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Optimal insurance and reinsurance policies chosen jointly in the individual risk model

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The paper studies the so-called individual risk model where both a policy of per-claim insurance and a policy of reinsurance are chosen jointly by the insurer in order to maximize his/her expected utility. The insurance and reinsurance premiums are defined by the expected value principle. The problem is solved under additional constraints on the reinsurer's risk and the residual risk of the insured. It is shown that the solution to the problem is the following: The optimal reinsurance is a modification of stop-loss reinsurance policy, so-called stop-loss reinsurance with an upper limit; the optimal insurer's indemnity is a combination of stop-loss- and deductible policies. The results are illustrated by a numerical example for the case of exponential utility function. The effects of changing model parameters on optimal insurance and reinsurance policies are considered.

Keywords: optimal risk-sharing; insurance; reinsurance; utility function

1. Introduction

This paper is devoted to a problem of optimal risk-sharing in the individual risk model (see, e.g. Daykin *et al.* (1994)) that includes here an insurance company (insurer), n potential insurance purchasers (insureds), and a reinsurance company (reinsurer). The decision maker is the insurer maximizing his/her expected utility, interests of the other $n + 1$ agents are reflected by imposing additional constraints on their risks that are supposed to be met with probability one.

The problem of optimal risk-sharing in static insurance models was analyzed in numerous papers. In his pioneering work, Arrow (1971) found that a policy with a deductible is optimal for the insured in the model with a premium containing a fixed percentage loading. Arrow's result has been extended in many papers. Raviv (1979) found Pareto-optimal policies for a model with a more general function of operating expenses. Further, models with additional constraints imposed on an insurance contract were also studied. For instance, Cummins & Mahul (2004) pointed out that the real-world insurance and reinsurance markets typically impose limits on coverage. In the paper, they investigated an influence of different type utility functions on deductible level under an upper limit on insurer's indemnity imposed with probability one. In Zhou & Wu (2008), the insured aims to maximize the expected utility of his/her terminal wealth, under the constraint that the expected loss of the insurer's terminal wealth is maintained below

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some prescribed level. Zhou *et al.* (2010) studied a similar problem with, however, an upper bound imposed on ceded losses. An optimal reinsurance problem from the perspective of the insurer is investigated in Chi & Lin (2014), who generalized the previous settings of constraints on a reinsurance contract. A criterion to be optimized has a general form of a functional that preserves the convex order that includes, in particular, an expected utility of terminal insurer's wealth. An optimal policy turns out to be a stop-loss reinsurance or a two-layer reinsurance, depending upon the amount of the reinsurance premium. A model close to ours is that in Golubin *et al.* (2009) in which, however, only per-claim reinsurance and insurance are allowed under an upper limit imposed on the mean value of insured's residual risk. As was proved, the latter constraint leads to existence of a whole class (typically uncountable) of optimal insurance policies.

Another direction of research is that employing the 'Value at Risk' (VaR) notion as either a constraint or a criterion to be minimized. Chi & Tan (2013) studied a model of reinsurance for quite a general class of reinsurance premium and showed that two-parameter layer reinsurance is always optimal with respect to Var and conditional VaR measures. The VaR might be one of the most popular risk measures, however, it suffers several shortcomings. In particular, VaR is defined as the 'possible maximum loss within a fixed confidence level,' so VaR is mostly concerned about the loss frequency, and disregards the loss magnitude – in distinction to the use of a utility function.

The present paper is an attempt to solve the problem of insurer's optimal risk-sharing in the individual risk model 'at full extent,' i.e. in a situation where the decision-maker (insurer) is allowed to use all admissible insurance and reinsurance policies jointly. The paper differs from the previous works by two main points. First, the insurance optimization problem is formulated for a group of insureds, not for a single policy holder. This, as we will see, leads to employing a conditional expectation in an optimality equation that determines an optimal insurance policy. Second, the insurer has at hand two tools for risk-sharing: he/she chooses jointly a policy of individual risk insurance and a policy of reinsurance. Unlike Zhou & Wu (2008), Zhou *et al.* (2010), and Chi & Lin (2014), the constraints on the reinsurer's and insured's retained risks are set with probability one – similar to that in Cummins & Mahul (2004).

The interest to reinsurance policies of aggregate risk stemmed from a result in Raviv (1979, section V) which establishes that the optimal insurer's policy of risk-sharing with the client having multiple losses is an arrangement of a pool of individual risks. As we will show, an analogous result holds for the problem of optimal choice of a reinsurance policy. In this context, the model, where policies of per-claim insurance and summary risk reinsurance are chosen jointly under natural constraints on insured's and reinsurer's risks, seems an interesting object for investigation.

The rest of the paper is organized as follows. Section 2 gives a formal description of the model. Section 3 deals with a variant of the studied problem, maximization of insurer's utility over reinsurance policies only. It is proved that the optimum is an aggregate risk reinsurance of the form of a stop-loss policy with an upper limit. In Section 4 we study the general case where both insurance and reinsurance are applied under upper limits on the insured's residual risk and reinsurer's indemnity imposed. We show that the solution to the problem is the following: The optimal reinsurance is still a stop-loss policy with an upper limit, the optimal insurer's indemnity is a combination of stop-loss- and deductible policies. Optimality equations determining two unknown parameters in the policies are derived. We also compare the optimal contracts in the case of per-claim reinsurance and in our case of aggregate risk reinsurance. In Section 5, the obtained results are illustrated by a numerical example involving truncated normal distribution of summary insurer's indemnity and exponential utility function. The effects of changing model parameters (such as the reinsurer's loading coefficient and upper limit of the residual insured's risk) on parameters of optimal insurance and reinsurance policies are discussed. Finally, Section 6 provides some concluding remarks. The proofs of all Lemmas, Propositions, and Theorems are placed in the Appendix.

2. The model setup

Consider a market consisting of $n+2$ agents: an insurer, a group of n insureds ($n \geq 2$), and a reinsurer. The individual insureds' losses X_j, $j = 1, \dots, n$ are non-negative independent stochastic variables defined on the same probability space $(\Omega, \mathcal{F}, P)$. The group is assumed to be homogeneous, that is, risks X_j are identically distributed, $F(x) \stackrel{def}{=} P\{X_j \leq x\}$. The essential supremum of X_j or, roughly saying, the greatest possible value of X_j is denoted by $T = Ess\,\sup(X_j) \leq \infty$.

The insurer's preferences are represented by his/her utility function $u(x)$, meaning that $X \succ Y$ if and only if $E\,u(X) > E\,u(Y)$. We assume the smooth increasing and strictly concave utility function, more exactly, $u'(x) > 0$ and $u''(x) < 0$ in the relevant domain.

