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On Maps Of Bounded p-Variation With $p > 1$

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Abstract. This paper addresses properties of maps of bounded p-variation ($p > 1$) in the sense of N. Wiener, which are defined on a subset of the real line and take values in metric or normed spaces. We prove the structural theorem for these maps and study their continuity properties. We obtain the existence of a Hölder continuous path of minimal p-variation between two points and establish the compactness theorem relative to the p-variation, which is an analog of the well-known Helly selection principle in the theory of functions of bounded variation. We prove that the space of maps of bounded p-variation with values in a Banach space is also a Banach space. We give an example of a Hölder continuous of exponent $0 < \gamma < 1$ set-valued map with no continuous selection. In the case $p = 1$ we show that a compact absolutely continuous set-valued map from the compact interval into subsets of a Banach space admits an absolutely continuous selection.

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1. Introduction

The purpose of this paper is to obtain properties of maps of bounded p-variation in the classical sense of Norbert Wiener (cf. Wiener, 1924, and Young, 1937). Consider a map $f : E \to X$ of bounded p-variation (see Sec. 2) defined on the nonempty subset E of the reals $\mathbb{R}$ with values in the metric or normed space X. If $p = 1$, the properties of the variation in the sense of C. Jordan (cf. Schwartz, 1967) were recently studied by the first author (Chistyakov, 1992 and 1997). Here we concentrate mainly on the case where $p > 1$. If $X = \mathbb{R}$, $p = 1$ and E is a closed bounded interval $[\,a, b\,]$ or an open interval $]\,a, b\,[$, then, $f : E \to \mathbb{R}$ is a function of bounded variation if and only if it is the difference of two bounded nondecreasing functions (Jordan's decomposition); see, e.g., Natanson (1965), Ch. 8. However, this criterion is inapplicable if $p > 1$, to say nothing of the case where X is a metric or a normed vector space.

If $p > 1$ and X is a metric space, we show that $f : E \to X$ is a map of bounded p-variation if and only if it is the composition of a bounded nondecreasing function

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$\varphi : E \to \mathbb{R}$ and an X-valued map defined on the image of φ and satisfying a Hölder condition of exponent $\gamma = 1/p$ and the Hölder constant ≤ 1 (Sec. 3). We point out that no special structure of the domain E, such as connectedness (open and closed intervals, etc.), is needed to obtain the properties of maps of bounded p-variation. In this way, we establish the general properties of these maps in Sec. 2 and the continuity properties in Sec. 4.

With the decomposition theorem at hand, in the case of the compact metric space X we prove that there always exist Hölderian geodesic paths (relative to the p-variation) between two points of X if there is at least one path of finite p-variation connecting these points (Sec. 5), and that any infinite family of maps of uniformly bounded p-variation admits a sequence which converges pointwise to a map of bounded p-variation (Helly's selection principle, Sec. 6).

In Sec. 7 we obtain additional properties of maps of bounded p-variation with values in normed vector spaces. In particular, we prove that if X is a Banach space, then the space of maps of bounded p-variation is a Banach space as well.

Finally, in Sec. 8 we treat set-valued maps (or multifunctions) of bounded p-variation. We show that a Hölder continuous of exponent $0 < \gamma < 1$ set-valued map may have no continuous selection. In the case $p = 1$ we prove that any compact absolutely continuous set-valued map from the compact interval into subsets of a Banach space admits an absolutely continuous selection.

2. Main Properties Of The p-Variation

2.1. NOTATION AND DEFINITION

Throughout this paper we exploit the following notation: $\emptyset \neq E \subset \mathbb{R}$, $E_t^- = \{ s \in E : s \leq t \}$ and $E_t^+ = \{ s \in E : t \leq s \}$ if $t \in E$, $E_a^b = E_a^+ \cap E_b^- = (E_a^+)_b^-$ if $a, b \in E$, $a \leq b$ (in particular, $[\, a, b\,] = \mathbb{R}_a^b$), X is a metric space with a fixed metric $d = d(\cdot\,, \cdot)$, X^E is the set of all maps $f : E \to X$ from E into X. Given a map $f \in X^E$, we denote by $f(E) = \{ f(t) : t \in E \}$ the image of f in X and by $\omega(f, E) = \sup\{ d(f(t), f(s)) : t,\ s \in E \}$ the diameter of the image $f(E)$ (or the oscillation of f on E). The composition $f \circ \varphi : E_1 \to X$ of two maps $f : E \to X$ and $\varphi : E_1 \to E$ is defined as usual by $(f \circ \varphi)(\tau) = f(\varphi(\tau))$ for all $\tau \in E_1$. In what follows, p is a fixed number, $1 < p < \infty$. We write $A := B$ or $B =: A$ to indicate that A is defined by means of B.

DEFINITION 2.1 *We denote by*

$$\mathcal{T}(E) = \{ T = \{t_i\}_{i=0}^m \subset E : m \in \mathbb{N} \cup \{0\},\ t_{i-1} \leq t_i,\ i = 1, \ldots, m \}$$

the set of all partitions of E by finite ordered collections of points from E. Given a map $f : E \to X$ and a partition $T = \{t_i\}_{i=0}^m \in \mathcal{T}(E)$, we set

$$V_p[f, T] = \sum_{i=1}^m d\big(f(t_i), f(t_{i-1})\big)^p,$$

where

$$d\big(f(t_i), f(t_{i-1})\big)^p = \Big(d\big(f(t_i), f(t_{i-1})\big)\Big)^p,$$

and define $V_p(f, E)$ *by*

$$V_p(f, E) = \sup\{ V_p[f, T] : T \in \mathcal{T}(E) \}.$$

The quantity $V_p(f, E) \in [0, \infty]$ *is called the total* p*-variation of* f *on* E. *If* $V_p(f, E) < \infty$, *the map* f *is said to be of bounded* p*-variation. The set of all maps of bounded* p*-variation from* E *into* X *is denoted by* $\mathcal{V}_p(E; X)$. *If* $\emptyset \neq A \subset E$, *we set* $V_p(f, A) = V_p(f|_A, A)$, *where* $f|_A$ *is the restriction of* f *to* A. *We also set* $\mathcal{T}(\emptyset) = \emptyset$ *and* $V_p(f, \emptyset) = 0$ *(so that* $\sup \emptyset = 0$*). The map* $V_p : X^E \times 2^E \to \mathbb{R}_0^+ \cup \{\infty\}$ *is called a* p*-variation.*

The above definition of $V_p(f, E)$ was introduced by Wiener (1924); if $p = 1$, it is classical and is due to C. Jordan (see Schwartz, 1967, Ch. 4, Sec. 9). Note that this definition is also suitable for maps defined on any linearly ordered set E. A number of results of this paper are valid in the case, where $\le$ is a linear ordering on E.

Now we list the general properties of the p-variation and deduce some of their consequences (if $p = 1$, see Chistyakov, 1997, Sec. 2).

2.2. GENERAL PROPERTIES OF THE p-VARIATION

Let $f : E \to X$ be an arbitrary map. We have

(P1) *minimality*: if $t, s \in E$, then $d(f(t), f(s))^p \le \omega(f, E)^p \le V_p(f, E)$;

(P2) *monotonicity*: if $a, t, s, b \in E$ and $a \le t \le s \le b$, then $V_p(f, E_t^-) \le V_p(f, E_s^-)$, $V_p(f, E_s^+) \le V_p(f, E_t^+)$, and $V_p(f, E_t^s) \le V_p(f, E_a^b)$;

(P3) *semi-additivity*: if $t \in E$, then
$$2^{1-p} V_p(f, E) \le V_p(f, E_t^-) + V_p(f, E_t^+) \le V_p(f, E);$$

(P4) *change of a variable*: if $E_1 \subset \mathbb{R}$ and $\varphi : E_1 \to E$ is a (not necessarily strictly) monotone function, then $V_p(f, \varphi(E_1)) = V_p(f \circ \varphi, E_1)$;

(P5) *regularity*: $V_p(f, E) = \sup\{ V_p(f, E_a^b) : a, b \in E, a \le b\}$;

(P6) *limit properties*: if $s = \sup E \in \mathbb{R} \cup \{\infty\}$ and $i = \inf E \in \mathbb{R} \cup \{-\infty\}$, we have

($P6_1$) if $s \notin E$, then $V_p(f, E) = \lim_{E \ni t \to s} V_p(f, E_t^-)$,

($P6_2$) if $i \notin E$, then $V_p(f, E) = \lim_{E \ni t \to i} V_p(f, E_t^+)$,

($P6_3$) if $s \notin E$ and $i \notin E$, then, in addition to ($P6_1$) and ($P6_2$), we have

$$V_p(f, E) = \lim_{\substack{E \ni a \to i \\ E \ni b \to s}} V_p(f, E_a^b) = \lim_{E \ni b \to s} \lim_{E \ni a \to i} V_p(f, E_a^b) =$$
$$= \lim_{E \ni a \to i} \lim_{E \ni b \to s} V_p(f, E_a^b);$$

(P7) *lower semi-continuity*: if the sequence of maps $\{f_n\}_{n=1}^{\infty} \subset X^E$ converges pointwise to f as $n \to \infty$ (i.e., $\lim_{n\to\infty} d(f_n(t), f(t)) = 0$ for all $t \in E$), then $V_p(f, E) \le \liminf_{n\to\infty} V_p(f_n, E)$.

Proof of (P1)–(P7). Properties (P1) and (P2) are obvious.

(P3): Step 1. First, we prove the following assertion:

Let $T = \{t_i\}_{i=0}^m \in \mathcal{T}(E)$ and $t \in E$. We have

(a) *if $t \le t_0$ or $t_m \le t$, then $V_p[f, T] \le V_p[f, T \cup \{t\}]$;*

(b) *if $t_{k-1} \le t \le t_k$ for some $1 \le k \le m$, then*

$$V_p[f, T] \le 2^{p-1} V_p[f, T \cup \{t\}].$$

Since (a) is clear, we turn to (b). Setting $\Sigma_1 = \sum_{i=1}^{k-1} d(f(t_i), f(t_{i-1}))^p$ and $\Sigma_2 = \sum_{i=k+1}^{m} d(f(t_i), f(t_{i-1}))^p$, we have

$$\begin{aligned} V_p[f, T] &= \Sigma_1 + d(f(t_k), f(t_{k-1}))^p + \Sigma_2 \le \\ &\le \Sigma_1 + \Big(d(f(t), f(t_{k-1})) + d(f(t_k), f(t)) \Big)^p + \Sigma_2 \le \\ &\le \Sigma_1 + 2^{p-1}\Big(d(f(t), f(t_{k-1}))^p + d(f(t_k), f(t))^p \Big) + \Sigma_2 \le \\ &\le 2^{p-1} V_p[f, T \cup \{t\}]. \end{aligned}$$

Here we have used the triangle inequality and the inequality $(\alpha+\beta)^p \le 2^{p-1}(\alpha^p + \beta^p)$, $\alpha \ge 0$, $\beta \ge 0$, $p \ge 1$. Note that in case (b) we also have the obvious equality

$$\begin{aligned} V_p[f, T \cup \{t\}] \quad &= V_p[f, T] + d(f(t), f(t_{k-1}))^p + \\ &\quad + d(f(t_k), f(t))^p - d(f(t_k), f(t_{k-1}))^p. \end{aligned} \tag{2.1}$$

Step 2. Let $T = \{t_i\}_{i=0}^m \in \mathcal{T}(E)$. Set $S = T \cup \{t\}$. We have two cases as in (a) and (b) above. If $t \le t_0$ or $t_m \le t$, then

$$V_p[f, T] \le V_p[f, S] \le V_p(f, E_t^-) + V_p(f, E_t^+).$$

If $t_{k-1} \le t \le t_k$, then, by virtue of the above assertion, we have

$$\begin{aligned} V_p[f, T] &\le 2^{p-1} V_p[f, S] = 2^{p-1}(V_p[f, S_t^-] + V_p[f, S_t^+]) \le \\ &\le 2^{p-1}(V_p(f, E_t^-) + V_p(f, E_t^+)). \end{aligned}$$

Taking the supremum over all $T \in \mathcal{T}(E)$, we arrive at the left hand side inequality in (P3).

