LIPSCHITZIAN SUPERPOSITION OPERATORS BETWEEN SPACES OF FUNCTIONS OF BOUNDED GENERALIZED VARIATION WITH WEIGHT

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Abstract

We present some properties of real valued functions of bounded generalized variation of Riesz-Orlicz type including weight and characterize Lipschitzian superposition Nemytskii operators which map between spaces (in fact, Banach algebras) of these functions.

1 Introduction

Let $I \subset \mathbb{R}$ be an interval, $\mathbb{R}^I$ the algebra of all functions $f : I \to \mathbb{R}$ under the usual pointwise operations and $h : I \times \mathbb{R} \to \mathbb{R}$ a given function of two variables, $h = h(t, x)$. The mapping $H = H_h : \mathbb{R}^I \to \mathbb{R}^I$ defined by

$$(Hf)(t) \equiv H(f)(t) = h(t, f(t)), \qquad t \in I, \quad f \in \mathbb{R}^I, \tag{1}$$

is called a *h-generated superposition Nemytskii operator*. Let $F(I) \subset \mathbb{R}^I$ be a Banach function space with the norm $|\cdot|_F$. In order to solve the functional equation $f(t) = h(t, f(t))$, $t \in I$, also written as $f = Hf$, with respect to $f \in F(I)$, one can try the classical Banach fixed point theorem, in which case the operator $H : F(I) \to F(I)$ should satisfy the following Lipschitz condition:

$$|Hf - Hg|_F \le \mu |f - g|_F, \qquad f, g \in F(I), \tag{2}$$

where μ is a constant, $0 < \mu < 1$. However, as was observed by Matkowski [10] in the case of Lipschitz functions $F(I) = \mathrm{Lip}(I)$ with $I = [a, b]$, condition (2) implies that the generating function h of the operator H has to be of the form:

$$h(t, x) = h_0(t) + h_1(t)x, \qquad t \in I, \quad x \in \mathbb{R}, \tag{3}$$

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where h_0, $h_1 \in F(I)$. Consequently, Banach's contraction principle cannot be applied directly in $F(I)$ if h is a "nonlinear" function in the second variable (and hence a more powerful tool must be invoked, such as the Schauder fixed point theorem). Subsequently, similar results have been established by Matkowski and Miś [10], Merentes [13] and Merentes and Rivas [14] in the spaces of functions of bounded variation in the sense of C. Jordan and F. Riesz.

In this paper we characterize Lipschitzian superposition operators of the above kind (see Sec. 3) in the space of functions of bounded generalized variation of Riesz-Orlicz type including weight as a continuation of the studies in [4] and [5]. The presence of the weight function σ in the definition of the variation (see Sec. 2) implies that functions of bounded variation in this sense can be defined on unbounded intervals and that they are no longer of bounded Jordan variation and a fortiori absolutely continuous in general. We have chosen the basic case of real valued functions since normed linear space valued functions (and even set-valued functions) and superposition operators on them can be studied following the general outline of [3] and [5].

2 Generalized variation with weight

Let $\mathcal{N}$ be the set of all convex continuous functions Φ from $\mathbb{R}^+ = [0, \infty)$ into itself such that $\Phi(\rho) = 0$ if and only if $\rho = 0$, and $\lim_{\rho\to\infty} \Phi(\rho)/\rho = \infty$. Let $I \subset \mathbb{R}$ be an arbitrary (i.e. closed, half-closed, open, bounded or unbounded) fixed interval and $\sigma : I \to \mathbb{R}$ a fixed continuous strictly increasing function called a *weight.* If $\Phi \in \mathcal{N}$, we define the (*total*) *generalized* Φ-*variation* $\mathsf{V}_\Phi(f) \equiv \mathsf{V}_\Phi(f, I, \sigma)$ of the function $f : I \to \mathbb{R}$ with respect to the weight function σ in two steps as follows (cf. [3]). If $I = [a, b]$ is a closed interval and $T = \{t_i\}_{i=0}^m$ is a partition of I (i.e. $m \in \mathbb{N}$ and $a = t_0 < t_1 < \ldots < t_{m-1} < t_m = b$), we set

$$V_\Phi(f, T, \sigma) = \sum_{i=1}^{m} \bigl(\sigma(t_i) - \sigma(t_{i-1})\bigr)\,\Phi\left(\frac{|f(t_i) - f(t_{i-1})|}{\sigma(t_i) - \sigma(t_{i-1})}\right)$$

and, denoting by $\mathcal{T}_a^b$ the set of all partitions of $[a, b]$, we set

$$\mathsf{V}_\Phi(f) \equiv \mathsf{V}_\Phi(f, [a, b], \sigma) = \sup\{\, V_\Phi(f, T, \sigma) \mid T \in \mathcal{T}_a^b \,\}.$$

If I is any interval in $\mathbb{R}$, we put

$$\mathsf{V}_\Phi(f) \equiv \mathsf{V}_\Phi(f, I, \sigma) = \sup\{\, \mathsf{V}_\Phi(f, [a, b], \sigma) \mid [a, b] \subset I \,\}.$$

The set of all functions *of bounded generalized* Φ-*variation* with weight σ will be denoted by $BV_\Phi(I) \equiv BV_\Phi(I, \sigma) = \{\, f : I \to \mathbb{R} \mid \mathsf{V}_\Phi(f, I, \sigma) < \infty \,\}$.