Suppose the insurer to choose a per-claim insurance of I and a reinsurance of A. That is, if the loss of jth insured is $X_j = x$ then he/she is paid $I(x)$; if the the vector $\bar{X}^I \stackrel{def}{=} (I(X_1), \dots, I(X_n))$ of insurer's indemnity payments is $(z_1, \dots, z_n)$ then the insurer pays $A(z_1, \dots, z_n)$, the rest $\sum_1^n z_j - A(\bar{z})$ is covered by the reinsurer. The functions $A(\bar{x})$ and $\sum_{j=1}^n x_j - A(\bar{x})$ are often called the retained and ceded loss functions, respectively. According to the expected value principle (see, e.g. Daykin *et al.* (1994)) that widely used in insurance literature and, in particular, in works dealing with optimal risk-sharing (see, e.g. Arrow (1971), Zhou *et al.* (2010), Chi & Tan (2013)), a premium received by insurer from each insured is $P = (1+\alpha)E\,I(X_1)$ and a premium paid to reinsurer is $P_1 = (1+\alpha_1)E\,[X^I - A(\bar{X}^I)]$, where $X^I = \sum_{j=1}^n I(X_j)$ is the summary insurer's indemnity before reinsurance, the constants $0 < \alpha < \alpha_1$ are insurer's and reinsurer's loading coefficients, respectively.

Functions (I, A) are called admissible policies if they are Borel-measurable functions defined, respectively, on R_+ and R_+^n, and satisfying the inequalities $0 \leq I(x) \leq x$ and $0 \leq A(\bar{x}) \leq \sum_{j=1}^n x_j$. The latter means that the insurer's reimbursement is always non-negative and does not exceed the summary loss. We also impose quite natural additional constraints on the admissible policies: Let q and Q be positive constants chosen, correspondingly, by the insured and the reinsurer so that the size of each insured's loss (after insurance) $X_j - I(X_j)$ must not be greater than q almost surely (a.s.), and the size of summary reinsurer's loss $X^I - A(\bar{X}^I)$ must not be greater than Q a.s. These constraints reflect their desire to protect themselves from great potential losses. Taking into account these limitations, the insurer narrows the set of admissible policies

I and A: now they also satisfy the inequalities $Ess\sup(X_1 - I(X_1)) \le q$ and $Ess\sup(X^I - A(\bar{X}^I)) \le Q$.

As is easily seen, equivalent forms of the additional constraints in terms of the coverage functions are $I(x) \ge x - q$ and $A(\bar{x}) \ge \sum_{j=1}^n x_j - Q$ for $x \in [0, \infty)$ and $\bar{x} \in R_+^n$.

Once a pair of policies (I, A) is chosen, the expected utility of insurer's wealth is

$$J[I, A] \stackrel{def}{=} E\, u\left(nP - P_1 - A(\bar{X}^I)\right). \tag{1}$$

Taking into account all the constraints on admissible policies (I, A), we have that the problem under consideration is as follows

$$\max_{I,A} J[I, A] \tag{2}$$

$$\text{subject to} \quad (x-q)_+ \le I(x) \le x \text{ and } ((\sum_{j=1}^n x_j) - Q)_+ \le A(\bar{x}) \le \sum_{j=1}^n x_j,$$

where we denote $(y)_+ = \max\{0, y\}$. To ensure that the expectation $J[I, A]$ is finite under any admissible policies I and A, it suffices to suppose $E\,X_1 < \infty$ and, as $u(x)$ is an increasing function, $E\,|u(-n(1+\alpha_1)E\,X_1 - X)| < \infty$, where $X \stackrel{def}{=} \sum_1^n X_j$. In the proofs below, we will also need an assumption that $E\,\{u'(-n(1+\alpha_1)E\,X_1 - X)X\} < \infty$. Remark in conclusion that coincidence of policies $(I, A) = (I', A')$ is understood with probability one, $I(X_1) = I'(X_1)$ and $A(\bar{X}^I) = A'(\bar{X}^{I'})$ a.s.

3. Optimization by reinsurance

Consider a variation of the above-described model where a choice of risk exchange between the insurer and an insured is not allowed or, in our notation, the coverage function of the insurer is set to be $I(x) \equiv x$. Now the optimization problem (2) takes the form

$$\text{maximize} \quad J[A] \equiv E\, u\,(nP - P_1 - A(X_1, \dots, X_n)), \tag{3}$$

where $P = (1+\alpha)E\,X_1$ and $P_1 = (1+\alpha_1)\{nE\,X_1 - E\,A(X_1, \dots, X_n)\}$.

The following proposition shows that in optimum we come to a reinsurance of summary risk, $A^*(\bar{x}) = A^*(\sum_1^n x_j)$. The proof differs from that in Raviv's paper (1979) and partially uses the reasonings in Buhlmann (1984).

Proposition 1 *If $A^*(\bar{x})$ is a solution to (3) then $A^*(X_1, \dots, X_n) = A^*(X_1 + \dots + X_n)$ a.s. for some admissible $A(x)$.*

By confining the analysis of (3) to consideration of summary risk reinsurance policies $(x - Q)_+ \le A(x) \le x$ for $x \in [0, \infty)$, we prove the following proposition establishing the form of optimal reinsurance policy.

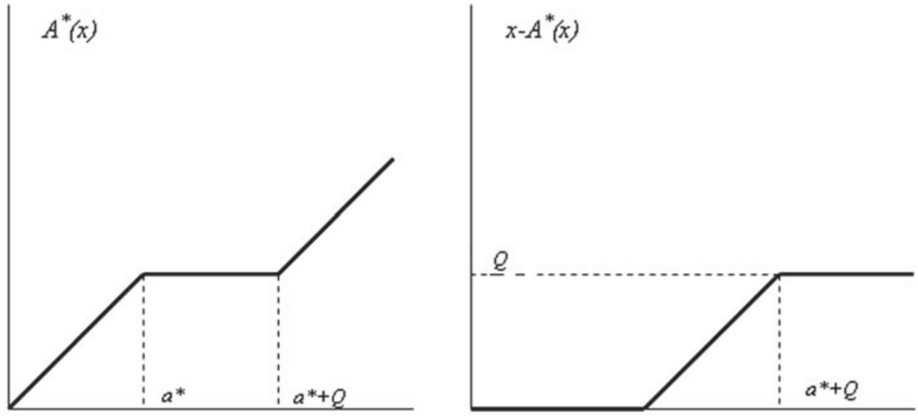

Figure 1. The share $A^*(X)$ paid by insurer and the rest of summary risk $X - A^*(X)$ paid by reinsurer.

PROPOSITION 2 *Problem* (3) *has a unique solution, a Stop-loss reinsurance with an upper limit* $A^*(x) = (x \wedge a^*) \vee (x - Q)$, *where* $x \wedge y$ *stands for* $\min\{x, y\}$ *and* $x \vee y$ *stands for* $\max\{x, y\}$ *(see Figure 1 below). The level*

$$a^* = \begin{cases} a^0 & \text{if } a^0 < nT, \\ nT & \text{otherwise,} \end{cases}$$

where $a^0 > 0$ *is the minimal root of the equation* $\psi(a) = 0$ *with*

$$\psi(a) \stackrel{def}{=} (1 + \alpha_1) E\, u'(P_a - A_a(X)) - u'(P_a - a), \tag{4}$$

where $P_a \stackrel{def}{=} (1 + \alpha_1) E\, A_a(X) - n(\alpha_1 - \alpha) E\, X_1,\ A_a(x) \stackrel{def}{=} (x \wedge a) \vee (x - Q),$ *and* $X \stackrel{def}{=} \sum_{j=1}^n X_j$.

The optimal reinsurance $A^*(x)$ found in Proposition 2 is a two-threshold policy (see Figure 1 below) that we call here a stop-loss reinsurance with an upper limit (the limit Q is the greatest sum the reinsurer is ready to pay). The 'tail' of the summary risk distribution is left with the insurer so that the reinsurer's indemnity $X - A^*(X) = (X - a^*)_+ \wedge Q$ is a 'medium' share of summary risk.

