.*

Proof Let G be a connected $\{fork, hammer\}$-free graph containing an induced P_5, and H be a graph obtained from G by deleting any vertex x^* of P_5. We may assume that $\gamma(G) \geq 10$. Hence, $\gamma(H) \geq 9$. Let V_1 be the set of vertices having a neighbor in $V_0 \triangleq V(P_5)$ and a neighbor outside $\bigcup\limits_{x \in V_0} N(x)$. Clearly, any element of V_1 must be adjacent to all vertices of P_5, as G is $\{fork, hammer\}$-free. Let $V_2 \triangleq \bigcup\limits_{x \in V_0} N(x) \setminus (V_0 \cup V_1)$ and $V_3 \triangleq V(G) \setminus (V_1 \cup V_2 \cup V_0)$. Similarly, each element of V_1 must be adjacent to every element of V_2 that does not adjacent to all vertices of P_5 and

$V_3 = \bigcup_{x \in V_1} N(x) \setminus (V_1 \cup V_2 \cup V_0)$. Moreover, any two elements of $z_1, z_2 \in V_3$ with $N_{V_1}(z_1) \nsubseteq N_{V_1}(z_2)$ and $N_{V_1}(z_2) \nsubseteq N_{V_1}(z_1)$ must be nonadjacent.

Let D_G and D_H be minimum dominating sets of G and H, respectively. Clearly, $D_G \cap V_3$ and $D_H \cap V_3$ are independent sets of H. Moreover, $N_{V_1}(x_1) \nsubseteq N_{V_1}(x_2)$ and $N_{V_1}(x_2) \nsubseteq N_{V_1}(x_1)$ for any $x_1, x_2 \in D_G \cap V_3$ and $x_1, x_2 \in D_H \cap V_3$. Hence, some $|D_G \cap V_3|$ elements of V_1 plus $D_G \cap V_1$ dominates V_3. Obviously, $D_G \cap (V_2 \cup V_0)$ has at most six vertices, as $|D_G \cap V_2| - 1$ elements of $D_G \cap V_2$ can be changed to some elements of $V_1 \cup V_0$ to keep the minimality of D_G. The last both facts are true for H. If $D_H \cap V_1$ is empty, then some $|D_H \cap V_3|$ vertices of V_1 dominates V_3. As $|D_H \cap V_3| \geq 3$, then they must constitute a clique, and this clique must dominate $V_1 \cup V_3$, since G is *hammer*-free. Therefore, there is a minimum dominating set of H containing elements of V_1. This result holds for G.

Let D'_G and D'_H be dominating sets of G and H containing elements in V_1. If $x^* \notin D'_G$, then D'_G is a dominating set of H. Otherwise, $(D'_G \setminus \{x^*\}) \cup \{y^*\}$ is a dominating set of H, where y^* is an arbitrary element of $V_0 \setminus \{x^*\}$. As $D'_H \cap V_1 \neq \emptyset$, D'_H is a dominating set of G. Hence, $\gamma(G) = \gamma(H)$. So, we have a polynomial-time reduction to $\{P_5, fork, hammer\}$-free graphs by deleting vertices in induced copies of P_5. □

Theorem 6 *The dominating set problem is polynomial-time solvable for* $\{fork, bull\}$*-free and* $\{fork, hammer\}$*-free graphs.*

Proof A proof follows from Lemmas 14–15, Theorem 3, and the fact that AT-free graphs constitute an easy case for the dominating set problem (Kratsch 2000). □

7 One more boundary class for the dominating set problem

Recall that there are known only three boundary classes for the dominating set problem (Alekseev et al. 2004). They are $\mathcal{S}, \mathcal{T}$, and $\mathcal{Q}$. The fourth boundary class will be presented in this section by revisiting a construction from the paper of Alekseev et al. (2004).