If $\sigma(t) = \mathrm{id}(t) = t$, $t \in I = [a, b]$, and $\Phi(\rho) = \rho^q$, $\rho \geq 0$, $q > 1$, the Φ-variation $\mathsf{V}_\Phi(f, I, \sigma)$, also written as $\mathsf{V}_q(f)$, is the classical *q-variation* of f in the sense of Riesz [16], who has proved that $\mathsf{V}_q(f) < \infty$ if and only if $f \in AC(I)$

(i.e. $f : I \to \mathbb{R}$ is absolutely continuous) and its almost everywhere derivative f' is Lebesgue q-summable on I. Recall that, as is well known, the space $BV_\Phi(I)$ with I, Φ and σ as above endowed with the norm $|f|_q = |f(a)| + \big(\mathsf{V}_q(f)\big)^{1/q}$ is a Banach algebra for all $q \geq 1$.

Riesz's criterion was extended by Medvedev [12]: if $\Phi \in \mathcal{N}$, then $f \in BV_\Phi(I)$ if and only if $f \in AC(I)$ and $\int_I \Phi(|f'(t)|)\,dt < \infty$. Functions of bounded generalized Φ-variation with $\Phi \in \mathcal{N}$ and $\sigma = \mathrm{id}$ (also called functions of bounded Riesz-Orlicz Φ-variation) were studied by Cybertowicz and Matuszewska [6]. They showed that if $f \in BV_\Phi(I)$, then $\mathsf{V}_\Phi(f) = \int_I \Phi\big(|f'(t)|\big)\,dt$, and that the space $GV_\Phi(I) = \{f \in \mathbb{R}^I$ such that $\lim_{\lambda \to +0} \mathsf{V}_\Phi(\lambda f) = 0\}$ is a semi-normed linear space with the Luxemburg-Nakano seminorm given by $p_\Phi(f) = \inf\{r > 0 \mid \mathsf{V}_\Phi(f/r) \leq 1\}$. Later Maligranda and Orlicz [9] proved that the space $GV_\Phi(I)$ equipped with the norm $\|f\|_\Phi = \sup_{t \in I} |f(t)| + p_\Phi(f)$ is a Banach algebra.

The notion of the generalized Φ-variation of a function is interesting due to the following result which is also valid for metric space valued functions [2, Theorem 6.7]: $f \in AC(I)$ if and only if there exists $\Phi \in \mathcal{N}$ such that $f \in BV_\Phi(I)$; in other words, $AC(I) = \bigcup_{\Phi \in \mathcal{N}} BV_\Phi(I, \mathrm{id})$.

Throughout this paper the weight function σ will be fixed, and so, as a rule, it won't be explicitly written.

In what follows we will use some facts which we present now as lemmas. The first lemma lists the main properties of the (generalized) Φ-variation.

Lemma 1 ([2, 3]) *Let* $f : I \to \mathbb{R}$ *and* $\Phi \in \mathcal{N}$. *We have*:
(a) *if* J *is a subinterval of* I, *then* $\mathsf{V}_\Phi(f, J) \leq \mathsf{V}_\Phi(f, I)$;
(b) *if* $t \in I$, *then* $\mathsf{V}_\Phi(f, I) = \mathsf{V}_\Phi(f, (-\infty, t] \cap I) + \mathsf{V}_\Phi(f, [t, \infty) \cap I)$;
(c) *if* $f_n : I \to \mathbb{R}$, $n \in \mathbb{N}$, *and* $\lim_{n\to\infty} f_n(t) = f(t)$ *for all* $t \in I$, *then*

$$\mathsf{V}_\Phi(f) \leq \liminf_{n\to\infty} \mathsf{V}_\Phi(f_n);$$

(d) *if* $f \in BV_\Phi(I, \sigma)$, *then* f *is absolutely continuous with respect to* σ, *and hence, continuous on* I.

Sets $BV_\Phi(I)$ corresponding to different functions Φ are related as follows.

Lemma 2 *Suppose that* Φ, $\Psi \in \mathcal{N}$ *and the function* σ *is bounded. Then* $BV_\Phi(I) \subset BV_\Psi(I)$ *if and only if* $\limsup_{\rho\to\infty} \Psi(\rho)/\Phi(\rho) < \infty$.

Proof. First we note that the condition of Lemma including the limit superior is equivalent to the following one: there exist constants $C > 0$ and $\rho_0 > 0$ such that $\Psi(\rho) \leq \Phi(\rho)$ for all $\rho \geq \rho_0$. Taking this into account, we find that *sufficiency* follows from the inequality:

$$\mathsf{V}_\Psi(f) \leq \Psi(\rho_0)|\sigma(I)| + C\,\mathsf{V}_\Phi(f), \qquad f \in BV_\Phi(I),$$

where $|\sigma(I)| = \sup_{t \in I} \sigma(t) - \inf_{t \in I} \sigma(t)$ is finite by the assumption.

If the *necessity* part is wrong, then there exists an increasing sequence $\{\rho_n\}_{n=1}^{\infty}$ of positive numbers such that $\lim_{n\to\infty}\rho_n=\infty$ and $\Psi(\rho_n)>2^n\Phi(\rho_n)$ for all $n\in\mathbb{N}$. Set $t_0=\inf I$ and $\sigma(t_0)=\inf_{t\in I}\sigma(t)$. Define the increasing sequence $\{t_n\}_{n=1}^{\infty}\subset I$ inductively as follows:

$$\sigma(t_n)-\sigma(t_{n-1})=2^{-n}|\sigma(I)|\Phi(\rho_1)/\Phi(\rho_n),\qquad n\in\mathbb{N}.$$