Now we prove the right hand side inequality. If $V_p(f, E_t^-) = \infty$ or $V_p(f, E_t^+) = \infty$, then $V_p(f, E) = \infty$, since, by the monotonicity, $V_p(f, E)$ is greater or equal to $V_p(f, E_t^-)$ and $V_p(f, E_t^+)$. Let $V_p(f, E_t^-) < \infty$ and $V_p(f, E_t^+) < \infty$. Then for every $\varepsilon > 0$ there are partitions $T_1 \in \mathcal{T}(E_t^-)$ and $T_2 \in \mathcal{T}(E_t^+)$ such that

$$V_p(f, E_t^-) \le V_p[f, T_1] + \frac{\varepsilon}{2} \quad \text{and} \quad V_p(f, E_t^+) \le V_p[f, T_2] + \frac{\varepsilon}{2}.$$

It follows that

$$\begin{aligned}V_p(f, E_t^-) + V_p(f, E_t^+) &\le V_p[f, T_1] + V_p[f, T_2] + \varepsilon \le \\ &\le V_p[f, T_1 \cup \{t\}] + V_p[f, T_2 \cup \{t\}] + \varepsilon = \\ &= V_p[f, T_1 \cup \{t\} \cup T_2] + \varepsilon \le \\ &\le V_p(f, E) + \varepsilon,\end{aligned}$$

and it remains to take into account the arbitrariness of $\varepsilon > 0$.

(P4): We shall prove that the right hand side is not greater than the left hand side, and vice versa. If $T_1 = \{\tau_i\}_{i=0}^m \in \mathcal{T}(E_1)$ and $T := \{t_i\}_{i=0}^m$ with $t_i := \varphi(\tau_i)$, then $T \in \mathcal{T}(\varphi(E_1))$ by the monotonicity of φ, and

$$\begin{aligned}V_p[f \circ \varphi, T_1] &= \sum_{i=1}^m d(f(\varphi(\tau_i)), f(\varphi(\tau_{i-1})))^p = \\ &= \sum_{i=1}^m d(f(t_i), f(t_{i-1}))^p = V_p[f, T] \le \\ &\le V_p(f, \varphi(E_1)).\end{aligned}$$

On the other hand, if a partition $T = \{t_i\}_{i=0}^m$ of $\varphi(E_1)$ is such that $t_{i-1} < t_i$ for $i = 1, \dots, m$, then there exist $\tau_i \in E_1$ such that $t_i = \varphi(\tau_i)$ and, again by the monotonicity of φ, $T_1 := \{\tau_i\}_{i=0}^m \in \mathcal{T}(E_1)$, so that, as above, we have

$$V_p[f, T] = V_p[f \circ \varphi, T_1] \le V_p(f \circ \varphi, E_1).$$

(P5): By the monotonicity of V_p, it is clear that the left hand side is not less than the right hand side. On the other hand, for any number $\alpha < V_p(f, E)$ there is a partition $T = \{t_i\}_{i=0}^m \in \mathcal{T}(E)$ such that $V_p[f, T] \ge \alpha$. Since, actually, $T \in \mathcal{T}(E_{t_0}^{t_m})$, it follows that $V_p(f, E_{t_0}^{t_m}) \ge V_p[f, T] \ge \alpha$, which was to be proved.

($\mathrm{P6}_1$): Since $s = \sup E \notin E$, s is a limit point of E, so that the filter base $E \ni t \to s$ is well defined. By virtue of (P2), the function $E \ni t \mapsto V_p(f, E_t^-) \in [0, \infty]$ is nondecreasing, and, hence, the limit in ($\mathrm{P6}_1$) exists (in $[0, \infty]$). Clearly, this limit is $\le V_p(f, E)$. On the other hand, due to (P5), for any $\alpha < V_p(f, E)$ there are $a, b \in E$, $a \le b < s$, such that $V_p(f, E_a^b) \ge \alpha$, which implies that $V_p(f, E_t^-) \ge V_p(f, E_a^b) \ge \alpha$ for any $t \in E \cap]\, b, s\, [\ne \emptyset$, and the equality in ($\mathrm{P6}_1$) follows.

($\mathrm{P6}_2$) and the first equality in ($\mathrm{P6}_3$) can be proved similarly.

($\mathrm{P6}_3$): In order to prove the second equality in ($\mathrm{P6}_3$), we apply ($\mathrm{P6}_1$) and ($\mathrm{P6}_2$):

$$\begin{aligned}V_p(f, E) &= \lim_{E \ni b \to s} V_p(f, E_b^-) = \lim_{E \ni b \to s} \lim_{E \ni a \to i} V_p(f, (E_b^-)_a^+) = \\ &= \lim_{E \ni b \to s} \lim_{E \ni a \to i} V_p(f, E_a^b).\end{aligned}$$

The last equality in ($\mathrm{P6}_3$) can be proved similarly.

(P7): Let $\alpha \in \mathbb{R}$ be such that $\alpha < V_p(f, E)$. From the definition of $V_p(f, E)$, for any $\alpha < \beta < V_p(f, E)$ there exists a partition $T = \{t_i\}_{i=0}^m$ of E such that $V_p[f, T] \geq \beta$. Consider the function

$$v_p : X^{m+1} = \underbrace{X \times \cdots \times X}_{m+1} \to \mathbb{R}$$

defined by

$$v_p(x_0, x_1, \ldots, x_m) = \sum_{i=1}^{m} d(x_i, x_{i-1})^p, \quad x_0, x_1, \ldots, x_m \in X.$$

Since v_p is continuous at every point of X^{m+1} (to see this, it suffices to employ the inequality $|a^p - b^p| \leq p|a - b|a^{p-1}$, $a \geq 0$, $b \geq 0$), it is continuous at the point $(x_0, x_1, \ldots, x_m) = (f(t_0), f(t_1), \ldots, f(t_m))$, so that for $\varepsilon = \beta - \alpha > 0$ there is a $\delta = \delta(\varepsilon) > 0$ such that if $(y_0, y_1, \ldots, y_m) \in X^{m+1}$ and $d(x_i, y_i) \leq \delta$, $i = 0, 1, \ldots, m$, then

$$|v_p(x_0, x_1, \ldots, x_m) - v_p(y_0, y_1, \ldots, y_m)| \leq \varepsilon = \beta - \alpha.$$

From the pointwise convergence of f_n to f one can find an integer $N = N(\delta) \in \mathbb{N}$ such that

$$d(f_n(t_i), f(t_i)) \leq \delta \quad \forall n \geq N, \quad \forall i = 0, 1, \ldots, m.$$

Setting $(y_0, y_1, \ldots, y_m) = (f_n(t_0), f_n(t_1), \ldots, f_n(t_m))$ and noting that

$$v_p(x_0, x_1, \ldots, x_m) = V_p[f, T], \qquad v_p(y_0, y_1, \ldots, y_m) = V_p[f_n, T],$$

we have

$$\beta \leq V_p[f, T] \leq V_p[f_n, T] + \varepsilon \leq V_p(f_n, E) + (\beta - \alpha) \quad \forall n \geq N.$$

Therefore, $\inf_{n \geq N} V_p(f_n, E) \geq \alpha$, and, hence, $\liminf_{n\to\infty} V_p(f_n, E) \geq \alpha$. It remains to let α go to $V_p(f, E)$. □

REMARK 2.1 *If* $f : E \to \mathbb{R}$ *is a* bounded monotone *function, then*

$$V_p(f, E) = \omega(f, E)^p = \Big(\sup_{t\in E} f(t) - \inf_{t\in E} f(t)\Big)^p.$$

To see this, let $a, b \in E$, $a \leq b$. *Clearly,* $V_p(f, E_a^b) \geq |f(b) - f(a)|^p$, *by (P1). On the other hand, if* $T = \{t_i\}_{i=0}^m \in \mathcal{T}(E_a^b)$ *and* $t \in E_a^b$ *is such that* $t_{k-1} \leq t \leq t_k$ *for some* $1 \leq k \leq m$, *then, by virtue of the monotonicity of* f, *we have*

$$|f(t_k) - f(t_{k-1})| = |f(t_k) - f(t)| + |f(t) - f(t_{k-1})|,$$

so that the equality (2.1) *implies that* $V_p[f, T \cup \{t\}] \leq V_p[f, T]$. *By induction, it follows that*

$$V_p[f, T] \leq V_p[f, \{a, b\}] = |f(b) - f(a)|^p \quad \forall\, T \in \mathcal{T}(E_a^b),$$

and, hence,

$$V_p(f, E_a^b) \leq |f(b) - f(a)|^p \leq \omega(f, E)^p \quad \forall\, a, b \in E,\, a \leq b.$$

Applying (P5), we arrive at $V_p(f, E) \leq \omega(f, E)^p$, *and, since the reverse inequality is always true, by (P1), we are through.*

REMARK 2.2 *The constants in (P3) are sharp: indeed, if* $f : [\,0, 1\,] \to \mathbb{R}$ *is defined by* $f(t) = t$ *for* $0 \leq t \leq 1$, *then we have* $2^{1-p} \leq t^p + (1-t)^p \leq 1$, *and the left and right hand side equalities are attained at* $t = 1/2$ *and* $t = 0, 1$, *respectively. Another example is the function* $f : [\,0, 1\,] \to \mathbb{R}$ *defined by* $f(t) = 0$ *if* $0 \leq t < 1/3$, $f(t) = a$ *if* $1/3 \leq t \leq 2/3$ *and* $f(t) = a + b$ *if* $2/3 < t \leq 1$, *for appropriately chosen* $a > 0$ *and* $b > 0$.

It should be noted that, in general, $V_p(f, T) > V_p[f, T]$ *for a finite set* T *if* $p > 1$. *To see this, in the above example of the function* $f(t) = t$ *on* $[0, 1]$ *consider* $T = \{\,0, t, 1\,\}$ *with* $0 < t < 1$. *Then (see Remark 2.1):*

$$V_p[f, T] = t^p + (1-t)^p < 1 = V_p(f, T).$$

REMARK 2.3 *Property (P6_1) is not true in general if* $s \in E$, *since* $V_p(f, E) = V_p(f, E_s^-)$; *consider, for instance,* $f : [\,0, 1\,] \to \mathbb{R}$ *such that* $f = 0$ *on* $[\,0, 1\,[$ *and* $f(1) = 1$. *A similar remark applies to (P6_2) and (P6_3).*

REMARK 2.4 *The inequality* $\leq$ *in (P7) cannot be replaced by the equality even if the convergence of* f_n *to* f *is uniform; for example, if* $f_n(t) = |\sin(2\pi n t)|/n^{1/p}$, $t \in [\,0, 1\,]$, *then* f_n *converges uniformly to* $f \equiv 0$, *but* $V_p(f_n, [\,0, 1\,]) = 4$. *Note that the sequence* $\{f_n\}$ *does not converge to* $f = 0$ *in the* normed vector space $\mathcal{V}_p([\,0, 1\,]; \mathbb{R})$, *defined in section 7.*

PROPOSITION 2.1 (minimality of V_p). *Suppose that the map*

$$W_p : X^E \times 2^E \to [\,0, \infty]$$

satisfies, for all $f : E \to X$ *and* $\emptyset \neq A \subset E$, *the following conditions* ($W_p(f, \emptyset) = 0$):

(a) $d(f(t), f(s))^p \leq W_p(f, A)$ *for all* $t, s \in A$;
(b) $W_p(f, A_t^s) \leq W_p(f, A)$ *for all* $t, s \in A$ *such that* $t \leq s$;
(c) $W_p(f, A_t^-) + W_p(f, A_t^+) \leq W_p(f, A)$ *for all* $t \in A$.