Remark 1 Concerning expression (4) for the function $\psi(a)$ in Proposition 2, note that the term $E\, A_a(X)$ can be rewritten as a Riemann integral

$$E\, A_a(X) = \int_0^\infty P\{A_a(X) > x\} dx = \int_0^a \bar{F}_\Sigma(x) dx + \int_a^\infty \bar{F}_\Sigma(x + Q) dx,$$

where $\bar{F}_\Sigma(x) = 1 - F_\Sigma(x)$ and $F_\Sigma(x)$ is the distribution function of summary risk X. Equation $\psi(a) = 0$ may have no root on $[0, nT]$ (here $nT = Ess\sup(X)$), that is, $\psi(a) > 0$ on $[0, nT]$. In this case the level a^* is set to be nT, which means that the insurer does not apply for reinsurance.

Remark 2 The form of optimal $A^*(x)$ is quite expectable: In the absence of the additional constraint, the second order dominance principle gives the stop-loss policy $A^*(x) = x \wedge a_*$ with some a_*. For our case, the solution 'glues' to the lower boundary $(x - Q)_+$ for $x \geq a^* + Q$. The way of the proof (see Appendix 1) differs from a common approach to solving an optimization problem of risk-sharing in insurance (see, e.g. Raviv (1979), Zhou & Wu (2008), Zhou *et al.* (2010)) that uses the Hamiltonian for the corresponding optimal control problem under a fixed premium. Instead, we find the directional derivatives at an optimum and then convert a condition on their sign into a problem of integral maximization which is analyzed via Neyman–Pearson lemma. Methodologically, this is justified by the following: first, the loss distribution $F(x)$ of a general form (even without density) is admissible within our approach; second, we will further apply it to a more complicated setting of the optimization problem, where both insurance and reinsurance are allowable.

4. The general case: optimal choice of insurance and reinsurance policies jointly

In this section we study problem (2) at full extent, i.e. allowing both per-claim insurance and aggregate risk reinsurance policies to be applied,

$$\text{maximize} \quad J[I, A] \equiv E\,u\left(nP - P_1 - A\left(\sum_{j=1}^{n} I(X_j)\right)\right) \tag{5}$$

subject to $(x - q)_+ \leq I(x) \leq x$ and $(x - Q)_+ \leq A(x) \leq x$.

Below we will need a two-threshold insurance policy $I(x) = (x \wedge k) \vee (x - q)$, which is a combination of a stop-loss policy $x \wedge k$ and a deductible policy $(x - q)_+$ (see Figure 2). As we will see, this kind of insurance policy, along with an upper-limit stop-loss reinsurance (see Figure 1), turns out to be optimal in (5). Under such a policy, the insurer takes the 'tail' of loss distribution and leaves a 'medium' share of insured's risk $X_1 - I(X_1) = (X_1 - k)_+ \wedge q$.

Denote the final insurer's wealth under these policies by $S = P_{k,a} - A_a(X_k^I)$, where $P_{k,a} = (1+\alpha_1)E\,A_a(X_k^I) - n(\alpha_1 - \alpha)E\,I_k(X_1)$, $A_a(x) = (x \wedge a) \vee (x - Q)$, and $X_k^I = \sum_{j=1}^{n} I_k(X_j)$ with $I_k(x) = (x \wedge k) \vee (x - q)$. Denote also $S_a = P_{k,a} - a$.

THEOREM 1 *Optimal policies (I^*, A^*) in problem (5) are of the following form: a combination of stop-loss- and deductible policies, $I^*(x) = (x \wedge k^*) \vee (x - q)$, and a stop-loss reinsurance with an upper limit, $A^*(x) = (x \wedge a^*) \vee (x - Q)$. The levels $k^* = k^0 \wedge T$ and $a^* = a^0 \wedge (nT)$ are such that $0 < k^* < a^*$, parameters k^0 and a^0 satisfy a pair of equations: $\psi_1(k, a) = 0$ and $\psi_2(k, a) = 0$ with*

$$\psi_1(k, a) \stackrel{def}{=} (1 + \alpha)E\,u'(S) - E\left[u'(S)|X_1 = k\right], \tag{6}$$

$$\psi_2(k, a) \stackrel{def}{=} (1 + \alpha_1)E\,u'(S) - u'(S_a). \tag{7}$$

The insurance policy I^* found in Theorem 1 is applied to each individual risk X_j, $j = 1, \ldots, n$, while the reinsurance policy A^* is applied to the summary insurer's indemnity

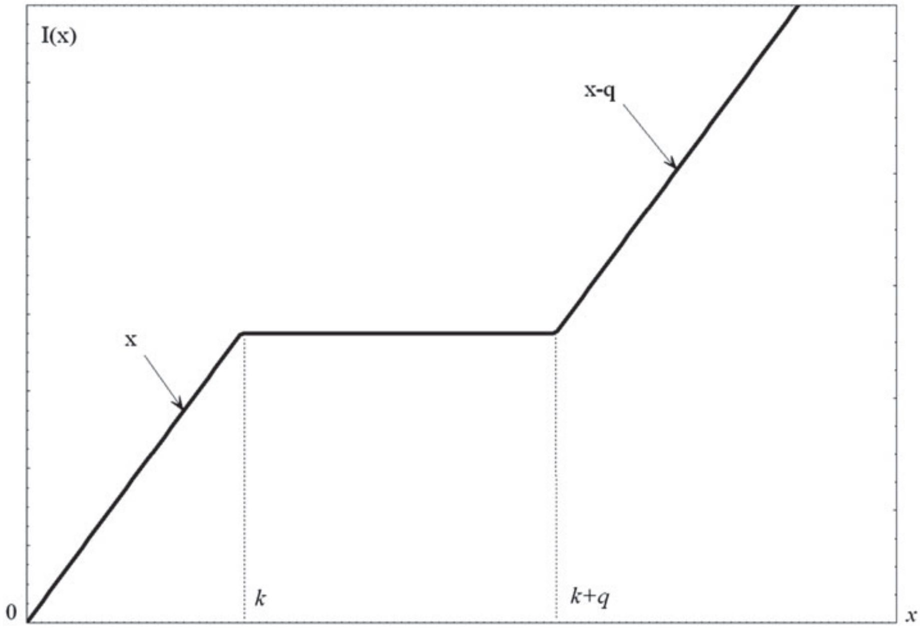

Figure 2. An insurance policy $I(x) = (x \wedge k) \vee (x - q)$.

$X^* = \sum_{j=1}^{n} I^*(X_j)$. This is a principal difference from the model with per-claim reinsurance (see, e.g. Golubin *et al.* (2009)) in which the final indemnity of insurer (after reinsurance) is $\sum_{j=1}^{n} A^*(I^*(X_j))$. If one chooses an appropriate approximation (for definiteness, say, normal) for $\sum_{j=1}^{n} I^*(X_j)$ then, in our case, the insurer's indemnity turns to be a piece-wise linear function $A^*(\cdot)$ of a normally distributed stochastic value X^*. In the case of per-claim reinsurance, the normal approximation should be directly applied to the final insurer's indemnity $\sum_{j=1}^{n} A^*(I^*(X_j))$.