Denote by $\sigma(I)=\{\sigma(t)\mid t\in I\}$ the image of σ (which is an interval) and define the function $\chi:\sigma(I)\to\mathbb{R}$ by $\chi(s)=\rho_n$ if $\sigma(t_{n-1})\le s<\sigma(t_n)$, $n\in\mathbb{N}$, and $\chi(s)=0$ otherwise. Setting $f(t)=\int_{\sigma(t_0)}^{\sigma(t)}\chi(s)\,ds$, $t\in I$, we claim that $f\in BV_\Phi(I)$ and $f\notin BV_\Psi(I)$. Indeed, using Lemma 1(b) we have:

$$\mathsf{V}_\Phi(f)=\sum_{n=1}^{\infty}\big(\sigma(t_n)-\sigma(t_{n-1})\big)\Phi(\rho_n)=|\sigma(I)|\Phi(\rho_1).$$

On the other hand, for any $m\in\mathbb{N}$ we have:

$$\begin{aligned}\mathsf{V}_\Psi(f)\ &\ge\ \sum_{n=1}^{m}\big(\sigma(t_n)-\sigma(t_{n-1})\big)\Psi\left(\frac{|f(t_n)-f(t_{n-1})|}{\sigma(t_n)-\sigma(t_{n-1})}\right)=\\ &=\ |\sigma(I)|\Phi(\rho_1)\sum_{n=1}^{m}2^{-n}\Psi(\rho_n)/\Phi(\rho_n)\ge m|\sigma(I)|\Phi(\rho_1).\end{aligned}$$ □

Lemma 3 *For $\Phi\in\mathcal{N}$ and bounded σ, $BV_\Phi(I,\sigma)$ is a linear space if and only if Φ satisfies the Δ_2-condition near infinity, i.e.* $\limsup_{\rho\to\infty}\Phi(2\rho)/\Phi(\rho)<\infty$.

Proof of Lemma 3 is the same as that of Proposition 6.1 in [2]. □

By Lemma 3, the convex set $BV_\Phi(I)$ is not a linear space in general: more precisely, it may happen so that $2f\notin BV_\Phi(I)$ for some $f\in BV_\Phi(I)$. For instance, let $\Phi(\rho)=e^\rho-1$, $\rho\ge0$, $\sigma=\mathrm{id}$, and define $f:[0,1]\to\mathbb{R}$ by $f(t)=t(1-\log t)/2$ if $0<t\le1$ and $f(0)=0$. Then we have:

$$\mathsf{V}_\Phi(f)=\int_0^1\Phi\big(|f'(t)|\big)\,dt=1\qquad\text{and}\qquad\mathsf{V}_\Phi(2f)=\infty.$$

We introduce the space $GV_\Phi(I)=GV_\Phi(I,\sigma)$ as follows: $f\in GV_\Phi(I)$ if there exists a constant $r>0$ (depending on f) such that $f/r\in BV_\Phi(I)$. Clearly,

$$BV_\Phi(I)\subset GV_\Phi(I)\subset(\text{space of continuous functions } I\to\mathbb{R}).$$

Moreover, the set $GV_\Phi(I)$ is a *linear space*. In fact, if $f_j\in GV_\Phi(I)$, then $\mathsf{V}_\Phi(f_j/r_j)<\infty$ for some $r_j>0$, $j=1$, 2, so that the convexity of the functional $\mathsf{V}_\Phi(\,\cdot\,)$ implies

$$\mathsf{V}_\Phi\left(\frac{f_1+f_2}{r_1+r_2}\right)\le\frac{r_1}{r_1+r_2}\mathsf{V}_\Phi(f_1/r_1)+\frac{r_2}{r_1+r_2}\mathsf{V}_\Phi(f_2/r_2),$$

whence $f_1 + f_2 \in GV_\Phi(I)$. Obviously, $\lambda f \in GV_\Phi(I)$ if $\lambda \in \mathbb{R}$ and $f \in GV_\Phi(I)$.

We define the *norm* $|\cdot|_\Phi$ on $GV_\Phi(I)$ by

$$|f|_\Phi = |f(a)| + p_\Phi(f), \qquad f \in GV_\Phi(I), \tag{4}$$

where $a \in I$ is arbitrary and fixed and $p_\Phi(\,\cdot\,)$ is the Luxemburg-Nakano *seminorm* (cf. [8, Sec. 2.4 and Remark 2 in Sec. 1]) given by

$$p_\Phi(f) = \inf\{r > 0 \mid \mathsf{V}_\Phi(f/r) \le 1\}.$$

Let Φ^{-1} designates the inverse function of $\Phi \in \mathcal{N}$, and set $\omega_\Phi(\rho) = \rho\Phi^{-1}(1/\rho)$, $\rho > 0$. Note that the function ω_Φ is continuous, subadditive, concave and $\lim_{\rho\to+0}\omega_\Phi(\rho) = \lim_{r\to\infty} r/\Phi(r) = 0$ since $\Phi \in \mathcal{N}$.

Some elementary properties of p_Φ are gathered in the following

Lemma 4 (cf. [5]) *Let $\Phi \in \mathcal{N}$ and $f \in GV_\Phi(I)$. We have*:
(a) *if t, $s \in I$, then $|f(t) - f(s)| \le \omega_\Phi\bigl(|\sigma(t) - \sigma(s)|\bigr)p_\Phi(f)$;*
(b) *if $p_\Phi(f) > 0$, then $\mathsf{V}_\Phi\bigl(f/p_\Phi(f)\bigr) \le 1$;*
(c) *if $r > 0$, we have: $p_\Phi(f) \le r$ if and only if $\mathsf{V}_\Phi(f/r) \le 1$; if $\mathsf{V}_\Phi(f/r) = 1$, then $p_\Phi(f) = r$ (but not vice versa in general)*;
(d) *if the sequence $\{f_n\}_{n=1}^\infty \subset GV_\Phi(I)$ converges to f pointwise on I as $n \to \infty$, then $p_\Phi(f) \le \limsup_{n\to\infty} p_\Phi(f_n)$.*