Then $V_p(f, A) \leq W_p(f, A)$ *for all* $f : E \to X$ *and* $A \subset E$.

Proof. Clearly, the p-variation V_p satisfies the properties (a)–(c). Thus, if $f: E \to X$, $\emptyset \neq A \subset E$ and $T = \{t_i\}_{i=0}^m \in \mathcal{T}(A)$, we have

$$V_p[f,T] = \sum_{i=1}^m d(f(t_i), f(t_{i-1}))^p \overset{\text{(a)}}{\leq} \sum_{i=1}^m W_p(f, A_{t_{i-1}}^{t_i}) \overset{\text{(c)}}{\leq}$$
$$\leq W_p(f, A_{t_0}^{t_m}) \overset{\text{(b)}}{\leq} W_p(f, A),$$

and the proposition follows upon taking the supremum over all partitions T of A. $\square$

Property (P1) implies that a map of bounded p-variation is a bounded map in the sense that its image has a finite diameter. The following proposition is a refinement of this property.

PROPOSITION 2.2 *If $f \in \mathcal{V}_p(E;X)$, then the image $f(E) \subset X$ is totally bounded and separable. If, in addition, X is complete, then $f(E)$ is precompact* (*i.e., the closure of $f(E)$ in X is compact*).

Proof. In order to prove that $f(E)$ is totally bounded, we have to show that for every $\varepsilon > 0$ the set $f(E)$ can be covered by a finite number of balls from X of radius ε centered at $f(E)$. On the contrary, let $\varepsilon > 0$ be such that $f(E)$ cannot be covered by finitely many balls of radius ε. Choose a sequence $\{x_n\}_{n=0}^\infty \subset E$ inductively as follows: begin with any $t_0 \in E$ and set $x_0 = f(t_0)$, and having chosen $x_0, x_1, \dots, x_{n-1} \in f(E)$, pick $x_n \in f(E) \setminus \bigcup_{j=1}^{n-1} B_\varepsilon(x_j)$ where $B_\varepsilon(x_j) = \{y \in X : d(y, x_j) < \varepsilon\}$. Let $t_n \in E$ be such that $x_n = f(t_n)$, $n \in \mathbb{N}$. Since $d(x_n, x_k) \geq \varepsilon$ for $n \neq k$, we have $t_n \neq t_k$. Without loss of generality we can suppose that $t_{n-1} < t_n$ for all $n \in \mathbb{N}$. Then, for $T_m := \{t_i\}_{i=0}^m \in \mathcal{T}(E)$, we have

$$V_p(f,E) \geq V_p[f,T_m] = \sum_{i=1}^m d(f(t_i), f(t_{i-1}))^p = \sum_{i=1}^m d(x_i, x_{i-1})^p \geq m\varepsilon^p.$$

Since $m \in \mathbb{N}$ is arbitrary, we infer that $V_p(f,E) = \infty$, which is a contradiction.

A totally bounded set in a metric space is known to be separable, and precompact if the metric space is complete. $\square$

REMARK 2.5 *If $1 \leq p \leq q$, then $\mathcal{V}_1(E;X) \subset \mathcal{V}_p(E;X) \subset \mathcal{V}_q(E;X)$ since, if $T = \{t_i\}_{i=0}^m$ is a partition of E, then we have*

$$V_q[f,T] = \sum_{i=1}^m d(f(t_i), f(t_{i-1}))^q = \sum_{i=1}^m \Big(d(f(t_i), f(t_{i-1}))^p \Big)^{q/p} \leq$$
$$\leq \left(\sum_{i=1}^m d(f(t_i), f(t_{i-1}))^p \right)^{q/p} \leq (V_p(f,E))^{q/p}.$$

PROPOSITION 2.3 *If $f \in \mathcal{V}_{p_0}(E; X)$ for some $p_0 \geq 1$, then*

$$\lim_{p\to\infty} \big(V_p(f, E)\big)^{1/p} = \omega(f, E) < \infty.$$

Proof. Since the function $v(p) := \big(V_p(f, E)\big)^{1/p}$, $p \geq p_0$, is bounded from below ($v(p) \geq \omega(f, E)$, by (P1)) and nonincreasing (by Remark 2.5), it has a limit as $p \to \infty$ which we denote by $v(\infty)$, so that $v(\infty) \geq \omega(f, E)$. To prove the reverse inequality, we note that for $p > p_0$ and $T = \{t_i\}_{i=0}^m \in \mathcal{T}(E)$ we have:

$$\begin{aligned} V_p[f, T] &\leq \big(\omega(f, E)\big)^{p-p_0} \sum_{i=1}^{m} d(f(t_i), f(t_{i-1}))^{p_0} \leq \\ &\leq \big(\omega(f, E)\big)^{p-p_0} V_{p_0}(f, E), \end{aligned}$$

which yields (after taking the supremum over $T \in \mathcal{T}(E)$)

$$V_p(f, E) \leq \big(\omega(f, E)\big)^{p-p_0} V_{p_0}(f, E).$$

It remains to pass to the limit as $p \to \infty$ in the inequality

$$v(p) = \big(V_p(f, E)\big)^{1/p} \leq \big(\omega(f, E)\big)^{1-(p_0/p)} \cdot \big(V_{p_0}(f, E)\big)^{1/p}.$$

□

3. A Structural Theorem

We recall that a map $f : E \to X$ is *Hölderian of exponent* $0 < \gamma \leq 1$ if there exists a number $C \in \mathbb{R}_0^+$ such that $d(f(t), f(s)) \leq C|t - s|^\gamma$ for all $t, s \in E$. The least number C satisfying the above inequality is called the *Hölder constant* of f and is denoted by $H(f)$.

The main result of this section is the following structural theorem.

THEOREM 3.1 *The map $f : E \to X$ is of bounded p-variation if and only if there exist a bounded nondecreasing function $\varphi : E \to \mathbb{R}$ and a Hölderian map $g : \varphi(E) \to X$ of exponent $\gamma = 1/p$ and $H(g) \leq 1$ such that $f = g \circ \varphi$ on E.*

Moreover, if X is a Banach space, the map $g : \varphi(E) \to X$ can be extended to a Hölderian map $\overline{g} : \mathbb{R} \to X$ of the same exponent $\gamma = 1/p$ and the Hölder constant $H(\overline{g}) \leq 3^{1-\gamma} H(g)$.

The proof of this theorem is contained in the following three lemmas. The first lemma (sufficiency) gives a large number of examples of maps of bounded p-variation.

LEMMA 3.2 *If $\varphi : E \to \mathbb{R}$ is bounded monotone, $g : \varphi(E) \to X$ is Hölderian of exponent $\gamma = 1/p$ and $f = g \circ \varphi$, then $f \in \mathcal{V}_p(E; X)$.*

Proof. Suppose that φ is nondecreasing. Since

$$\varphi(E \cap [\,a, b\,]) = \varphi(E) \cap [\,\varphi(a), \varphi(b)\,], \quad a, b \in E, a \le b, \tag{3.1}$$

by virtue of (P4) we have

$$V_p(f, E_a^b) = V_p(g \circ \varphi, E_a^b) = V_p(g, \varphi(E_a^b)) = V_p(g, \varphi(E)_{\varphi(a)}^{\varphi(b)}).$$

If $T = \{t_i\}_{i=0}^m$ is a partition of the set in (3.1), then

$$V_p[g, T] \le H(g)^p \cdot \sum_{i=1}^m (t_i - t_{i-1}) \le H(g)^p \cdot (\varphi(b) - \varphi(a)),$$

which implies that

$$V_p(f, E_a^b) \le H(g)^p \cdot (\varphi(b) - \varphi(a)) \quad \forall\, a, b \in E, a \le b.$$

Now property (P5) and the monotonicity and boundedness of φ yield

$$V_p(f, E) \le H(g)^p \cdot \Big(\sup_{t\in E} \varphi(t) - \inf_{t \in E} \varphi(t)\Big) = H(g)^p \cdot \omega(\varphi, E) < \infty.$$

The proof is similar if φ is nonincreasing. $\square$

REMARK 3.1 *In particular, if $f : E \to X$ is Hölderian of exponent $\gamma = 1/p$ and $E \subset \mathbb{R}$ is a bounded set, then f is of bounded p-variation and*

$$V_p(f, E) \le H(f)^p \cdot (\sup E - \inf E).$$

In the second lemma (necessity) we obtain the canonical decomposition of a map of bounded p-variation.

LEMMA 3.3 *Let $f : E \to X$ be a map of bounded p-variation. Then there exist a bounded nondecreasing nonnegative function $\varphi : E \to \mathbb{R}$ and a Hölderian map $g : E_1 := \varphi(E) \to X$ of exponent $\gamma = 1/p$ and the Hölder constant $H(g) \le 1$ such that*

(a) $f = g \circ \varphi$ *on* E;
(b) $g(E_1) = f(E)$ *in* X;
(c) $V_p(g, E_1) = V_p(f, E)$.

Proof. The function $\varphi : E \to \mathbb{R}$ given by $\varphi(t) = V_p(f, E_t^-)$, $t \in E$, is well-defined, nonnegative, bounded ($\varphi(t) \le V_p(f, E)$) and nondecreasing, due to (P2). If $\tau \in E_1$, denote by $\varphi^{-1}(\tau) = \{\, t \in E : \varphi(t) = \tau\,\}$ the inverse image of the one-point set $\{\tau\}$ under the function φ. We define the map $g : E_1 \to X$ as follows: if $\tau \in E_1$ we set

$$g(\tau) = f(t) \quad \text{for any point} \quad t \in \varphi^{-1}(\tau). \tag{3.2}$$

This is correct, i.e., $f(t)$ is one and the same element of X for all $t \in \varphi^{-1}(\tau)$, since by virtue of (P1) and (P3), we have

$$d(f(s), f(t))^p \le V_p(f, E_t^s) \le \varphi(s) - \varphi(t), \quad t \in E, s \in E_t^+; \tag{3.3}$$

indeed, if $t, s \in \varphi^{-1}(\tau)$, $t \le s$, then $\varphi(t) = \tau = \varphi(s)$, so that (3.3) implies $f(t) = f(s)$.

Now, the representation of f in (a) follows from (3.2), for if $t \in E$, then $\tau := \varphi(t) \in E_1$ and $t \in \varphi^{-1}(\tau)$, so that (3.2) yields $f(t) = g(\tau) = g(\varphi(t)) = (g \circ \varphi)(t)$. The assertions in (b) and (c) follow from (a) and (P4).

