Homomorphisms and Congruence Relations for Games with Preference Relations

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Abstract In this paper we consider games with preference relations. The main optimality concept for such games is concept of equilibrium. We introduce a notion of homomorphism for games with preference relations and study a problem concerning connections between equilibrium points of games which are in a homomorphic relation. The main result is finding covariantly and contravariantly complete families of homomorphisms.

Keywords: homomorphism, equilibrium points, Nash equilibrium, game with preference relations.

1. Introduction

In this paper we study games in which a valuation structure by preference relations is given.

We can consider a n-person game with preference relations as a system of the form

$$G = \langle X_1, \ldots, X_n, A, \rho_1, \ldots, \rho_n, F \rangle \tag{1}$$

where X_i is a set of *strategies* of player i $(i = 1, \ldots, n)$, A is a set of *outcomes*, $\rho_i \subseteq A^2$ is a preference relation of player i $(i = 1, \ldots, n)$ and realization function F is a mapping of set of *situations* $X = X_1 \times \ldots \times X_n$ in set of outcomes A.

The main optimality concept for games of this class are various modifications of Nash equilibrium. We introduce a concept of equilibrium as a generalization of Nash equilibrium for games of the form (1). We consider equilibrium and Nash equilibrium as optimal solutions for games with preference relations. The basic subject of research in our paper are homomorphisms of certain types. It is important that homomorphisms preserve optimal solutions of some types. The main results of the present work are theorems concerning connections between optimal solutions of games which are in a homomorphic relation.

2. Preliminaries

2.1. Basic concepts for preference structures

A preference structure on a set A can be given as a pair $\langle A, \rho \rangle$ where ρ is arbitrary reflexive binary relation on A.

The condition $(a_1, a_2) \in \rho$ means that element a_1 is less preference than a_2. Given a preference relation $\rho \subseteq A^2$, we denote $\rho^s = \rho \cap \rho^{-1}$ its symmetric part and $\rho^* = \rho \backslash \rho^s$ its strict part.

We write

$a_1 \overset{\rho}{\lesssim} a_2$ instead of $(a_1, a_2) \in \rho$,
$a_1 \overset{\rho}{\sim} a_2$ instead of $(a_1, a_2) \in \rho^s$,
$a_1 \overset{\rho}{<} a_2$ instead of $(a_1, a_2) \in \rho^*$.

Remark 1. Conditions $a_1 \overset{\rho}{\lesssim} a_2$ and $a_2 \overset{\rho}{<} a_1$ are not compatible.

In this paper we consider some important types of preference structures: transitive, antisymmetric, linear, acyclic, ordinal.

Definition 1. A preference structure $\langle A, \rho \rangle$ is called

- *transitive* if for any $a_1, a_2, a_3 \in A$

$$(a_1, a_2) \in \rho \wedge (a_2, a_3) \in \rho \Rightarrow (a_1, a_3) \in \rho;$$

- *antisymmetric* if for any $a_1, a_2 \in A$

$$(a_1, a_2) \in \rho \wedge (a_2, a_1) \in \rho \Rightarrow a_1 = a_2;$$

- *linear* if for any $a_1, a_2 \in A$

$$(a_1, a_2) \in \rho \vee (a_2, a_1) \in \rho;$$

- *acyclic* if for any $n = 2, 3, \ldots$ and $a_1, \ldots, a_n \in A$

$$(a_1, a_2) \in \rho \wedge \ldots \wedge (a_{n-1}, a_n) \in \rho \wedge (a_n, a_1) \in \rho \Rightarrow a_1 = a_2 = \ldots = a_n;$$

- *ordinal* if axioms of transitivity and antisymmetry hold.

Remark 2. An ordinal preference structure $\langle A, \rho \rangle$ is a transitive and acyclic one and the converse is true.

Thus, transitive preference structure and acyclic one are a natural generalization of ordinal preference structure.

Definition 2. Let $\langle A, \rho \rangle$ be a preference structure and ε be an equivalence relation on A. Relation ρ is said to be *acyclic under* ε if for any $n = 2, 3, \ldots$ the implication

$$a_0 \overset{\rho}{\lesssim} a_1 \overset{\rho}{\lesssim} a_2 \overset{\rho}{\lesssim} \ldots \overset{\rho}{\lesssim} a_n \overset{\rho}{\lesssim} a_0 \Rightarrow a_0 \overset{\varepsilon}{\equiv} a_1 \overset{\varepsilon}{\equiv} \ldots \overset{\varepsilon}{\equiv} a_n$$

holds.

2.2. Homomorphisms of preference structures

Let $\langle A, \rho \rangle$ and $\langle B, \sigma \rangle$ be two preference structures.

Definition 3. A mapping $\psi \colon A \to B$ is called *a homomorphism* of the first structure into the second one if for any $a_1, a_2 \in A$ the condition

$$a_1 \overset{\rho}{\lesssim} a_2 \Rightarrow \psi(a_1) \overset{\sigma}{\lesssim} \psi(a_2) \tag{2}$$

holds.

A homomorphism $\psi\colon A \to B$ is said to be *a homomorphism "onto"* if ψ is a mapping of A onto B.

A homomorphism ψ is said to be *strict* if the following two conditions are satisfied:

$$a_1 \overset{\rho}{<} a_2 \Rightarrow \psi(a_1) \overset{\sigma}{<} \psi(a_2), \tag{3}$$

$$a_1 \overset{\rho}{\sim} a_2 \Rightarrow \psi(a_1) \overset{\sigma}{\sim} \psi(a_2). \tag{4}$$

A homomorphism ψ is called *regular* if the following two conditions

$$\psi(a_1) \overset{\sigma}{<} \psi(a_2) \Rightarrow a_1 \overset{\rho}{<} a_2, \tag{5}$$

$$\psi(a_1) \overset{\sigma}{\sim} \psi(a_2) \Rightarrow \psi(a_1) = \psi(a_2). \tag{6}$$

hold.