Remark 3 As follows from Theorem 1's statement, there are two possible degenerated cases: If the insurer's retention level $k^* \geq T = Ess\sup(X_1)$ then $I^*(X_1) = X_1$ a.s., so risk sharing between the insurer and insured is not profitable – insurer takes the entire risk. If $a^* \geq nT^* = n(T \wedge k^*) \vee (T - q) = Ess\sup(X^*)$, that is, the level a^* is not less than the greatest possible value of summary insurer's indemnity $X^* = \sum_{j=1}^{n} I^*(X_j)$, then reinsurance is not needed.

It seems interesting to compare the optimal insurer's indemnities in the case of per-claim reinsurance, where the insurer's wealth is $nP - nP^1 - \sum_{j=1}^{n} A(I(X_j))$ with an individual premium $P^1 = (1 + \alpha_1)E[I(X_1) - A(I(X_1))]$ paid to reinsurer, and in our case of summary risk reinsurance, where the wealth is $nP - P_1 - A(\sum_{j=1}^{n} I(X_j))$. For simplicity, consider a variant without additional constraints on insurance and reinsurance policies, so that admissible I and A are those $0 \leq I(x) \leq x$ and $0 \leq A(x) \leq x$. As was shown in Golubin *et al.* (2009, Theorem 1), the optimal policies in the first case are $I^*(x) = x \wedge k^*$ and $A^*(x) = x \wedge a^*$, with $k^* \leq a^*$ as long as $\alpha < \alpha_1$. Hence, the insurer's indemnity $A^*(I^*(X_1)) = I^*(X_1)$ a.s. – the insurer does not apply for reinsurance. However, it is, generally, not the case for the considered model with summary risk reinsurance. Indeed, formally setting $q = Q = \infty$ in Theorem 1, we have that $I^*(x) = x \wedge k^*$ and $A^*(x) = x \wedge a^*$, where $k^* < a^*$. Despite that the retention level k^* of individual risk insurance is not greater than a^*, the upper possible value $Ess\ \sup(X^*) = nk^*$ of summary risk under reinsurance $X^* = \sum_{j=1}^{n} X_j \wedge k^*$ may exceed the level a^*. Such a situation means applying for reinsurance, and the final insurer's indemnity is then $A^*(X^*) = X^* \wedge a^*$. The following proposition shows that this situation always takes place if reinsurance is not very expensive, i.e. the reinsurer's loading coefficient α_1 is 'not much greater' than the loading coefficient α of insurance. An explanation for this effect is the above-mentioned fact that the best insurer's policy of risk-sharing with the reinsurer is an arrangement of a pool of individual indemnities, $\sum_{j=1}^{n} I^*(X_j)$. Having the indemnities pooled in, the insurer should reinsure 'the tail' of the pool distribution even at a (moderately) higher price.

PROPOSITION 3 *For the problem* $\max_{I,A} J[I, A]$ *s.t.* $0 \le I(x) \le x$ *and* $0 \le A(x) \le x$ *in the non-degenerated case* $k^* < T$, *there exists* $\alpha' > \alpha$ *such that for any* $\alpha_1 \in [\alpha, \alpha')$ *the inequality* $nk^* > a^*$ *holds.*

Note in conclusion that if the constraint $0 \le A(x) \le (x - Q)_+$ is imposed on reinsurance policies (admissible I are still those $0 \le I(x) \le x$), assertion of Proposition 3 remains valid by a repetition of the reasonings in its proof. However, in such a situation, the reinsurance is somewhat less convenient for the insurer due to the presence of an additional constraint on admissible A, which, naturally, leads to a lower value of α' in the range $[\alpha, \alpha')$ of reinsurance loading coefficients. It seems interesting that in this case not a 'tail' of the insurer's indemnity, $(X^* - a^*)_+$, – the 'worst' share of his risk – but a 'medium' share of it, $(X^* - a^*)_+ \wedge Q$, is still transferred to the reinsurer.

5. Example

Consider the problem of finding a pair of optimal insurance and reinsurance policies (I^*, A^*) in the case of exponential utility function of the insurer, $u(x) = c^{-1}(1 - \exp(-cx))$ with a given risk aversion coefficient $c = 0.2$. Let the distribution function $F(x)$ of the insured's loss be an exponential distribution, $F(x) = 1 - \exp(-x)$. An important point for further analysis is that we employ a widely used (see, e.g. Daykin *et al.* (1994)) truncated normal distribution $\Phi^{tr}_{m,\sigma}(x)$ as a modeling distribution for aggregated insurer's indemnity $\sum_{j=1}^{n} I(X_j)$. The density of this kind of distributions is given by

$$\mathbf{1}\{x \ge 0\} \exp\left(-\frac{(x-m)^2}{2\sigma^2}\right) \left[\int_0^\infty \exp\left(-\frac{(x-m)^2}{2\sigma^2}\right) dx\right]^{-1}.$$

By Theorem 1, optimal policies in problem (5) of insurer's utility maximization are $I^*(x) = (x \wedge k^*) \vee (x - q)$ and $A^*(x) = (x \wedge a^*) \vee (x - Q)$. The levels k^* and a^* are determined by the pair of optimality equations: $\psi_1(k, a) = 0$ and $\psi_2(k, a) = 0$ (see (6) and (7)). Due to specifics of $u'(x) = e^{-cx}$, the equations take the form

Table 1. Parameters of optimal policies for a group of 20 insureds.

For $n = 20$						
			$q = 2$			
α_1	0.5	0.8	1.1	1.4	1.7	2.0
k^*	2.8817	2.3030	2.0901	1.9864	1.9285	1.8934
a^*	20.9969	21.7055	22.2800	22.8139	23.3135	23.7798
			$\alpha_1 = 0.5$			
q	2	3	4	5	6	7
k^*	2.8817	2.8388	2.8219	2.8155	2.8131	2.8123
a^*	20.9969	20.8737	20.8247	20.8061	20.7992	20.7966

Table 2. Parameters of optimal policies for a group of 5 insureds.

For $n = 5$						:
			$q = 2$			
α_1	0.5	0.8	1.1	1.4	1.7	2.0
k^*	2.0307	1.8712	1.8382	1.8303	1.8283	1.8278
a^*	6.6192	7.4998	8.2618	8.9272	9.5155	10.0422
			$\alpha_1 = 0.5$			
q	2	3	4	5	6	7
k^*	2.0307	1.9830	1.9642	1.9571	1.9545	1.9535
a^*	6.6192	6.5237	6.4854	6.4709	6.4654	6.4634

$$(1+\alpha)E\ \exp[cA_a(X_k^I)] - E\ \exp\left[cA_a\left(k + \sum_{j=2}^{n} I_k(X_j)\right)\right] = 0$$

$$(1+\alpha_1)E\ \exp\left[cA_a(X_k^I)\right] - \exp[ca] = 0,$$

where $A_a(x) = (x \wedge a) \vee (x - Q)$, $I_k(x) = (x \wedge k) \vee (x - q)$, and $X_k^I = \sum_{j=1}^n I_k(X_j)$. By supposition, the stochastic variables X_k^I and $\sum_{j=2}^n I_k(X_j)$ have, respectively, truncated normal distributions $\Phi_{m,\sigma}^{tr}(x)$ and $\Phi_{m_1,\sigma_1}^{tr}(x)$ with parameters $m = nE\ I_k(X_1)$, $\sigma^2 = n\mathrm{Var}\ I_k(X_1)$, and $m_1 = (n-1)E\ I_k(X_1)$, $\sigma_1^2 = (n-1)\mathrm{Var}\ I_k(X_1)$. Formulae for two first moments of $I_k(X_1) = (X_1 \wedge k) \vee (X_1 - q)$ are easily derived,

$$E\ I_k(X_1) = \int_0^k \bar{F}(x)dx + \int_{k+q}^{\infty} \bar{F}(x)dx,$$

$$E\ I_k^2(X_1) = 2[\int_0^k \bar{F}(x)xdx + \int_{k+q}^{\infty} \bar{F}(x)(x-q)dx],$$

where $\bar{F}(x) = 1 - F(x)$. Tables 1–2 present numerical results of solving the optimality equations after inserting the exponential distribution $F(x)$ under a fixed insurer's loading $\alpha = 0.2$ and upper limit on the reinsurer's risk $Q = 35$. The number n of insureds, the reinsurer's loading coefficient α_1, and upper limit q for the risk covered by the insured are varying parameters.