Let $G = (V, E)$ be a *subcubic graph*, i.e., graph having degrees of vertices at most three. Let V' be the set of degree three vertices of G and $V'' \triangleq V(G) \setminus V'$. Assume that G has no adjacent degree three vertices and V' is not a vertex cover of G. Hence, any vertex cover of G must contain at least one element of V''. We define a graph $G^* \triangleq Q^*(G)$ as follows. The set $V(G^*)$ coincides with $V'' \cup E$. A vertex $x \in V'$ is incident to edges $e_1(x), e_2(x), e_3(x)$ in the graph G. The set $E(G^*)$ coincides with $\{(v_i, v_j) : v_i, v_j \in V''\} \cup \{(v, e) : v \in V'', e \in E, v \text{ is incident to } e\} \cup \bigcup_{x \in V'} \{(e_1(x), e_2(x)), (e_1(x), e_3(x)), (e_2(x), e_3(x))\}$.

Let $G_i(x) \triangleq G \setminus e_i(x)$ and $G_i^*(x) \triangleq G^* \setminus e_i(x)$.

Lemma 16 *We have* $\beta(G) = \gamma(G^*)$ *and* $\beta(G_i(x)) = \gamma(G_i^*(x))$.

Proof If VC is a minimum vertex cover of G, then $(VC \cap V'') \cup \bigcup_{x \in V' \cap VC} \{e_1(x)\}$ is a dominating set of G. Otherwise, if D is a minimum dominating set of G^*, then

$(D \cap V'') \cup \bigcup_{x:\{e_1(x), e_2(x), e_3(x)\} \cap D \neq \emptyset} \{x\}$ is a vertex cover of G. Hence, $\beta(G) = \gamma(G^*)$. Similarly, $\beta(G_i(x)) = \gamma(G_i^*(x))$. □

Let $\mathcal{X}_0$ be the set of all subcubic graphs, $\mathcal{X} \subseteq \mathcal{X}_0$, $Q^*(\mathcal{X}) \triangleq \{Q^*(G) : G \in \mathcal{X}$ is a graph, whose set of degree three vertices is independent and is not a vertex cover$\}$. The *hereditary closure of a graph class* $\mathcal{X}$ denoted by $[\mathcal{X}]$ is the set of all induced subgraphs of all graphs in $\mathcal{X}$. Let $\mathcal{X}_i$ be the set of all graphs obtained from elements of $\mathcal{X}_0$ by $2i$-ary subdivision of each edge.

Lemma 17 *The class* $[Q^*(\mathcal{X}_1)]$ *is finitely defined.*

Proof By H_1 we denote the complement of the graph obtained from C_6 by adding an edge between two vertices lying at distance two. By H_2 we denote the graph obtained from two complete graphs with vertices v_1, v_2, v_3, v_4 and u_1, u_2, u_3, u_4 respectively by adding the edges (v_1, u_1), (v_2, u_2), (v_3, u_3), (v_4, u_4). Let H_3 be the graph obtained by coinciding a vertex of C_4 with an end vertex of P_3. Let H_4 be the graph obtained by coinciding an edge of C_4 with an edge of C_4. A graph H_5 can be obtained from a *sinker* by adding a new vertex and an edge incident to the new vertex and the degree two vertex of *sinker*. It is easy to see that all graphs $2P_3$, $2K_4$, $K_4 + P_3$, $K_{1,4}$, $K_{2,3}$, C_5, $\overline{K_2 + O_3}$, $\overline{K_3 + O_2}$, $\overline{C_4 + O_1}$, $\overline{P_2 + P_3}$, H_1, H_2, H_3, H_4, H_5 belong to $Forb([Q^*(\mathcal{X}_1)])$.

Assume that $Forb([Q^*(\mathcal{X}_1)])$ is infinite. Since there is a finite set of connected $\{2P_3, K_{1,4}, K_6\}$-free graphs (as they have bounded degrees of vertices and a bounded diameter), the set $Forb([Q^*(\mathcal{X}_1)]) \cap Free(\{K_6\})$ is finite. Let G be an element of $Forb([Q^*(\mathcal{X}_1)])$ having a maximum clique Q' with at least six vertices. Clearly, G is $\{2K_4, K_4{+}P_3, K_{2,3}, C_5, \overline{K_2 + O_3}, \overline{K_3 + O_2}, \overline{C_4 + O_1}, \overline{P_2 + P_3}, H_1, H_2, H_3, H_4, H_5\}$-free.