Remark 3. For any homomorphism the condition (4) holds. Indeed, let ψ be a homomorphism from A into B and $a_1 \overset{\rho}{\sim} a_2$ holds. The condition $a_1 \overset{\rho}{\sim} a_2$ means that $a_1 \overset{\rho}{\lesssim} a_2$ and $a_2 \overset{\rho}{\lesssim} a_1$. Hence, $\psi(a_1) \overset{\sigma}{\lesssim} \psi(a_2)$ and $\psi(a_2) \overset{\sigma}{\lesssim} \psi(a_1)$ hold, i.e. $\psi(a_1) \overset{\sigma}{\sim} \psi(a_2)$.

Remark 4. Any strict homomorphism is a homomorphism but the converse is false.

Let $\langle A, \rho \rangle$ be a preference structure and $\varepsilon \subseteq A^2$ an equivalence relation.

Definition 4. A factor-structure for preference structure $\langle A, \rho \rangle$ is a pair $\langle A/\varepsilon, \rho/\varepsilon \rangle$ where we denote for any $C_1, C_2 \in A/\varepsilon$:

$$(C_1, C_2) \in \rho/\varepsilon \overset{\text{def}}{\Longleftrightarrow} (\exists a_1 \in C_1, a_2 \in C_2)\ (a_1, a_2) \in \rho.$$

Lemma 1 (about homomorphisms of preference structures).
Let $\langle A, \rho \rangle$ be a preference structure, ε be an equivalence relation on A. Then

1. *a canonical mapping $\psi\colon a \to [a]_\varepsilon$ is a homomorphism from preference structure $\langle A, \rho \rangle$ onto factor-structure $\langle A/\varepsilon, \rho/\varepsilon \rangle$;*
2. *a canonical mapping ψ is a strict homomorphism if and only if condition*

$$\left.\begin{array}{l} a_1 \overset{\rho}{\lesssim} a_2 \\ a_1' \overset{\varepsilon}{\equiv} a_1 \\ a_2' \overset{\varepsilon}{\equiv} a_2 \\ a_2' \overset{\rho}{\lesssim} a_1' \end{array}\right\} \Rightarrow a_1 \overset{\rho}{\sim} a_2 \tag{7}$$

is satisfied;

3. *a canonical mapping ψ is a regular homomorphism if and only if conditions*

$$\left.\begin{array}{l} a_1 \overset{\varepsilon}{\not\equiv} a_2 \\ a_1 \overset{\rho}{<} a_2 \\ a'_1 \overset{\varepsilon}{\equiv} a_1 \\ a'_2 \overset{\varepsilon}{\equiv} a_2 \end{array}\right\} \Rightarrow a'_1 \overset{\rho}{<} a'_2, \tag{8}$$

$$\left.\begin{array}{l} a_1 \overset{\rho}{\lesssim} a_2 \\ a'_1 \overset{\varepsilon}{\equiv} a_1 \\ a'_2 \overset{\varepsilon}{\equiv} a_2 \\ a'_2 \overset{\rho}{\lesssim} a'_1 \end{array}\right\} \Rightarrow a_1 \overset{\varepsilon}{\equiv} a_2. \tag{9}$$

hold.

Proof (of lemma).

1. Suppose $a_1 \overset{\rho}{\lesssim} a_2$. Then according to definition of factor relation we have $[a_1]_\varepsilon \overset{\rho/\varepsilon}{\lesssim} [a_2]_\varepsilon$. Hence, ψ is a homomorphism. Since a canonical homomorphism is a homomorphism "onto", we obtain the proof of the part (1) of the Lemma.
2. Let a canonical homomorphism ψ be strict and the implication condition (17) is satisfied. Suppose $a_1 \overset{\rho}{<} a_2$. Since a canonical homomorphism is strict by condition of the Lemma then $[a_1]_\varepsilon \overset{\rho/\varepsilon}{<} [a_2]_\varepsilon$ holds. On the other hand from the condition $a'_2 \overset{\rho}{\lesssim} a'_1$ it follows that $[a'_2]_\varepsilon \overset{\rho/\varepsilon}{\lesssim} [a'_1]_\varepsilon$. As $[a_1]_\varepsilon = [a'_1]_\varepsilon$, $[a_2]_\varepsilon = [a'_2]_\varepsilon$ then

$$\begin{cases} [a_1]_\varepsilon \overset{\rho/\varepsilon}{<} [a_2]_\varepsilon, \\ [a_2]_\varepsilon \overset{\rho/\varepsilon}{\lesssim} [a_1]_\varepsilon. \end{cases}$$

