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INPUT-OUTPUT LINKAGES AND OPTIMAL PRODUCT DIVERSITY

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Input-output linkages and optimal product diversity

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Abstract

We derive a simple necessary and sufficient condition on preferences for the market outcome to be socially optimal under monopolistic competition with input-output (IO) linkages. Preferences that satisfy this condition are typically non-CES and display pro-competitive effects, although they converge to the CES when IO linkages become negligibly weak. We show that the equilibrium with pro-competitive effects may deliver both excessive and insufficient entry of firms in equilibrium.

JEL classification: D4, D6, L1.

Keywords: input-output linkages; optimum product diversity; monopolistic competition; pro-competitive effects.

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1 Introduction

This paper addresses the question of how agglomeration economies affect optimum product diversity. The discussion on optimum product diversity was launched by Spence (1976), who pointed out that love for variety and tougher competition push the economy, respectively, toward excessive and insufficient entry compared to the optimum. Hence, the comparison is generally ambiguous. The key message is that a decreasing elasticity of utility usually generates excess entry (Dixit and Stiglitz, 1977; Dhingra and Morrow, 2017) while the links between a decreasing elasticity of utility and pro-competitive effects have been studied by Bykadorov et al. (2015). They show that utilities with decreasing elasticity typically generate pro-competitive effects under monopolistic competition with additive preferences. Hence pro-competitive effects generally lead to excessive entry. We contribute to the literature by showing that taking IO linkages into account dramatically changes these results.

We show that the properties of the technological side of the economy are key for welfare effects. Contrary to the majority of the existing normative analysis, the CES case is no longer the border line between excess and insufficient entry in the presence of IO linkages. The reason is that IO linkages affect the market outcome in different ways. First, the existence of the intermediate sector increases variety.[1] Second, a higher substitution in production decreases love for variety which is a 'weighted average' of consumers' love for variety and love for variety in production. Hence, the higher the technological substitutability, the more likely a low love for variety in production is to dominate the love for variety in consumption, thus reducing variety. In other words, the two effects work in opposite directions. This leaves room for shifting the equilibrium towards insufficient entry. Hence, under high technological substitution IO linkages reduce excess entry in equilibrium with pro-competitive effects. As a result, the subclass of utilities for which the optimum and equilibrium coincide display pro-competitive markup behavior, i.e. markups decrease with the the mass of firms. Moreover, the equilibrium with pro-competitive effects may feature insufficient entry when that effect is strong enough. However, for given preferences with pro-competitive effects and relatively low technological substitution, both effects work in the same direction, so that IO linkages increase variety.

Thus, we revisit the role of agglomeration economies for optimum product diversity. In this respect, our results are related to those by Ethier (1982) and Benassy (1996), who study the role of external increasing returns to scale and consumption externalities, respectively. However, we diverge from these studies by employing a well-known micro-founded mechanism of IO linkages instead of 'black-box' assumptions on consumption externalities, thus complementing their results.

[1] Agglomeration economies intensify market interactions between firms via IO linkages which increase demand for varieties. Thus, an increase in product demand invites new entrants and drives firms to exploit the increasing returns to scale more heavily which may foster competition in the presence of pro-competitive effects. This effect is of paramount importance in the literature on international trade (Ethier, 1982) and economic growth (Romer, 1990; Grossman and Helpman, 1990).

We show that for any non-zero size of the intermediate good sector, there exists a utility function with pro-competitive markup behavior such that the market outcome coincides with the social optimum. Therefore, since pro-competitive effects generally lead to excessive entry, IO linkages may push the market outcome towards optimal levels of product diversity under the presence of pro-competitive effects.

2 The model

Consider an economy with a mass L of consumers each of whom supplies one unit of labor. There is one sector producing a horizontally differentiated good which involves a mass of varieties N. Each firm $k \in [0, N]$ produces a single variety, and each variety is produced by a single firm. In other words, our framework suggests monopolistic competition without scope economies.

2.1 Preferences and technology

We assume that consumers share unspecified (although identical and symmetric) additive preferences (Krugman, 1979; Vives, 1999; Zhelobodko et al., 2012) given by

$$U = \int_0^N u(x_k)dk, \tag{1}$$

where x_k is per capita consumption of variety k and $u(x_k)$ is the utility of its consumption. We assume that $u(\cdot)$ is thrice differentiable, increasing and concave, and $u(0) = 0$. Each consumer seeks to maximize her utility (1) subject to the budget constraint

$$\int_0^N p_k x_k dk = w, \tag{2}$$

where w is the wage. The first order conditions yield a demand function D_k^F for final consumption

$$D_k^F = L(u')^{-1}(\lambda p_i), \tag{3}$$

where λ is the Lagrange multiplier.