Remark 1. Estimate in Lemma 4(a) shows that any function $f \in GV_\Phi(I)$ is continuous on I (cf. also Lemma 1(d)). It shows also that the modulus of continuity of f (even) in the case $\sigma = \mathrm{id}$ is "finer" than the modulus of continuity from the embedding theorem for Sobolev-Orlicz spaces (cf. [1, Thm. 8.36]) since

$$\omega_\Phi(\rho) = \rho\Phi^{-1}(1/\rho) < \int_{+0}^{\rho} \Phi^{-1}(1/s)\,ds = \int_{1/\rho}^{\infty} \Phi^{-1}(\tau)/\tau^2\,d\tau, \qquad \rho > 0.$$

We will require certain partial-ordering relationships among functions from the set $\mathcal{N}$ (cf. [7, Secs. 3 and 13]). If Φ, $\Psi \in \mathcal{N}$, we say that Φ *dominates* Ψ *near infinity* (in symbols, $\Psi \preccurlyeq \Phi$) provided $\limsup_{\rho\to\infty} \Phi^{-1}(\rho)/\Psi^{-1}(\rho) < \infty$, or, equivalently, if there exist constants $C > 0$ and $\rho_0 > 0$ such that $\Psi(\rho) \le \Phi(C\rho)$ for all $\rho \ge \rho_0$. The two functions Φ and Ψ are *equivalent near infinity* if $\Psi \preccurlyeq \Phi$ and $\Phi \preccurlyeq \Psi$.

We say that Φ *increases essentially more slowly than* Ψ *near infinity* and write $\Phi \lhd \Psi$ if $\Phi \preccurlyeq \Psi$ and Φ and Ψ are not equivalent near infinity. This is exactly the case if and only if $\lim_{\rho\to\infty} \Phi(C\rho)/\Psi(\rho) = 0$ for all $C > 0$. Moreover, the relation $\Phi \lhd \Psi$ can be characterized as follows:

$$\Phi \lhd \Psi \qquad \text{if and only if} \qquad \lim_{\rho\to\infty} \Psi^{-1}(\rho)/\Phi^{-1}(\rho) = 0. \tag{5}$$

Theorem 5 *Let* Φ, $\Psi \in \mathcal{N}$ *and the function* σ *be bounded.*

(a) *The space* $GV_\Phi(I,\sigma)$ *equipped with the norm* (4) *is a Banach algebra, and for all* f, $g \in GV_\Phi(I)$ *the following inequality holds*:

$$|fg|_\Phi \le \gamma |f|_\Phi |g|_\Phi, \tag{6}$$

where $\gamma = \gamma(\Phi, |\sigma(I)|) = \max\{1, 2\omega_\Phi(|\sigma(I)|)\}$ *and* $|\sigma(I)| = \sup_{t\in I}\sigma(t) - \inf_{t\in I}\sigma(t)$.

(b) $GV_\Phi(I) \subset GV_\Psi(I)$ *if and only if* $\Psi \preccurlyeq \Phi$; *moreover, there exists a constant* $\kappa = \kappa(\Phi, \Psi) > 0$ *such that* $|f|_\Psi \le \kappa |f|_\Phi$ *for all* $f \in GV_\Phi(I)$.

Proof. (a) Since the function $\omega_\Phi(\rho) = \rho\Phi^{-1}(1/\rho)$ is nondecreasing for $\rho > 0$, by Lemma 4(a) for any function $f \in GV_\Phi(I)$ we have the estimate:

$$\|f\| \equiv \sup_{t\in I} |f(t)| \le |f(a)| + \omega_\Phi(|\sigma(I)|) p_\Phi(f). \tag{7}$$

Given f, $g \in GV_\Phi(I)$, let us prove the following inequality:

$$p(fg) \le p(f)\|g\| + \|f\| p(g), \tag{8}$$

where the subscript Φ is omitted in $p_\Phi(fg)$, $p_\Phi(f)$ and $p_\Phi(g)$ for the sake of brevity. Without loss of generality we may assume that the quantities $\|f\|$, $\|g\|$, $p(f)$ and $p(g)$ are strictly positive. Set $r = p(f)\|g\| + \|f\|p(g)$. If $T = \{t_i\}_{i=0}^m$ is a partition of I, then setting $\Delta f_i = f(t_i) - f(t_{i-1})$, $\Delta g_i = g(t_i) - g(t_{i-1})$, $\Delta\sigma_i = \sigma(t_i) - \sigma(t_{i-1})$ and using the monotonicity and convexity of Φ and applying Lemma 4(b) we have:

$$\begin{aligned}
\mathsf{V}_\Phi(fg/r, T) &= \sum_{i=1}^m \Delta\sigma_i \Phi\big(|(\Delta f_i)g(t_i) + f(t_{i-1})(\Delta g_i)|/(r\Delta\sigma_i)\big) \le \\
&\le \sum_{i=1}^m \Delta\sigma_i \Phi\big((|\Delta f_i| \cdot \|g\| + \|f\| \cdot |\Delta g_i|)/(r\Delta\sigma_i)\big) \le \\
&\le \big(p(f)\|g\|/r\big) \sum_{i=1}^m \Delta\sigma_i \Phi\big(|\Delta f_i|/(p(f)\Delta\sigma_i)\big) + \\
&\qquad + \big(\|f\|p(g)/r\big) \sum_{i=1}^m \Delta\sigma_i \Phi\big(|\Delta g_i|/(p(g)\Delta\sigma_i)\big) \le \\
&\le \big(p(f)\|g\|/r\big)\, \mathsf{V}_\Phi\big(f/p(f)\big) + \big(\|f\|p(g)/r\big)\, \mathsf{V}_\Phi\big(g/p(g)\big) \le \\
&\le \big(p(f)\|g\| + \|f\|p(g)\big)/r = 1.
\end{aligned}$$

Due to the arbitrariness of T, we get $\mathsf{V}_\Phi(fg/r) \le 1$, so that the definition of $p(fg)$ gives $p(fg) \le r$, which is (8). Now, inequality (6) follows from (4), (8) and (7).