The last system of conditions cannot be true (because of remark 1). Hence, our assumption is not true and since $a_1 \overset{\rho}{\lesssim} a_2$ we get $a_1 \overset{\rho}{\sim} a_2$.
Conversely, suppose that the condition (17) holds. We have to prove that a canonical homomorphism is strict. Indeed, take two elements a_1, a_2 for which $a_1 \overset{\rho}{<} a_2$ takes place, hence $a_1 \overset{\rho}{\lesssim} a_2$. By the part (1) of this Lemma $[a_1]_\varepsilon \overset{\rho/\varepsilon}{\lesssim} [a_2]_\varepsilon$ holds. We assume that $[a_2]_\varepsilon \overset{\rho/\varepsilon}{\lesssim} [a_1]_\varepsilon$. Then there exist elements a'_1, a'_2 such that $a'_1 \overset{\varepsilon}{\equiv} a_1$, $a'_2 \overset{\varepsilon}{\equiv} a_2$, condition $a'_2 \overset{\rho}{\lesssim} a'_1$ holds. In this case, all assumptions of condition (17) are satisfied and by (17) we have $a_1 \overset{\rho}{\sim} a_2$, which is contradictory to $a_1 \overset{\rho}{<} a_2$. Thus, $[a_2]_\varepsilon \overset{\rho/\varepsilon}{\lesssim} [a_1]_\varepsilon$ does not take place and we get $[a_1]_\varepsilon \overset{\rho/\varepsilon}{<} [a_2]_\varepsilon$. So, the first condition of homomorphism (3) for canonical homomorphism is satisfied. By remark 3 ψ is a strict homomorphism.

3. Suffice to verify that for regular homomorphism ψ its kernel ε_ψ satisfies (8) and (9). Suppose

$$\begin{cases} a_1 \overset{\varepsilon_\psi}{\not\equiv} a_2, \\ a_1 \overset{\rho}{<} a_2, \\ a_1' \overset{\varepsilon_\psi}{\equiv} a_1, \\ a_2' \overset{\varepsilon_\psi}{\equiv} a_2. \end{cases}$$

From $a_1 \overset{\rho}{<} a_2$ it follows that $a_1 \overset{\rho}{\lesssim} a_2$ then $\psi(a_1) \overset{\sigma}{\lesssim} \psi(a_2)$. Assume that $\psi(a_1) \overset{\sigma}{\sim} \psi(a_2)$; by using (11) we get $\psi(a_1) = \psi(a_2)$, i.e. $a_1 \overset{\varepsilon_\psi}{\equiv} a_2$ is in contradiction with our assumptions. Hence, $\psi(a_1) \overset{\sigma}{<} \psi(a_2)$ holds, i.e. $\psi(a_1') \overset{\sigma}{<} \psi(a_2')$. By (10) we obtain $a_1' \overset{\rho}{<} a_2'$ which was to be proved.
Now suppose conditions of (9) hold. Since ψ is a homomorphism we have

$$\begin{cases} \psi(a_1) \overset{\sigma}{\lesssim} \psi(a_2), \\ \psi(a_2') \overset{\sigma}{\lesssim} \psi(a_1'). \end{cases}$$

Hence, $\psi(a_1) \overset{\sigma}{\sim} \psi(a_2)$. By (11) we get $\psi(a_1) = \psi(a_2)$, i.e. $a_1 \overset{\varepsilon_\psi}{\equiv} a_2$.
Conversely, assume $[a_1]_\varepsilon \overset{\rho/\varepsilon}{<} [a_2]_\varepsilon$. Then $[a_1]_\varepsilon \overset{\rho/\varepsilon}{\lesssim} [a_2]_\varepsilon$ that is there exist such elements a_1', a_2' that $a_1' \overset{\varepsilon}{\equiv} a_1$, $a_2' \overset{\varepsilon}{\equiv} a_2$ and $a_1' \overset{\rho}{\lesssim} a_2'$. The condition $a_2' \overset{\rho}{\lesssim} a_1'$ does not hold otherwise $[a_2']_\varepsilon \overset{\rho/\varepsilon}{\lesssim} [a_1']_\varepsilon$, i.e. $[a_2]_\varepsilon \overset{\rho/\varepsilon}{\lesssim} [a_1]_\varepsilon$; it is contradiction (see remark 1). Hence $a_1' \overset{\rho}{<} a_2'$. The condition $a_1' \overset{\varepsilon}{\equiv} a_2'$ does not hold, hence the conditions

$$\begin{cases} a_1 \overset{\varepsilon}{\not\equiv} a_2, \\ a_1' \overset{\rho}{<} a_2', \\ a_1 \overset{\varepsilon}{\equiv} a_1', \\ a_2 \overset{\varepsilon}{\equiv} a_2'. \end{cases}$$

hold. According to (8) we obtain $a_1 \overset{\rho}{<} a_2$.
Now verify (11). Suppose $[a_1]_\varepsilon \overset{\rho/\varepsilon}{\sim} [a_2]_\varepsilon$, i.e. there exist elements $a_1', a_1'' \overset{\varepsilon}{\equiv} a_1$ and $a_2', a_2'' \overset{\varepsilon}{\equiv} a_2$ such that

$$\begin{cases} a_1' \overset{\rho}{\lesssim} a_2', \\ a_2'' \overset{\rho}{\lesssim} a_1''. \end{cases}$$

Then according to (9) we get $a_1' \overset{\varepsilon}{\equiv} a_2'$, i.e. $[a_1]_\varepsilon = [a_2]_\varepsilon$, which was to be proved. □

Lemma 2. *Let $\langle A, \rho\rangle$ be a preference structure, ε be an equivalence relation on A. Factor-structure of preferences $\langle A/\varepsilon, \rho/\varepsilon\rangle$ is transitive if and only if the inclusion*

$$\rho \circ \varepsilon \circ \rho \subseteq \varepsilon \circ \rho \circ \varepsilon \tag{10}$$

holds.

Proof (of lemma).