It remains to prove that g is Hölderian. As in (3.1), we have

$$(E_1)_\tau^- = \varphi(E_t^-) \quad \text{for any} \quad \tau \in E_1 \quad \text{and} \quad t \in \varphi^{-1}(\tau),$$

so that applying (P4), we arrive at

$$V_p(g, (E_1)_\tau^-) = V_p(g, \varphi(E_t^-)) = V_p(g \circ \varphi, E_t^-) = V_p(f, E_t^-) = \varphi(t) = \tau.$$

Hence, if $\alpha, \beta \in E_1$, $\alpha \le \beta$, then, by virtue of (P1) and (P3), we infer that

$$d(g(\beta), g(\alpha))^p \le V_p(g, (E_1)_\alpha^\beta) \le V_p(g, (E_1)_\beta^-) - V_p(g, (E_1)_\alpha^-) = \beta - \alpha.$$

□

REMARK 3.2 *Note that the map g in the proof of Lemma 3.3 satisfies the property: if $\alpha, \beta \in E_1$ and $t \in \varphi^{-1}(\alpha)$, $s \in \varphi^{-1}(\beta)$, then*

$$d(g(\alpha), g(\beta)) = d(g(\varphi(t)), g(\varphi(s))) = d(f(t), f(s)).$$

REMARK 3.3 *In the case, where $\varphi : E \to E_1$ is strictly increasing, it is a bijection, so that the equality $f = g \circ \varphi$ on E is equivalent to the equality $g = f \circ \varphi^{-1}$ on E_1 where $\varphi^{-1} : E_1 \to E$ is the inverse function of φ.*

REMARK 3.4 *An algebraic aspect in the construction of the map such as g in Lemma 3.3 was considered by* Chistyakov *(*1997*), Sec. 3.*

LEMMA 3.4 *Let X be a Banach space* (*over the field $\mathbb{R}$ or $\mathbb{C}$*) *and $g : E_1 \to X$ be a Hölderian map of exponent $0 < \gamma \le 1$. Then there exists a Hölderian map $\overline{g} : \mathbb{R} \to X$ of the same exponent γ and $H(\overline{g}) \le 3^{1-\gamma} H(g)$ such that the restriction of $\overline{g}$ to E_1 coincides with g.*

Proof. Since g is uniformly continuous on E_1, it admits an extension to the closure $\overline{E_1}$ of E_1, denoted by g_1, such that $g_1 : \overline{E_1} \to X$ is Hölderian of exponent γ and $H(g_1) \le H(g)$. We define $\overline{g}$ to be equal to g_1 on $\overline{E_1}$. The complement $\mathbb{R} \setminus \overline{E_1}$ of $\overline{E_1}$ in $\mathbb{R}$ is open, and, hence, it is at most a countable union of disjoint open intervals $]\,a_k, b_k[$. On intervals $]\,a_k, b_k\,[$ with $b_k - a_k < \infty$ we define $\overline{g}$ as follows:

$$\overline{g}(t) = g_1(a_k) + c_k(t - a_k)^\gamma, \quad c_k := \frac{g_1(b_k) - g_1(a_k)}{(b_k - a_k)^\gamma}, \quad t \in]\,a_k, b_k\,[.$$

If $a_k = -\infty$, we set $\overline{g}(t) = g_1(b_k)$ for all $t \in]-\infty, b_k]$, and if $b_k = \infty$, we set $\overline{g}(t) = g_1(a_k)$ for all $t \in [a_k, \infty[$.

If $\|\cdot\|$ denotes the norm in X, then $\|c_k\| \le H(g_1) \le H(g)$, and, hence, if $b_k - a_k < \infty$, then for all $t, s \in [a_k, b_k]$ we have

$$\|\overline{g}(t) - \overline{g}(s)\| = \|c_k\| \cdot |(t-a_k)^\gamma - (s-a_k)^\gamma| \le H(g) \cdot |t-s|^\gamma.$$

Note that if $a_k = -\infty$ (or $b_k = \infty$), then $\overline{g}$ is constant on $]-\infty, b_k]$ (or on $[a_k, \infty[$).

It remains to verify that $\overline{g}$ is Hölderian on $\mathbb{R}$. There are three cases: 1) $t \in \overline{E_1}$, $s \in \overline{E_1}$; 2) $t \in \overline{E_1}$, $s \notin \overline{E_1}$; 3) $t \notin \overline{E_1}$, $s \notin \overline{E_1}$. Case 1) is clear from the above. In case 2), suppose that $s \in]a_k, b_k[$ and $b_k \le t$. Using the triangle and Hölder's inequalities, we have

$$\begin{aligned}\|\overline{g}(t) - \overline{g}(s)\| &\le \|g_1(t) - g_1(b_k)\| + \|g_1(b_k) - \overline{g}(s)\| \le \\ &\le H(g) \cdot ((t-b_k)^\gamma + (b_k - s)^\gamma) \le \\ &\le 2^{1-\gamma} H(g)|t-s|^\gamma.\end{aligned}$$

In case 3), suppose that $t \in]a_m, b_m[$, $s \in]a_k, b_k[$ and $b_k \le a_m$. Again, using the triangle and Hölder's inequalities, we infer that

$$\begin{aligned}\|\overline{g}(t) - \overline{g}(s)\| &\le \|\overline{g}(t) - g_1(a_m)\| + \|g_1(a_m) - g_1(b_k)\| + \\ &\quad + \|g_1(b_k) - \overline{g}(s)\| \le \\ &\le H(g) \cdot ((t-a_m)^\gamma + (a_m - b_k)^\gamma + (b_k - s)^\gamma) \le \\ &\le 3^{1-\gamma} H(g)|t-s|^\gamma.\end{aligned}$$

The proof is complete. □

4. Continuity Properties

In this section we study continuity properties of maps of bounded p-variation and show that, in the large, these maps behave like maps of bounded variation with $p = 1$, cf. Chistyakov (1997), Sec. 4, however, the proof technique is entirely different.

In this section we assume that X is a metric space, $f : E \to X$ is a fixed map of bounded p-variation and the function $\varphi : E \to \mathbb{R}$ is defined by $\varphi(t) = V_p(f, E_t^-)$ for $t \in E$.

LEMMA 4.1 *Let $t \in E$ be a limit point of the set $E_t^\pm$ (in what follows, $+$ and $-$ are concordant). Then $d(f(t), f(s))$ has a limit in $[0, \infty[$ as $E \ni s \to t \pm 0$.*

If, moreover, X is complete, then, as $E \ni s \to t \pm 0$, $f(s)$ has a one-sided limit in X, denoted by $f(t\pm)$, and $d(f(s), f(t))$ tends to $d(f(t\pm), f(t))$ as $E \ni s \to t \pm 0$.

Proof. Let $t \in E$ be a limit point of E_t^-. If $s_1, s_2 \in E$, $s_1 \le s_2 < t$, then, by virtue of (P1) and (P3), we have

$$|d(f(t), f(s_1)) - d(f(t), f(s_2))|^p \le d(f(s_1), f(s_2))^p \le \\ \le V_p(f, E_{s_1}^{s_2}) \le V_p(f, E_{s_2}^-) - V_p(f, E_{s_1}^-) = \varphi(s_2) - \varphi(s_1). \tag{4.1}$$

Since the function φ is bounded and nondecreasing, the limit $\varphi(t-) := \lim_{E \ni s \to t-0} \varphi(s)$ exists and is equal to $\sup\{\varphi(s) : s \in E_t^-,\ s \ne t\}$. The existence of the limit of $d(f(t), f(s))$ as $E \ni s \to t - 0$ now follows from (4.1) and Cauchy's criterion in the complete metric space $\mathbb{R}$.

Now, let X be complete. If $s_1, s_2 \in E$, $s_1 \le s_2 < t$, then, as above, we have: $d(f(s_1), f(s_2))^p \le \varphi(s_2) - \varphi(s_1)$, and, hence, Cauchy's criterion of the existence of the limit $f(t-)$ applies in the complete metric space X. It remains to note that, as $E \ni s \to t - 0$,

$$|d(f(s), f(t)) - d(f(t-), f(t))| \le d(f(s), f(t-)) \to 0.$$

The case, where $t \in E$ is a limit point of the set E_t^+, is completely analogous. □

REMARK 4.1 *Note that if $t \in E$ is a limit point of E_t^-, then, applying property (P6_1) with the set $E_t^- \setminus \{t\}$ in place of E_t^-, we get*

$$V_p(f, E_t^- \setminus \{t\}) = \lim_{E \ni s \to t-0} V_p(f, (E_t^- \setminus \{t\})_s^-) = \\ = \lim_{E \ni s \to t-0} V_p(f, E_s^-) = \varphi(t-).$$

THEOREM 4.2 *Let $f : E \to X$ be a map of bounded p-variation. Then* (a) *f is continuous at the point $t \in E$ if and only if the function φ is continuous at t*; (b) *f is continuous on E outside, possibly, of a subset of E which is at most countable.*

Proof. (a) The case, where $t \in E$ is an isolated point of E, is obvious (and uninformative). Hence, in the rest of the proof we assume that $t \in E$ is a limit point of E; moreover, we assume, in addition, that $t \in E$ is a limit point of each of the sets E_t^- and E_t^+.

Sufficiency in (a) follows from the inequalities (cf. (4.1)):

$$d(f(t), f(s))^p \le V_p(f, E_s^t) \le \varphi(t) - \varphi(s), \quad s \in E_t^-,$$
$$d(f(s), f(t))^p \le V_p(f, E_t^s) \le \varphi(s) - \varphi(t), \quad s \in E_t^+.$$

In particular, these inequalities imply that

$$\lim_{E \ni s \to t-0} d(f(t), f(s)) \le (\varphi(t) - \varphi(t-))^{1/p},$$
$$\lim_{E \ni s \to t+0} d(f(s), f(t)) \le (\varphi(t+) - \varphi(t))^{1/p}.$$

Necessity in (a) follows from Lemma 4.3 below.

(b) This assertion is a consequence of the fact that a nondecreasing function on E has at most countably many points of discontinuity and that, by (a), the sets of discontinuity points of f and the nondecreasing function φ are the same. □

LEMMA 4.3 *Let $t \in E$ be a limit point of each of the sets E_t^- and E_t^+. Then*

$$\varphi(t+) - \varphi(t-) \le p\, M\Big(\lim_{E \ni s \to t-0} d(f(t),\, f(s)) + \lim_{E \ni s \to t+0} d(f(s),\, f(t))\Big), \tag{4.2}$$

where $M = M(f, p, E) := (V_p(f, E))^{\frac{p-1}{p}}$.

Before the proof of this Lemma a few remarks are in order.