One may, quite naturally, expect that as the reinsurer's loading coefficient α_1 increases, the retention level a^* of reinsurance increases, that is, the insurer makes his share of risk (after reinsurance, see Figure 1) greater. At the same time, to increase the profit, he/she might enlarge his indemnity $\sum_{j=1}^n I^*(X_j)$, i.e. increase the level k^*. However, it is not the case, as follows

Table 3. Parameters of optimal policies in the problem without constraints.

α_1	0.5	0.8	1.1	1.4	1.7	2.0
k^*	1.9530	1.8130	1.7898	1.7854	1.7846	1.7845
a^*	6.4623	7.3066	8.0634	8.7283	9.3168	9.8435

from Tables 1 and 2, the increase in α_1 leads to that the levels a^* and k^* change in opposite directions: while a^* becomes greater, the first retention level k^* of insurance decreases so that the insurer lessens his indemnity before reinsurance (see Figure 2) in order to avoid reinsurance of a larger share of the risk at a high price. When the upper limit q of the insured's residual risk grows, both the levels k^* and a^* decrease (within 3.5%), which is explained by a desire of the insurer to reduce his indemnity and to transfer a larger risk to the reinsurer at an appropriate price, i.e. under the reinsurance loading $\alpha_1 = 0.5$. A decrease in the number of clients n from 20 to 5 means a decrease in the summary risk of clients, which expectedly lessens a^*, the first level of insurer's indemnity after reinsurance.

Now return to the model without additional constraints considered in Proposition 3, where we set $q = Q = \infty$. Numerical results for the case $n = 5$ are presented in Table 3.

It is easily seen that $nk^* < a^*$, beginning with the value of reinsurer's loading coefficient $\alpha_1 = 1.7$. This means that the insurer does not apply for reinsurance under such high prices. A more detailed numerical analysis shows that in our case the interval (see Proposition 3) $[\alpha, \alpha')$ for the reinsurer's loading coefficient is $[\alpha, \alpha') = [0.2, 1.4962)$. So, applying for reinsurance is still profitable under a relatively costly reinsurance treaty.

6. Conclusions

In contrast to the existing literature, we have investigated the problem of optimality of insurance and reinsurance policies chosen jointly under natural constraints on residual insured's and reinsurer's risks. It is proved that the optimal policies have the form of limited stop-loss policies, the equations determining their parameters are obtained. We have also shown that, in distinction to per-claim reinsurance, it is profitable for the insurer to apply for reinsurance as long as the gap between insurance and reinsurance loading coefficients is not 'too big,' i.e. the reinsurance is not very expensive.

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References

Arrow, K. J. (1971). *Essays in the theory of risk-bearing*. Amsterdam: North-Holland.
Buhlmann, H. (1984). The general economic premium principle. *ASTIN Bulletin* **14**, 13–21.
Chi, Y. & Lin, X. S. (2014). Optimal reinsurance with limited ceded risk: a stochastic dominance approach. *ASTIN Bulletin* **44**, 103–126.
Chi, Y. & Tan, K. S. (2013). Optimal reinsurance with general premium principles. *Insurance: Mathematics and Economics* **52**, 180–189.

Cummins, J. D. & Mahul, O. (2004). The demand for insurance with an upper limit on coverage. *Journal of Risk and Insurance* **71**, 253–264.
Daykin, C. D., Pentikainen, T. & Pesonen, M. (1994). *Practical risk theory for actuaries*. London: Chapman and Hall.
Golubin, A. Y. (2006). Pareto-optimal insurance policies in the models with a premium based on the actuarial value. *Journal of Risk and Insurance* **73**(3), 469–487.
Golubin, A. Y., Gridin, V. N. & Gazov, A. I. (2009). Risk bearing in a statistical model with reinsurance. *Automation and Remote Control* **70**(8), 1385–1395.
Gut, A. (2005). *Probability: a graduate course*. New York: Springer.
Karlin, S. & Studden, W. J. (1966). *Tchebycheff systems: with applications in analysis and statistics*. New York: Interscience Publishers.
Lehmann, E. L. (1959). *Testing statistical hypotheses*. New York: Wiley and Sons.
Raviv, A. (1979). The design of an optimal insurance policy. *Amer. Economic Review* **69**, 84–96.
Tucker, H. G. (1967). *A graduate course in probability*. New York: Academic Press.
Zhou, C. & Wu, C. (2008). Optimal insurance under the insurer's risk constraint. *Insurance: Mathematics and Economics* **42**, 992–999.
Zhou, C., Wu, C. & Wu, W. (2010). Optimal insurance in the presence of insurer's loss limit. *Insurance: Mathematics and Economics* **46**, 300–307.

Appendix 1

The proof of Proposition 1 Denote the summary risk by $X = \sum_{i=j}^{n} X_j$ and the vector of risks by $\bar{X} = (X_1, \ldots, X_n)$. Define a stochastic value $\mathcal{A} = E\,[A^*(\bar{X})|X]$ – the conditional expectation of $A^*(\bar{X})$ with respect to a sigma-algebra $\sigma(X)$ generated by X (for the definition, see, e.g. Gut (2005, chapter 10, section 1)). The reinsurer's premium P_1^* remains unchanged as $E\,A^*(\bar{X}) = E\,\{E\,[A^*(\bar{X})|X]\} = E\,\mathcal{A}$. From concavity of $u(x)$ and Jensen's inequality we have

$$E\,[u(nP - P_1^* - A^*(\bar{X}))|X] \leq E\,[u(nP - P_1^* - \mathcal{A})|X] \text{ a.s.}$$
$$\text{Then } E\,u(nP - P_1^* - A^*(\bar{X})) \leq E\,u(nP - P_1^* - \mathcal{A}).$$

Since $u(x)$ is strictly concave, the latter inequality is strict unless $A^*(\bar{X}) = \mathcal{A}$ a.s.