Suppose $(a_1, a_3) \in \rho \circ \varepsilon \circ \rho$. According to the definition of composition of binary relations, then there exist such elements $a_2, a_2' \in A$ that $(a_1, a_2) \in \rho$, $(a_2, a_2') \in \varepsilon$, $(a_2', a_3) \in \rho$ hold. Denote by $C_1 = [a_1]_\varepsilon$, $C_2 = [a_2]_\varepsilon = [a_2']_\varepsilon$, $C_3 = [a_3]_\varepsilon$. According to the definition of factor-relation we have $(C_1, C_2) \in \rho/\varepsilon$, $(C_2, C_3) \in \rho/\varepsilon$; since the factor-relation is supposed to be transitive then $(C_1, C_3) \in \rho/\varepsilon$. It means that for some $a_1' \in C_1$, $a_3' \in C_3$, $(a_1', a_3') \in \rho$ is satisfied. As $a_1' \overset{\varepsilon}{\equiv} a_1$, $a_3' \overset{\varepsilon}{\equiv} a_3$ we get $(a_1, a_3) \in \varepsilon \circ \rho \circ \varepsilon$ which was to be proved.

Conversely, let the inclusion (10) be held. Let us take three classes $C_1, C_2, C_3 \in A/\varepsilon$ for which $(C_1, C_2) \in \rho/\varepsilon$, $(C_2, C_3) \in \rho/\varepsilon$. Then there exist the elements $a_1 \in C_1$, $a_2 \in C_2$, $a_2' \in C_2$, $a_3 \in C_3$ such that $(a_1, a_2) \in \rho$, $(a_2', a_3) \in \rho$. Since $a_2' \overset{\varepsilon}{\equiv} a_2$ we get $(a_1, a_3) \in \rho \circ \varepsilon \circ \rho$. Hence, according to (10), $(a_1, a_3) \in \varepsilon \circ \rho \circ \varepsilon$. It means that there exist the elements $\overline{a}_1, \overline{a}_3 \in A$ such that $(a_1, \overline{a}_1) \in \varepsilon$, $(\overline{a}_1, \overline{a}_3) \in \rho$, $(\overline{a}_3, a_3) \in \varepsilon$. Then $([\overline{a}_1]_\varepsilon, [\overline{a}_3]_\varepsilon) \in \rho/\varepsilon$ and as $[\overline{a}_3]_\varepsilon = [a_3]_\varepsilon = C_3$, $[\overline{a}_1]_\varepsilon = [a_1]_\varepsilon = C_1$ we get $(C_1, C_3) \in \rho/\varepsilon$ which was to be proved. □

Corollary 1. *Let $\langle A, \rho \rangle$ be a transitive preference structure, ε be an equivalence relation on A. If at least one of the conditions $\rho \circ \varepsilon \subseteq \varepsilon \circ \rho$ or $\varepsilon \circ \rho \subseteq \rho \circ \varepsilon$ or $\varepsilon \subseteq \rho$ holds then factor-structure $\langle A/\varepsilon, \rho/\varepsilon \rangle$ is transitive.*

Proof (of corollary).

1. Indeed, let for example the first inclusion $\rho \circ \varepsilon \subseteq \varepsilon \circ \rho$ be satisfied. Then $\rho \circ \varepsilon \circ \rho \subseteq (\rho \circ \varepsilon) \circ \rho \subseteq (\varepsilon \circ \rho) \circ \rho = \varepsilon \circ \rho^2 \subseteq \varepsilon \circ \rho \subseteq \varepsilon \circ \rho \circ \varepsilon$. According to Lemma 2 factor-structure $\langle A/\varepsilon, \rho/\varepsilon \rangle$ is transitive.

2. Now let $\varepsilon \subseteq \rho$ be satisfied. Multiplying the inclusion $\varepsilon \subseteq \rho$ by ρ to the left we have $\rho \circ \varepsilon \subseteq \rho \circ \rho = \rho^2 \subseteq \rho \subseteq \varepsilon \circ \rho$. Multiplying initial inclusion $\varepsilon \subseteq \rho$ by ρ to the right we obtain $\varepsilon \circ \rho \subseteq \rho \circ \rho = \rho^2 \subseteq \rho \circ \varepsilon$. From the inclusions proved we have $\rho \circ \varepsilon = \varepsilon \circ \rho$, i.e. relations ρ and ε commute. From part (1) of the proof of this corollary it follows that $\langle A/\varepsilon, \rho/\varepsilon \rangle$ is transitive. □

Lemma 3. *Let $\langle A, \rho \rangle$ be a preference structure, ε be an equivalence relation on A. Factor-structure $\langle A/\varepsilon, \rho/\varepsilon \rangle$ is acyclic if and only if $\rho \cup \varepsilon$ is acyclic under ε.*

Proof (of lemma).

Remark 5. It is easy to verify that conditions

$$a_0 \overset{\rho\cup\varepsilon}{\lesssim} a_1' \overset{\rho\cup\varepsilon}{\lesssim} a_1 \overset{\rho\cup\varepsilon}{\lesssim} \dots \overset{\rho\cup\varepsilon}{\lesssim} a_n \overset{\rho\cup\varepsilon}{\lesssim} a_0' \overset{\rho\cup\varepsilon}{\lesssim} a_0 \Rightarrow a_0 \overset{\varepsilon}{\equiv} a_1 \overset{\varepsilon}{\equiv} \dots \overset{\varepsilon}{\equiv} a_n \tag{11}$$

and

$$a_0 \overset{\rho}{\lesssim} a_1' \overset{\varepsilon}{\equiv} a_1 \overset{\rho}{\lesssim} a_2' \overset{\varepsilon}{\equiv} a_2 \overset{\rho}{\lesssim} \dots \overset{\varepsilon}{\equiv} a_n \overset{\rho}{\lesssim} a_0' \overset{\varepsilon}{\equiv} a_0 \Rightarrow a_0 \overset{\varepsilon}{\equiv} a_1 \overset{\varepsilon}{\equiv} \dots \overset{\varepsilon}{\equiv} a_n \tag{12}$$

are equivalent.