Any element of $V(G) \setminus Q'$ is adjacent to at most two vertices of Q', since G must be $\overline{K_2 + O_3}$-free. The clique Q' cannot have a vertex having two adjacent neighbors outside Q', since the vertex, those two neighbors and some two elements of Q' induce $\overline{C_4 + O_1}$. Similarly, each vertex of Q' cannot have three neighbors outside Q', since G is $K_{1,4}$-free and $|Q'| \geq 6$. If x and y are adjacent elements of $V(G) \setminus Q'$ both having neighbors in Q', then $|N_{Q'}(x)| = |N_{Q'}(y)| = 1$, since G is $\{\overline{P_2 + P_3}\}$-free. If x and y are non-adjacent elements of $V(G) \setminus Q'$ both having two neighbors in Q', then $N_{Q'}(x) \neq N_{Q'}(y)$, since G is $\{\overline{K_3 + O_2}\}$-free. The subgraph of G induced by elements of $V(G) \setminus Q'$ having a neighbor in Q' must be $\{P_3, K_4\}$-free, as G is $\{H_1, H_2, K_{2,3}\}$-free. Since G is $\{H_3, H_4\}$-free, $V(G) \setminus Q'$ has no four vertices x_1, x_2, x_3, x_4 such that each of them has a neighbor in Q', $(x_1, x_2) \in E(G)$, $(x_3, x_4) \in E(G)$, x_1 and x_3 have a common neighbor in Q'. Let x be an arbitrary vertex non-dominated by Q'. Since G is $\{2K_4, P_3 + K_4, C_5, H_5\}$-free, x belongs to a connected component of G inducing a clique with at most three vertices or its neighborhood forms a clique with at most two vertices having a vertex dominated by Q', and any such a vertex of the neighborhood has only one neighbor in Q'. Moreover, as $|Q'| \geq 6$ and G is $P_3 + K_4$-free, if $y \notin Q'$ is a neighbor of x with nonempty $N_{Q'}(y)$, then there is no a neighbor z of y such that $z \notin Q' \cup N(x)$ and $N_{Q'}(z) \neq \emptyset$. Hence, G must belong to $[Q^*(\mathcal{X}_1)]$. We have a contradiction with the assumption. So, $[Q^*(\mathcal{X}_1)]$ is finitely defined. □

A class $\mathcal{Q}^*$ is the set $[Q^*(\mathcal{S})]$.

Theorem 7 *The class $\mathcal{Q}^*$ is boundary for the dominating set problem.*

Proof Clearly, the graph $Q^*(G)$ can be constructed in polynomial time for each graph $G \in \mathcal{X}_0$. On the other hand, for any graph $H \in Q(\mathcal{X}_0)$ its inverse image (i.e., a graph $G \in \mathcal{X}_0$ such that $H = Q^*(G)$) can be constructed in polynomial time. Indeed, one may consider only the connected case, i.e., when H is connected. Determine in H a maximal under inclusion clique with at most five vertices if one exists. This clique must correspond to $V''(H)$, and the ends of its "hairs" correspond to $E(H)$. The case of the absence a five-clique is trivial, as $|V''(H)| \leq 4$ and $|E(H)| \leq 12$.

The class $\mathcal{X}_0$ is a hard case for the vertex cover problem (Garey and Johnson 1979). A double subdivision of any edge of each graph increases its vertex cover number by one (Alekseev et al. 2007). Hence, for each i, the vertex cover problem is NP-complete for $\mathcal{X}_i$. The mapping $Q^*(\cdot)$ defined above can be applied to any member of $\mathcal{X}_i$ for $i > 0$. Hence, by Lemma 16, the vertex cover problem for $\mathcal{X}_i$ is polynomially equivalent to the dominating set problem for $Q^*(\mathcal{X}_i)$. Therefore, the last problem is NP-complete for $Q^*(\mathcal{X}_i)$. Moreover, by Lemma 16, for any strongly hereditary class $\mathcal{X} \subseteq \mathcal{X}_0$, the dominating set problem for $[Q^*(\mathcal{X})]$ is polynomially equivalent to the vertex cover problem for $\mathcal{X}$. Since $[\mathcal{X}_1] \supset [\mathcal{X}_2] \supset \dots, \bigcap_{i=1}^{\infty} [\mathcal{X}_i] = \mathcal{S}$ and $\bigcap_{i=1}^{\infty} [Q^*(\mathcal{X}_i)] = \mathcal{Q}^*$, then $\mathcal{Q}^*$ is a limit class for the dominating set problem.