To prove that $GV_\Phi(I)$ is complete, suppose that $\{f_n\}_{n=1}^\infty$ is a Cauchy sequence in $GV_\Phi(I)$, i.e.

$$|f_n - f_m|_\Phi = |f_n(a) - f_m(a)| + p_\Phi(f_n - f_m) \to 0 \quad \text{as} \quad n,\, m \to \infty.$$

By Lemma 4(a) it follows that $\{f_n(t)\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$ for all $t \in I$ and therefore there exists a function $f : I \to \mathbb{R}$ such that f_n converges to f pointwise on I as $n \to \infty$. Lemma 4(d) yields:

$$|f_n - f|_\Phi \le \limsup_{m\to\infty} |f_n - f_m|_\Phi = \lim_{m\to\infty} |f_n - f_m|_\Phi \in \mathbb{R}^+, \qquad n \in \mathbb{N}.$$

Since $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $GV_\Phi(I)$, we have:

$$\limsup_{n\to\infty} |f_n - f|_\Phi \le \lim_{n\to\infty} \lim_{m\to\infty} |f_n - f_m|_\Phi = 0.$$

Hence $|f_n - f|_\Phi \to 0$ as $n \to \infty$. It follows that there exists $n_0 \in \mathbb{N}$ such that $|f_{n_0} - f|_\Phi \le 1$, and so $|f|_\Phi \le |f - f_{n_0}|_\Phi + |f_{n_0}|_\Phi \le 1 + |f_{n_0}|_\Phi < \infty$. Therefore, $f \in GV_\Phi(I)$, which was to be proved.

(b) Suppose that $\Psi \preccurlyeq \Phi$ and $f \in GV_\Phi(I)$, so that $\mathsf{V}_\Phi(f/r) < \infty$ for some $r > 0$. Using the equivalent condition for the relation $\Psi \preccurlyeq \Phi$ (see p. 5), we have:

$$\mathsf{V}_\Psi\big(f/(rC)\big) \le \Psi(\rho_0)|\sigma(I)| + \mathsf{V}_\Phi(f/r),$$

so that $f \in GV_\Psi(I)$.

If the relation $\Psi \preccurlyeq \Phi$ does not hold, then there exists an increasing sequence $\{\rho_n\}_{n=1}^{\infty}$ of positive numbers such that $\lim_{n\to\infty} \rho_n = \infty$ and $\Psi(\rho_n) > \Phi(n2^n\rho_n)$ for all $n \in \mathbb{N}$. Setting $\theta = 1/2^n$ and $\rho = n2^n\rho_n$ in the inequality $\Phi(\theta\rho) \le \theta\Phi(\rho)$ we find that $\Phi(n2^n\rho_n) \ge 2^n\Phi(n\rho_n)$, and so

$$\Psi(\rho_n) > 2^n\Phi(n\rho_n), \qquad n \in \mathbb{N}. \tag{9}$$

Set $t_0 = \inf I$, $\sigma(t_0) = \inf_{t\in I} \sigma(t)$ and define the increasing sequence $\{t_n\}_{n=1}^{\infty} \subset I$ inductively as follows:

$$\sigma(t_n) - \sigma(t_{n-1}) = 2^{-n}|\sigma(I)|\Phi(\rho_1)/\Phi(n\rho_n), \qquad n \in \mathbb{N}.$$

If $\sigma(I)$ is the image of σ, define the function $\chi : \sigma(I) \to \mathbb{R}$ by $\chi(s) = n\rho_n$ if $\sigma(t_{n-1}) \le s < \sigma(t_n)$, $n \in \mathbb{N}$, and $\chi(s) = 0$ otherwise. If the function $f : I \to \mathbb{R}$ is given by $f(t) = \int_{\sigma(t_0)}^{\sigma(t)} \chi(s)\,ds$, $t \in I$, then $f \in BV_\Phi(I) \setminus GV_\Psi(I)$. In fact,

$$\mathsf{V}_\Phi(f) = \sum_{n=1}^{\infty} \big(\sigma(t_n) - \sigma(t_{n-1})\big)\Phi(n\rho_n) = |\sigma(I)|\Phi(\rho_1).$$

On the other hand, let us show that $\mathsf{V}_\Psi(f/r) = \infty$ for all $r \ge 1$. Taking into consideration (9), for any $m \in \mathbb{N}$ such that $m \ge r$, we have:

$$\begin{aligned} \mathsf{V}_\Psi(f/r) \;&\ge\; \sum_{n=m}^{2m} \big(\sigma(t_n) - \sigma(t_{n-1})\big)\Psi\left(\frac{|f(t_n) - f(t_{n-1})|}{(\sigma(t_n) - \sigma(t_{n-1}))r}\right) \ge \\ &\ge\; \sum_{n=m}^{2m} \big(\sigma(t_n) - \sigma(t_{n-1})\big)\Psi(\rho_n) \ge m|\sigma(I)|\Phi(\rho_1). \end{aligned}$$

Therefore $f \notin GV_\Psi(I)$.