Let the condition of the implication (12) be held. Put $C_0 = [a_0]_\varepsilon = [a_0']_\varepsilon$, $C_1 = [a_1]_\varepsilon = [a_1']_\varepsilon, \dots, C_n = [a_n]_\varepsilon = [a_n']_\varepsilon$. According to the definition of factor-relation we have $(C_0, C_1) \in \rho/\varepsilon$, $(C_1, C_2) \in \rho/\varepsilon, \dots, (C_n, C_0) \in \rho/\varepsilon$. Since factor-relation is supposed to be acyclic then $C_0 = C_1 = \dots = C_n$. It means that $a_0 \overset{\varepsilon}{\equiv} a_1 \overset{\varepsilon}{\equiv} \dots \overset{\varepsilon}{\equiv} a_n$.

Conversely, let (12) be satisfied. Let us take classes $C_0, C_1, \ldots, C_n \in A/\varepsilon$, for which $(C_0, C_1) \in \rho/\varepsilon$, $(C_1, C_2) \in \rho/\varepsilon, \ldots, (C_n, C_0) \in \rho/\varepsilon$. Then there exist elements $a_0 \in C_0$, $a'_1 \in C_1$, $a_1 \in C_1$, $a'_2 \in C_2, \ldots, a_n \in C_n$, $a'_0 \in C_0$ such that $(a_0, a'_1) \in \rho$, $(a_1, a'_2) \in \rho, \ldots, (a_n, a'_0) \in \rho$; since $a'_i \overset{\varepsilon}{\equiv} a_i$ $(i = 0, 1, \ldots, n)$ we get $a_0 \overset{\varepsilon}{\equiv} a_1 \overset{\varepsilon}{\equiv} \ldots \overset{\varepsilon}{\equiv} a_n$. It means that $[a_0]_\varepsilon = [a_1]_\varepsilon = \ldots = [a_n]_\varepsilon$. As $C_0 = [a_0]_\varepsilon$, $C_1 = [a_1]_\varepsilon, \ldots, C_n = [a_n]_\varepsilon$ we obtain $C_0 = C_1 = \ldots = C_n$. This completes the proof of Lemma 3. □

3. Games with preference relations

3.1. Homomorphisms of games with preference relations

Consider two games with preference relations for players $\{1, \ldots, n\}$:

$$G = \langle X_1, \ldots, X_n, A, \rho_1, \ldots, \rho_n, F\rangle \text{ and } \Gamma = \langle U_1, \ldots, U_n, B, \sigma_1, \ldots, \sigma_n, \Phi\rangle.$$

Definition 5. A $(n+1)$ system of mappings $f = (\varphi_1, \ldots, \varphi_n, \psi)$ where for any $i = 1, \ldots, n$, $\varphi_i \colon X_i \to U_i$ and $\psi \colon A \to B$ is called *a homomorphism* from game G into game Γ if the following two conditions are satisfied:

for any $i = 1, \ldots, n$ and $a_1, a_2 \in A$

$$a_1 \overset{\rho_i}{\lesssim} a_2 \Rightarrow \psi(a_1) \overset{\sigma_i}{\lesssim} \psi(a_2), \tag{13}$$

$$\psi \circ F = \Phi \circ (\varphi_1 \square \ldots \square \varphi_n). \tag{14}$$

Remark 6. For any situation $x = (x_1, \ldots, x_n)$ of game G condition (14) means that $\psi(F(x_1, \ldots, x_n)) = \Phi(\varphi_1(x_1), \varphi_2(x_2), \ldots, \varphi_n(x_n))$.

A homomorphism f is said to be *strict homomorphism* if system of the conditions

$$a_1 \overset{\rho_i}{<} a_2 \Rightarrow \psi(a_1) \overset{\sigma_i}{<} \psi(a_2), \quad (i = 1, \ldots, n) \tag{15}$$

$$a_1 \overset{\rho_i}{\sim} a_2 \Rightarrow \psi(a_1) \overset{\sigma_i}{\sim} \psi(a_2) \quad (i = 1, \ldots, n) \tag{16}$$

holds instead of condition (13).

A homomorphism f is said to be *regular homomorphism* if for any $i = 1, \ldots, n$, mapping ψ is a regular homomorphism between the preference structures $\langle A, \rho_i\rangle$ and $\langle B, \sigma_i\rangle$, that is the following two conditions

$$\psi(a_1) \overset{\sigma_i}{<} \psi(a_2) \Rightarrow a_1 \overset{\rho_i}{<} a_2, \tag{17}$$

$$\psi(a_1) \overset{\sigma_i}{\sim} \psi(a_2) \Rightarrow \psi(a_1) = \psi(a_2). \tag{18}$$

hold.

A homomorphism f is said to be *homomorphism "onto"*, if each φ_i $(i = 1, \ldots, n)$ is a mapping "onto"; *an isomorphic inclusion map*, if each φ_i $(i = 1, \ldots, n)$ is one-to-one function; *an isomorphism*, if for any $i = 1, \ldots, n$, φ_i is one-to-one function and mapping ψ is an isomorphism between $\langle A, \rho_i\rangle$ and $\langle B, \sigma_i\rangle$, that is the following equivalence

$$a_1 \overset{\rho_i}{\lesssim} a_2 \Leftrightarrow \psi(a_1) \overset{\sigma_i}{\lesssim} \psi(a_2) \tag{19}$$

holds.