On the supply side, we assume a technology à la Krugman and Venables (1995) – the whole range of differentiated varieties is used both in final consumption and in production of the differentiated good. Hence, the total cost function is Cobb-Douglas over labor and intermediates:

$$C(q_k) = (F + cq_k)w^\alpha P^{1-\alpha}, \tag{4}$$

where q_k is output of firm k, α is a share of labor in production, and P is the CES price index,

$$P = \left(\int_0^N p_k^{1-\sigma} \mathrm{d}k \right)^{\frac{1}{1-\sigma}},$$

and $\sigma > 1$ is the elasticity of technological substitution across intermediate varieties. We assume that final and intermediate goods are traded on the same market, therefore, in equilibrium, the price for each variety is the same for both types of buyers.

The total demand D_k for each variety k is given by

$$D_k(p_k) = D_k^F + D_k^I, \tag{5}$$

where D_k^I is the demand for variety k as the intermediate good. Firms' total spending on intermediates is given by $(1-\alpha)C(q_k)$ due to the Cobb-Douglas technology (4), therefore, D_k^I takes the form

$$D_k^I = N \cdot \frac{p_k^{-\sigma}}{P^{1-\sigma}} \cdot (1-\alpha) \cdot C(q_k). \tag{6}$$

2.2 Equilibrium

Since both production costs and demand schedules are identical across firms, we suppress the firm index k and study the symmetric equilibrium. The price elasticity $\varepsilon_p(D)$ of demand for each variety takes the standard form of a weighted average

$$\varepsilon_p(D) = \frac{\frac{D^F}{r_u(x)} + \sigma D^I}{D^F + D^I}, \tag{7}$$

where $r_u(x)$ is the elasticity of inverse demand for the final consumption given by

$$r_u(x) = -\frac{xu''(x)}{u'(x)}.$$

Using the zero-profit condition $pq = C(q)$ and the firm's budget constraint $(1-\alpha)C(q) = p \cdot D^I$, we obtain that, in equilibrium, the shares of the total output used for final and intermediate consumption are constant and equal, respectively, to α and $1-\alpha$. As a consequence, using (7) the markup $m = 1/\varepsilon_p(D)$ takes the form

$$m(x) = \frac{1}{\frac{\alpha}{r_u(x)} + \sigma(1-\alpha)}. \tag{8}$$

Equation (8) shows that, similar to the case without IO linkages, we can represent the equilibrium markup (8) as a function of per capita consumption x only. Moreover, similar to Zhelobodko et al. (2012), preferences exhibit pro-competitive behavior of markups, i.e. $m'(x) > 0$, if the elasticity $r_u(x)$ of inverse demand is an increasing function.

At a symmetric outcome, the equilibrium price index is given by:

$$P = N^{\frac{1}{1-\sigma}} p, \tag{9}$$

whence the equilibrium price is

$$p = \frac{wc^{\frac{1}{\alpha}} N^{\frac{1-\alpha}{\alpha(1-\sigma)}}}{(1-m)^{\frac{1}{\alpha}}}. \tag{10}$$

Plugging (10) into the zero profit condition $pq = C(q)$ and using $Lx = \alpha q$, we obtain

$$\frac{xm}{1-m} = \alpha \cdot \frac{F}{cL}. \tag{11}$$

Finally, plugging (8) in (11) we get the formula

$$\frac{cLx}{cLx+F} = \frac{(\sigma(1-\alpha)-1)\, r_u(x) + \alpha}{\left(\sigma(1-\alpha) - 1 + \frac{1}{\alpha}\right) r_u(x) + \alpha}, \tag{12}$$

which pins down the equilibrium individual consumption x_{eq}.