By construction, the 'risk-sharing' $\mathcal{A}$ is measurable with respect to a sigma-algebra $\sigma(X)$ generated by X and satisfies the inequalities $\sum_1^n X_j \geq \mathcal{A} \geq ([\sum_1^n X_j] - Q)_+$ a.s. Therefore, the stochastic value $\mathcal{A}$ can be represented (see Tucker (1967, Th. 1.1)) as $\mathcal{A} = A^*(X)$ a.s. for some Borel-measurable function $A^*(x)$ which is an admissible reinsurance policy. □

The proof of Proposition 2 By Proposition 1, we confine ourselves to functions $A(x)$ of a scalar argument. Due to concavity of the utility function $u(\cdot)$, the functional $J[A]$ is also concave in A. Therefore, a policy A^* is optimal if and only if the directional derivative

$$\frac{d}{d\lambda} J[\lambda A^* + (1-\lambda)A]\bigg|_{\lambda=1} \geq 0$$

for any admissible A. Calculating the derivative (recall that $J[A] \equiv E\,u\,(nP - (1+\alpha_1)E[X - A(X)] - A(X))$), we have that the latter inequality means

$$\int_0^{nT} \eta(x)(A^*(x) - A(x))dF_\Sigma(x) \geq 0, \qquad \text{where the function}$$
$$\eta(x) = (1+\alpha_1)Eu'(nP - P_1^* - A^*(X)) - u'(nP - P_1^* - A^*(x)) \tag{8}$$

and $F_\Sigma(x)$ denotes the distribution function of summary risk X. Finiteness of an integral $\int_0^{nT} \eta(x)A(x)dF_\Sigma(x)$ for any admissible $A^*(x)$ and $A(x)$ is guaranteed by the assumption $E\{u'(-n(1+\alpha_1)EX_1 - X)X\} < \infty$ (see Section 2, see page 183) because $u'(x)$ is a positive decreasing function and, thus, $0 \le u'(nP - P_1^* - A^*(x)) \le u'(-n(1+\alpha_1)EX_1 - x)$ for all $x \in [0, nT]$. Actually, $\eta(x)$ coincides, up to the sign, with the Gateaux derivative of functional $J[A]$ calculated at the point A^*.

So A^* is optimal if and only if

$$\max_A \int_0^{nT} \eta(x)A(x)dF_\Sigma(x) \tag{9}$$

over the set of Borel-measurable functions $\{A(x) : (x - Q)_+ \le A(x) \le x\}$ is attained at A^*. Note that the unknown $A^*(\cdot)$ enters the expression for $\eta(x)$ (see (8)). A problem of maximizing an integral over a set of bounded functions is solved by Neyman–Pearson lemma (see, e.g. Lehmann (1959, Chapter 3, section 6, Th. 5) and Karlin & Studden (1966, Chapters VIII, XIV), applications to optimization in insurance problems can be found in Golubin (2006)). We present this lemma in a slightly modified version.

Lemma 1 (Neyman–Pearson lemma) *Let $l(x)$, $L(x)$, and $\mu(x)$ be Borel-measurable functions on $[0, \infty)$ such that $0 \le l(x) \le L(x)$, and $F(x)$ be a distribution function on $[0, \infty)$. Let the integrals $\int_0^\infty \mu(x)l(x)dF(x)$ and $\int_0^\infty \mu(x)L(x)dF(x)$ be finite.*

A function $y^*(x)$ from the set of Borel-measurable functions $Y = \{y(x) : l(x) \le y(x) \le L(x)$ for $x \in [0, \infty)\}$ is a maximizer in the problem

$$\max_y \int_0^\infty \mu(x)y(x)dF(x) \quad \text{subject to } y \in Y$$

if and only if

$$y^*(x) = \begin{cases} l(x), & \mu(x) < 0, \\ L(x), & \mu(x) > 0, \end{cases}$$

up to a set of zero F-measure.

Thus, an admissible policy A^* is a maximizer in (9) if and only if

$$A^*(x) = \begin{cases} (x - Q)_+, & \text{if } \eta(x) < 0, \\ x, & \text{if } \eta(x) > 0 \end{cases} \tag{10}$$

up to a set of zero F_Σ-measure. Since $u'(\cdot)$ is a decreasing function, it is easily seen that $\eta(0) > 0$. Then, as x increases from $x = 0$, we have by (10) that $A^*(x)$ remains equal to x up to a point a^0 at which $\eta(x)$ first becomes equal to zero – note that $\eta(x)$ is decreasing with $A^*(x) = x$. On $[a^0, a^0 + Q]$ the function $\eta(x)$ can neither decrease nor increase, since otherwise we would have a contradiction with (10).

Thus, $\eta(x)$ is decreasing on the interval $[0, a^0]$ with $A^*(x) = x$, and is identically (with respect to F_Σ-measure) equal to zero on the interval $[a^0, a^0 + Q]$ under $A^*(x) = a^0$. As x increases from the point $a^0 + Q$, the values of $A^*(x)$ lie on the lower boundary $x - Q$. Indeed, if $A^*(x) > x - Q$ then $\eta(x)$ decreases from $0 = \eta(a + Q)$ and, by (10), we would have $A^*(x) = x - Q$. Finally, the optimal policy is an upper-limit Stop-loss reinsurance $(x \wedge a^0) \vee (x - Q)$ on $[0, \infty)$. In order to determine a^0, the point at which $\eta(x)$ first becomes zero, we insert into $\eta(x)$ an expression $(X \wedge a) \vee (X - Q)$ instead of $A^*(X)$ and a instead of $A^*(x)$. An equation for determining a^0 takes thus the form $\psi(a) = 0$, where the function $\psi(a)$ is specified in (4). Taking into account a degenerated case $\psi(a) > 0$ on $[0, nT)$, where $nT = Ess\sup(X)$, we come to the final formula for the optimal policy, $A^*(x) = (x \wedge a^*) \vee (x - Q)$, with a^* uniquely defined in Proposition 2.

By reasoning backward, we obtain that if $A^*(x)$ is chosen as defined in Proposition 2 then, by a repetition of the above-given arguments, $A^*(x)$ satisfies (10). Hence, A^* is a solution to problem (9), which, in turn, implies optimality of A^* in (3). To sum up, problem (3) has a solution $A^*(x)$ that is unique and determined in Proposition 2. □

The proof of Theorem 1 Note, first of all, that the goal functional $J[I, A]$ is not concave in (I, A). In this connection, we begin with establishing the existence of a solution in problem (5) and then employ a necessary condition for optimality dealing with the sign of directional derivatives of $J[I, A]$.

LEMMA 2 *There exists a solution to problem* (5).

Proof Denote by $\{I_s, A_s\}$ a sequence of admissible policies maximizing (5), that is, $\lim_{s\to\infty} J[I_s, A_s] = J^* \stackrel{def}{=} \sup_{I,A} J[I, A]$. We will show that J^* is finite and attained at a pair of admissible policies (I^*, A^*). By Helly's selection theorem (see, e.g. Gut (2005, Th. 8.1, p. 232)), there exists a subsequence $\{I_m(X_1), A_m(X^{I_m})\}$, where $X^{I_m} = \sum_{j=1}^n I_m(X_j)$, such that the corresponding distribution functions $F_m^I(x)$ and $F_m^A(x)$ converge to some 'pseudo-distribution' functions $F^*(x)$ and $F_A^*(x)$ at all continuity points x of them (the term 'pseudo' means that $\lim_{x\to\infty} F^*(x) \le 1$ and $\lim_{x\to\infty} F_A^*(x) \le 1$).