Suppose that there is a limit class $\mathcal{Y} \subset \mathcal{Q}^*$ for the dominating set problem. Then there exists a graph $G \in \mathcal{S}$ such that $\mathcal{Y} \subseteq \mathcal{Q}^* \cap Free(\{Q^*(G)\})$. By Lemma 17, any monotonically decreasing sequence $\{\mathcal{Y}_i\}$ of hard classes for the dominating set problem converging to $\mathcal{Y}$ must contain a member $\mathcal{Y}_{i^*} \subseteq [Q^*(\mathcal{X}_1)] \cap Free(\{Q^*(G)\})$. Clearly, $[Q^*(\mathcal{X}_1)] \cap Free(\{Q^*(G)\}) \subseteq [Q^*(\mathcal{X}_0)] \cap Free(\{Q^*(G)\}) \subseteq [Q^*(\mathcal{X}_0 \cap Free_s(\{G\}))]$, where $Free_s(G)$ is the set of all graphs that do not contain G as a subgraph not necessarily induced. The class $\mathcal{X}_0 \cap Free_s(\{G\})$ is strongly hereditary, as $\mathcal{X}_0$ holds this property. Hence, the dominating set problem for $\mathcal{Y}_{i^*}$ can be polynomially reduced to the vertex cover and the independent set problems for $\mathcal{X}_0 \cap Free_s(\{G\})$. The independent set problem can be solved in polynomial time for any strongly hereditary graph class that does not include $\mathcal{S}$ (Alekseev 2003). Hence, the dominating set problem for $\mathcal{Y}_{i^*}$ is also polynomial-time solvable. We have a contradiction, unless $P = NP$. So, $\mathcal{Q}^*$ must be minimal limit, i.e., boundary.

A possible application of Theorem 7 is to prove NP-completeness of the dominating set problem for some graph classes that Theorem 1 and $\mathcal{S}, \mathcal{T}, \mathcal{Q}$ do not give. For example, $Free(\{2P_3, K_{1,4}\})$ includes no one of $\mathcal{S}, \mathcal{T}, \mathcal{Q}$, but it includes $\mathcal{Q}^*$. Hence, by Theorem 1, it is hard for the problem. □

Acknowledgments The article was prepared within the framework of the Academic Fund Program at the National Research University Higher School of Economics (HSE) in 2015–2016 (Grant 15-01-0010) and supported within the framework of a subsidy granted to the HSE by the Government of the Russian Federation for the implementation of the Global Competitiveness Program.

References

AbouEisha H, Hussain S, Lozin V, Monnot J, Ries B (2014) A dichotomy for upper domination in monogenic classes. In: Combinatorial optimization and applications. Lecture notes in computer science, vol 8881. Springer, pp 258–267

Alekseev V (1983) On the local restrictions effect on the complexity of finding the graph independence number. In: Combinatorial-algebraic methods in applied mathematics. Gorkiy University Press, Gorkiy, pp 3–13 [in russian]

Alekseev V (1999) A polynomial algorithm for finding the largest independent sets in fork-free graphs. Diskretn Analiz i Issled Oper Ser 1 6(4):3–19 [in russian]

Alekseev V (2003) On easy and hard hereditary classes of graphs with respect to the independent set problem. Discret Appl Math 132(1–3):17–26

Alekseev V, Boliac R, Korobitsyn D, Lozin V (2007) NP-hard graph problems and boundary classes of graphs. Theor Comput Sci 389(1–2):219–236