Definition 6. A $(n+1)$ system of equivalence relations $\bar{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_n, \varepsilon)$ where $\varepsilon_i \subseteq X_i^2$ $(i = 1, \dots, n)$, $\varepsilon \subseteq A^2$ is called *congruence* in game G if consistency condition for realization function holds, i.e.

$$\left.\begin{array}{l} x_1' \overset{\varepsilon_1}{\equiv} x_1 \\ x_2' \overset{\varepsilon_2}{\equiv} x_2 \\ \dots \\ x_n' \overset{\varepsilon_n}{\equiv} x_n \end{array}\right\} \Rightarrow F(x_1', \dots, x_n') \overset{\varepsilon}{\equiv} F(x_1, \dots, x_n). \tag{20}$$

Congruence $\bar{\varepsilon}$ in game G is said to be *str-congruence* if consistency condition for preference relations for any $i = 1, \dots, n$

$$\left.\begin{array}{l} a_1 \overset{\rho_i}{\lesssim} a_2 \\ a_1' \overset{\varepsilon}{\equiv} a_1 \\ a_2' \overset{\varepsilon}{\equiv} a_2 \\ a_2' \overset{\rho_i}{\lesssim} a_1' \end{array}\right\} \Rightarrow a_1 \overset{\rho_i}{\sim} a_2 \tag{21}$$

holds.

Congruence $\bar{\varepsilon}$ in game G is said to be *reg-congruence* if the following two conditions for any $i = 1, \dots, n$

$$\left.\begin{array}{l} a_1 \overset{\varepsilon}{\not\equiv} a_2 \\ a_1 \overset{\rho_i}{<} a_2 \\ a_1' \overset{\varepsilon}{\equiv} a_1 \\ a_2' \overset{\varepsilon}{\equiv} a_2 \end{array}\right\} \Rightarrow a_1' \overset{\rho_i}{<} a_2', \tag{22}$$

$$\left.\begin{array}{l} a_1 \overset{\rho_i}{\lesssim} a_2 \\ a_1' \overset{\varepsilon}{\equiv} a_1 \\ a_2' \overset{\varepsilon}{\equiv} a_2 \\ a_2' \overset{\rho_i}{\lesssim} a_1' \end{array}\right\} \Rightarrow a_1 \overset{\varepsilon}{\equiv} a_2. \tag{23}$$

hold.

Definition 7. Let $f = (\varphi_1, \dots, \varphi_n, \psi)$ be a homomorphism from game G into game Γ. A $(n+1)$ system of equivalence relations $\bar{\varepsilon}_f = (\varepsilon_{\varphi_1}, \dots, \varepsilon_{\varphi_n}, \varepsilon_\psi)$ where for any $i = 1, \dots, n$, ε_{φ_i} is kernel of φ_i and ε_ψ is kernel of ψ, is called *kernel* of homomorphism f.

Theorem 1. *Let G be a game with preference relations of the form (1) and $\bar{\varepsilon}$ be a congruence in game G. Then we can define a factor-game $G/\bar{\varepsilon}$ with preference relations by*

$$G/\bar{\varepsilon} = \langle X_1/\varepsilon_1, \dots, X_n/\varepsilon_n, A/\varepsilon, \rho_1/\varepsilon, \dots, \rho_n/\varepsilon, F_\varepsilon \rangle$$

where realization function $F_\varepsilon([x_1]_{\varepsilon_1}, \dots, [x_n]_{\varepsilon_n}) \overset{df}{=} [F(x_1, \dots, x_n)]_\varepsilon$.

1. *Canonical homomorphism $f_{\bar{\varepsilon}} = (\varphi_{\varepsilon_1}, \dots, \varphi_{\varepsilon_n}, \psi_\varepsilon)$ where for any $i = 1, \dots, n$, $\varphi_{\varepsilon_i} : X_i \to X_i/\varepsilon_i$ and $\psi_\varepsilon : A \to A/\varepsilon$ is a homomorphism from game G onto game $G/\bar{\varepsilon}$.*

2. *Canonical homomorphism $f_{\overline{\varepsilon}}$ is strict if and only if congruence $\overline{\varepsilon}$ is str-congruence.*
3. *Canonical homomorphism $f_{\overline{\varepsilon}}$ is regular if and only if congruence $\overline{\varepsilon}$ is reg-congruence.*

The proof of this Theorem is based on Lemma 1.

Theorem 2. *Let G and Γ be two games with preference relations and a $(n+1)$ system of mappings $f = (\varphi_1, \dots, \varphi_n, \psi)$ be a homomorphism of game G onto game Γ. Then*

1. *for a $(n+1)$–tuple of equivalence relations $\overline{\varepsilon}_f = (\varepsilon_{\varphi_1}, \dots, \varepsilon_{\varphi_n}, \varepsilon_\psi)$, where $\overline{\varepsilon}_f$ is kernel of homomorphism f, consistency condition (20) holds. Hence, we can construct factor-game $G/\overline{\varepsilon}_f$;*
2. *there exists a $(n+1)$ system of mappings $\overline{\theta} = (\theta_1, \dots, \theta_n, \theta)$ from game $G/\overline{\varepsilon}_f$ into game Γ which is an isomorphic inclusion map from $G/\overline{\varepsilon}_f$ into Γ.*

Proof (of theorem).