It remains to prove the inequality in (b). Since $\Psi \preccurlyeq \Phi$, the identity operator Id given by $\mathrm{Id}(f) = f$ maps $GV_\Phi(I)$ into $GV_\Psi(I)$ and is closed (by virtue of (4) and (7)), so that, by the closed graph theorem, it is continuous, and it suffices to define the constant $\kappa > 0$ as the operator norm of the identity operator $\mathrm{Id} : GV_\Phi(I) \to GV_\Psi(I)$. □

Remark 2. If the right derivative $\Phi'(+0) > 0$, the assumption that σ is bounded is redundant in Theorem 5(a). To see this, note that ω_Φ is nondecreasing and

$$\sup_{\rho>0} \omega_\Phi(\rho) = \lim_{\rho\to\infty} \rho\Phi^{-1}(1/\rho) = \lim_{r\to+0} r/\Phi(r) = 1/\Phi'(+0) \in (0,\infty).$$

It follows from Lemma 4(a) that inequality (7) can be replaced by

$$\sup_{t\in I} |f(t)| \le |f(a)| + p_\Phi(f)/\Phi'(+0),$$

and so $\gamma = \gamma(\Phi) = \max\{1, 2/\Phi'(+0)\}$ in (6).

Remark 3. From the theory of Banach algebras it is well known that the norm (4) with the property (6) can always be replaced by an equivalent norm $\|\cdot\|_\Phi$ on $GV_\Phi(I)$ such that $\|fg\|_\Phi \le \|f\|_\Phi \|g\|_\Phi$ for all $f,\ g \in GV_\Phi(I)$.

Remark 4. In the proofs of Lemma 2 and Theorem 5(b) we have used certain ideas from the theory of Orlicz spaces (cf. [7, Secs. 8.3 and 13.1] and [8, Sec. 3]).

3 Lipschitzian superposition operators

Theorem 6 *Suppose that* $h : I \times \mathbb{R} \to \mathbb{R}$, $H = H_h : \mathbb{R}^I \to \mathbb{R}^I$ *is the* h*-generated superposition Nemytskii operator* (*see* (1)) *and* $\Phi,\ \Psi \in \mathcal{N}$.

(a) *If* H *maps* $GV_\Phi(I)$ *into* $GV_\Psi(I)$ *and is Lipschitzian in the sense that there exists a constant* $\mu > 0$ *such that*

$$|Hf - Hg|_\Psi \le \mu |f-g|_\Phi \qquad \forall\, f,\ g \in GV_\Phi(I), \tag{10}$$

then there exists a function $\mu_0 : I \to \mathbb{R}^+$ *such that*

$$|h(t,x) - h(t,y)| \le \mu_0(t)|x-y|, \qquad t\in I, \quad x,\ y \in \mathbb{R}, \tag{11}$$

and there exist two functions $h_0,\ h_1 \in GV_\Psi(I)$ *such that* (3) *holds.*

If, in addition, $\Phi \lhd \Psi$, *then* $h(t,x) = h(t,0)$ *for all* $t \in I$ *and* $x \in \mathbb{R}$.

(b) *Conversely, if* $\Psi \preccurlyeq \Phi$, *the function* σ *is bounded and there are two functions* $h_0,\ h_1 \in GV_\Psi(I)$ *such that* (3) *holds, then the superposition operator* H *maps* $GV_\Phi(I)$ *into* $GV_\Psi(I)$ *and is Lipschitzian.*

Proof. (a) Inequality (10) and definition (4) imply that if f, $g \in GV_\Phi(I)$, then $p_\Psi(Hf - Hg) \le \mu|f-g|_\Phi$ which, in the case when $|f-g|_\Phi > 0$, is, by Lemma 4(c), equivalent to

$$\mathsf{V}_\Psi\left(\frac{Hf - Hg}{\mu|f-g|_\Phi}\right) \le 1.$$

Taking into account definitions of V_Ψ and H, for all α, $\beta \in I$, $\alpha < \beta$, we have:

$$\big(\sigma(\beta) - \sigma(\alpha)\big)\Psi\left(\frac{|h(\beta, f(\beta)) - h(\beta, g(\beta)) - h(\alpha, f(\alpha)) + h(\alpha, g(\alpha))|}{\mu|f-g|_\Phi\big(\sigma(\beta) - \sigma(\alpha)\big)}\right) \le 1,$$

which yields

$$\begin{aligned} |h(\beta, f(\beta)) - h(\beta, g(\beta)) - h(\alpha, f(\alpha)) + h(\alpha, g(\alpha))| &\le \\ \le \mu|f-g|_\Phi \omega_\Psi\big(\sigma(\beta) - \sigma(\alpha)\big) \end{aligned} \tag{12}$$

for all f, $g \in GV_\Phi(I)$ and all α, $\beta \in I$, $\alpha < \beta$.

For α and β as above define functions $\eta_{\alpha,\beta} : I \to [0,1]$ by

$$\eta_{\alpha,\beta}(s) = \begin{cases} 0 & \text{if} \quad s \le \alpha, \\ \dfrac{\sigma(s) - \sigma(\alpha)}{\sigma(\beta) - \sigma(\alpha)} & \text{if} \quad \alpha \le s \le \beta, \\ 1 & \text{if} \quad \beta \le s. \end{cases}$$

Without loss of generality we may assume that the point $a \in I$ in (4) is an interior point of I.