Alekseev V, Korobitsyn D, Lozin V (2004) Boundary classes of graphs for the dominating set problem. Discret Math 285(1–3):1–6

Bacsó G, Tuza Z (1990) Dominating cliques in P_5-free graphs. Period Math Hung 21:303–308

Bertossi A (1984) Dominating sets for split and bipartite graphs. Inform Process Lett 19(1):37–40

Brandstädt A, Dragan F, Le H-O, Mosca R (2005) New graph classes of bounded clique-width. Theory Comput Syst 38(5):623–645

Broersma H, Golovach P, Paulusma D, Song J (2012) Updating the complexity status of coloring graphs without a fixed induced linear forest. Theor Comput Sci 414(1):9–19

Clark B, Colbourn C, Johnson D (1990) Unit disk graphs. Ann Discret Math 48:165–177

Courcelle B, Makowsky J, Rotics U (2000) Linear time optimization problems on graphs of bounded clique width. Theory Comput Syst 33(2):125–150

Garey M, Johnson D (1979) Computers and intractability: a guide to the theory of NP-completeness. Freeman, New York 1979

Golovach P, Paulusma D (2014) List coloring in the absence of two subgraphs. Discret Appl Math 166:123–130

Golovach P, Paulusma D, Ries B (2015) Coloring graphs characterized by a forbidden subgraph. Discret Appl Math 180(1):101–110

Golovach P, Paulusma D, Song J (2013) 4-coloring H-free graphs when H is small. Discret Appl Math 161(1–2):140–150

Graphs: 5-vertices. Information system on graph classes and their inclusions. http://www.graphclasses.org/smallgraphs.html#nodes5. Accessed 27 Jan 2015

Graphclass: AT-free. Information system on graph classes and their inclusions. http://www.graphclasses.org/classes/gc_61.html. Accessed 27 Jan 2015

Korobitsyn D (1992) On the complexity of domination number determination in monogenic classes of graphs. Discret Math Appl 2(2):191–199

Kral' D, Kratochvil J, Tuza Z, Woeginger G (2001) Complexity of coloring graphs without forbidden induced subgraphs. Lect. Notes Comput Sci 2204:254–262

Kratsch D (2000) Domination and total domination in asteroidal triple-free graphs. Discret Appl Math 99(1–3):111–123

Kratsch S, Schweitzer P (2012) Graph isomorphism for graph classes characterized by two forbidden induced subgraphs. Lect Notes Comput Sci 7551:34–45

Lokshtantov D, Vatshelle M, Villanger Y (2014) Independent set in P_5-free graphs in polynomial time. In: Proceedings of the ACM-SIAM Symposium on Discrete Algorithms, pp 570–581

Lozin V (2008) Boundary classes of planar graphs. Comb Probab Comput 17(2):287–295

Lozin V, Malyshev D (2015) Vertex coloring of graphs with few obstructions. Discret Appl Math. doi:10.1016/j.dam.2015.02.015

Lozin V, Mosca R (2004) Independent sets in extensions of $2K_2$-free graphs. Discret Appl Math 146(1):74–80

Malyshev D (2013) The complexity of the 3-colorability problem in the absence of a pair of small forbidden induced subgraphs. Discret Math (submitted)

Malyshev D (2014) The complexity of the edge 3-colorability problem for graphs without two induced fragments each on at most six vertices. Sib Electron Math Rep 11:811–822

Malyshev D (2014) Classes of graphs critical for the edge list-ranking problem. J Appl Ind Math 8(2): 245–255
Problem: Domination. Information system on graph classes and their inclusions. http://www.graphclasses.org/classes/problem_Domination.html. Accessed 27 Jan 2015
Schweitzer P (2014) Towards an isomorphism dichotomy for hereditary graph classes. arXiv, arXiv:1411.1977
Yannakakis M, Gavril F (1980) Edge dominating sets in graphs. SIAM J Appl Math 38(3):364–372
Zverovich I (2003) The domination number of (K_p, P_5)-free graphs. Australas J Comb 27:95–100