To prove that $F^*(x)$ and $F_A^*(x)$ are proper distribution functions, note that $I_m(X_1) \le X_1$ and $A_m(X^{I_m}) \le \sum_{j=1}^n X_j$ a.s. so that $F_m^I(x) \ge F(x)$ and $F_m^A(x) \ge F_\Sigma(x)$, therefore the limits $\lim_{x\to\infty} F^*(x) = \lim_{x\to\infty} F_A^*(x) = 1$. Thus, the subsequences $\{I_m(X_1)\}$ and $\{A_m(X^{I_m})\}$ converge in distribution to a pair (δ, ε) of proper stochastic values defined on the same probability space $(\Omega, \mathcal{F}, P)$.

Since $(X_1 - q)_+ \le I_m(X_1) \le X_1$ a.s. and $I_m(X_1)$ are measurable with respect to a sigma-algebra $\sigma(X_1)$ generated by X_1, the stochastic value δ is measurable with respect to $\sigma(X_1)$ and can be represented (see, e.g. Tucker (1967, Th. 1.1)) as $\delta = I^*(X_1)$ a.s. for some Borel-measurable function $I^*(x)$ which is an admissible insurance policy . Analogously, the other stochastic value ε can be represented as $\varepsilon = A^*(\sum_{j=1}^n I^*(X_j))$ a.s. for some admissible reinsurance policy $A^*(x)$.

To prove the convergence $J[I_m, A_m] \to J[I^*, A^*]$ as $m \to \infty$, note first that $I_m(X_1) \le X_1$ and $A_m(X^{I_m}) \le X$ a.s., where $EX_1 < \infty$ and $EX < \infty$. By Th. 4.4 on p.216 in Gut (2005), we have that these sequences are uniformly integrable. Then, by Th. 5.9 on p. 224

in Gut (2005), $E\, I_m(X_1) \to E\, I^*(X_1)$ and $E\, A_m(X^{I_m}) \to E\, A^*(X^*)$ as $m \to \infty$. From continuity of the utility function u and the continuous mapping theorem (see Gut (2005, Th. 10.4)) it follows that $u\left(nP^m - P_1^m - A_m(X^{I_m})\right) \to u\left(nP^* - P_1^* - A^*(X^*)\right)$ in distribution. By the assumption $E\,|u(-n(1+\alpha_1)E\,X_1 - X)| < \infty$ (see page 183), we have that conditions of Th. 4.4 and Th. 5.9 in Gut (2005) are satisfied in our case. Therefore, the expectations $J[I_m, A_m] \to J[I^*, A^*] = J^* < \infty$. $\square$

Let (I^*, A^*) be a solution to (5), that does exist by Lemma 2. The policy A^* solves the problem $\max\limits_A J[I^*, A]$, therefore $A^*(x) = (x \wedge a^*) \vee (x - Q)$ for an appropriate a^* as established by Proposition 2 in which now the summary risk X is replaced with $X^* = \sum_{j=1}^n I^*(X_j)$.

Consider a problem $\max_I J[I, A^*]$. Like in the Proposition 2's proof, define $I_\rho = \rho I^* + (1-\rho)I$ then a necessary condition for optimality of I^* is

$$\frac{d}{d\rho}\, J[I_\rho, A^*]\Big|_{\rho=1} \geq 0$$

for any admissible I. After differentiating $J[I_\rho, A^*]$ (note that $J[I, A^*] = E\,u(nP - P_1^* - A^*(X^I))$ with $P_1^* = (1+\alpha_1)E\,[X^I - A^*(X^I)]$), we get this inequality in the form

$$\int_0^T \ldots \int_0^T \xi(x_1, \ldots, x_n)\left(\sum_1^n I^*(x_j) - \sum_1^n I(x_j)\right) dF(x_1)\ldots dF(x_n) \geq 0,$$

where the function

$$\begin{aligned}\xi(\bar{x}) = \mathbf{1}(\bar{x})[(1+\alpha_1)E\,u'(nP^* - P_1^* - A^*(X^*)) \\ - u'(nP^* - P_1^* - A^*(\sum_1^n I^*(x_j))] - (\alpha_1 - \alpha)E\,u'(nP^* - P_1^* - A^*(X^*)),\end{aligned}$$

here $\mathbf{1}(\bar{x}) = \mathbf{1}\{\sum_1^n I^*(x_j) \in [0, a^*] \cup [a^* + Q, \infty)\}$ and $\mathbf{1}\{\cdot\}$ denotes the indicator function. Similar to the Proposition 2's proof, the integral above is convergent due to the assumption $E\,\{u'(-n(1+\alpha_1)E\,X_1 - X)X\} < \infty$ (page 183). Indeed,

$$\int_0^T \ldots \int_0^T \xi(x_1, \ldots, x_n)\sum_1^n I(x_j)dF(x_1)\ldots dF(x_n)$$

is finite for any admissible I, since $E\,\{u'(nP^* - P_1^* - A^*(X^*))X^I\} \leq E\,\{u'(-n(1+\alpha_1)E\,X_1 - X)X\} < \infty$. In order to analyze the optimality condition above, note that $\xi(\bar{x})$ depends on $\bar{x}$ only through $\sum_1^n I^*(x_j)$. So we can introduce a new function $\xi_\Sigma(\cdot)$ of a scalar argument related in some sense to a risk-sharing model with a single 'collective' insured. The function is defined as $\xi(\bar{x}) \equiv \xi_\Sigma(\sum_1^n I^*(x_j))$. Now we can rewrite the inequality above in the form

$n \int_0^T \theta(x)(I^*(x) - I(x))dF(x) \geq 0$, where

$$\theta(x) = E\,\xi_\Sigma\left(I^*(x) + \sum_{j=2}^{n} I^*(X_j)\right), \tag{11}$$

and the expectation is taken with respect to $X_2, \ldots, X_n$. In other words, if I^* is optimal in the problem $\max_I J[I, A^*]$ then

$$\max_I \int_0^T \theta(x)I(x)dF(x) \quad \text{s.t. } (x-q)_+ \leq I(x) \leq x \tag{12}$$

is attained at I^*. Here, like in the Proposition 2's proof, the integrant $\theta(x)$ depends on $I^*(\cdot)$ (see (11)). Now we prove that such a function is unique and has the form $I^*(x) = (x \wedge k^*) \vee (x - q)$, the level $0 < k^* < a^*$.

Applying Neyman–Pearson lemma, we have

$$I^*(x) = \begin{cases} (x-q)_+, & \theta(x) < 0, \\ x, & \theta(x) > 0, \end{cases} \tag{13}$$

up to a set of zero F-measure. The function $\theta(x)$ defined in (11) is the averaging of $\xi_\Sigma(I^*(x)+z)$ with respect to a distribution $G(z)$ of stochastic variable $Z = \sum_2^n I^*(X_j)$. After the averaging over z is taken, we have the following:

$$\begin{aligned}\theta(0) = {} & (1+\alpha)Eu'(nP^* - P_1^* - A^*(X^*)) \\ & - E\left[u'(nP^* - P_1^* - A^*(x + \sum_2^n I^*(X_j))\right]\Big|_{x=0} > 0\end{aligned}$$