1. Let the condition of the implication (20) be held. Since $\overline{\varepsilon}_f$ is kernel of homomorphism f then for any $i = 1, \dots, n$
$$x_i' \overset{\varepsilon_{\varphi_i}}{\equiv} x_i \Leftrightarrow \varphi_i(x_i') = \varphi_i(x_i),$$
$$a' \overset{\varepsilon_\psi}{\equiv} a \Leftrightarrow \psi(a') = \psi(a)$$
hold.
Let us prove that the equality $\psi(F(x_1', \dots, x_n')) = \psi(F(x_1, \dots, x_n))$ is true. Since f is homomorphism, then $\psi(F(x_1', \dots, x_n')) = \Phi(\varphi_1(x_1'), \dots, \varphi_n(x_n'))$ and $\psi(F(x_1, \dots, x_n)) = \Phi(\varphi_1(x_1), \dots, \varphi_n(x_n))$.
Thus, the equality $\Phi(\varphi_1(x_1'), \dots, \varphi_n(x_n')) = \Phi(\varphi_1(x_1), \dots, \varphi_n(x_n))$ is obvious.
By using Theorem 1 we can construct factor-game $G/\overline{\varepsilon}_f$ and canonical homomorphism is a homomorphism from game G onto game $G/\overline{\varepsilon}_f$.
2. We define isomorphic inclusion map $\overline{\theta} = (\theta_1, \dots, \theta_n, \theta)$ from game $G/\overline{\varepsilon}_f$ into game Γ by $\theta_i([x_i]_{\varepsilon_{\varphi_i}}) = \varphi_i(x_i)$ for any $i = 1, \dots, n$ and $\theta([a]_{\varepsilon_\psi}) = \psi(a)$.
First, we prove that all mappings $\theta_1, \dots, \theta_n, \theta$ are one-to-one functions. For example, we verify that θ_1 is one-to-one function. We write
$$\theta_1([x_1']_{\varepsilon_{\varphi_1}}) = \theta_1([x_1]_{\varepsilon_{\varphi_1}}) \Leftrightarrow \varphi_1(x_1') = \varphi_1(x_1) \Leftrightarrow x_1' \overset{\varepsilon_{\varphi_1}}{\equiv} x_1 \Leftrightarrow [x_1']_{\varepsilon_{\varphi_1}} = [x_1]_{\varepsilon_{\varphi_1}}.$$
Now we prove that $\overline{\theta} = (\theta_1, \dots, \theta_n, \theta)$ is a homomorphism from game $G/\overline{\varepsilon}_f$ into game Γ. Suppose $([a_1]_{\varepsilon_\psi}, [a_2]_{\varepsilon_\psi}) \in \rho_i/\dot{\varepsilon}_\psi$ then there exist $a_1' \overset{\varepsilon_\psi}{\equiv} a_1, a_2' \overset{\varepsilon_\psi}{\equiv} a_2$ (i.e. $\psi(a_1') = \psi(a_1)$, $\psi(a_2') = \psi(a_2)$) such that $(a_1', a_2') \in \rho_i$. Since f is a homomorphism, it follows that $(\psi(a_1'), \psi(a_2')) \in \sigma_i$, that is $(\psi(a_1), \psi(a_2)) \in \sigma_i$. By definition θ, we get $(\theta([a_1]_{\varepsilon_\psi}), \theta([a_2]_{\varepsilon_\psi})) \in \sigma_i$. Hence, condition of homomorphism (13) for $\overline{\theta}$ holds.
Now we verify condition (14). We write
$$\theta(F_\varepsilon([x_1]_{\varepsilon_{\varphi_1}}, \dots, [x_n]_{\varepsilon_{\varphi_n}})) = \theta([F(x_1, \dots, x_n)]_{\varepsilon_\psi}) = \psi(F(x_1, \dots, x_n)).$$
Since f is a homomorphism then
$$\psi(F(x_1, \dots, x_n)) = \Phi(\varphi_1(x_1), \dots, \varphi_n(x_n)) = \Phi(\theta_1([x_1]_{\varepsilon_{\varphi_1}}), \dots, \theta_n([x_n]_{\varepsilon_{\varphi_n}})).$$

Thus, $\overline{\theta} = (\theta_1, \dots, \theta_n, \theta)$ is an isomorphic inclusion map from factor-game $G/\overline{\varepsilon}_f$ into game Γ. This completes the proof of Theorem 2.

□

Theorem 3. *Let G be a game with preference relations of the form (1) and $\overline{\varepsilon}$ be a congruence in game G. A factor-game $G/\overline{\varepsilon}$ is a game with transitive preference structure if and only if for any $i = 1, \dots, n$ the condition*

$$\rho_i \circ \varepsilon \circ \rho_i \subseteq \varepsilon \circ \rho_i \circ \varepsilon$$

holds.

The proof of this Theorem is based on Lemma 2.

Theorem 4. *Let G be a game with transitive preference structure, $\overline{\varepsilon}$ be a congruence in game G. If for any $i = 1, \dots, n$ at least one of the conditions $\rho_i \circ \varepsilon \subseteq \varepsilon \circ \rho_i$ or $\varepsilon \circ \rho_i \subseteq \rho_i \circ \varepsilon$ or $\varepsilon \subseteq \rho_i$ holds then a factor-game $G/\overline{\varepsilon}$ is a game with transitive preference structure.*

The proof of this Theorem is based on Corollary 1 of Lemma 2.

Theorem 5. *Let G be a game with preference relations of the form (1) and $\overline{\varepsilon}$ be a congruence in game G. A factor-game $G/\overline{\varepsilon}$ is a game with acyclic preference structure if and only if for any $i = 1, \dots, n$, $\rho_i \cup \varepsilon$ is acyclic under ε, i.e. the implication*

$$a_0 \overset{\rho_i \cup \varepsilon}{\lesssim} a_1 \overset{\rho_i \cup \varepsilon}{\lesssim} \dots \overset{\rho_i \cup \varepsilon}{\lesssim} a_n \overset{\rho_i \cup \varepsilon}{\lesssim} a_0 \Rightarrow a_0 \overset{\varepsilon}{\equiv} a_1 \overset{\varepsilon}{\equiv} \dots \overset{\varepsilon}{\equiv} a_n$$

holds.