2.3 Optimality

Using the duality principle, (4) may be represented by a production function given by

$$q = \frac{1}{c} \cdot \left(\frac{1}{N} \cdot \frac{L^{\alpha} Y^{1-\alpha}}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}} - F \right), \tag{13}$$

where $Y \equiv \left(\int_0^N y_k^{\frac{\sigma-1}{\sigma}} \mathrm{d}k \right)^{\frac{\sigma}{\sigma-1}}$ is the CES aggregator over manufacturing varieties while y_k is the output for intermediate consumption. Under symmetric allocation of varieties it becomes $Y = yN^{\frac{\sigma}{\sigma-1}}$. Thus, the first-best optimum is the solution to the following problem

$$\max_{N,x} U = Nu(x),$$

$$\text{s.t.} \qquad Lx + Ny \leq \frac{1}{c} \cdot \left(\frac{1}{N} \cdot \frac{L^{\alpha} Y^{1-\alpha}}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}} - F \right).$$

After simplifications (see Appendix A for details), the problem takes the form

$$\max_{x} U = C \cdot \frac{u(x)}{(Lcx+F)^{\frac{\alpha(\sigma-1)}{\sigma+\alpha-2}}}, \tag{14}$$

where C is a positive constant. The first-order condition of (14) yields

$$\frac{Lcx}{Lcx+F} = \frac{\sigma+\alpha-2}{\alpha(\sigma-1)} \cdot \varepsilon_u(x), \tag{15}$$

where $\varepsilon_u(x) = xu'(x)/u(x)$ is the elasticity of utility function. Hence, the optimal per capita consumption x_{opt} is a solution to (15). Therefore, combining (12) and (15) we obtain the following Lemma.

Lemma 1. *A necessary and sufficient condition on preferences for the market outcome to be socially optimal is given by*

$$\frac{(\sigma(1-\alpha)-1)\, r_u(x)+\alpha}{\left(\sigma(1-\alpha)-1+\frac{1}{\alpha}\right) r_u(x)+\alpha} = \frac{\sigma+\alpha-2}{\alpha(\sigma-1)} \cdot \varepsilon_u(x). \tag{16}$$

In the case without IO linkages, $\alpha \to 1$ so that (16) boils down to the standard "private markup = social markup" condition $r_u(x) = 1 - \varepsilon_u(x)$ (Kuhn and Vives, 1999). However, (16) shows that taking IO linkages into account dramatically changes this condition, thus, we end up with our key result.

Proposition. *For any given size of the intermediate sector* $1-\alpha > 0$*, there exists a set of non-CES additive preferences with pro-competitive behavior (*$m'(x) > 0$*) satisfying (16) so that the market outcome is socially optimal.*

Proof. See Appendix B.

Agglomeration economies give rise to new effects. First, the existence of the intermediate good sector leads to (i) additional demand stemming from producers, hence, more variety compared to the case without IO linkages; and (ii) a complementary 'social value' by decreasing production costs (4) through a drop in the CES price index. Second, for any positive size of the intermediate sector, $1-\alpha > 0$, elasticity of substitution $1/m$ is a 'weighted average' of final consumption elasticity $1/r_u(x)$ and technological elasticity σ. Higher substitution in production σ increases 'average' elasticity which reduces variety. In other words, higher technological substitution outweighs consumers' love for variety. Thus, the two effects work in opposite directions.

Lemma 2. *For a given additive preferences with pro-competitive effects there exists a threshold value* $\hat{\sigma}$ *such that for any* $\sigma > \hat{\sigma}$ *the second effect dominates the first one.*

Proof. See Appendix C.

Under the condition $\sigma > \hat{\sigma}$, IO linkages reduce excess entry in equilibrium with pro-competitive effects and the equilibrium may reach the optimal product diversity. Furthermore, the equilibrium with pro-competitive effects may even feature insufficient entry if the effect is strong enough. However, when the technological substitutability σ is relatively low, the first effect dominates the second, so that IO linkages amplify variety. Consequently, in this case variety is broader compared to the equilibrium without IO linkages due to high love for variety in production.

Note also that we rely on the case with pro-competitive effects as the more plausible one. Even though, for each given size of the intermediate sector $1-\alpha$ there also exists a set of preferences with anti-competitive behavior, i.e. $m'(x) < 0$, such that the market outcome coincides with the optimum.

Last, to illustrate our findings, we study the special case when $\sigma(1-\alpha) = 1$ and show that the differential equation (16) can be solved in closed form (see Appendix D). In this case, the market outcome coincides with the social optimum when preferences are given by

$$u(x) = x^{\beta} - \gamma x^{\delta}, \tag{17}$$

where $0 < \beta < 1$, $\gamma \geq 0$, and $\delta \geq 1$ are constants. Unless $\gamma = 0$, this family of preferences generate pro-competitive markup behavior.