REMARK 4.2 *If X is a complete metric space, then* (*4.2*) *assumes the form*

$$\varphi(t+) - \varphi(t-) \le p\, M(\, d(f(t),\, f(t-)) + d(f(t+),\, f(t))\,).$$

If, moreover, $p = 1$, then $\varphi(t+) - \varphi(t-) \le d(f(t),\, f(t-)) + d(f(t+),\, f(t))$. In this case, the last inequality is, actually, the equality (cf. Chistyakov, 1997, *Lemma 5.2(a,b)).*

REMARK 4.3 *If $t = \inf E \in E$ is a limit point of E, then the inequality* (*4.2*) *holds if we replace the first limit by zero and $\varphi(t-)$ by $\varphi(t)$. If $t \in E$ is a limit point of E_t^-, then* (*4.2*) *holds as well, if we replace the second limit by zero and $\varphi(t+)$ by $\varphi(t)$. In particular, this remark and Lemma 4.3 imply that if $f : [\,a, b\,] \to X$ is a continuous map of bounded p-variation, then the function $\varphi : [\,a, b\,] \to \mathbb{R}$, defined by $\varphi(t) = V_p(f, [\,a, t\,])$ for $t \in [\,a, b\,]$, is also continuous.*

REMARK 4.4 *The estimate* (*4.2*) *is "sharp" as the following example shows. For $0 < \varepsilon < 1$ define $f_\varepsilon : [\,0, 2\,] \to \mathbb{R}$ by: $f_\varepsilon(t) = \varepsilon t$ if $0 \le t < 1$ and $f_\varepsilon(t) = 1$ if $1 \le t \le 2$, and set $\varphi_\varepsilon(t) = V_p(f_\varepsilon, [\,0, t\,])$ for $t \in [\,0, 2\,]$. Then $M = M(f_\varepsilon, p, [\,0, 2\,]) = 1$ and*

$$\frac{\varphi_\varepsilon(1+) - \varphi_\varepsilon(1-)}{|f_\varepsilon(1) - f_\varepsilon(1-)| + |f_\varepsilon(1+) - f_\varepsilon(1)|} = \frac{1 - \varepsilon^p}{1 - \varepsilon} \quad \underset{\varepsilon \to 1-0}{\longrightarrow} \quad p = p\, M.$$

In order to prove Lemma 4.3, we need one more lemma.

LEMMA 4.4 *If $a, s, b \in E$, then*

$$d(f(b),\, f(a))^p \le d(f(s),\, f(a))^p + p\, M d(f(b),\, f(s)),$$

where M is the same as in Lemma 4.3.

Proof. Since $d(f(b),\, f(a))^{p-1} \le M$ by (P1), it suffices to prove that

$$d(f(b),\, f(a))^p \le d(f(s),\, f(a))^p + p\, d(f(b),\, f(a))^{p-1} \cdot d(f(b),\, f(s)),$$

or, equivalently, that

$$d(f(b),\, f(a))^{p-1} \cdot [\, d(f(b),\, f(a)) - p\, d(f(b),\, f(s))\,] \le d(f(s),\, f(a))^p. \tag{4.3}$$

If the expression in square brackets in (4.3) is < 0, then we are through. Now suppose that the $[\dots] \geq 0$. We note that the left hand side in (4.3) is less than or equal to

$$\begin{aligned}&\Big(d(f(b), f(s)) + d(f(s), f(a))\Big)^{p-1} \cdot \Big[d(f(b), f(s)) + \\ &\quad + d(f(s), f(a)) - p\, d(f(b), f(s))\Big].\end{aligned} \tag{4.4}$$

Setting $u = d(f(b), f(s))$ and $v = d(f(s), f(a))$ and taking into account that

$$(u+v)^p - v^p = \int_v^{u+v} p\,\xi^{p-1}\,d\xi \leq p\,u\,(u+v)^{p-1},$$

we have the inequality

$$(u+v)^{p-1}[\,u + v - p\,u\,] \leq v^p,$$

which, together with (4.4), proves (4.3). □

Proof of Lemma 4.3. Set

$$A = \lim_{E \ni s \to t-0} d(f(t), f(s)), \quad B = \lim_{E \ni s \to t+0} d(f(s), f(t)).$$

Let $\varepsilon > 0$ be fixed. Choose $a_0, b_0 \in E$, $a_0 < t < b_0$, such that

$$|d(f(t), f(s)) - A| \leq \varepsilon \qquad \forall\, s \in E,\, a_0 \leq s < t, \tag{4.5}$$

$$|d(f(s), f(t)) - B| \leq \varepsilon \qquad \forall\, s \in E,\;\; t < s \leq b_0. \tag{4.6}$$

Let $T = \{t_0 < t_1 < \dots < t_{m-1} < t_m\}$ be a partition of the set $E_{b_0}^-$ with the property (from the definition of $\varphi(b_0) = V_p(f, E_{b_0}^-)$)

$$\varphi(b_0) - \varepsilon \leq V_p[f, T]. \tag{4.7}$$

First of all, we consider the case, where $t_0 < t < t_m$. There are two cases: I) $t \notin T$, and II) $t \in T$.

I) Let $t \notin T$. There is a $k \in \{1, \dots, m\}$ such that $t_{k-1} < t < t_k$, so that we have

$$\begin{aligned}V_p[f, T] &= \sum_{i=1}^{k-1} d(f(t_i), f(t_{i-1}))^p + d(f(t_k), f(t_{k-1}))^p + \\ &\quad + \sum_{i=k+1}^{m} d(f(t_i), f(t_{i-1}))^p \leq \\ &\leq \varphi(t_{k-1}) + d(f(t_k), f(t_{k-1}))^p + V_p(f, E_{t_k}^{b_0}) \leq \\ &\leq \varphi(t_{k-1}) + d(f(t_k), f(t_{k-1}))^p + \varphi(b_0) - \varphi(t+);\end{aligned} \tag{4.8}$$

here we have used that $V_p(f, E_{t_k}^{b_0}) \le \varphi(b_0) - \varphi(t_k) \le \varphi(b_0) - \varphi(t+)$. We have two cases: a) $a_0 \le t_{k-1}$, and b) $t_{k-1} < a_0$.

a) If $a_0 \le t_{k-1}$, then, taking into account (4.5) and (4.6), we have

$$\begin{aligned} &d(f(t_k), f(t_{k-1}))^p = d(f(t_k), f(t_{k-1}))^{p-1} \cdot d(f(t_k), f(t_{k-1})) \le \\ &\le M[\, d(f(t), f(t_{k-1})) + d(f(t_k), f(t)) \,] \le M(A + \varepsilon + B + \varepsilon). \end{aligned}$$

By virtue of (4.8), it follows that

$$V_p[f, T] \le \varphi(t-) + M(A + B + 2\varepsilon) + \varphi(b_0) - \varphi(t+). \tag{4.9}$$

b) If $t_{k-1} < a_0$, then, using Lemma 4.4 with $a = t_{k-1}$, $s = a_0$ and $b = t_k$, and using (4.5) and (4.6), we have

$$\begin{aligned} &d(f(t_k), f(t_{k-1}))^p \le d(f(a_0), f(t_{k-1}))^p + p\, M d(f(t_k), f(a_0)) \le \\ &\quad \le d(f(a_0), f(t_{k-1}))^p + p\, M[\, d(f(t), f(a_0)) + d(f(t_k), f(t)) \,] \le \\ &\quad \le d(f(a_0), f(t_{k-1}))^p + p\, M(A + B + 2\varepsilon). \end{aligned}$$

By virtue of (4.8), it follows that

$$\begin{aligned} V_p[f, T] \;&\le\; \varphi(t_{k-1}) + d(f(a_0), f(t_{k-1}))^p + p\, M(A + B + 2\varepsilon) + \\ &\quad + \varphi(b_0) - \varphi(t+) \le \\ &\le\; \varphi(a_0) + p\, M(A + B + 2\varepsilon) + \varphi(b_0) - \varphi(t+) \le \\ &\le\; \varphi(t-) + p\, M(A + B + 2\varepsilon) + \varphi(b_0) - \varphi(t+). \end{aligned} \tag{4.10}$$

II) Now we consider the second case: suppose that $t \in T$. There is a $k \in \{1, \ldots, m-1\}$ such that $t = t_k$, so that we have

$$\begin{aligned} V_p[f, T] \;&=\; \sum_{i=1}^{k-1} d(f(t_i), f(t_{i-1}))^p + d(f(t), f(t_{k-1}))^p + \\ &\quad + d(f(t_{k+1}), f(t))^p + \sum_{i=k+2}^{m} d(f(t_i), f(t_{i-1}))^p \le \\ &\le\; \varphi(t_{k-1}) + d(f(t), f(t_{k-1}))^p + \\ &\quad + d(f(t_{k+1}), f(t))^{p-1} \cdot d(f(t_{k+1}), f(t)) + \varphi(b_0) - \varphi(t+) \le \\ &\le\; \varphi(t_{k-1}) + d(f(t), f(t_{k-1}))^p + M(B + \varepsilon) + \varphi(b_0) - \varphi(t+), \end{aligned} \tag{4.11}$$

where in the last inequality we have used (4.6). As above, we have two cases: a) $a_0 \le t_{k-1}$, and b) $t_{k-1} < a_0$.

a) If $a_0 \le t_{k-1}$, then, taking into account (4.5), we have

$$d(f(t), f(t_{k-1}))^p = d(f(t), f(t_{k-1}))^{p-1} \cdot d(f(t), f(t_{k-1})) \le M(A + \varepsilon).$$

By virtue of (4.11), it follows that

$$V_p[f, T] \le \varphi(t-) + M(A + B + 2\varepsilon) + \varphi(b_0) - \varphi(t+). \tag{4.12}$$

b) If $t_{k-1} < a_0$, then, using Lemma 4.4 with $a = t_{k-1}$, $s = a_0$ and $b = t_k = t$, and using (4.5), we have

$$\begin{aligned} d(f(t), f(t_{k-1}))^p &\le d(f(a_0), f(t_{k-1}))^p + p\,M d(f(t), f(a_0)) \le \\ &\le d(f(a_0), f(t_{k-1}))^p + p\,M(A + \varepsilon). \end{aligned}$$

By virtue of (4.11), it follows that

$$\begin{aligned} V_p[f, T] \;&\le\; \varphi(t_{k-1}) + d(f(a_0), f(t_{k-1}))^p + p\,M(A + \varepsilon) + \\ &\quad + M(B + \varepsilon) + \varphi(b_0) - \varphi(t+) \le \\ &\le\; \varphi(t-) + p\,M(A + B + 2\varepsilon) + \varphi(b_0) - \varphi(t+). \end{aligned} \tag{4.13}$$

Therefore, from (4.9), (4.10), (4.12), and (4.13), we infer that in both cases I) and II) we have the inequality

$$V_p[f, T] \le \varphi(b_0) + \varphi(t-) - \varphi(t+) + p\,M(A + B + 2\varepsilon).$$

Taking into account (4.7), we find that

$$\varphi(t+) - \varphi(t-) \le p\,M(A + B) + \varepsilon(1 + 2p\,M) \qquad \forall\,\varepsilon > 0.$$

Now, it is clear from the above that the last inequality can be similarly proved if $t \le t_0$ or $t_m \le t$. □

5. Paths Of Minimal p-Variation

Let $E = [\,a, b\,]$ be a compact interval in $\mathbb{R}$. We denote by $\mathcal{T}_a^b$ the set

$$\{\, T = \{t_i\}_{i=0}^m \subset [\,a, b\,] : m \in \mathbb{N},\; a = t_0 < t_1 < \cdots < t_{m-1} < t_m = b \,\}$$

of all partitions of $[\,a, b\,]$ containing points a and b and we set

$$V_{a,p}^b(f) = \sup\{\, V_p[f, T] : T \in \mathcal{T}_a^b \,\}.$$

Clearly, $V_{a,p}^b(f) = V_p(f, [\,a, b\,])$.

We denote by $\mathcal{C}([\,a, b\,]; X)$ the set of all continuous maps from $[\,a, b\,]$ into the metric space X. A *path* in X is a continuous map $f : [\,a, b\,] \to X$; its *trajectory* is the image $f([\,a, b\,])$ which, as is well known, is a compact subset of X. The domain $[\,a, b\,]$ of f is called a *set of parameters* on (of) the path, in which case we also say that the path is *parametrized* by the interval $[\,a, b\,]$. Two points $x, y \in X$ are said to be *connected by a path* in X if there exists a path $f : [\,a, b\,] \to X$ such that $f(a) = x$ and $f(b) = y$, in which case we say that f is a *path between x and y*.