The proof of this Theorem is based on Lemma 3.
It is easy to see that the following results are true.

Theorem 6. *A $(n+1)$-tuple of equivalence relations $\overline{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_n, \varepsilon)$ in game G is kernel of some homomorphism from game G into a game with preference relations if and only if $\overline{\varepsilon}$ is a congruence in game G.*

Theorem 7. *A $(n+1)$-tuple of equivalence relations $\overline{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_n, \varepsilon)$ in game G is kernel of some strict homomorphism from game G into a game with preference relations if and only if $\overline{\varepsilon}$ is a str-congruence in game G.*

Theorem 8. *A $(n+1)$-tuple of equivalence relations $\overline{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_n, \varepsilon)$ in game G is kernel of some regular homomorphism from game G into a game with preference relations if and only if $\overline{\varepsilon}$ is a reg-congruence in game G.*

3.2. Equilibrium points in games with preference relations

Let G be a game with preference relations of the form (1). Any situation $x \in X$ can be given in the form $x = (x_i)_{i=1,\dots,n}$, where x_i is the i–th component of x. For $x'_i \in X_i$, we denote by $x \parallel x'_i$ a situation whose i–th component is x'_i and other components are the same as in x.

Definition 8. A situation $x \in X$ is called *an equilibrium point* in game G if such $i = 1, \dots, n$ and $x_i' \in X_i$ for which the condition

$$F(x) \overset{\rho_i}{<} F(x \parallel x_i')$$

holds do not exist.

Nash equilibrium point is an equilibrium point x for which the outcomes $F(x)$ and $F(x \parallel x_i')$ are comparable under preference relation ρ_i for any $i = 1, \dots, n$. In this case it satisfies

$$F(x \parallel x_i') \overset{\rho_i}{\lesssim} F(x).$$

Let K and $\mathcal{K}$ be two arbitrary classes of games with preference relations. Fix in these classes certain optimality concepts and let $Opt\, G$ be the set of optimal solutions of any game $G \in$ K, $Opt\, \Gamma$ the set of optimal solutions of any game $\Gamma \in \mathcal{K}$. If f is a homomorphism from G into Γ, then a correspondence between outcomes (and also between strategies and between situations) of these games is given; we denote this correspondence also by f.

Definition 9. A homomorphism f is said to be *covariant*, if f–image of any optimal solution in G is an optimal solution in Γ that is $f(Opt\, G) \subseteq Opt\, \Gamma$.

A homomorphism f is said to be *contravariant*, if f–preimage of any optimal solution in Γ is an optimal solution in G that is $f^{-1}(Opt\, \Gamma) \subseteq Opt\, G$.

Now suppose that for each $j \in J$ a homomorphism f_j of game $G \in$ K into some game $\Gamma_j \in \mathcal{K}$ is given.

Definition 10. A family of homomorphisms $(f_j)_{j \in J}$ is said to be *covariantly complete* if for each $x \in Opt\, G$ there exists such index $j \in J$ that $f_j(x) \in Opt\, \Gamma_j$.

A family of homomorphisms $(f_j)_{j \in J}$ is said to be *contravariantly complete* if the condition $f_j(x) \in Opt\, \Gamma_j$ for all $j \in J$ implies $x \in Opt\, G$.

Lemma 4. *1. A family of homomorphisms $(f_j)_{j \in J}$ is a covariantly complete family of contravariant homomorphisms if and only if*

$$Opt\, G = \bigcup_{j \in J} f_j^{-1}(Opt\, \Gamma_j). \tag{24}$$

2. A family of homomorphisms $(f_j)_{j \in J}$ is a conravariantly complete family of covariant homomorphisms if and only if

$$Opt\, G = \bigcap_{j \in J} f_j^{-1}(Opt\, \Gamma_j). \tag{25}$$

Proof (of lemma).

We prove, for example, assertion 1. Since for each $j \in J$, f_j is a conravariant homomorphism then by definition we get $f_j^{-1}(Opt\, \Gamma_j) \subseteq Opt\, G$. Hence, for arbitrary family of conravariant homomorphisms

$$\bigcup_{j \in J} f_j^{-1}(Opt\, \Gamma_j) \subseteq Opt\, G$$

is satisfied. Since $(f_j)_{j\in J}$ is covariantly complete family of homomorphisms then there exists such index $j \in J$ that $f_j(Opt\,G) \subseteq Opt\,\Gamma_j$, i.e. $Opt\,G \subseteq f_j^{-1}(Opt\,\Gamma_j)$. Hence

$$Opt\,G \subseteq \bigcup_{j\in J} f_j^{-1}(Opt\,\Gamma_j).$$

Thus,

$$Opt\,G = \bigcup_{j\in J} f_j^{-1}(Opt\,\Gamma_j).$$

It is easy to verify that the converse is true. This completes the proof of Lemma 4. □

Now consider the case when an optimality concept is the concept of equilibrium. It is easy to verify that the following result is true.

Theorem 9. *1. For equilibrium any strict homomorphism is a contravariant homomorphism.*

2. For equilibrium any regular homomorphism is a covariant homomorphism.

3. For Nash equilibrium any homomorphism "onto" is a covariant homomorphism.

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