In order to prove claim (11), consider the following three cases for the point $t \in I$: i) $t > a$; ii) $t < a$; iii) $t = a$.

i) Let $t > a$ and α, $\beta \in I$, $a \le \alpha < \beta$. Define two functions

$$f(s) = \eta_{\alpha,\beta}(s)x, \quad g(s) = \eta_{\alpha,\beta}(s)y, \qquad s \in I, \quad x,\, y \in \mathbb{R}.$$

To compute the norm $|f-g|_\Phi$, let $x \ne y$ and, applying Lemma 1(b), let us choose a number $r > 0$ such that

$$\mathsf{V}_\Phi\big((f-g)/r\big) = \big(\sigma(\beta) - \sigma(\alpha)\big)\Phi\left(\frac{|x-y|}{\big(\sigma(\beta) - \sigma(\alpha)\big)r}\right) = 1.$$

Then Lemma 4(c) gives:

$$p_\Phi(f-g) = r = |x-y|/\omega_\Phi(\sigma(\beta) - \sigma(\alpha)), \qquad x,\, y \in \mathbb{R}.$$

Substituting functions f and g into inequality (12) and noting that $f(\beta) = x$, $g(\beta) = y$ and $f(\alpha) = g(\alpha) = 0$, we get:

$$|h(\beta, x) - h(\beta, y)| \le \mu|x-y|\omega_\Psi\big(\sigma(\beta) - \sigma(\alpha)\big)/\omega_\Phi\big(\sigma(\beta) - \sigma(\alpha)\big). \tag{13}$$

Setting $\alpha = a$ and $\beta = t$ we obtain (11) with a suitably chosen number $\mu_0(t)$.

ii) Let $t < a$ and α, $\beta \in I$, $\alpha < \beta \le a$. Substituting functions

$$f(s) = \big(1 - \eta_{\alpha,\beta}(s)\big)x, \quad g(s) = \big(1 - \eta_{\alpha,\beta}(s)\big)y, \qquad s \in I, \quad x,\, y \in \mathbb{R}, \tag{14}$$

into (12) and noting that $f(\beta) = g(\beta) = 0$, $f(\alpha) = x$ and $g(\alpha) = y$, we have as above:

$$|h(\alpha, x) - h(\alpha, y)| \le \mu|x - y|\omega_\Psi\big(\sigma(\beta) - \sigma(\alpha)\big)/\omega_\Phi\big(\sigma(\beta) - \sigma(\alpha)\big). \tag{15}$$

Setting $\alpha = t$ and $\beta = a$ we obtain (11) with an obvious choice of $\mu_0(t)$.

iii) Let $t = a$. Since it is an interior point of I, fix $\beta \in I$ such that $a < \beta$. Substituting functions (14) with $\alpha = a$ into (12) and noting that $|f(a) - g(a)| = |x - y|$, we arrive at

$$|h(a, x) - h(a, y)| \le \mu|x - y|\left(1 + \frac{1}{\omega_\Phi\big(\sigma(\beta) - \sigma(a)\big)}\right)\omega_\Psi\big(\sigma(\beta) - \sigma(a)\big). \tag{16}$$

Therefore, we are through with inequality (11).

Now we prove that $h(t, x)$ is of the form (3). For α, $\beta \in I$, $\alpha < \beta$, set

$$f(s) = \eta_{\alpha,\beta}(s)x + y, \quad g(s) = \eta_{\alpha,\beta}(s)x, \qquad s \in I, \quad x,\, y \in \mathbb{R},$$

and observe that $f(\beta) = x + y$, $g(\beta) = x$, $f(\alpha) = y$, $g(\alpha) = 0$ and $f - g \equiv y$. Hence, inequality (12) provides the estimate:

$$|h(\beta, x + y) - h(\beta, x) - h(\alpha, y) + h(\alpha, 0)| \le \mu|y|\omega_\Psi\big(\sigma(\beta) - \sigma(\alpha)\big). \tag{17}$$

Since H maps $GV_\Phi(I)$ into $GV_\Psi(I)$ and constant functions belong to $GV_\Phi(I)$, the function $h(\,\cdot\,, x) = H(x)$ is in $GV_\Psi(I)$ for all $x \in \mathbb{R}$, and so it is continuous on I according to Lemma 4(a). Given $t \in I$, letting $\beta - \alpha$ tend to zero in (17) in such a way that $[\alpha, \beta] \ni t$, we get:

$$h(t, x + y) - h(t, x) - h(t, y) + h(t, 0) = 0, \qquad t \in I, \quad x,\, y \in \mathbb{R}.$$

It follows that $h(t, x + y) - 2h(t, x) + h(t, x - y) \equiv 0$ and hence

$$\lim_{y \to 0} \frac{h(t, x + y) - 2h(t, x) + h(t, x - y)}{y^2} = 0, \qquad t \in I, \quad x \in \mathbb{R}, \tag{18}$$

i.e. the second symmetric derivative of $h(t, \,\cdot\,)$ (which is defined by the left hand side of (18)) vanishes at any point $x \in \mathbb{R}$. Since, by (11), the function $h(t, \,\cdot\,)$ is continuous on $\mathbb{R}$, this implies (cf. [15, Ch. 10, Sec. 5, Thm. 1]) that $h(t, x)$ is of the form (3) for some functions h_0, $h_1 \in \mathbb{R}^I$. Taking into account the equalities $h_0 = h(\,\cdot\,, 0) = H(0)$ and $h_1 = h(\,\cdot\,, 1) - h(\,\cdot\,, 0) = H(1) - H(0)$, we conclude that h_0, $h_1 \in GV_\Psi(I)$. This completes the proof of the representation (3).

Now suppose that $\Phi \lhd \Psi$, and let $t \in I$. If $t > a$, we set $\beta = t$, $y = 0$ and let $\alpha \to t - 0$ in (13). If $t < a$, we set $\alpha = t$, $y = 0$ and let $\beta \to t + 0$ in (15). If $t = a$, we set $y = 0$ and let $\beta \to a + 0$ in (16). Noting that

$$\frac{\omega_\Psi\big(\sigma(\beta) - \sigma(\alpha)\big)}{\omega_\Phi\big(\sigma(\beta) - \sigma(\alpha)\big)} = \frac{\Psi^{-1}\big(1/(\sigma(\beta) - \sigma(\alpha))\big)}{\Phi^{-1}\big(1/(\sigma(\beta) - \sigma(\alpha))\big)}$$

and taking into account (5) and the continuity of $h(\,\cdot\,, x)$, $x \in \mathbb{R}$, we find that $h(t, x) = h(t, 0)$ for all $t \in I$ and $x \in \mathbb{R}$ where $h(\,\cdot\,, 0) \in GV_\Psi(I)$. In particular, we see that H is a *constant* operator.