The following theorem asserts the existence of a Hölderian geodesic path between two points with respect to the p-variation, and it extends the results which were previously given for paths in the case $p = 1$ (see Busemann, 1955, Ch. 1, (5.18), and, for a more precise statement, Chistyakov, 1997, Theorem 6.1).

THEOREM 5.1 *Let K be a compact subset of X and $x, y \in K$. If there is a path in K between x and y of finite p-variation, then the points x and y can be connected in K by a Hölderian path of exponent $\gamma = 1/p$ of minimal p-variation.*

Proof. The theorem is clear if $x = y$. Hence we suppose that $x \neq y$. Since any path $f : [a, b] \to X$ can be replaced by a path of the same p-variation (and the same trajectory) and the set of parameters $[0, 1]$ (see (P4)), it suffices to restrict our consideration to paths defined on $[0, 1]$. Thus, consider the set of paths in K defined on $[0, 1]$ and connecting the points x and y:

$$W(x, y) = \{ f \in \mathcal{C}([0, 1]; K) : f(0) = x,\ f(1) = y \},$$

and set

$$\ell = \inf\{ V_{0,p}^1(f) : f \in W(x, y) \}.$$

By the assumption, $W(x, y)$ contains a path f_0 of finite p-variation, so that $\ell \le V_{0,p}^1(f_0)$ is finite. On the other hand, by virtue of (P1), for any $f \in W(x, y)$ we have

$$V_{0,p}^1(f) \ge d(f(0), f(1))^p = d(x, y)^p > 0, \tag{5.1}$$

so that $\ell \ge d(x, y)^p$. Since $\ell < \infty$, there exists a sequence $\{f_n\}_{n=1}^\infty$ in $W(x, y)$ such that

$$\lim_{n\to\infty} \ell_n = \ell, \quad \text{where} \quad \ell_n = V_{0,p}^1(f_n) > 0 \quad \text{by (5.1)}.$$

The existence of the last limit implies that if $L = \sup_{n\in\mathbb{N}} \ell_n$, then L is finite > 0, so that the sequence $\{f_n\}$ is of uniformly bounded p-variation. By Lemma 3.3, for any $n \in \mathbb{N}$ there exists a path $g_n : [0, \ell_n] \to X$ with the properties

$$d(g_n(\alpha), g_n(\beta)) \le |\alpha - \beta|^{1/p}, \quad \alpha, \beta \in [0, \ell_n],$$

$$f_n = g_n \circ \varphi_n \quad \text{on} \quad [0, 1], \quad \text{where} \quad \varphi_n(t) = V_{0,p}^t(f_n),\ t \in [0, 1],$$

and, in particular, $g_n(0) = f_n(0) = x$, $g_n(\ell_n) = f_n(1) = y$, $g_n([0, \ell_n]) = f_n([0, 1]) \subset K$ and $V_{0,p}^{\ell_n}(g_n) = V_{0,p}^1(f_n) = \ell_n$. If we set $h_n(\tau) = g_n(\tau\ell_n)$, $\tau \in [0, 1]$, then we have

$$h_n \in W(x, y),$$
$$V_{0,p}^1(h_n) = \ell_n \to \ell \quad \text{as} \quad n \to \infty \quad \text{(by (P4))},$$
$$d(h_n(\alpha), h_n(\beta)) \le (\ell_n)^{1/p}|\alpha - \beta|^{1/p} \le L^{1/p}|\alpha - \beta|^{1/p}, \quad \alpha, \beta \in [0, 1].$$

It follows that the sequence $\{h_n\}_{n=1}^{\infty} \subset \mathcal{C}([0,1]; K)$ is equicontinuous, so that by Ascoli-Arzelà's theorem (cf. Folland, 1984, p. 131, Theorem (4.44)), it has a subsequence $\{h_{n_k}\}_{k=1}^{\infty}$ which converges uniformly on $[0,1]$ to a map $h \in \mathcal{C}([0,1]; K)$. It is clear that $h \in W(x,y)$, and h is Hölderian of exponent $\gamma = 1/p$ and $H(h) \le L^{1/p}$. From (P7) we infer that

$$V_{0,p}^{1}(h) \le \liminf_{k\to\infty} V_{0,p}^{1}(h_{n_k}) = \lim_{k\to\infty} \ell_{n_k} = \ell.$$

It remains to note that from the definition of ℓ we have $\ell \le V_{0,p}^{1}(h)$, so that $\ell = V_{0,p}^{1}(h)$, which was to be proved. □

6. Helly's Selection Principle

The main result of this section is the following compactness theorem relative to the p-variation, which, in the theory of mappings of bounded variation (i.e., when $p = 1$), is known as *E. Helly's selection principle* (cf. Natanson, 1965, Ch. 8, Sec. 4, Helly's theorem, and more recently Chistyakov, 1997, Sec. 7, theorem 7.1).

THEOREM 6.1 *Let K be a compact subset of the metric space X and $\mathcal{F} \subset \mathcal{C}([a,b]; K)$ be an infinite family of continuous maps from the interval $[a,b]$ into K of uniformly bounded p-variation, that is,* $\sup_{f\in\mathcal{F}} V_{a,p}^{b}(f) < \infty$. *Then there exists a sequence $\{f_n\}_{n=1}^{\infty}$ of maps from $\mathcal{F}$ which converges pointwise on $[a,b]$ to a map $f : [a,b] \to K$ of bounded p-variation.*

Moreover, if X is a Banach space, then the assumption of continuity of the family $\mathcal{F}$ is redundant.

Proof. Step 1 (*common part*). By Theorem 3.1, any map $f \in \mathcal{F}$ can be written in the form $f = g_f \circ \varphi_f$ on $[a,b]$, where $\varphi_f(t) = V_{a,p}^{t}(f)$, $a \le t \le b$, and $g_f : E_{1f} = \varphi_f([a,b]) \to K$ is Hölderian of exponent $\gamma = 1/p$ and $H(g_f) \le 1$. Note that φ_f is nondecreasing, nonnegative and $\varphi_f(a) = 0$. The family of nondecreasing functions $\{\varphi_f : f \in \mathcal{F}\}$ is infinite and uniformly bounded on $[a,b]$, since $\omega(\varphi_f, [a,b]) = \varphi_f(b) = V_{a,p}^{b}(f)$, and, hence, by the well known fact (Natanson, 1965, Ch. 8, Sec. 4, Lemma 2), it contains a sequence of functions $\{\varphi_n\}_{n=1}^{\infty}$, corresponding to the decompositions $f_n = g_n \circ \varphi_n$ (i.e., $\varphi_n = \varphi_{f_n}$ and $g_n = g_{f_n}$) for all $n \in \mathbb{N}$, which converges pointwise on $[a,b]$ to a nondecreasing (and bounded) function $\varphi : [a,b] \to \mathbb{R}$. Let $\ell = V_{a,p}^{b}(\varphi) = \varphi(b)$. Then $0 \le \ell < \infty$, and if $\ell_n = V_{a,p}^{b}(f_n) = V_{a,p}^{b}(\varphi_n) = \varphi_n(b)$, then $\ell_n \to \ell$ as $n \to \infty$.

Step 2. Suppose that the family $\mathcal{F}$ consists of continuous maps. Since $f_n \in \mathcal{F}$ is continuous, φ_n is continuous as well, so that the Hölderian map g_n is defined on $E_{1n} = \varphi_n([a,b]) = [0,\ell_n]$. If $\ell_n \ge \ell$, then we consider g_n only on the segment $[0,\ell]$, and if $\ell_n < \ell$, then we extend g_n to $]\ell_n,\ell]$ by setting $g_n(\tau) = g_n(\ell_n)$ for all $\tau \in]\ell_n,\ell]$. By Ascoli-Arzelà's theorem, the sequence of Hölderian maps $g_n : [0,\ell] \to K$ of exponent $\gamma = 1/p$ and $H(g_n) \le 1$ is precompact in $\mathcal{C}([0,\ell]; K)$, so that it has a uniformly convergent subsequence $\{g_{n_k}\}_{k=1}^{\infty}$. Let

g be the uniform limit of $\{g_{n_k}\}$. Clearly, $g : [\,0, \ell\,] \to K$ is Hölderian of exponent $\gamma = 1/p$ and $H(g) \le 1$, so that, by virtue of Lemma 3.2, the composed map $f = g \circ \varphi : [\,a, b\,] \to X$ is of bounded p-variation. Now, if $t \in [\,a, b\,]$, we have

$$\begin{aligned} d(f_{n_k}(t), f(t)) &= d((g_{n_k} \circ \varphi_{n_k})(t), (g \circ \varphi)(t)) \le \\ &\le d(g_{n_k}(\varphi_{n_k}(t)), g_{n_k}(\varphi(t))) + d(g_{n_k}(\varphi(t)), g(\varphi(t))) \le \\ &\le |\varphi_{n_k}(t) - \varphi(t)|^{1/p} + d(g_{n_k}(\varphi(t)), g(\varphi(t))). \end{aligned}$$

Since the terms in the last sum tend to zero as $k \to \infty$, the sequence $\{f_{n_k}\}_{k=1}^{\infty} \subset \mathcal{F}$ converges pointwise on $[\,a, b\,]$ to f.

Step 3. Let X be a Banach space and $\mathcal{F}$ be an infinite family of maps from $[\,a, b\,]$ into K of uniformly bounded p-variation. We again use the reasoning of step 1. Note that in this case $E_{1n} = \varphi_n([\,a, b\,]) \subset [\,0, \ell_n]$. If $L = \sup_{n \in \mathbb{N}} \ell_n$, then $0 \le L < \infty$ and $\ell = \lim_{n\to\infty} \ell_n \le L$. Denote by $\widetilde{g_n}$ the restriction to $[\,0, L\,]$ of the map $\overline{g_n}$ given by Lemma 3.4. By Ascoli-Arzelà's theorem, the sequence of Hölderian maps $\widetilde{g_n} : [\,0, L\,] \to K$ of exponent $\gamma = 1/p$ and $H(\widetilde{g_n}) \le 3^{1-\gamma}$ has a uniformly convergent subsequence $\{\widetilde{g_{n_k}}\}_{k=1}^{\infty}$, whose uniform limit we denote by $\widetilde{g}$. It is clear that $\widetilde{g} : [\,0, L\,] \to K$ is Hölderian of exponent $\gamma = 1/p$ and $H(\widetilde{g}) \le 3^{1-\gamma}$. Let $E_1 = \varphi([\,a, b\,])$, and let g be the restriction of $\widetilde{g}$ to E_1. By virtue of Lemma 3.2, the map $f = g \circ \varphi : [\,a, b\,] \to K$ is of bounded p-variation. Now, if $t \in [\,a, b\,]$, then, as at the end of step 2, we have

$$\begin{aligned} d(f_{n_k}(t), f(t)) &= d(\widetilde{g_{n_k}}(\varphi_{n_k}(t)), \widetilde{g}(\varphi(t))) \le \\ &\le 3^{1-(1/p)} |\varphi_{n_k}(t) - \varphi(t)|^{1/p} + d(\widetilde{g_{n_k}}(\varphi(t)), \widetilde{g}(\varphi(t))), \end{aligned}$$

which completes the proof. □

REMARK 6.1 *We do not know if the condition "$\mathcal{F} \subset K^{[\,a,b\,]}$ where $K \subset X$ is compact" in Theorem 6.1 can be replaced by a weaker condition: "for every $t \in [\,a, b\,]$ the section $\mathcal{F}(t) = \{\, f(t) \in X : f \in \mathcal{F} \,\}$ is precompact in X".*