because $u'(\cdot)$ is decreasing and $A^*(\cdot)$ is a non-decreasing function (of the form $A^*(x) = (x \wedge a^*) \vee (x - Q)$). As x increases from $x = 0$, the function $\theta(x)$ decreases from a positive value $\theta(0) > 0$ under $I^*(x) = x$ as follows from (13). Let us prove that a point k^0 at which $\theta(x)$ first vanishes is less than a^*. In the optimal reinsurance $A^*(x)$, the level $a^* = a^0 \wedge (nT)$, where (see the Proposition 2's proof) a^0 is a point at which the function $\eta(x)$ defined in (8), where now X replaced by X^*, first becomes zero under $A^*(x) = x$ for $x \leq a^0$. Since the loading $\alpha_1 > \alpha$, it can be easily shown that $\eta(x) > \xi_\Sigma(x) = (1+\alpha)Eu'(nP^* - P_1^* - A^*(X^*)) - u'(nP^* - P_1^* - A^*(x))$ as x goes from 0 up to a point K_Σ, where $\xi_\Sigma(x)$ vanishes. Hence, $K_\Sigma < a^0$ and, clearly, the least zero of $\xi_\Sigma(x+z)$ with a fixed $z > 0$ is $K_\Sigma - z < a^0$ (see Figure 3 below). Then, after the expectation $\theta(x) = E\,\xi_\Sigma(I^*(x) + Z)$ is taken, we still have that k^0, the least zero of $\theta(x)$, is less than $K_\Sigma < a^0$.

As x runs over $[k^0, k^0 + q)$, the function $\theta(x)$ can neither decrease nor increase, so that $I^*(x)$ remains equal to k^0. Indeed, if $I^*(x)$ decreases then, as follows from definition in (11), $\theta(x)$ increases from zero and, by (13), $I^*(x) = x$ – we have a contradiction. Suppose $I^*(x)$ to increase, then $\theta(x)$ decreases from $\theta(k^0) = 0$ so that $\theta(x) < 0$ and, by (13), $I^*(x) = (x - q)_+ < I^*(k^0) = k^0$ so we have a contradiction with the increase in $I^*(x)$.

As x increases from $k^0 + q$, the policy $I^*(x)$ equals $x - q$, the lower boundary of admissible policy values. Suppose the contrary: $I^*(x) > x - q$. As was proved above, this increase from $I^*(k^0 + q) = k^0$ leads to inequality $\theta(x) < 0$. Then (13) gives that $I^*(x)$ must coincide with $x - q$.

Thus, we can conclude that the only optimal policy in (12) is $I^*(x) = (x \wedge k^0) \vee (x - q)$ on $[0, \infty)$, where $0 < k^0 < a^* = a^0 \wedge (nT)$.

In order to determine the point k^0 at which $\theta(x)$ first becomes zero, we insert into $\theta(x)$ (see (11)) the found expression $(X_j \wedge k) \vee (X_j - q)$ instead of $I^*(X_j)$, $j = 2, \dots, n$, and k instead of $I^*(x)$. Then k^0 is the minimal root of an equation $\psi_1(k, a^*) = 0$, where the function $\psi_1(k, a)$ is given in (6). It is left to consider a degenerated case: If $\psi_1(k, a^*) > 0$ on $[0, T)$, which means $I^*(X_1) = X_1$ a.s. (so that the insurer takes the entire insured's risk), we set $k^* = T$.

To find the parameter a^* in the optimal reinsurance policy $A^*(x) = (x \wedge a^*) \vee (x - Q)$, we return to Proposition 2. As was established there for the reinsurance optimization problem, $a^* = a^0 \wedge (nT)$ and a^0 is a root of the equation $\psi(a) = 0$, where $\psi(a)$ is defined in (4). In our case of using the insurance of the form $I_k(x) = (x \wedge k) \vee (x - q)$, the summary risk $X = \sum_{j=1}^n X_j$ should be replaced in (4) with $X_k^I = \sum_{j=1}^n I_k(X_j)$. This results in an equation $\psi_2(k, a) = 0$, where $\psi_2(k, a)$ is given in (7).

To sum up, we have proved: Problem (5) does have a solution, (I^*, A^*). The function A^* necessarily has the form $A^*(x) = (x \wedge a^*) \vee (x - Q)$. Under a given a^*, the policy $I^*(x) = (x \wedge k^*) \vee (x - q)$ is uniquely determined by equation $\psi_1(k, a^*) = 0$. The parameters a^* and k^* satisfy equations $\psi_1(k, a) = 0$ and $\psi_2(k, a) = 0$ with $\psi_i(k, a)$ defined in (6) and (7). □

The proof of Proposition 3 According to Theorem 1's proof, $k^* < T$ is the least root of the equation $\theta(x) = 0$, where (see (11)) $\theta(x) = E\,\xi_\Sigma(I^*(x) + Z)$ with $Z = \sum_2^n I^*(X_j)$ and, in our case without additional constraints, $I^*(x) = x \wedge k^*$. It was shown that as long as $\alpha_1 > \alpha$, the level $k^* < K_\Sigma < a^*$ where K_Σ is a point at which $\xi_\Sigma(x)$ vanishes. On the other hand, k^* can

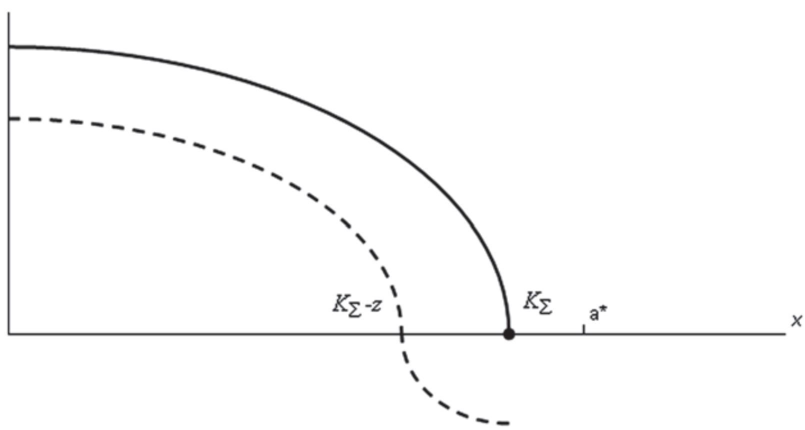

Figure 3. Function $\xi_\Sigma(x)$ (solid line) and function $\xi_\Sigma(x + z)$ (dashed line) for $0 \le x \le K_\Sigma$ ($< a^*$).

also be defined as the least zero of the function $E\,\xi_\Sigma(x+Z)$ (see a graph of $\xi_\Sigma(x+z)$ on the previous page).

Let us prove that $nk^* > K_\Sigma$. Suppose the contrary, $nk^* \leq K_\Sigma$. Since $I^*(x) = x \wedge k^*$, we have $Ess\,\sup(Z) = (n-1)k^*$ and, by supposition, $Ess\,\sup(Z) \leq K_\Sigma - k^*$. As was noted, k^* is the least zero of $E\,\xi_\Sigma(x+Z)$, therefore $k^* > K_\Sigma - Ess\,\sup(Z) \geq K_\Sigma - (K_\Sigma - k^*) = k^*$. This contradiction proves the needed inequality $nk^* > K_\Sigma$.

Taking into account the fact that, by construction of $\xi_\Sigma(x)$, the point K_Σ coincides with a^* if $\alpha_1 = \alpha$, and using a continuity argument, we obtain the existence of $\alpha' > \alpha$ such that $nk^* > a^*$ under any $\alpha_1 \in [\alpha, \alpha')$. $\square$