3 Conclusion

We contribute to the literature on optimal product diversity by showing that, in the presence of input-output linkages, the family of CES preferences is no longer the border line between excess and insufficient entry. Hence, pro-competitive behavior of preferences is compatible with both excess and insufficient entry of firms in equilibrium. This result confirms the existence of *non-CES additive preferences that deliver optimum product diversity* under monopolistic competition with IO linkages.[2] Therefore, agglomeration economies may push the market outcome towards optimal levels of product diversity in the presence of pro-competitive effects.

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[2] Note that Parenti et al. (2017) construct examples of non-CES preferences which capture the pro-competitive effect of entry and yield optimum product diversity under monopolistic competition even in the absence of IO linkages. Contrary to (17), those preferences are neither additive nor homothetic.

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4 Appendices

4.1 Appendix A

The social planner's program is given by

$$\max_{N,\,(x_k)} U = \int_0^N u(x_k)\mathrm{d}k$$

s.t.

$$Lx + \int_0^N y_k \mathrm{d}k \leq \frac{1}{c}\left(\frac{1}{N}\frac{L^\alpha Y^{1-\alpha}}{\alpha^\alpha(1-\alpha)^{1-\alpha}} - F\right), \tag{18}$$

where Y is the total demand for the intermediate good:

$$Y \equiv \left(\int_0^N y_k^{\frac{\sigma-1}{\sigma}} \mathrm{d}k\right)^{\frac{\sigma}{\sigma-1}},$$

y_k is the demand for the variety k.

In the symmetric outcome we have

$$Y = N^{\frac{\sigma}{\sigma-1}} y.$$

Using symmetry and monotonicity, restate the constraint (18) as follows:

$$Lcx + Ncy + F = \frac{L^{\alpha} N^{\frac{1-\alpha\sigma}{\sigma-1}} y^{1-\alpha}}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}}, \tag{19}$$

which pins down per capita consumption x as a function of both mass of firms N and intermediate consumption y. Hence, the first order condition for a given mass of firms N yields

$$y = \frac{(1-\alpha)L}{\alpha c^{\frac{1}{\alpha}} N^{\frac{(1+\alpha)\sigma-2}{\alpha(\sigma-1)}}}. \tag{20}$$

Plugging (20) into (19) and solving for N, we get:

$$N = \frac{C}{(Lcx+F)^{\frac{\alpha(\sigma-1)}{\sigma+\alpha-2}}}, \tag{21}$$

where

$$C = \left(\frac{L}{c^{\frac{1-\alpha}{\alpha}}}\right)^{\frac{\alpha(\sigma-1)}{\sigma+\alpha-2}}.$$

Finally, in symmetric outcomes the objective functional of the social planner's program is expressed by $Nu(x)$. Combining this with (21), we reduce the social optimum analysis to solving the following univariate unconstrained optimization problem:

$$\max_x U = \frac{Cu(x)}{(Lcx+F)^{\frac{\alpha(\sigma-1)}{\sigma+\alpha-2}}}. \tag{22}$$

The first order condition for (22) is given by

$$\frac{u'(x)}{(Lcx+F)^{\frac{\alpha(\sigma-1)}{\sigma+\alpha-2}}} - \frac{\alpha(\sigma-1)}{\sigma+\alpha-2} \frac{u(x)}{(Lcx+F)^{\frac{\alpha(\sigma-1)}{\sigma+\alpha-2}+1}} Lc = 0,$$

which yields after simplification:

$$\frac{Lcx}{Lcx+F} = \frac{\sigma+\alpha-2}{\alpha(\sigma-1)} \cdot \varepsilon_x(u),$$

where $\varepsilon_x(u) = xu'(x)/u(x)$ is the elasticity of utility function. The last equation pins down the optimal per capita consumption level x_{opt}.