REMARK 6.2 *Note that the continuity of the family $\mathcal{F}$ does not, in general, imply that the resulting map of bounded p-variation f is continuous.*

7. Maps Valued In A Normed Vector Space

In this section we assume that X is a normed vector space over the field $\mathbb{K} = (\mathbb{R}$ or $\mathbb{C})$ with the norm $\|\cdot\|$ and, as usual, $\emptyset \ne E \subset \mathbb{R}$. Naturally, X^E becomes a vector space (over $\mathbb{K}$) with respect to the pointwise operations:

$$(f + g)(t) = f(t) + g(t), \quad (cf)(t) = cf(t), \quad f, g \in X^E, \ c \in \mathbb{K}, \ t \in E.$$

The space X^E is endowed with the functional

$$\|\, f\,\|_p^* = \|\, f(a)\,\| + (V_p(f, E))^{1/p}, \quad a \in E \ \text{fixed}, \quad f \in X^E.$$

PROPOSITION 7.1 (a) *The functional* $\| \cdot \|_p^* : X^E \to [0, \infty]$ *is a* pseudonorm *on* X^E (*i.e., it satisfies the axioms of a norm and possibly takes infinite values*);

(b) *If* $\{f_n\}_{n=1}^{\infty} \subset X^E$, $f \in X^E$ *and* $\| f_n - f \|_p^* \to 0$ *as* $n \to \infty$, *then* $V_p(f_n, E)$ *tends to* $V_p(f, E)$ *as* $n \to \infty$; *if, moreover,* $\{f_n\}_{n=1}^{\infty} \subset \mathcal{V}_p(E; X)$, *then* $\sup_{n\in\mathbb{N}} V_p(f_n, E) < \infty$ *and* $f \in \mathcal{V}_p(E; X)$.

Proof. (a) Let $f, g \in X^E$ and $c \in \mathbb{K}$. First, note that (P1) implies that

$$\| f(t) \| \le \| f(a) \| + (V_p(f, E))^{1/p} = \| f \|_p^* \quad \forall\, t \in E. \tag{7.1}$$

The other two axioms of a norm are consequences of the (in)equalities

$$\left.\begin{aligned} (V_p[cf, T])^{1/p} &= |c|(V_p[f, T])^{1/p}, \\ (V_p[f+g, T])^{1/p} &\le (V_p[f, T])^{1/p} + (V_p[g, T])^{1/p}, \end{aligned}\right. \qquad T \in \mathcal{T}(E),$$

the last one following from the triangle and Minkowski's inequalities.

(b) For all $t \in E$ we have $\| f_n(t) - f(t) \| \le \| f_n - f \|_p^* \to 0$ as $n \to \infty$, so that

$$(V_p(f, E))^{1/p} \le \liminf_{n\to\infty} (V_p(f_n, E))^{1/p}, \quad \text{by (P7).}$$

On the other hand, for all $n \in \mathbb{N}$

$$\begin{aligned} (V_p(f_n, E))^{1/p} &\le (V_p(f_n - f, E))^{1/p} + (V_p(f, E))^{1/p} \le \\ &\le \| f_n - f \|_p^* + (V_p(f, E))^{1/p}, \end{aligned} \tag{7.2}$$

whence

$$\limsup_{n\to\infty} (V_p(f_n, E))^{1/p} \le \lim_{n\to\infty} \| f_n - f \|_p^* + (V_p(f, E))^{1/p} = (V_p(f, E))^{1/p}.$$

Therefore, $(V_p(f, E))^{1/p} = \lim_{n\to\infty} (V_p(f_n, E))^{1/p}$.

Suppose now that $f_n \in \mathcal{V}_p(E; X)$, $n \in \mathbb{N}$. From (7.2) with $f = f_k$ we have, as $n, k \to \infty$,

$$|(V_p(f_n, E))^{1/p} - (V_p(f_k, E))^{1/p}| \le \| f_n - f \|_p^* + \| f - f_k \|_p^* \to 0,$$

so that $\{(V_p(f_n, E))^{1/p}\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$, and, hence it is bounded and convergent. The inclusion $f \in \mathcal{V}_p(E; X)$ is then obvious. □

PROPOSITION 7.2 *The restriction of* $\| \cdot \|_p^*$ *to* $\mathcal{V}_p(E; X)$ *is a norm on* $\mathcal{V}_p(E; X)$, *and* $V_p(\,\cdot\,, E)$ *is a continuous functional on* $\mathcal{V}_p(E; X)$. *If, in addition,* X *is a Banach space, then* $\mathcal{V}_p(E; X)$ *is also a Banach space with respect to* $\| \cdot \|_p^*$.

Proof. It suffices to prove that $\mathcal{V}_p(E; X)$ is complete. Let $\{f_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $\mathcal{V}_p(E; X)$. From (7.1) we have $\| f_n(t) - f_k(t) \| \le \| f_n - f_k \|_p^*$ for all $t \in E$ and $n, k \in \mathbb{N}$. Since X is complete, there exists a map $f \in X^E$ such that f_n

converges to f pointwise on E as $n \to \infty$. Since $f_n - f_k \to f_n - f$ as $k \to \infty$, by virtue of (P7), we have

$$\| f_n - f \|_p^* \le \liminf_{k\to\infty} \| f_n - f_k \|_p^* = \lim_{k\to\infty} \| f_n - f_k \|_p^* \quad \forall\, n \in \mathbb{N}.$$

Thus, using the fact that $\{f_n\}_{n=1}^{\infty}$ is Cauchy, we have

$$\limsup_{n\to\infty} \| f_n - f \|_p^* \le \lim_{n\to\infty} \lim_{k\to\infty} \| f_n - f_k \|_p^* = 0,$$

so that $\| f_n - f \|_p^* \to 0$ as $n \to \infty$. By Proposition 7.1(b) it follows that $f \in \mathcal{V}_p(E; X)$. □

PROPOSITION 7.3 *Let $f \in \mathcal{V}_p([\,a, b\,]; X)$. Then, for all $h \in [\,0, b-a\,]$, we have*

$$\int_a^{b-h} \| f(t+h) - f(t) \|^p \, dt = \int_{a+h}^{b} \| f(t) - f(t-h) \|^p \, dt \le h V_{a,p}^b(f).$$

Proof. Let $h \in [\,0, b-a\,]$. Since $a \le t \le t+h \le b$ for all $t \in [\,a, b-h\,]$, by virtue of (P1), (P3), and (P2), we have

$$\| f(t+h) - f(t) \|^p \le V_{t,p}^{t+h}(f) \le V_{a,p}^{t+h}(f) - V_{a,p}^{t}(f) \le V_{a,p}^{b}(f).$$

It follows that the function $[\,a, b-h\,] \ni t \mapsto \| f(t+h) - f(t) \|^p \in \mathbb{R}_0^+$ is bounded, and continuous almost everywhere (due to Theorem 4.2(b)), so that it is Riemann integrable on $[\,a, b-h\,]$ thanks to Lebesgue's criterion. Now it suffices to integrate the second inequality above:

$$\begin{aligned}\int_a^{b-h} \| f(t+h) - f(t) \|^p \, dt \ &\le \int_{a+h}^{b} V_{a,p}^t(f)\, dt - \int_a^{b-h} V_{a,p}^t(f)\, dt \le \\ &\le \int_{b-h}^{b} V_{a,p}^t(f)\, dt \le h V_{a,p}^b(f). \qquad \square\end{aligned}$$

REMARK 7.1 *Proposition 7.3 means that a map f of bounded p-variation is continuous in L^p. If $p = 1$, the inequality in Proposition 7.3 gives weak (almost everywhere) differentiability of maps of bounded variation if X is a reflexive Banach space (see* Barbu and Precupanu, 1978, *Ch. 1, Sec. 3, and* Komura, 1967*).*

8. Selections Of Set-Valued Maps

We start by recalling definitions of the *Hausdorff distance* and *the set-valued maps* (for detailed exposition see Aubin and Cellina, 1984, Ch. 1, Sec. 1, Sec.5, and Castaing and Valadier, 1977, Ch. 2, Sec. 1).

Given two nonempty subsets A, $B \subset X$ of a metric space (X, d), the *excess of A over B* is defined by

$$\mathrm{e}(A, B) = \sup_{x \in A} \mathrm{dist}(x, B) = \sup_{x \in A} \inf_{y \in B} d(x, y) \in [\,0, \infty\,]$$

and the *Hausdorff distance between A and B* is defined by

$$d_H(A, B) = \max\{\, \mathrm{e}(A, B), \mathrm{e}(B, A)\,\}.$$

If A, B, $C \subset X$ are nonempty, then we have $\mathrm{e}(A, B) = 0$ if and only if A is contained in the closure of B, and $\mathrm{e}(A, B) \le \mathrm{e}(A, C) + \mathrm{e}(C, B)$, and, hence, d_H is a *pseudometric* on the set of all nonempty closed subsets of X, i.e., d_H satisfies the usual axioms of a metric and possibly takes infinite values. The map d_H is a *metric* on the set of all nonempty closed bounded subsets of X, on the set of all nonempty compact subsets of X and, if X is bounded, also on the set of all nonempty closed subsets of X.

Let E and X be two metric spaces, 2^X be the set of all subsets of X and $\dot{2}^X = 2^X \setminus \{\emptyset\}$. A *set-valued map from E into X* is a map $F : E \to 2^X$, so that $F(t) \subset X$ for every $t \in E$. The *graph of F* is the set $\mathrm{Gr}(F) = \{\,(t, x) \in E \times X : x \in F(t)\,\}$ and the *range of F* is the set $R(F) = \bigcup_{t \in E} F(t)$.

The set-valued map $F : E \to \dot{2}^X$ is said to be

(a) *Hausdorff continuous at* $t_0 \in E$ if for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $d_H(F(t), F(t_0)) \le \varepsilon$ for all $t \in E$ with $d_E(t, t_0) \le \delta$; *Hausdorff continuous on E* if it is so at every $t_0 \in E$;

(b) *Hölder continuous of exponent* $0 < \gamma \le 1$ *on E* if, for some $L \ge 0$ and all $t, s \in E$, $d_H(F(t), F(s)) \le L(d_E(t, s))^\gamma$; the least L is called the *Hölder constant of F* and is denoted by $H(F)$. If $\gamma = 1$, F is also called *Lipschitz continuous on E* and $H(F)$ is called the *Lipschitz constant* and is denoted by $\mathrm{Lip}(F)$;

(c) *compact-valued* if $F(t)$ is a compact subset of X for every $t \in E$;

(d) *compact* if its graph $\mathrm{Gr}(F)$ is compact in $E \times X$ (and, hence, F is compact-valued, but not vice versa);

(e) *of bounded p-variation on $E = [\,a, b\,] \subset \mathbb{R}$* if

$$V_{a,p}^b(F) := \sup\{\, V_{H,p}[F, T] : T \in \mathcal{T}_a^b\,\} < \infty,$$

where

$$V_{H,p}[F, T] = \sum_{i=1}^m \Big(d_H(F(t_i), F(t_{i-1}))\Big)^p, \quad T = \{t_i\}_{i=0}^m \in \mathcal{T}_a^b;$$

if $p = 1$, F is also called a map *of bounded variation*;

(f) *absolutely continuous on* $E = [\,a, b\,] \subset \mathbb{R}$ if for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that if $\{\,]\,a_n, b_n\,[\,\}_{n=1}^{N}$ is a finite collection of *disjoint* subintervals of $[\,a, b\,]$ with the property $\sum_{n=1}^{N}(b_n - a_n) \le \delta$, then

$$\sum_{n=1}^{N} d_H(F(b_n), F(a_n)) \le \varepsilon.$$

The map $f : E \to X$ is said to be a (*regular*) *selection* of a set-valued map $F : E \to \dot{2}^X$ if $f(t) \in F(t)$ for all $t \in E$.