(b) Since $\Psi \preccurlyeq \Phi$, then $GV_\Phi(I) \subset GV_\Psi(I)$ by Theorem 5(b), and since the operator H is given according to assumption (3) by

$$(Hf)(t) = h_0(t) + h_1(t)f(t), \qquad t \in I, \quad f \in GV_\Phi(I),$$

and $GV_\Psi(I)$ is an algebra by Theorem 5(a), it follows that H maps the space $GV_\Phi(I)$ into $GV_\Psi(I)$. Now, for all f, $g \in GV_\Phi(I)$, inequality (6) and Theorem 5(b) yield the estimate

$$|Hf - Hg|_\Psi \le \gamma(\Psi, |\sigma(I)|)\kappa(\Phi, \Psi)|h_1|_\Psi|f - g|_\Phi, \tag{19}$$

which shows that H is a Lipschitzian operator. □

Remark 5. If $\Phi(\rho) = \rho^p$, $\Psi(\rho) = \rho^q$, $\rho \ge 0$, $p > 1$, $q > 1$, and $\sigma(t) = t$, $t \in [a, b]$, Theorem 6 gives the results of Merentes and Rivas [14]. It suffices to note only that $\Psi \preccurlyeq \Phi$ if and only if $q \le p$, and $\Phi \lhd \Psi$ if and only if $p < q$.

Remark 6. Given h_0, $h_1 \in GV_\Phi(I)$, one can easily find conditions on the function h_1 in order to solve the “linear” functional equation $x = h_0 + h_1 x$ with respect to $x \in GV_\Phi(I)$ by using the classical Banach fixed point theorem.

Corollary 7 *For $\Phi \in \mathcal{N}$ define*

$$\gamma_\Phi = \begin{cases} \gamma(\Phi, |\sigma(I)|) \text{ as in Theorem 5(a)} & \text{if } \Phi'(+0) = 0 \text{ and } \sigma \text{ bounded,} \\ \gamma(\Phi) \text{ as in Remark 2} & \text{if } \Phi'(+0) > 0. \end{cases}$$

If f, $g \in GV_\Phi(I)$ and $|1 - g|_\Phi < 1/\gamma_\Phi$, then $f/g \in GV_\Phi(I)$.

Proof. Apply Banach's contraction principle in $GV_\Phi(I)$ to solve the functional equation $x = (1 - g)x + f$ with respect to the unknown function $x \in GV_\Phi(I)$ (see also estimate (19) with $\Psi = \Phi$ and $h_1 = 1 - g$). □

Given $n \in \mathbb{N}$, let $(\mathbb{R}^n)^I = (\mathbb{R}^I)^n$ be the algebra of all functions $f : I \to \mathbb{R}^n$, $h : I \times \mathbb{R}^n \to \mathbb{R}$ a function of $n + 1$ variables, $h = h(t, x_1, \ldots, x_n)$, and let $H : (\mathbb{R}^I)^n \to \mathbb{R}^I$ be the (h-generated) *superposition operator* defined by

$$H(f)(t) = h(t, f_1(t), \ldots, f_n(t)), \qquad t \in I, \quad f = (f_1, \ldots, f_n) \in (\mathbb{R}^n)^I. \tag{20}$$

If $\Phi = (\Phi_1, \ldots, \Phi_n) \in \mathcal{N}^n$, we endow the Cartesian product

$$GV_\Phi(I) = GV_{\Phi_1}(I) \times \cdots \times GV_{\Phi_n}(I)$$

with the product norm $|f|_\Phi = \sum_{i=1}^n |f_i|_{\Phi_i}$, $f = (f_1, \ldots, f_n) \in GV_\Phi(I)$. Clearly, $GV_\Phi(I)$ is a Banach algebra with respect to componentwise operations.

If $\Phi = (\Phi_1, \ldots, \Phi_n) \in \mathcal{N}^n$ and $\Psi \in \mathcal{N}$, we write $\Psi \preccurlyeq \Phi$ provided $\Psi \preccurlyeq \Phi_i$ for all $i = 1, \ldots, n$, and $\Phi \lhd \Psi$—provided $\Phi_i \lhd \Psi$ for all $i = 1, \ldots, n$.

Corollary 8 *Let $H : (\mathbb{R}^I)^n \to \mathbb{R}^I$ be the superposition operator generated by the function $h : I \times \mathbb{R}^n \to \mathbb{R}$ according to* (20)*, and let $\Phi \in \mathcal{N}^n$ and $\Psi \in \mathcal{N}$.*

(a) *If $\Psi \preccurlyeq \Phi$ and σ is bounded, then H maps $GV_\Phi(I)$ into $GV_\Psi(I)$ and is Lipschitzian if and only if $h(t, x_1, \ldots, x_n) = h_0(t) + \sum_{i=1}^n h_i(t)x_i$, $t \in I$, $(x_1, \ldots, x_n) \in \mathbb{R}^n$, for some functions $h_i \in GV_\Psi(I)$, $i = 1, \ldots, n$.*

(b) *If $\Phi \lhd \Psi$ and $H : GV_\Phi(I) \to GV_\Psi(I)$ is Lipschitzian, then H is constant.*

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