4.2 Appendix B

One can note that the left-hand side of the optimal condition (16) is a decreasing function of $r_u(x)$ while the right-hand side increases with $\varepsilon_u(x)$. Hence, the solution to (16) is a function with either increasing $r_u(x)$ and decreasing $\varepsilon_u(x)$ or vice versa. In addition, the solution to (16) does not satisfy standard utility properties when $\sigma < 2-\alpha$ since the the right-hand side of (16) is negative. Let us rewrite the (16) as

$$\frac{(\sigma(1-\alpha)-1)\, r_u + \alpha}{\frac{(1-\alpha)(\alpha\sigma+1)}{\alpha} r_u + \alpha} = \frac{\sigma+\alpha-2}{\alpha(\sigma-1)} \cdot \varepsilon_u \tag{23}$$

and denote $a = \sigma(1-\alpha)-1$, $b = \frac{(1-\alpha)(\alpha\sigma+1)}{\alpha}$ and $d = \frac{\sigma+\alpha-2}{\alpha(\sigma-1)}$. Note that $b > 0$, $b > a$, while $d > 1$ if $\sigma > 2$. Hence, in the following analysis we distinguish between two cases: (i) when $\sigma \geq 2$, and (ii) when $2-\alpha < \sigma < 2$.

Using notations, (23) takes the form

$$\alpha(1 - d\cdot\varepsilon_u) = r_u(db\cdot\varepsilon_u - a).$$

Therefore, $\frac{a}{b}\cdot\frac{1}{d} < \varepsilon_u < \frac{1}{d}$. Using the expression $r_u = 1 - \varepsilon_u - \frac{x\varepsilon_u'}{\varepsilon_u}$ we get

$$\left(\varepsilon_u - \frac{a}{bd}\right)\frac{x\varepsilon_u'}{\varepsilon_u} = -\left(\varepsilon_u^2 - \frac{bd+a+\alpha d}{bd}\cdot\varepsilon_u + \frac{a+\alpha}{bd}\right). \tag{24}$$

Let ξ_1 and ξ_2 be the solutions to

$$\varepsilon_u^2 - \frac{bd+a+\alpha d}{bd}\cdot\varepsilon_u + \frac{a+\alpha}{cb} = 0. \tag{25}$$

Hence, (24) takes the form

$$\frac{\mathrm{d}\varepsilon_u}{dx}\cdot\frac{x}{\varepsilon_u} = -\frac{(\varepsilon_u-\xi_1)(\varepsilon_u-\xi_2)}{\left(\varepsilon_u - \frac{a}{bd}\right)}. \tag{26}$$

One can show that the right-hand side of the (24) is negative and increases for $\varepsilon_u = \frac{a}{bd}$. We start with the case when $\sigma > 2$. Then $0 < \xi_1 < 1 < \xi_2$ and the left-hand side of the (25) is negative when $\varepsilon_u = 1/d$. Therefore, (26) implies that $\mathrm{d}\varepsilon_u/dx < 0$ at the interval $(a/bd,\, \xi_1]$ and $\mathrm{d}\varepsilon_u/dx > 0$ when $\varepsilon_u \in (\xi_1,\, 1/d]$. Hence, for the former case the solution to (16) is an increasing and concave utility function with pro-competitive markup behavior whereas for the latter case it demonstrates anti-competitive behavior. Consequently, the subclass of additive preferences which delivers optimal product diversity features variable elasticity of substitution. Note that the CES function with $\rho = \xi_1$ belongs to this subclass.

We turn to the case when $2-\alpha < \sigma < 2$. Then $0 < \xi_1 < \xi_2 < 1$. Note however that the right-hand side of the (25) reaches its maximum at

$$\varepsilon_u^* = \frac{bd + a + \alpha d}{2bd}.$$

One can show that the right-hand side of the (25) is positive for $\sigma = 2$ at the maximum point $\varepsilon_u = \varepsilon_u^*$ and negative for $\sigma = 2 - \alpha$. Therefore, there exists a threshold value $\bar{\sigma}$ such that for $\bar{\sigma} < \sigma < 2$: (i) $\mathrm{d}\varepsilon_u/dx < 0$ when $\varepsilon_u \in (a/bd,\, \xi_1] \cup [\xi_2,\, 1/d]$, and (ii) $\mathrm{d}\varepsilon_u/dx > 0$ when $\varepsilon_u \in (\xi_1,\, \xi_2)$. In other words, similar to the $\sigma > 2$ case utilities which feature both pro-competitive and anti-competitive behavior could be solutions. Whereas in the case $2 - \alpha < \sigma < \bar{\sigma}$ the solution to the (16) is a function which exhibits pro-competitive behavior since $\mathrm{d}\varepsilon_u/dx < 0$.