It is known that a compact-valued set-valued map $F : E \to \dot{2}^X$ is Hausdorff continuous on E if and only if it is both upper semi-continuous and lower semi-continuous at every point $t_0 \in E$ (cf. Aubin and Cellina, 1984, Ch. 1, Sec. 5, Corollary 1).

In what follows, we assume that $E = [\,a, b\,] \subset \mathbb{R}$. To contrast our first result of this section (Proposition 8.2) with the known case $p = 1$, we recall the following theorem (in order not to break the exposition we postpone comments to Theorem 8.1 until Remark 8.2, p. 44):

THEOREM 8.1 *Let X be a Banach space, $F : [\,a, b\,] \to \dot{2}^X$ be a compact set-valued map, $t_0 \in [\,a, b\,]$, $x_0 \in F(t_0)$ and $p = 1$. Then*

(a) *if F is Lipschitz continuous on $[\,a, b\,]$, there exists a Lipschitzian map $f : [\,a, b\,] \to X$, a selection of F, such that $f(t_0) = x_0$ and* $\mathrm{Lip}(f) \le \mathrm{Lip}(F)$;

(b) *if F is continuous on $[\,a, b\,]$ of bounded variation, there exists a continuous map $f : [\,a, b\,] \to X$ of bounded variation, a selection of F, such that $f(t_0) = x_0$ and $V_{a,1}^{b}(f) \le V_{a,1}^{b}(F)$* ;

(c) *if F is of bounded variation on $[\,a, b\,]$ and the range $R(F)$ of F is contained in a convex compact subset of X, there exists a map $f : [\,a, b\,] \to X$ of bounded variation, a selection of F, such that $f(t_0) = x_0$ and $V_{a,1}^{b}(f) \le V_{a,1}^{b}(F)$.* □

Opposite to Lipschitz continuous maps, Hölder continuous maps of exponent $0 < \gamma < 1$ *do not*, in general, have continuous selections as the following proposition shows.

PROPOSITION 8.2 *There exists a Hölder continuous, of exponent γ for every $0 < \gamma < 1$, compact set-valued map $F : [-1, 1\,] \to \dot{2}^{\mathbb{R}^2}$ (and, hence, F is Hausdorff continuous of bounded p-variation with $p = 1/\gamma$) which admits no continuous selection.*

Proof. The example below is a modification of Example 1 from (Aubin and Cellina, 1984, Ch. 1, Sec. 6). Let $C = \{\,(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\,\}$ be the unit circumference in $\mathbb{R}^2$ and

$$A(t) = \{\,(x, y) \in \mathbb{R}^2 : x = \cos\theta,\ y = \sin\theta,\ \alpha(t) < \theta < \alpha(t) + 2\beta(t)\,\},$$

where $\alpha(t) = 1/|t|$ and $\beta(t) = e^{-1/|t|}$ for $t \neq 0$. Define $F : [-1, 1\,] \to \dot{2}^{\mathbb{R}^2}$ by

$$F(t) = \begin{cases} C \setminus A(t) & \text{if } \ t \neq 0, \\ \quad C & \text{if } \ t = 0. \end{cases}$$

For $t \neq 0$, $F(t)$ is the unit circumference in $\mathbb{R}^2$ from which a section from the angle $\alpha(t)$ to the angle $\alpha(t)+2\beta(t)$ is removed. As t gets smaller, the arclength of the hole decreases while the initial angle increases as $1/|t|$, i.e., the hole spins around the origin with increasing angular speed. Any continuous selection $f(t) = (x(t), y(t))$ defined on $[-1, 0\,[$ or on $]\,0, 1\,]$ (for instance, $x(t) = \cos(1/|t|)$, $y(t) = \sin(1/|t|)$) cannot be continuously extended to the whole $[-1, 1\,]$. In fact, the hole in the circumference would force this selection to rotate around the origin with an angle $\theta_0(t)$ between $\alpha(t)+2\beta(t)$ and $\alpha(t)+2\pi$ and the limits $\lim_{t\to\pm 0} f(t)$ cannot exist.

However, F is Hölder continuous on $[-1, 1\,]$ of exponent γ for every $0<\gamma<1$. To see this, let $0 < s < t \leq 1$. Since the length of the chord is less than the length of an arc it spans, we have the estimate

$$d_H(F(t), F(s)) \leq \min\{\,\alpha(s) - \alpha(t), \beta(t)\,\}.$$

The inequality $\alpha(s)-\alpha(t) \leq \beta(t)$ is equivalent to the inequality $s \geq t/(1 + t\beta(t)) =: s_0(t)$, so that

$$d_H(F(t), F(s)) \leq \begin{cases} \ \beta(t) & \text{if } \ 0 < s \leq s_0(t), \\ \dfrac{1}{s} - \dfrac{1}{t} & \text{if } \ s_0(t) \leq s \leq t. \end{cases}$$

If $0 < s \leq s_0(t)$, then we have

$$\begin{aligned} \frac{d_H(F(t), F(s))}{(t-s)^\gamma} &\leq \frac{\beta(t)}{(t-s)^\gamma} \leq \frac{\beta(t)}{(t-s_0(t))^\gamma} \\ &= \frac{(\beta(t))^{1-\gamma}}{t^{2\gamma}}(1+t\beta(t))^\gamma. \end{aligned} \tag{8.1}$$

If $s_0(t) \leq s \leq t$, then we have

$$\begin{aligned} \frac{d_H(F(t), F(s))}{(t-s)^\gamma} &\leq \frac{t-s}{ts(t-s)^\gamma} \leq \frac{(t-s_0(t))^{1-\gamma}}{ts_0(t)} \\ &= \frac{(\beta(t))^{1-\gamma}}{t^{2\gamma}}(1+t\beta(t))^\gamma. \end{aligned} \tag{8.2}$$

The function in the right hand sides of (8.1) and (8.2) tends to zero as $t \to +0$, hence it is bounded for $0 < t \leq 1$ by a constant $M(\gamma)$ depending on γ.

Now, if $0 < t \leq 1$, then

$$\frac{d_H(F(t), F(0))}{t^\gamma} \leq \frac{\beta(t)}{t^\gamma} \leq M_1(\gamma).$$

The cases $-1 \leq t < s < 0$ and $-1 \leq t < 0$ are similar. $\square$

REMARK 8.1 *It is an open question whether a set-valued map of bounded p-variation with $p > 1$ (in particular, a Hölder continuous map of exponent $0 < \gamma = 1/p < 1$) admits a selection of bounded p-variation. The result of Proposition 8.2 might mean that the answer to this question is negative.*

To end this section, we are going to supplement Theorem 8.1 with one more result on the existence of selections which was not explicitly given in (Chistyakov, 1997, Sec. 9).

THEOREM 8.3 *Under the conditions of Theorem* 8.1*, if F is absolutely continuous on $[a, b]$, then there exists an absolutely continuous map $f : [a, b] \to X$, a selection of F, such that $f(t_0) = x_0$.*

Proof. First, we note that if $\varphi(t) = V^t_{a,1}(F)$, $t \in [a, b]$, then φ is absolutely continuous on $[a, b]$. In fact, given $\varepsilon > 0$, there is $\delta = \delta(\varepsilon) > 0$ such that if $\{]a_n, b_n[\}^n_{n=1}$ is a finite collection of disjoint subintervals of $[a, b]$ with the property $\sum_{n=1}^N (b_n - a_n) \le \delta$, then $\sum_{n=1}^N d_H(F(b_n), F(a_n)) \le \varepsilon/2$. For each $n \in \{1, \dots, N\}$ let $T_n = \{t_{n,i}\}_{i=0}^{k_n} \in \mathcal{T}_{a_n}^{b_n}$ be a partition of $[a_n, b_n]$ such that

$$V_{a_n,1}^{b_n}(F) \le V_{H,1}[F, T_n] + (\varepsilon/2N).$$

Since $\sum_{n=1}^N \sum_{i=1}^{k_n}(t_{n,i} - t_{n,i-1}) = \sum_{n=1}^N (b_n - a_n) \le \delta$, we infer that

$$\sum_{n=1}^N |\varphi(b_n) - \varphi(a_n)| = \sum_{n=1}^N V_{a_n,1}^{b_n}(F) \le \sum_{n=1}^N V_{H,1}[F, T_n] + (\varepsilon/2) =$$

$$= \sum_{n=1}^N \sum_{i=1}^{k_n} d_H(F(t_{n,i}), F(t_{n,i-1})) + (\varepsilon/2) \le (\varepsilon/2) + (\varepsilon/2) = \varepsilon.$$

Now, by virtue of Lemma 3.3 (with $p = 1$) we have the decomposition $F = G \circ \varphi$ on $[a, b]$, where $G : [0, \ell] = \varphi([a, b]) \to \dot{2}^X$ is a Lipschitz continuous set-valued map such that $\ell = V^b_{a,1}(F) < \infty$ and $\mathrm{Lip}(G) \le 1$. Since F is compact, G is compact as well. Noting that $x_0 \in F(t_0) = G(\tau_0)$ with $\tau_0 = \varphi(t_0)$, by Theorem 8.1(a) we find a Lipschitzian map $g : [0, \ell] \to X$ such that $g(\tau_0) = x_0$, $g(\tau) \in G(\tau)$ for all $\tau \in [0, \ell]$, and $\mathrm{Lip}(g) \le \mathrm{Lip}(G) \le 1$.

Set $f = g \circ \varphi$. Since φ is absolutely continuous and g is Lipschitzian, $f : [a, b] \to X$ is absolutely continuous, $f(t_0) = x_0$ and $f(t) = g(\varphi(t)) \in G(\varphi(t)) = F(t)$ for all $t \in [a, b]$. □

REMARK 8.2 *Theorem 8.1 generalizes the results of* Hermes *(*1971*) and* Kikuchi and Tonita *(*1971*) to nonconvex-valued infinite-dimensional set-valued maps. Part (a) in Theorem 8.1 is due to* Mordukhovich *(*1988*), Supplement, Theorem 1.8, where he also proved that under conditions of part (b) there exists a* continuous *selection f of F. Part (b) in Theorem 8.1, which refines the last assertion, and part (c) were proved by* Chistyakov *(*1997*), Theorem 9.1. Theorem 8.3 is a generalization of*

Lemma 1 by Zhu Qiji *(*1986*), which was proved in the finite-dimensional case $X = \mathbb{R}^d$ ($d \in \mathbb{N}$). Finally, recall that there are compact continuous set-valued maps $F : [0, 1] \to \dot{2}^{\mathbb{R}^d}$ without continuous selections (see* Hermes *(*1971*),* Aubin and Cellina *(*1984*), Ch. 1, Sec. 6, and Proposition 8.2 above).*

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