4.3 Appendix C

Combining (8) and (11), we obtain the following representation for per capita consumption x:

$$x = \alpha \cdot \frac{F}{cL}\left(\frac{\alpha}{r_u(x)} + \sigma(1-\alpha) - 1\right).$$

Therefore, the changes in x with respect to substitution σ in production are given by

$$\left(1 + \frac{\alpha^2 F}{cL} \cdot \frac{r_u'}{r_u^2}\right) \cdot \frac{\mathrm{d}x}{\mathrm{d}\sigma} = (1-\alpha) \cdot \frac{\alpha F}{cL}. \tag{27}$$

It is readily verified that under pro-competitive effects, i.e. $r_u' > 0$, higher production substitutability leads to higher consumption level x. Plugging (10) into the budget constraint (2) we get

$$cN^{\frac{\alpha\sigma-1}{(\sigma-1)}} \cdot \frac{x^\alpha}{(1-m)} = 1. \tag{28}$$

Higher elasticity of substitution σ results in an increase in x whereas the impact on the markup is ambiguous. Indeed, the first term in the denominator of (8) decreases under pro-competitive effects whereas the second term increases. Hence, higher σ decreases variety if the second term $x^\alpha/(1-m)$ in (28) increases. Taking the derivative and using (27), we get

$$\frac{\mathrm{d}\left(x^\alpha/(1-m)\right)}{\mathrm{d}\sigma} = x_\sigma' \frac{\alpha - m}{\alpha(1-m)^{\frac{1}{\alpha}}}.$$

The last derivative is positive when $\alpha > m$, substituting for m, we show that it holds when $\sigma > \hat{\sigma}$, where $\hat{\sigma}$ is given by

$$\hat{\sigma} = \frac{1}{\alpha(1-\alpha)}\left(1 - \frac{\alpha^2}{r_u(x)}\right).$$

4.4 Appendix D

Setting $\sigma = 1/(1-\alpha)$ in (16) we obtain

$$\frac{\alpha^4}{\alpha-(1-\alpha)^2} \cdot \frac{1}{\varepsilon_u} - \alpha^2 = r_u.$$

Plugging $r_u = 1 - \varepsilon_u - \frac{x\varepsilon'_u}{\varepsilon_u}$ into the last equation, we get

$$\frac{\alpha^4}{\alpha-(1-\alpha)^2} \cdot \frac{1}{\varepsilon_u} - \alpha^2 = 1 - \varepsilon_u - \frac{x\varepsilon'_u}{\varepsilon_u}$$

$$x\varepsilon'_u = -\left[\varepsilon_u^2 - (1+\alpha^2)\varepsilon_u + \frac{\alpha^4}{\alpha-(1-\alpha)^2}\right] = (\varepsilon_u - \xi_1)(\xi_2 - \varepsilon_u), \tag{29}$$

where ξ_1 and ξ_2 are solutions to

$$\xi^2 - (1+\alpha^2)\xi + \frac{\alpha^4}{\alpha-(1-\alpha)^2} = 0.$$

Note that when $\alpha \in (1/2,\ 1)$, we have $0 < \xi_1 < 1 < \xi_2$. We can restate (29) as follows

$$\frac{\mathrm{d}\varepsilon_u}{(\varepsilon_u - \xi_1)(\xi_2 - \varepsilon_u)} = \frac{dx}{x}. \tag{30}$$

Integrating the last equation leads to

$$\ln \frac{\xi_1 - \varepsilon_u}{\xi_2 - \varepsilon_u} + C = (\xi_2 - \xi_1) \ln x,$$

where C is constant. Taking the exponential on both sides, we come to

$$\gamma \cdot \frac{\xi_2}{\xi_1} \cdot x^{\xi_2 - \xi_1} = \frac{1 - \varepsilon_u/\xi_1}{1 - \varepsilon_u/\xi_2},$$

where $\gamma \equiv \exp(-C)$ is an arbitrary positive constant. Solving for ε_u yields

$$\varepsilon_u(x) = \xi_2 \cdot \frac{\xi_1/\xi_2 - \gamma x^{\xi_2 - \xi_1}}{1 - \gamma x^{\xi_2 - \xi_1}}, \quad x \le \overline{x} \equiv \left(\frac{\xi_1}{\gamma \xi_2}\right)^{\frac{1}{\xi_2 - \xi_1}}.$$

It is readily verified that this elasticity is associated with the following sub-utility:

$$u(x) = x^{\xi_1} - \gamma x^{\xi_2},$$

which exhibits pro-competitive behavior, i.e. $r'_u(x) > 0$. Setting $\xi_1 = \beta$ and $\xi_2 = \delta$ we obtain (17).

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